ON CERTAIN DUALITY OF NÉRON-SEVERI LATTICES OF SUPERSINGULAR $K3$ SURFACES AND ITS APPLICATION TO GENERIC SUPERSINGULAR $K3$ SURFACES

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ABSTRACT. Let X and Y be supersingular $K3$ surfaces defined over an algebraically closed field. Suppose that the sum of their Artin invariants is 11. Then there exists a certain duality between their Néron-Severi lattices. We investigate geometric consequences of this duality. As an application, we classify genus one fibrations on supersingular $K3$ surfaces with Artin invariant 10 in characteristic 2 and 3, and give a set of generators of the automorphism group of the nef cone of these supersingular $K3$ surfaces. The difference between the automorphism group of a supersingular $K3$ surface X and the automorphism group of its nef cone is determined by the period of X. We define the notion of genericity for supersingular $K3$ surfaces in terms of the period, and prove the existence of generic supersingular $K3$ surfaces in odd characteristics for each Artin invariant larger than 1.

1. INTRODUCTION

A $K3$ surface defined over an algebraically closed field k is said to be *supersingular* (in the sense of Shioda) if its Picard number is 22. Supersingular $K3$ surfaces exist only when k is of positive characteristic. Let X be a supersingular $K3$ surface in characteristic $p > 0$, and let S_X denote its Néron-Severi lattice. Artin [1] showed that the discriminant group of S_X is a p-elementary abelian group of rank 2σ, where σ is an integer such that $1 \leq \sigma \leq 10$. This integer σ is called the *Artin invariant* of X. The isomorphism class of the lattice S_X depends only on p and σ (Rudakov and Shafarevich [27]). Moreover supersingular $K3$ surfaces with Artin invariant σ form a $(\sigma-1)$-dimensional family, and a supersingular $K3$ surface with Artin invariant 1 in characteristic p is unique up to isomorphisms (Ogus [24], [25], Rudakov and Shafarevich [27]).

Recently many studies of supersingular $K3$ surfaces in small characteristics with Artin invariant 1 have appeared. For example, for $p = 2$, Dolgachev and Kondo [8], Katsura and Kondo [12], Elkies and Schütt [11]; for $p = 3$, Katsura and Kondo [13], Kondo and Shimada [18], Sengupta [28]; and for $p = 5$, Shimada [33]. On the other hand, geometric properties of supersingular $K3$ surfaces with big Artin invariant are not so much known (e.g. Rudakov and Shafarevich [26], [27], Shioda [35], Shimada [31], [32]).

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In this paper, we present some methods to investigate supersingular $K3$ surfaces with big Artin invariant by means of the following simple observation. Let $X_{p,\sigma}$ be a supersingular $K3$ surface in characteristic p with Artin invariant σ, and let $S_{p,\sigma}$ denote its Néron-Severi lattice.

Lemma 1.1. *Suppose that $\sigma + \sigma' = 11$. Then $S_{p,\sigma'}$ is isomorphic to $S^{\vee}_{p,\sigma}(p)$, where $S^{\vee}_{p,\sigma}(p)$ is the lattice obtained from the dual lattice $S^{\vee}_{p,\sigma}$ of $S_{p,\sigma}$ by multiplying the symmetric bilinear form with p.*

Lemma 1.1 is proved in Section 3. We use this duality between $S_{p,\sigma}$ and $S_{p,\sigma'}$ in the study of genus one fibrations and the automorphism groups of supersingular $K3$ surfaces.

First, we apply Lemma 1.1 to the classification of genus one fibrations. Note that the Néron-Severi lattice S_Y of a $K3$ surface Y is a hyperbolic lattice. The orthogonal group $\mathrm{O}(S_Y)$ of S_Y contains the stabilizer subgroup $\mathrm{O}^+(S_Y)$ of a positive cone of $S_Y \otimes \mathbb{R}$ as a subgroup of index 2.

Definition 1.2. Let Y be a $K3$ surface, and let $\phi : Y \to \mathbb{P}^1$ be a genus one fibration. We denote by $f_\phi \in S_Y$ the class of a fiber of ϕ. Let $\psi : Y \to \mathbb{P}^1$ be another genus one fibration on Y. We say that ϕ and ψ are Aut-*equivalent* if there exist $g \in \mathrm{Aut}(Y)$ and $\bar{g} \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\phi \circ g = \bar{g} \circ \psi$ holds, while we say that ϕ and ψ are *lattice equivalent* if there exists $g \in \mathrm{O}^+(S_Y)$ such that $f_\phi^g = f_\psi$. We denote by $\mathbb{E}(Y)$ the set of lattice equivalence classes of genus one fibrations on Y, and by $[\phi] \in \mathbb{E}(Y)$ the lattice equivalence class containing ϕ.

Many combinatorial properties of a genus one fibration $\phi : Y \to \mathbb{P}^1$ depend only on the lattice equivalence class $[\phi]$. See Proposition 4.1. Moreover, when $\sigma = 10$, the classification of genus one fibrations by Aut-equivalence seems to be too fine, as is suggested by Proposition 9.2. Therefore, we concentrate upon the study of lattice equivalence classes.

Using Lemma 1.1, we prove the following:

Theorem 1.3. *Suppose that $\sigma + \sigma' = 11$. Then there exists a canonical one-to-one correspondence*

$$[\phi] \mapsto [\phi']$$

between $\mathbb{E}(X_{p,\sigma})$ and $\mathbb{E}(X_{p,\sigma'})$.

We say that a genus one fibration is *Jacobian* if it admits a section.

Theorem 1.4. *Suppose that a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ is a Jacobian fibration, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a genus one fibration on $X_{p,\sigma'}$ with $\sigma' = 11 - \sigma$ such that $[\phi'] \in \mathbb{E}(X_{p,\sigma'})$ corresponds to $[\phi] \in \mathbb{E}(X_{p,\sigma})$ by Theorem 1.3. Then ϕ' does not admit a section.*

Elkies and Schütt [11] proved the following:

Theorem 1.5 ([11]). *Any genus one fibration on $X_{p,1}$ admits a section.*

Therefore we obtain the following:

Corollary 1.6. *There exist no Jacobian fibrations on* $X_{p,10}$.

By an ADE*-type*, we mean a finite formal sum of the symbols A_i $(i \geq 1)$, D_i $(j \geq 4)$ and E_k $(k = 6, 7, 8)$ with non-negative integer coefficients. For a genus one fibration $\phi : Y \to \mathbb{P}^1$ on a $K3$ surface Y, we have the ADE-type of reducible fibers of ϕ. This ADE-type depends only on the lattice equivalence class $[\phi] \in \mathbb{E}(Y)$ (see Proposition 4.1). Therefore we can use $R_{[\phi]}$ to denote the ADE-type of the reducible fibers of ϕ.

From the classification of lattice equivalence classes of genus one fibrations of $X_{2,1}$ by Elkies and Schütt [11], and that of $X_{3,1}$ by Sengupta [28], we obtain the classification of lattice equivalence classes of genus one fibrations on $X_{2,10}$ and $X_{3,10}$. In particular, we obtain the list of ADE-types $R_{[\phi']}$ of the reducible fibers of genus one fibrations ϕ' on $X_{2,10}$ and $X_{3,10}$. See Theorems 4.8 and 4.9.

In Elkies and Schütt [11] and Sengupta [28] mentioned above, they also obtained explicit defining equations of the Jacobian fibrations. Besides [11] and [28], there have been many works on the classification of Aut-equivalence classes and lattice equivalence classes of Jacobian fibrations on a $K3$ surface (e.g. Oguiso [23], Nishiyama [22], Shimada and Zhang [34], Shimada [29], Kloosterman [16]). In particular, the lattice equivalence classes of all *extremal* (quasi-) elliptic fibrations (i.e., Jacobian fibrations with Mordell-Weil rank zero) on supersingular $K3$ surfaces are classified in Shimada [30].

As the second application of Lemma 1.1, we investigate the automorphism group of the nef cone of a supersingular $K3$ surface. For a $K3$ surface Y, let $\mathrm{Nef}(Y) \subset S_Y \otimes \mathbb{R}$ denote the nef cone. We denote by $Aut(\mathrm{Nef}(Y)) \subset \mathrm{O}^+(S_Y)$ the group of isometries of S_Y that preserve $\mathrm{Nef}(Y)$. Since $\mathrm{Aut}(X_{p,\sigma})$ acts on $S_{p,\sigma}$ faithfully (Rudakov and Shafarevich [27, Section 8, Proposition 3]), we have

$$\mathrm{Aut}(X_{p,\sigma}) \subset Aut(\mathrm{Nef}(X_{p,\sigma})) \subset \mathrm{O}^+(S_{p,\sigma}). \tag{1.1}$$

Using the description of $\mathrm{Aut}(X_{2,1})$ by Dolgachev and Kondo [8], and that of $\mathrm{Aut}(X_{3,1})$ by Kondo and Shimada [18], we give a set of generators of $Aut(\mathrm{Nef}(X_{2,10}))$ and $Aut(\mathrm{Nef}(X_{3,10}))$ in Theorems 6.4 and 6.9, respectively.

Suppose that p is odd. We fix a lattice N isomorphic to $S_{p,\sigma}$. Then a quadratic space (N_0, q_0) of dimension 2σ over $\mathbb{F}_p$ is defined by

$$N_0 := pN^\vee/pN \quad \text{and} \quad q_0(px \bmod pN) := px^2 \bmod p \quad (x \in N^\vee). \tag{1.2}$$

We fix a *marking* $\eta : N \xrightarrow{\sim} S_{p,\sigma}$ for a supersingular $K3$ surface $X := X_{p,\sigma}$ defined over k. Then $Aut(\mathrm{Nef}(X))$ acts on (N_0, q_0), and the *period* $K_{(X,\eta)} \subset N_0 \otimes k$ of the marked supersingular $K3$ surface (X, η) is defined as the Frobenius pull-back of the kernel of the natural homomorphism

$$N \otimes k \to S_X \otimes k \to H^2_{\mathrm{DR}}(X/k)$$

(see Section 7). In virtue of Torelli theorem for supersingular $K3$ surfaces by Ogus [24], [25], the subgroup $\mathrm{Aut}(X)$ of $Aut(\mathrm{Nef}(X))$ is equal to the stabilizer subgroup of the period $K_{(X,\eta)}$. In particular, the index of $\mathrm{Aut}(X_{p,\sigma})$ in $Aut(\mathrm{Nef}(X_{p,\sigma}))$ is finite. On the other hand, the classification of 2-reflective lattices due to Nikulin [21] implies that $Aut(\mathrm{Nef}(X_{p,\sigma}))$ is infinite. Hence, at least when p is odd, $\mathrm{Aut}(X_{p,\sigma})$ is an infinite group. See Sections 5 and 7 for details. Moreover, Lieblich and Maulik [19] proved that, if $p > 2$, then $\mathrm{Aut}(X_{p,\sigma})$ is finitely generated and its action on $\mathrm{Nef}(X_{p,\sigma})$ has a rational polyhedral fundamental domain.

We say that a supersingular $K3$ surface X is *generic* if there exists a marking $\eta : N \xrightarrow{\sim} S_X$ such that the isometries of (N_0, q_0) that preserve the period $K_{(X,\eta)} \subset N_0 \otimes k$ are only scalar multiplications (see Definition 7.5). Using the surjectivity of the period mapping proved by Ogus [25], we prove the following:

Theorem 1.7. *Suppose that p is odd and $\sigma > 1$. Then there exist an algebraically closed field k and a supersingular $K3$ surface X with Artin invariant σ defined over k that is generic.*

Suppose that $X_{3,10}$ is generic. From the generators of $Aut(\mathrm{Nef}(X_{3,10}))$ given in Theorem 6.9, we can obtain a finite set of generators of $\mathrm{Aut}(X_{3,10})$. However, the computation would be very heavy. See Remarks 7.7 and 7.8.

As the third application, we show by an example that a lattice equivalence class of genus one fibrations on $X_{3,10}$ can contain a very large number of Aut-equivalence classes, provided that $X_{3,10}$ is generic. An analogous result for a generic complex Enriques surface was obtained by Barth and Peters [2].

This paper is organized as follows. In Section 2, we fix notation and terminologies about lattices and $K3$ surfaces. In Section 3, Lemma 1.1 is proved by means of the fundamental results of Rudakov and Shafarevich [27] on the Néron-Severi lattices of supersingular $K3$ surfaces. In Section 4, we study genus one fibrations on supersingular $K3$ surfaces, and prove Theorems 1.3 and 1.4. Moreover, the bijections $\mathbb{E}(X_{p,1}) \cong \mathbb{E}(X_{p,10})$ for $p = 2$ and 3 are given explicitly in Tables 4.1 and 4.2. In Section 5, we review the classical method to investigate the orthogonal group of a hyperbolic lattice by means of a chamber decomposition of the associated hyperbolic space, and fix some notation and terminologies. We then apply this method to the nef cone of a supersingular $K3$ surface. In Section 6, we give a set of generators of $Aut(\mathrm{Nef}(X_{2,10}))$ and $Aut(\mathrm{Nef}(X_{3,10}))$. In Section 7, we review the theory of the period mapping and Torelli theorem for supersingular $K3$ surfaces in odd characteristics due to Ogus [24], [25], and describe the relation between $\mathrm{Aut}(X_{p,\sigma})$ and $Aut(\mathrm{Nef}(X_{p,\sigma}))$. In Section 8, we prove Theorem 1.7. In the last section, we illustrate that the number of Aut-equivalence classes of genus one fibrations on $X_{3,10}$ is intractably large if $X_{3,10}$ is generic.

Convention. We use Aut to denote automorphism groups of lattice theoretic objects, and Aut to denote automorphism groups of geometric objects on $K3$ surfaces.

2. Preliminaries

2.1. **Lattices.** A $\mathbb{Q}$-*lattice* is a free $\mathbb{Z}$-module L of finite rank equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_L : L \times L \to \mathbb{Q}$. We omit the subscript L in $\langle \cdot, \cdot \rangle_L$ if no confusions will occur. If $\langle \cdot, \cdot \rangle_L$ takes values in $\mathbb{Z}$, we say that L is a *lattice*. For $x \in L \otimes \mathbb{R}$, we call $x^2 := \langle x, x \rangle$ the *norm* of x. A vector in $L \otimes \mathbb{R}$ of norm n is sometimes called an n-*vector*. A lattice L is said to be *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in L$.

Let L be a free $\mathbb{Z}$-module of finite rank. A submodule M of L is *primitive* if L/M is torsion free. A non-zero vector $v \in L$ is *primitive* if the submodule of L generated by v is primitive.

Let L be a $\mathbb{Q}$-lattice of rank r. For a non-zero rational number m, we denote by $L(m)$ the free $\mathbb{Z}$-module L with the symmetric bilinear form $\langle x, y \rangle_{L(m)} := m\langle x, y \rangle_L$. The signature of L is the signature of the real quadratic space $L \otimes \mathbb{R}$. We say that L is *negative definite* if the signature of L is $(0, r)$, and L is *hyperbolic* if the signature is $(1, r-1)$. A *Gram matrix* of L is an $r \times r$ matrix with entries $\langle e_i, e_j \rangle$, where $\{e_1, \ldots, e_r\}$ is a basis of L. The determinant of a Gram matrix of L is called the *discriminant* of L.

For an even lattice L, the set of (-2)-vectors is denoted by $\mathcal{R}(L)$. A *negative* definite even lattice L is called a *root lattice* if L is generated by $\mathcal{R}(L)$. Let R be an ADE-type. The root lattice of type R is the root lattice whose Gram matrix is the Cartan matrix of type R. Suppose that L is negative definite. By the ADE-*type of* $\mathcal{R}(L)$, we mean the ADE-type of the root sublattice $\langle \mathcal{R}(L) \rangle$ of L generated by $\mathcal{R}(L)$. (See, for example, Ebeling [10] for the classification of root lattices.)

Let L be an even lattice and let $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ be identified with a submodule of $L \otimes \mathbb{Q}$ with the extended symmetric bilinear form. We call this $\mathbb{Q}$-lattice $L^\vee$ the *dual lattice* of L. The *discriminant group* of L is defined to be the quotient $L^\vee/L$, and is denoted by A_L. We define the *discriminant quadratic form* of L

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$$

by $q_L(x \bmod L) := x^2 \bmod 2\mathbb{Z}$. The order of A_L is equal to the discriminant of L up to sign. We say that L is *unimodular* if A_L is trivial, while L is p-*elementary* if A_L is p-elementary. An even 2-elementary lattice L is said to be *of type* I if $q_L(x \bmod L) \in \mathbb{Z}/2\mathbb{Z}$ holds for any $x \in L^\vee$. Note that L is p-elementary if and only if pG_L^{-1} is an integer matrix, where G_L is a Gram matrix of L.

Let $\mathrm{O}(L)$ denote the orthogonal group of a lattice L, that is, the group of isomorphisms of L preserving $\langle \cdot, \cdot \rangle_L$. We assume that $\mathrm{O}(L)$ acts on L from *right*, and the action of $g \in \mathrm{O}(L)$ on $v \in L \otimes \mathbb{R}$ is denoted by $v \mapsto v^g$. Similarly $\mathrm{O}(q_L)$ denotes the group of isomorphisms of A_L preserving q_L. There is a natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(q_L)$.

Let L be a hyperbolic lattice. A *positive cone* of L is one of the two connected components of

$$\{\, x \in L \otimes \mathbb{R} \mid x^2 > 0 \,\}.$$

Let $\mathcal{P}_L$ be a positive cone of L. We denote by $\mathrm{O}^+(L)$ the group of isometries of L that preserve $\mathcal{P}_L$. We have $\mathrm{O}(L) = \mathrm{O}^+(L) \times \{\pm 1\}$. For a vector $v \in L \otimes \mathbb{R}$ with $v^2 < 0$, we put

$$(v)^\perp := \{\, x \in \mathcal{P}_L \mid \langle x, v \rangle = 0 \,\},$$

which is a real hyperplane of $\mathcal{P}_L$. An isometry $g \in \mathrm{O}^+(L)$ is called a *reflection with respect to* v or a *reflection into* $(v)^\perp$ if g is of order 2 and fixes each point of $(v)^\perp$. An element r of $\mathcal{R}(L)$ defines a reflection

$$s_r : x \mapsto x + \langle x, r \rangle r$$

with respect to r. We denote by $W(L)$ the subgroup of $\mathrm{O}^+(L)$ generated by the set of these reflections $\{s_r \mid r \in \mathcal{R}(L)\}$. It is obvious that $W(L)$ is normal in $\mathrm{O}^+(L)$.

2.2. $K3$ **surfaces.** Let Y be a $K3$ surface, and let S_Y denote the Néron-Severi lattice of Y. A smooth rational curve on Y is called a (-2)-*curve*. We denote by $\mathcal{P}(Y) \subset S_Y \otimes \mathbb{R}$ the positive cone containing an ample class of Y. Recall that the *nef cone* $\mathrm{Nef}(Y)$ of Y is defined by

$$\mathrm{Nef}(Y) := \{\, x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \ \text{ for any curve } C \text{ on } Y \,\},$$

where $[C] \in S_Y$ is the class of a curve $C \subset Y$. Then $\mathrm{Nef}(Y)$ is contained in the closure $\overline{\mathcal{P}}(Y)$ of $\mathcal{P}(Y)$ in $S_Y \otimes \mathbb{R}$. We put

$$\mathrm{Nef}^\circ(Y) := \mathrm{Nef}(Y) \cap \mathcal{P}(Y) = \{\, x \in \mathrm{Nef}(Y) \mid x^2 > 0 \,\}.$$

The following is well-known. See, for example, Rudakov and Shafarevich [27, Section 3].

Proposition 2.1. (1) *We have*

$$\mathrm{Nef}(Y) = \{\, x \in S_Y \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \ \textit{for any } (-2)\textit{-curve } C \textit{ on } Y \,\}.$$

(2) *If* $v \in S_Y$ *is contained in* $\overline{\mathcal{P}}(Y)$*, then there exists* $g \in W(S_Y)$ *such that* $v^g \in \mathrm{Nef}(Y)$.

3. Néron-Severi lattices of supersingular $K3$ surfaces

Let $X_{p,\sigma}$ be a supersingular $K3$ surface with Artin invariant σ in characteristic $p > 0$. Then the isomorphism class of the Néron-Severi lattice $S_{p,\sigma}$ of $X_{p,\sigma}$ depends only on p and σ, and is characterized as follows (see Rudakov-Shafarevich [27, Sections 3,4 and 5] for the proof).

Theorem 3.1 ([27])**.** (1) *The lattice* $S_{p,\sigma}$ *is an even hyperbolic* p*-elementary lattice of rank* 22 *with discriminant* $-p^{2\sigma}$*. Moreover,* $S_{2,\sigma}$ *is of type* I.

(2) *Suppose that* N *is an even hyperbolic* p*-elementary lattice of rank* 22 *with discriminant* $-p^{2\sigma}$*. When* $p = 2$*, we further assume that* N *is of type* I*. Then* N *is isomorphic to* $S_{p,\sigma}$*.*

Using this theorem, we can prove Lemma 1.1 easily.

Proof of Lemma 1.1 . It is enough to show that $S^{\vee}_{p,\sigma}(p)$ is an even p-elementary lattice of discriminant $-p^{2\sigma'}$, and that $S^{\vee}_{2,\sigma}(2)$ is of type I. Since $S_{p,\sigma}$ is p-elementary, we have $pS^{\vee}_{p,\sigma} \subset S_{p,\sigma}$. Therefore $S^{\vee}_{p,\sigma}(p)$ is a lattice. Let $G_{p,\sigma}$ be a Gram matrix of $S_{p,\sigma}$. Then the determinant of the Gram matrix $pG^{-1}_{p,\sigma}$ of $S^{\vee}_{p,\sigma}(p)$ is equal to $p^{22} \cdot \det(G_{p,\sigma})^{-1} = -p^{2\sigma'}$. Therefore the discriminant of $S^{\vee}_{p,\sigma}(p)$ is $-p^{2\sigma'}$. Since $p(pG^{-1}_{p,\sigma})^{-1} = G_{p,\sigma}$ is an integer matrix, $S^{\vee}_{p,\sigma}(p)$ is p-elementary. Suppose that p is odd. Then, for any $\xi \in S^{\vee}_{p,\sigma}$, we have $p\xi \in S_{p,\sigma}$ and hence $\langle p\xi, p\xi\rangle_{S_{p,\sigma}} = p\langle \xi, \xi\rangle_{S^{\vee}_{p,\sigma}(p)}$ is even. Therefore $S^{\vee}_{p,\sigma}(p)$ is even. Suppose that $p = 2$. Then, for any $\xi \in S^{\vee}_{2,\sigma}$, we have $\langle \xi, \xi\rangle_{S^{\vee}_{2,\sigma}} \in \mathbb{Z}$, because $S_{2,\sigma}$ is of type I. Therefore $S^{\vee}_{2,\sigma}(2)$ is even. Moreover, for any $\eta \in (S^{\vee}_{2,\sigma}(2))^{\vee} = S_{2,\sigma}(1/2)$, we have $\langle \eta, \eta\rangle_{S_{2,\sigma}(1/2)} \in \mathbb{Z}$, because $S_{2,\sigma}$ is even. Therefore $S^{\vee}_{2,\sigma}(2)$ is of type I. □

Corollary 3.2. *Suppose that $\sigma + \sigma' = 11$. Then there exists an embedding of $\mathbb{Z}$-modules*

$$j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$$

that induces an isomorphism of lattices $S^{\vee}_{p,\sigma}(p) \cong S_{p,\sigma'}$. This embedding induces an isomorphism

$$j_* : \mathrm{O}(S_{p,\sigma}) \xrightarrow{\sim} \mathrm{O}(S_{p,\sigma'}).$$

Moreover such an embedding j is unique up to compositions with elements of $\mathrm{O}(S_{p,\sigma'})$.

Remark 3.3. Suppose that $v \in S_{p,\sigma}$ satisfies $v^2 \geq 0$. Then, by Proposition 2.1(2), we can choose $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ in Corollary 3.2 in such a way that $j(v)$ is contained in $\mathrm{Nef}(X_{p,\sigma'})$.

4. Genus one fibrations

Let Y be a $K3$ surface defined over an algebraically closed field of arbitrary characteristic. Recall that $f_\phi \in S_Y$ is the class of a fiber of a genus one fibration $\phi : Y \to \mathbb{P}^1$, $\mathbb{E}(Y)$ is the set of lattice equivalence classes of genus one fibrations on Y, and $[\phi] \in \mathbb{E}(Y)$ is the class containing ϕ. We summarize properties of a genus one fibration $\phi : Y \to \mathbb{P}^1$ that depends only on the class $[\phi]$. See Sections 3 and 4 of Rudakov and Shafarevich [27], and Shioda [36] for the proof.

(1) The fibration ϕ admits a section if and only if there exists a (-2)-vector $z \in S_Y$ such that $\langle f_\phi, z\rangle = 1$.

(2) Note that $f_\phi \in S_Y$ is primitive of norm 0, and that $\langle f_\phi\rangle^\perp/\langle f_\phi\rangle$ is an even negative definite lattice, where $\langle f_\phi\rangle^\perp$ is the orthogonal complement in S_Y of the lattice $\langle f_\phi\rangle$ of rank 1 generated by f_ϕ. The ADE-type of the reducible fibers of ϕ is equal to the ADE-type of the set $\mathcal{R}(\langle f_\phi\rangle^\perp/\langle f_\phi\rangle)$ of (-2)-vectors in $\langle f_\phi\rangle^\perp/\langle f_\phi\rangle$.

(3) Suppose that ϕ admits a section $Z \subset Y$. Then f_ϕ and $[Z] \in S_Y$ generate an even unimodular hyperbolic lattice U_ϕ of rank 2 in S_Y. Let K_ϕ denote the orthogonal complement of U_ϕ in S_Y. We have an orthogonal direct-sum decomposition

$$S_Y = U_\phi \oplus K_\phi,$$

and the lattice $\langle f_\phi\rangle^\perp/\langle f_\phi\rangle$ is isomorphic to K_ϕ. Then the Mordell-Weil group of ϕ is isomorphic to $K_\phi/\langle\mathcal{R}(K_\phi)\rangle$, where $\langle\mathcal{R}(K_\phi)\rangle$ is the root sublattice of K_ϕ generated by the (-2)-vectors in K_ϕ.

(4) In characteristic 2 or 3, ϕ is quasi-elliptic if and only if $\langle\mathcal{R}(K_\phi)\rangle$ is p-elementary of rank 20.

As a corollary, we obtain the following:

Proposition 4.1. *Suppose that genus one fibrations $\phi : Y \to \mathbb{P}^1$ and $\psi : Y \to \mathbb{P}^1$ on Y are lattice-equivalent. Then the following hold.*

(1) *The fibration ϕ admits a section if and only if so does ψ.*

(2) *The ADE-type of the reducible fibers of ϕ is equal to that of ψ.*

(3) *Suppose that ϕ and ψ admit a section. Then the Mordell-Weil groups for ϕ and for ψ are isomorphic.*

(4) *In characteristic 2 or 3, the fibration ϕ is quasi-elliptic if and only if so is ψ.*

Definition 4.2. For a hyperbolic lattice S, we put

$$\widetilde{\mathcal{E}}(S) := \{v \in S\otimes\mathbb{Q} \mid v \neq 0, v^2 = 0\}/\mathbb{Q}^\times \quad \text{and} \quad \mathcal{E}(S) := \widetilde{\mathcal{E}}(S)/\mathrm{O}(S).$$

Remark 4.3. Let a positive cone $\mathcal{P}_S$ of S be fixed. Then each element of $\widetilde{\mathcal{E}}(S)$ is represented by a unique non-zero primitive vector $v \in S$ of norm 0 that is contained in the closure $\overline{\mathcal{P}}_S$ of $\mathcal{P}_S$ in $S\otimes\mathbb{R}$.

In Sections 3 and 4 of Rudakov and Shafarevich [27], the following is proved:

Proposition 4.4. *Let v be a non-zero vector of S_Y. Then there exists a genus one fibration $\phi : Y \to \mathbb{P}^1$ such that $v = f_\phi$ if and only if v is primitive, $v^2 = 0$, and $v \in \mathrm{Nef}(Y)$.*

Combining Propositions 2.1, 4.4 and Remark 4.3, we obtain the following:

Corollary 4.5. *The map $\phi \mapsto f_\phi$ induces a bijection from $\mathbb{E}(Y)$ to $\mathcal{E}(S_Y)$.*

From now on, we work over an algebraically closed field of characteristic $p > 0$.

Proof of Theorem 1.3. Consider the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ in Corollary 3.2. Then j is unique up to $\mathrm{O}(S_{p,\sigma'})$, induces a bijection from $\widetilde{\mathcal{E}}(S_{p,\sigma})$ to $\widetilde{\mathcal{E}}(S_{p,\sigma'})$, and induces an isomorphism $\mathrm{O}(S_{p,\sigma}) \cong \mathrm{O}(S_{p,\sigma'})$. Hence it induces a canonical bijection from $\mathcal{E}(S_{p,\sigma})$ to $\mathcal{E}(S_{p,\sigma'})$. $\square$

We denote this canonical one-to-one correspondence from $\mathbb{E}(X_{p,\sigma})$ to $\mathbb{E}(X_{p,\sigma'})$ by $[\phi] \mapsto [\phi']$.

Remark 4.6. Let a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ be given, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a representative of $[\phi']$. Then we can choose the embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ inducing $S^\vee_{p,\sigma}(p) \cong S_{p,\sigma'}$ in such a way that $j(f_\phi)$ is a scalar multiple of $f_{\phi'}$ by a positive integer.

Theorem 4.7. *Suppose that a genus one fibration $\phi : X_{p,\sigma} \to \mathbb{P}^1$ admits a section. Then the corresponding genus one fibration $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ does not admit a section. Moreover the ADE-type of the reducible fibers of ϕ' is equal to the ADE-type of $\mathcal{R}(K^\vee_\phi(p))$.*

No.	R_N	$\sigma=1$			$\sigma=10$
		$R_{[\phi]}$	$\mathrm{MW}_{\mathrm{tor}}$	$\mathrm{rank}(\mathrm{MW})$	$R_{[\phi']}$
1	$4A_5+D_4$	$4A_5$	$[3,6]$	0	0
2	$6D_4$	$5D_4$	$[2,2,2,2]$	0	0
3	$2A_7+2D_5$	$2A_7+D_5$	$[8]$	1	A_1
4	$2A_9+D_6$	$2A_1+2A_9$	$[10]$	0	$2A_1$
5	$4D_6$	$2A_1+3D_6$	$[2,2,2]$	0	$2A_1$
6	$A_{11}+D_7+E_6$	$A_{11}+D_7$	$[4]$	2	A_2
7	$A_{11}+D_7+E_6$	$A_3+A_{11}+E_6$	$[6]$	0	$3A_1$
8	$4E_6$	$3E_6$	$[3]$	2	A_2
9	$3D_8$	D_4+2D_8	$[2,2]$	0	$4A_1$
10	$A_{15}+D_9$	$A_{15}+D_5$	$[4]$	0	$5A_1$
11	$A_{17}+E_7$	$3A_1+A_{17}$	$[6]$	0	A_3
12	$D_{10}+2E_7$	$3A_1+D_{10}+E_7$	$[2,2]$	0	A_3
13	$D_{10}+2E_7$	D_6+2E_7	$[2]$	0	$6A_1$
14	$2D_{12}$	D_8+D_{12}	$[2]$	0	$8A_1$
15	$D_{16}+E_8$	D_4+D_{16}	$[2]$	0	D_4
16	$D_{16}+E_8$	$D_{12}+E_8$	$[1]$	0	$12A_1$
17	$3E_8$	D_4+2E_8	$[1]$	0	D_4
18	D_{24}	D_{20}	$[1]$	0	$20A_1$

TABLE 4.1. Genus one fibrations on $X_{2,1}$ and $X_{2,10}$

Proof. Let $z\in S_{p,\sigma}$ be the class of a section of ϕ. We choose $j:S_{p,\sigma}\hookrightarrow S_{p,\sigma'}$ as in Remark 4.6. Since $U_\phi^\vee=U_\phi$, we see that $j(f_\phi)$ is primitive in $S_{p,\sigma'}$ and hence $j(f_\phi)=f_{\phi'}$. We have an isomorphism $S_{p,\phi'}\cong U_\phi(p)\oplus K_\phi^\vee(p)$ such that $f_{\phi'}$ and $j(z)$ form a basis of $U_\phi(p)$. Since there are no vectors $v\in U_\phi(p)\oplus K_\phi^\vee(p)$ with $\langle v,f_{\phi'}\rangle=1$, the fibration ϕ' does not admit a section. Moreover the lattice $\langle f_{\phi'}\rangle^\perp/\langle f_{\phi'}\rangle$ is isomorphic to $K_\phi^\vee(p)$. □

The list of lattice equivalence classes of genus one fibrations on $X_{2,1}$ and $X_{3,1}$ were obtained by Elkies and Schütt [11] and by Sengupta [28], respectively. From their results, we obtain the following results on supersingular $K3$ surfaces with Artin invariant 10:

Theorem 4.8. *There exist* 18 *lattice equivalence classes of genus one fibrations on* $X_{2,10}$. *The* ADE*-type* $R_{[\phi']}$ *of the reducible fibers of each* $[\phi']\in\mathbb{E}(X_{2,10})$ *is given at the last column of Table 4.1.*

Theorem 4.9. *There exist* 52 *lattice equivalence classes of genus one fibrations on* $X_{3,10}$. *The* ADE*-type* $R_{[\phi']}$ *of the reducible fibers of each* $[\phi']\in\mathbb{E}(X_{3,10})$ *is given at the last column of Table 4.2.*

No.	R_N	$\sigma = 1$			$\sigma = 10$
		$R_{[\phi]}$	$\mathrm{MW}_{\mathrm{tor}}$	rank(MW)	$R_{[\phi']}$
1	$12A_2$	$10A_2$	$[3,3,3,3]$	0	0
2	$8A_3$	$6A_3$	$[4,4]$	2	0
3	$6A_4$	$2A_1 + 4A_4$	$[5]$	2	0
4	$6D_4$	$4D_4$	$[2,2]$	4	0
5	$4A_5 + D_4$	$A_2 + 3A_5$	$[3]$	3	0
6	$4A_5 + D_4$	$3A_5 + D_4$	$[2,6]$	1	A_1
7	$4A_5 + D_4$	$2A_2 + 2A_5 + D_4$	$[2]$	2	0
8	$4A_6$	$3A_6$	$[7]$	2	A_1
9	$4A_6$	$2A_3 + 2A_6$	$[1]$	2	0
10	$2A_7 + 2D_5$	$4A_1 + 2A_7$	$[2,4]$	2	0
11	$2A_7 + 2D_5$	$A_1 + A_7 + 2D_5$	$[4]$	2	A_1
12	$2A_7 + 2D_5$	$2A_1 + A_4 + A_7 + D_5$	$[2]$	2	0
13	$2A_7 + 2D_5$	$2A_4 + 2D_5$	$[1]$	2	0
14	$3A_8$	$A_2 + 2A_8$	$[3]$	2	A_1
15	$3A_8$	$2A_5 + A_8$	$[1]$	2	0
16	$4D_6$	$3D_6$	$[2,2]$	2	$2A_1$
17	$4D_6$	$2A_3 + 2D_6$	$[2,2]$	2	0
18	$2A_9 + D_6$	$2A_9$	$[5]$	2	$2A_1$
19	$2A_9 + D_6$	$A_3 + A_9 + D_6$	$[2]$	2	A_1
20	$2A_9 + D_6$	$A_3 + A_6 + A_9$	$[1]$	2	0
21	$2A_9 + D_6$	$2A_6 + D_6$	$[1]$	2	0
22	$4E_6$	$A_2 + 3E_6$	$[3]$	0	A_2
23	$4E_6$	$4A_2 + 2E_6$	$[3,3]$	0	0
24	$A_{11} + D_7 + E_6$	$A_2 + A_{11} + D_7$	$[4]$	0	A_2
25	$A_{11} + D_7 + E_6$	$A_{11} + E_6$	$[3]$	3	$2A_1$
26	$A_{11} + D_7 + E_6$	$2A_2 + A_{11} + D_4$	$[6]$	1	0
27	$A_{11} + D_7 + E_6$	$A_5 + D_7 + E_6$	$[1]$	2	A_1
28	$A_{11} + D_7 + E_6$	$2A_2 + A_8 + D_7$	$[1]$	1	0
29	$A_{11} + D_7 + E_6$	$A_8 + D_4 + E_6$	$[1]$	2	0
30	$2A_{12}$	$A_6 + A_{12}$	$[1]$	2	A_1
31	$2A_{12}$	$2A_9$	$[1]$	2	0
32	$3D_8$	$2A_1 + 2D_8$	$[2,2]$	2	$2A_1$
33	$3D_8$	$2D_5 + D_8$	$[2]$	2	0
34	$A_{15} + D_9$	$A_3 + A_{15}$	$[4]$	2	$2A_1$
35	$A_{15} + D_9$	$A_9 + D_9$	$[1]$	2	A_1
36	$A_{15} + D_9$	$A_{12} + D_6$	$[1]$	2	0

(to be continued)

No.	R_N		$\sigma = 1$		$\sigma = 10$
37	$A_{17} + E_7$	$A_2 + A_{17}$	[3]	1	$A_1 + A_2$
38	$A_{17} + E_7$	$A_{11} + E_7$	[1]	2	A_1
39	$A_{17} + E_7$	$A_5 + A_{14}$	[1]	1	0
40	$D_{10} + 2E_7$	$A_2 + D_{10} + E_7$	[2]	1	$A_1 + A_2$
41	$D_{10} + 2E_7$	$2A_5 + D_{10}$	$[2, 2]$	0	0
42	$D_{10} + 2E_7$	$D_4 + 2E_7$	[2]	2	$2A_1$
43	$D_{10} + 2E_7$	$A_5 + D_7 + E_7$	[2]	1	0
44	$2D_{12}$	$D_6 + D_{12}$	[2]	2	$2A_1$
45	$2D_{12}$	$2D_9$	[1]	2	0
46	$3E_8$	$2A_2 + 2E_8$	[1]	0	$2A_2$
47	$3E_8$	$2E_6 + E_8$	[1]	0	0
48	$D_{16} + E_8$	$2A_2 + D_{16}$	[2]	0	$2A_2$
49	$D_{16} + E_8$	$D_{10} + E_8$	[1]	2	$2A_1$
50	$D_{16} + E_8$	$D_{13} + E_6$	[1]	1	0
51	A_{24}	A_{18}	[1]	2	A_1
52	D_{24}	D_{18}	[1]	2	$2A_1$

TABLE 4.2. Genus one fibrations on $X_{3,1}$ and $X_{3,10}$

In Table 4.1 (resp. Table 4.2), the lists $\mathbb{E}(X_{2,1})$ and $\mathbb{E}(X_{2,10})$ (resp. $\mathbb{E}(X_{3,1})$ and $\mathbb{E}(X_{3,10})$) are presented. Two lattice equivalence classes in the same row are the pair of $[\phi] \in \mathbb{E}(X_{p,1})$ and its corresponding partner $[\phi'] \in \mathbb{E}(X_{p,10})$. The ADE-type $R_{[\phi]}$ of the reducible fibers of ϕ, and the torsion $\mathrm{MW}_{\mathrm{tor}}$ and the rank of the Mordell-Weil group of ϕ are also given. (Recall that ϕ is Jacobian for any $[\phi] \in \mathbb{E}(X_{p,1})$ by Elkies and Schütt [11].) The meaning of the entry R_N is explained in the proof of Theorems 4.8 and 4.9.

Proof of Theorems 4.8 and 4.9. By Theorem 4.7, it is enough to calculate the ADE-type of $\mathcal{R}(K_\phi^\vee(p))$ for $p = 2, 3$ and $[\phi] \in \mathbb{E}(X_{p,1})$. The lattices K_ϕ are calculated in Elkies and Schütt [11] and Sengupta [28] by Nishiyama's method [22]. We put

$$T := \text{the root lattice of type} \begin{cases} D_4 & \text{if } p = 2, \\ 2A_2 & \text{if } p = 3. \end{cases}$$

Then, for each $[\phi] \in \mathbb{E}(X_{p,1})$, there exist a Niemeier lattice N_ϕ and a primitive embedding of T into N_ϕ such that K_ϕ is isomorphic to the orthogonal complement of T in N_ϕ. The entry R_N in Tables 4.1 and 4.2 indicates the ADE-type of $\mathcal{R}(N_\phi)$. From a Gram matrix of K_ϕ, we can calculate the ADE-type of $\mathcal{R}(K_\phi^\vee(p))$ by the algorithm described in [32, Section 4] or [33, Section 3]. □

Corollary 4.10. *There exist no quasi-elliptic fibrations on $X_{3,10}$.*

Remark 4.11. Rudakov and Shafarevich [27, Section 5] showed that there exists a quasi-elliptic fibration on $X_{2,\sigma}$ for any σ. The quasi-elliptic fibration on $X_{2,10}$ (No. 18 of Table 4.1) was discovered by Rudakov and Shafarevich [26, Section 4].

5. Chamber decomposition of a positive cone

Let S be an even hyperbolic lattice, and let $\mathcal{P}_S \subset S \otimes \mathbb{R}$ be a positive cone. In this section, we review a general method to find a set of generators of a subgroup of $\mathrm{O}^+(S)$ by means of a chamber decomposition of $\mathcal{P}_S$, which was developed by Vinberg [37], [38], Conway [7] and Borcherds [3], [4].

Any real hyperplane in $\mathcal{P}_S$ is written in the form $(v)^\perp$ by some vector $v \in S \otimes \mathbb{R}$ with negative norm. We denote by $\mathcal{H}_S$ the set of real hyperplanes in $\mathcal{P}_S$, which is canonically identified with

$$\{\ v \in S \otimes \mathbb{R}\ \mid\ v^2 < 0\ \}/\mathbb{R}^\times.$$

For a subset V of $\{v \in S \otimes \mathbb{R} \mid v^2 < 0\}$, we denote by $V^* \subset \mathcal{H}_S$ the image of V by $v \mapsto (v)^\perp$. A closed subset D of $\mathcal{P}_S$ is called a *chamber* if the interior D° of D is non-empty and there exists a set Δ_D of vectors $v \in S \otimes \mathbb{R}$ with $v^2 < 0$ such that

$$D = \{\ x \in \mathcal{P}_S\ \mid\ \langle x, v \rangle \geq 0\ \text{ for all }\ v \in \Delta_D\ \}.$$

A hyperplane $(v)^\perp$ of $\mathcal{P}_S$ is called a *wall* of D if $D^\circ \cap (v)^\perp = \emptyset$ and $D \cap (v)^\perp$ contains an open subset of $(v)^\perp$. When D is a chamber, we always assume that the set Δ_D is minimal in the sense that, for any $v \in \Delta_D$, there exists a point $x \in \mathcal{P}_S$ such that $\langle x, v \rangle < 0$ and $\langle x, v' \rangle \geq 0$ for any $v' \in \Delta_D \setminus \{v\}$, that is, the projection $\Delta_D \to \Delta_D^*$ is bijective and every hyperplane $(v)^\perp \in \Delta_D^*$ is a wall of D.

For a chamber D, we put

$$Aut(D) := \{\ g \in \mathrm{O}^+(S)\ \mid\ D^g = D\ \}.$$

A chamber D is said to be *fundamental* if the following hold:

(i) $\mathcal{P}_S$ is the union of all D^g, where g runs through $\mathrm{O}^+(S)$, and
(ii) if $D^\circ \cap D^g \neq \emptyset$, then $g \in Aut(D)$.

Let $\mathcal{F}$ be a family of hyperplanes in $\mathcal{P}_S$ with the following properties:

(a) $\mathcal{F}$ is locally finite in $\mathcal{P}_S$, and
(b) $\mathcal{F}$ is invariant under the action of $\mathrm{O}^+(S)$ on $\mathcal{H}_S$.

Then the closure of each connected component of

$$\mathcal{P}_S \setminus \bigcup_{\mathcal{F}} (v)^\perp$$

is a chamber, which we call an *$\mathcal{F}$-chamber*.

Suppose that D is an $\mathcal{F}$-chamber. Then D^g is also an $\mathcal{F}$-chamber for any $g \in \mathrm{O}^+(S)$ by the property (b) of $\mathcal{F}$, and D satisfies the property (ii) in the definition of fundamental chambers. Moreover, D satisfies the property (i) if and only if every $\mathcal{F}$-chamber is equal to D^g for some $g \in \mathrm{O}^+(S)$.

For each wall $(v)^\perp \in \Delta_D^*$ of an $\mathcal{F}$-chamber D, there exists a unique $\mathcal{F}$-chamber D' distinct from D such that $D \cap D' \cap (v)^\perp$ contains an open subset of $(v)^\perp$. We say that D' is *adjacent to D along $(v)^\perp$*, and that $(v)^\perp$ is the *wall between the adjacent chambers D and D'*.

Proposition 5.1. *An $\mathcal{F}$-chamber D is fundamental if and only if, for each $v \in \Delta_D$, there exists $g_v \in \mathrm{O}^+(S)$ such that D^{g_v} is adjacent to D along $(v)^\perp$.*

Proof. The 'only if' part is obvious. We prove the 'if' part. It is enough to show that, for an arbitrary $\mathcal{F}$-chamber D', there exists $g \in \mathrm{O}^+(S)$ such that $D' = D^g$. Since the family $\mathcal{F}$ of hyperplanes is locally finite in $\mathcal{P}_S$, there exists a finite chain of $\mathcal{F}$-chambers $D_0 = D, D_1, \dots, D_N = D'$ such that D_i and D_{i+1} are adjacent. We show, by induction on N, that there exists a sequence of vectors $v_1, \dots, v_N$ in Δ_D such that $D_i = D^{g_{v_i} \cdots g_{v_1}}$ holds for $i = 1, \dots, N$. The case $N = 0$ is trivial. Suppose that $N > 0$. Let $(w)^\perp$ be the wall between D_{N-1} and D_N, and let $v_N \in \Delta_D$ be the vector such that the wall $(v_N)^\perp$ of D is mapped to the wall $(w)^\perp$ of D_{N-1} by $g_{v_{N-1}} \cdots g_{v_1}$. Then we have $D_N = D^{g_{v_N} \cdots g_{v_1}}$. □

Remark 5.2. If an $\mathcal{F}$-chamber is fundamental, then any $\mathcal{F}$-chamber is fundamental.

Let $\mathcal{G}$ be a subset of $\mathcal{F}$ that is invariant under the action of $\mathrm{O}^+(S)$. Then $\mathcal{G}$ is locally finite, and any $\mathcal{G}$-chamber is a union of $\mathcal{F}$-chambers. If an $\mathcal{F}$-chamber is fundamental, then any $\mathcal{G}$-chamber is also fundamental.

Proposition 5.3. *Let D be an $\mathcal{F}$-chamber and let C be a $\mathcal{G}$-chamber such that $D \subset C$. Suppose that D is fundamental. For $v \in \Delta_D$, let $g_v \in \mathrm{O}^+(S)$ be an isometry such that D^{g_v} is adjacent to D along $(v)^\perp$. We put*

$$\Gamma := \{\, g_v \mid v \in \Delta_D, (v)^\perp \notin \mathcal{G} \,\}.$$

Then $Aut(C)$ is generated by $Aut(D)$ and Γ.

Proof. If $g_v \in \Gamma$, then D^{g_v} is contained in C because the wall $(v)^\perp$ between D and D^{g_v} does not belong to $\mathcal{G}$, and hence $g_v \in Aut(C)$. Therefore the subgroup $\langle Aut(D), \Gamma \rangle$ of $\mathrm{O}^+(S)$ generated by $Aut(D)$ and Γ is contained in $Aut(C)$. To prove $Aut(C) \subset \langle Aut(D), \Gamma \rangle$, it is enough to show that, for any $g \in Aut(C)$, there exists a sequence $\gamma_1, \dots, \gamma_N$ of elements of Γ such that $D^g = D^{\gamma_N \cdots \gamma_1}$. There exists a sequence of $\mathcal{F}$-chambers $D_0 = D, D_1, \dots, D_N = D^g$ such that each D_i is contained in C and that D_{i+1} is adjacent to D_i for $i = 0, \dots, N-1$. Suppose that we have constructed $\gamma_1, \dots, \gamma_i \in \Gamma$ such that $D_i = D^{\gamma_i \cdots \gamma_1}$ holds. The wall $(w)^\perp$ between D_i and D_{i+1} does not belong to $\mathcal{G}$. Let v_{i+1} be an element of Δ_D such that the wall $(v_{i+1})^\perp$ of D is mapped to the wall $(w)^\perp$ of D_i by $\gamma_i \dots \gamma_1$. Since $\mathcal{G}$ is invariant under the action of $\mathrm{O}^+(S)$, we have $(v_{i+1})^\perp \notin \mathcal{G}$ and hence $\gamma_{i+1} := g_{v_{i+1}}$ is an element of Γ. Then $D_{i+1} = D^{\gamma_{i+1}\gamma_i \cdots \gamma_1}$ holds. □

Remark 5.4. Let D and C be as in Proposition 5.3. Let v and v' be elements of Δ_D. Suppose that the wall $(v)^\perp$ of D is mapped to the wall $(v')^\perp$ of D by $h \in Aut(D)$. Then $D^{hg_{v'}h^{-1}}$ is adjacent to D along $(v)^\perp$. Let Δ'_D be a subset of Δ_D such that the subset Δ'^*_D of Δ^*_D is a complete set of representatives of the orbit decomposition of Δ^*_D by the action of $Aut(D)$. Then $Aut(C)$ is generated by $Aut(D)$ and $\{g_v \,|\, v \in \Delta'_D, (v)^\perp \notin \mathcal{G}\}$.

Considering the case $\mathcal{G} = \emptyset$, we obtain the following:

Corollary 5.5. *Let D be an $\mathcal{F}$-chamber. If D is fundamental, then $\mathrm{O}^+(S)$ is generated by $Aut(D)$ and the isometries g_v that map D to its adjacent chambers.*

Example 5.6. Recall that $W(S) \subset \mathrm{O}^+(S)$ is the subgroup generated by $\{s_r \,|\, r \in \mathcal{R}(S)\}$. Any $\mathcal{R}(S)^*$-chamber is fundamental, because every $r \in \mathcal{R}(S)$ defines a reflection s_r. It follows that $\mathrm{O}^+(S)$ is equal to the semi-direct product of $W(S)$ and the automorphism group $Aut(D)$ of an $\mathcal{R}(S)^*$-chamber D. In particular, we have

$$Aut(D) \cong \mathrm{O}^+(S)/W(S).$$

Let L be an even *unimodular* hyperbolic lattice, and let $\iota : S \hookrightarrow L$ be a primitive embedding. Let $\mathcal{P}_L$ be the positive cone of L that contains $\iota(\mathcal{P}_S)$. We denote by T_ι the orthogonal complement of S in L, and by

$$v \mapsto v_S$$

the orthogonal projection $L \otimes \mathbb{R} \to S \otimes \mathbb{R}$. Since L is a submodule of $S^\vee \oplus T_\iota^\vee$, the image of L by $v \mapsto v_S$ is contained in $S^\vee$. We assume the following:

$$\text{the natural homomorphism } \mathrm{O}(T_\iota) \to \mathrm{O}(q_{T_\iota}) \text{ is surjective.} \tag{5.1}$$

Then we have the following:

Proposition 5.7. *For any $g \in \mathrm{O}^+(S)$, there exists $\tilde{g} \in \mathrm{O}^+(L)$ such that $\iota(v^g) = \iota(v)^{\tilde{g}}$ holds for any $v \in S \otimes \mathbb{R}$.*

Proof. See Nikulin [20, Proposition 1.6.1]. □

A hyperplane $(r)^\perp$ of $\mathcal{P}_L$ defined by a (-2)-vector $r \in \mathcal{R}(L)$ intersects $\iota(\mathcal{P}_S)$ if and only if $r_S^2 < 0$. We put

$$\mathcal{R}(L, \iota) := \{\, r_S \;\mid\; r \in \mathcal{R}(L) \;\text{ and }\; r_S^2 < 0 \,\} \quad\subset\quad S^\vee.$$

Since T_ι is negative definite, we have $-2 \le r_S^2$ for any $r \in \mathcal{R}(L)$. Since $S^\vee$ is discrete in $S \otimes \mathbb{R}$, the family of hyperplanes $\mathcal{R}(L, \iota)^*$ is locally finite in $\mathcal{P}_S$. By Proposition 5.7, if $r \in \mathcal{R}(L)$ satisfies $r_S \in \mathcal{R}(L, \iota)$, then, for any $g \in \mathrm{O}^+(S)$, we have $r_S^g = (r^{\tilde{g}})_S \in \mathcal{R}(L, \iota)$. Therefore $\mathcal{R}(L, \iota)$ is invariant under the action of $\mathrm{O}^+(S)$. Note that $\mathcal{R}(L, \iota)$ contains $\mathcal{R}(S)$, and that $\mathcal{R}(S)$ is obviously invariant under the action of $\mathrm{O}^+(S)$. Therefore, by Proposition 5.3, we can obtain a set of generators of the automorphism group $Aut(C)$ of an $\mathcal{R}(S)^*$-chamber C if we find an $\mathcal{R}(L, \iota)^*$-chamber D contained in C, show that D is fundamental, calculate the group $Aut(D)$, and find isometries of S that map D to its adjacent chambers.

Let L_{26} denote an even hyperbolic unimodular lattice of rank 26, which is unique up to isomorphisms by Eichler's theorem (see, for example, Cassels [6]). The walls of an $\mathcal{R}(L_{26})^*$-chamber $\mathcal{D} \subset L_{26} \otimes \mathbb{R}$ and the group $Aut(\mathcal{D}) \subset \mathrm{O}^+(L_{26})$ were determined by Conway [7]. Then Borcherds [3], [4] determined the structure of $\mathrm{O}^+(S)$ for some even hyperbolic lattices S of rank < 26 by embedding S into L_{26} in such a way that T_ι is a root lattice.

Kondo [17] applied the Conway-Borcherds method to the study of the automorphism group of a generic Jacobian Kummer surface. Then Keum and Kondo [14] applied it to Kummer surfaces associated with the product of two elliptic curves, Dolgachev and Keum [9] applied it to quartic Hessian surfaces, Dolgachev and Kondo [8] applied it to $X_{2,1}$, and Kondo and Shimada [18] applied it to $X_{3,1}$.

We say that an even hyperbolic lattice S is 2*-reflective* if the index of $W(S)$ in $\mathrm{O}^+(S)$ is finite, or equivalently, if the automorphism group of an $\mathcal{R}(S)^*$-chamber is finite (see Example 5.6). Nikulin [21] classified all 2-reflective lattices of rank ≥ 5. It turns out that there are no 2-reflective lattices of rank > 19.

Let Y be a $K3$ surface with the Néron-Severi lattice S_Y and the positive cone $\mathcal{P}(Y)$ containing an ample class. Then the closed subset $\mathrm{Nef}^\circ(Y) = \mathrm{Nef}(Y) \cap \mathcal{P}(Y)$ of $\mathcal{P}(Y)$ is an $\mathcal{R}(S_Y)^*$-chamber by Proposition 2.1(1), and hence we have

$$Aut(\mathrm{Nef}(Y)) = Aut(\mathrm{Nef}^\circ(Y)) \cong \mathrm{O}^+(S_Y)/W(S_Y).$$

Combining this fact with Nikulin's classification of 2-reflective lattices, we obtain the following:

Corollary 5.8. *For any supersingular $K3$ surface $X_{p,\sigma}$, the group $Aut(\mathrm{Nef}(X_{p,\sigma}))$ is infinite.*

6. The groups $Aut(\mathrm{Nef}(X_{2,10}))$ and $Aut(\mathrm{Nef}(X_{3,10}))$

6.1. **The group $Aut(\mathrm{Nef}(X_{2,10}))$.** By Lemma 1.1, the result of Dolgachev and Kondo [8], and the method of the previous section, we obtain a set of generators of $Aut(\mathrm{Nef}(X_{2,10}))$.

First we recall the results of [8]. As a projective model of $X_{2,1}$, we consider the minimal resolution X of the inseparable double cover $Y \to \mathbb{P}^2$ of $\mathbb{P}^2$ defined by

$$w^2 = x_0 x_1 x_2 (x_0^3 + x_1^3 + x_2^3).$$

Note that the projective plane $\mathbb{P}^2(\mathbb{F}_4)$ defined over $\mathbb{F}_4$ contains 21 points $p_1, \dots, p_{21}$ and 21 lines $\ell_1, \dots, \ell_{21}$. The inseparable double cover Y has 21 ordinary nodes over the 21 points in $\mathbb{P}^2(\mathbb{F}_4)$ and hence X has 21 disjoint (-2)-curves. We denote by $e_1, \dots, e_{21} \in S_{2,1}$ the classes of these (-2)-curves, by $h \in S_{2,1}$ the class of the pullback of a line on $\mathbb{P}^2$, and by $f_1, \dots, f_{21} \in S_{2,1}$ the classes of the proper transforms of the 21 lines in $\mathbb{P}^2(\mathbb{F}_4)$. Then $S_{2,1}$ is generated by the (-2)-vectors $e_1, \dots, e_{21}, f_1, \dots, f_{21}$. The vector

$$w_M := \frac{1}{3}\sum_{i=1}^{21}(e_i + f_i)$$

has the property

$$w_M \in S_X,\ w_M^2 = 14,\ \langle w_M, e_i\rangle = \langle w_M, f_i\rangle = 1.$$

The complete linear system associated with the line bundle corresponding to w_M defines an embedding of X into $\mathbb{P}^2 \times \mathbb{P}^2$, and its image $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$ is defined by

$$\begin{cases} x_0y_0^2 + x_1y_1^2 + x_2y_2^2 &= 0, \\ x_0^2y_0 + x_1^2y_1 + x_2^2y_2 &= 0. \end{cases}$$

Six points on $\mathbb{P}^2(\mathbb{F}_4)$ are said to be *general* if no three points of them are collinear. There exist 168 sets of general six points in $\mathbb{P}^2(\mathbb{F}_4)$. If $I = \{p_{i_1}, \dots, p_{i_6}\}$ is a set of general six points, then the (-1)-vector

$$c_I := h - \frac{1}{2}(e_{i_1} + \cdots + e_{i_6})$$

is contained in $S_{2,1}^\vee$. Note that each c_I defines a reflection

$$x \mapsto x + 2\langle x, c_I\rangle c_I$$

in $\mathrm{O}^+(S_{2,1})$ because $c_I \in S_{2,1}^\vee$. Let $P(X_{2,1})$ be the positive cone of $S_{2,1}$ containing an ample class. and let $\Delta(X_{2,1})$ be the set consisting of $e_1, \dots, e_{21}, f_1, \dots, f_{21}$ and the (-1)-vectors c_I defined above. We define a chamber $D(X_{2,1})$ in $P(X_{2,1})$ by

$$D(X_{2,1}) := \{\ x \in P(X_{2,1}) \ \mid\ \langle x, v\rangle \geq 0 \ \text{ for all } \ v \in \Delta(X_{2,1})\ \}.$$

Then, for each $v \in \Delta(X_{2,1})$, the hyperplane $(v)^\perp$ is a wall of $D(X_{2,1})$. Moreover the ample class w_M is contained in the interior of $D(X_{2,1})$. Recall that L_{26} is the even unimodular hyperbolic lattice of rank 26. There exists a primitive embedding $\iota : S_{2,1} \hookrightarrow L_{26}$, which is unique up to $\mathrm{O}(L_{26})$. The orthogonal complement T_ι of $S_{2,1}$ in L_{26} is isomorphic to the root lattice of type D_4, and hence satisfies the hypothesis (5.1).

Proposition 6.1. *The chamber $D(X_{2,1})$ is an $\mathcal{R}(L_{26}, \iota)^*$-chamber contained in the $\mathcal{R}(S_{2,1})^*$-chamber* $\mathrm{Nef}^\circ(X_{2,1})$. *An isometry $g \in \mathrm{O}^+(S_{2,1})$ belongs to $Aut(D(X_{2,1}))$ if and only if $w_M^g = w_M$.*

Thus we can apply Proposition 5.3 to the pair of chambers $D(X_{2,1})$ and $\mathrm{Nef}^\circ(X_{2,1})$ for the study of $Aut(\mathrm{Nef}(X_{2,1}))$ and $\mathrm{Aut}(X_{2,1})$.

We have the following elements in $\mathrm{Aut}(X_{2,1})$ and $\mathrm{O}^+(S_{2,1})$. Since $\mathrm{Aut}(X_{2,1})$ is naturally embedded in $\mathrm{O}^+(S_{2,1})$, we use the same letter to denote an element of $\mathrm{Aut}(X_{2,1})$ and its image in $\mathrm{O}^+(S_{2,1})$.

- The action of $\mathrm{PGL}(3, \mathbb{F}_4)$ on $\mathbb{P}^2$ induces automorphisms of the inseparable double cover Y of $\mathbb{P}^2$, and hence automorphisms of $X_{2,1}$. Their action on $S_{2,1}$ preserves $D(X_{2,1})$.
- The interchange of the two factors of $\mathbb{P}^2 \times \mathbb{P}^2$ preserves $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$, and hence it induces an involution $\mathrm{sw} \in \mathrm{Aut}(X_{2,1})$, which we call the *switch*. Its action on $S_{2,1}$ preserves $D(X_{2,1})$.
- For each set I of general six points in $\mathbb{P}^2(\mathbb{F}_4)$, the linear system of plane curves of degree 5 that pass through the points of I and are singular at each point of I defines a birational involution of $\mathbb{P}^2$, and this involution lifts to an involution of Y. Hence we obtain an involution $\mathrm{Cr}_I \in \mathrm{Aut}(X_{2,1})$, which we call a *Cremona automorphism* of $X_{2,1}$. The action of Cr_I on $S_{2,1}$ is the reflection with respect to $c_I \in S_{2,1}^{\vee}$.
- The Frobenius action of $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ on X_M induces an isometry Fr of $S_{2,1}$, which preserves $D(X_{2,1})$.
- We have the reflections s_{e_i} and s_{f_i} with respect to the (-2)-vectors e_i and f_i.

By the reflections Cr_I, s_{e_i} and s_{f_i}, we see that the chamber $D(X_{2,1})$ is fundamental.

Theorem 6.2 ([8]). (1) *The projective automorphism group* $\mathrm{Aut}(X_{2,1}, w_M)$ *of* $X_M \subset \mathbb{P}^2 \times \mathbb{P}^2$ *is generated by* $\mathrm{PGL}(3, \mathbb{F}_4)$ *and the switch* sw.

(2) *The group* $Aut(D(X_{2,1}))$ *is generated by* $\mathrm{Aut}(X_{2,1}, w_M)$ *and* Fr.

(3) *The automorphism group* $\mathrm{Aut}(X_{2,1})$ *is generated by* $\mathrm{Aut}(X_{2,1}, w_M)$ *and the* 168 *Cremona automorphisms* Cr_I.

(4) *The group* $Aut(\mathrm{Nef}(X_{2,1}))$ *is generated by* $\mathrm{Aut}(X_{2,1})$ *and* Fr.

(5) *The group* $\mathrm{O}^+(S_{2,1})$ *is generated by* $Aut(\mathrm{Nef}(X_{2,1}))$ *and the* $21 + 21$ *reflections* s_{e_i} *and* s_{f_i}.

We then study $Aut(\mathrm{Nef}(X_{2,10}))$. By Corollary 3.2, we have an embedding

$$j : S_{2,1} \hookrightarrow S_{2,10}$$

that induces $S_{2,1}^{\vee}(2) \cong S_{2,10}$. Composing j with some element of $W(S_{2,10}) \times \{\pm 1\}$, we can assume that $j(w_M)$ is contained in $\mathrm{Nef}(X_{2,10})$ (Proposition 2.1(2)). The isomorphism $j_* : \mathrm{O}^+(S_{2,1}) \xrightarrow{\sim} \mathrm{O}^+(S_{2,10})$ induced by j is denoted by

$$g \mapsto g'.$$

The $j(\mathcal{R}(L_{26}, \iota))^*$-chamber $j(D(X_{2,1}))$ is fundamental, and we have

$$Aut(j(D(X_{2,1}))) = Aut(D(X_{2,1}))'.$$

Lemma 6.3. *The set* $j(\mathcal{R}(L_{26}, \iota))$ *contains* $\mathcal{R}(S_{2,10})$. *Hence the* $j(\mathcal{R}(L_{26}, \iota))^*$*-chamber* $j(D(X_{2,1}))$ *is contained in the* $\mathcal{R}(S_{2,10})^*$*-chamber* $\mathrm{Nef}^{\circ}(X_{2,10})$.

Proof. It is enough to show that, if $v \in S_{2,1}^{\vee}$ satisfies $v^2 = -1$, then $v \in \mathcal{R}(L_{26}, \iota)$, that is, there exists $u \in T_{\iota}^{\vee}$ such that $u^2 = -1$ and that $u + v$ is contained in the submodule L_{26} of $S_{2,1}^{\vee} \oplus T_{\iota}^{\vee}$. By

Nikulin [20, Proposition 1.4.1], the submodule $L_{26}/(S_{2,1}\oplus T_\iota)$ of $(S_{2,1}^\vee\oplus T_\iota^\vee)/(S_{2,1}\oplus T_\iota)=A_{S_{2,1}}\oplus A_{T_\iota}$ is the graph of an isomorphism

$$q_{S_{2,1}}\cong -q_{T_\iota}.$$

Hence it is enough to show that, for any $\bar{u}\in A_{T_\iota}$ with $q_{T_\iota}(\bar{u})=1$, there exists $u\in T_\iota^\vee$ such that $u^2=-1$ and $u \bmod T_\iota=\bar{u}$. Since T_ι is a root lattice of type D_4, we can confirm this fact by direct computation. The set of (-1)-vectors in $T_\iota^\vee$ consists of 24 vectors, and its image by the natural projection $T_\iota^\vee\to A_{T_\iota}$ is the set of all non-zero elements of $A_{T_\iota}\cong\mathbb{F}_2^2$. □

The set of walls of $j(D(X_{2,1}))$ is equal to

$$\{(j(e_i))^\perp \mid i=1,\dots,21\}\ \cup\ \{(j(f_i))^\perp \mid i=1,\dots,21\}\ \cup$$
$$\{(j(c_I))^\perp \mid I \text{ is a set of general six points}\}.$$

Note that the $21+21$ vectors $j(e_i)$ and $j(f_i)$ are of norm -4 and the 168 vectors $j(c_I)$ are of norm -2. Note also that neither $(j(e_i))^\perp$ nor $(j(f_i))^\perp$ are contained in $\mathcal{R}(S_{2,10})^*$, because there are no rational numbers λ such that $(-4)\lambda^2=-2$. By Proposition 5.3, Theorem 6.2 and Lemma 6.3, we obtain the following:

Theorem 6.4. *The group $Aut(\mathrm{Nef}(X_{2,10}))$ is generated by $\mathrm{PGL}(3,\mathbb{F}_4)'$, sw', Fr', s'_{e_i} and s'_{f_i}.*

6.2. **The group $Aut(\mathrm{Nef}(X_{3,10}))$.** By the same argument as above, we obtain a set of generators of $Aut(\mathrm{Nef}(X_{3,10}))$ from the result of Kondo and Shimada [18].

We consider the Fermat quartic surface

$$X_{\mathrm{FQ}}: x_0^4+x_1^4+x_2^4+x_3^4=0$$

in characteristic 3. Then X_{FQ} is isomorphic to $X_{3,1}$. The surface X_{FQ} contains 112 lines, and their classes $l_1,\dots,l_{112}$ span $S_{3,1}$. We denote by $h_{\mathrm{FQ}}\in S_{3,1}$ the class of a hyperplane section of X_{FQ}.

There exists a primitive embedding $\iota: S_{3,1}\hookrightarrow L_{26}$, which is unique up to $\mathrm{O}(L_{26})$. The orthogonal complement T_ι is isomorphic to the root lattice of type $2A_2$, and hence satisfies the hypothesis (5.1). We calculated an $\mathcal{R}(L_{26},\iota)^*$-chamber $D(X_{3,1})$ that contains h_{FQ} in its interior, and found

$$648 \text{ vectors } u_j\in S_{3,1}^\vee \text{ of norm } -4/3, \quad\text{and}\quad 5184 \text{ vectors } w_k\in S_{3,1}^\vee \text{ of norm } -2/3$$

such that the walls of $D(X_{3,1})$ consist of the 112 hyperplanes $(l_i)^\perp$, the 648 hyperplanes $(u_j)^\perp$ and the 5184 hyperplanes $(w_k)^\perp$. Note that the $\mathcal{R}(L_{26},\iota)^*$-chamber $D(X_{3,1})$ is contained in the $\mathcal{R}(S_{3,1})^*$-chamber $\mathrm{Nef}^\circ(X_{3,1})$, because $h_{\mathrm{FQ}}\in D(X_{3,1})^\circ$. Moreover, since $28\,h_{\mathrm{FQ}}=\sum l_i$, the following holds:

Proposition 6.5. *An isometry $g\in\mathrm{O}^+(S_{3,1})$ belongs to $Aut(D(X_{3,1}))$ if and only if $h_{\mathrm{FQ}}^g=h_{\mathrm{FQ}}$.*

We have the following elements in $\mathrm{Aut}(X_{3,1})$ and $\mathrm{O}^+(S_{3,1})$. Note that, for a polarization $h\in S_{3,1}$ of degree 2, we have the deck transformation $\tau(h)\in\mathrm{Aut}(X_{3,1})$ of the generically finite morphism $X_{3,1}\to\mathbb{P}^2$ of degree 2 induced by the the complete linear system associated with h.

- The subgroup $\mathrm{PGU}(4, \mathbb{F}_9)$ of $\mathrm{PGL}(4, k) = \mathrm{Aut}(\mathbb{P}^3)$ acts on X_{FQ}. Its action on $S_{3,1}$ preserves $D(X_{3,1})$. Moreover, the action of $\mathrm{PGU}(4, \mathbb{F}_9)$ on $S_{3,1}^{\vee}$ is transitive on each of the set of 112 vectors l_i, the set of 648 vectors u_j and the set of 5184 vectors w_k.
- There exists a polarization $h_{648} \in S_{3,1}$ of degree 2 such that the deck transformation $\tau(h_{648}) \in \mathrm{Aut}(X_{3,1})$ maps $D(X_{3,1})$ to an $\mathcal{R}(L_{26}, \iota)^*$-chamber adjacent to $D(X_{3,1})$ along one of the 648 walls $(u_j)^{\perp}$.
- There exists a polarization $h_{5184} \in S_{3,1}$ of degree 2 such that the deck transformation $\tau(h_{5184}) \in \mathrm{Aut}(X_{3,1})$ maps $D(X_{3,1})$ to an $\mathcal{R}(L_{26}, \iota)^*$-chamber adjacent to $D(X_{3,1})$ along one of the 5184 walls $(w_k)^{\perp}$.
- The Frobenius action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ on X_{FQ} gives rise to an element $\mathrm{Fr} \in Aut(D(X_{3,1}))$ of order 2.
- We have the reflections s_{l_i} with respect to the classes l_i of the 112 lines on X_{FQ}.

Remark 6.6. The actions of the involutions $\tau(h_{648})$ and $\tau(h_{5184})$ on $S_{3,1}$ are *not* reflections.

Thus $D(X_{3,1})$ is fundamental, and hence we have the following:

Theorem 6.7 ([18])**.** (1) *The projective automorphism group* $\mathrm{Aut}(X, h_{\mathrm{FQ}})$ *of the Fermat quartic surface* $X_{\mathrm{FQ}} \subset \mathbb{P}^3$ *is equal to* $\mathrm{PGU}(4, \mathbb{F}_9)$.

(2) *The group* $Aut(D(X_{3,1}))$ *is generated by* $\mathrm{Aut}(X, h_{\mathrm{FQ}})$ *and* Fr.

(3) *The automorphism group* $\mathrm{Aut}(X_{3,1})$ *is generated by* $\mathrm{Aut}(X, h_{\mathrm{FQ}})$ *and the two involutions* $\tau(h_{648})$ *and* $\tau(h_{5184})$.

(4) *The group* $Aut(\mathrm{Nef}(X_{3,1}))$ *is generated by* $\mathrm{Aut}(X_{3,1})$ *and* Fr.

(5) *The group* $\mathrm{O}^+(S_{3,1})$ *is generated by* $Aut(\mathrm{Nef}(X_{3,1}))$ *and the* 112 *reflections* s_{l_i}.

By Corollary 3.2, we have an embedding

$$j : S_{3,1} \hookrightarrow S_{3,10}$$

that induces $S_{3,1}^{\vee}(3) \cong S_{3,10}$. By Proposition 2.1(2), we can assume that $j(h_{\mathrm{FQ}})$ is contained in $\mathrm{Nef}(X_{3,10})$. The isomorphism $j_* : \mathrm{O}^+(S_{3,1}) \xrightarrow{\sim} \mathrm{O}^+(S_{3,10})$ induced by j is denoted by $g \mapsto g'$. The $j(\mathcal{R}(L_{26}, \iota))^*$-chamber $j(D(X_{3,1}))$ is fundamental, and $Aut(j(D(X_{3,1})))$ is equal to $Aut(D(X_{3,1}))'$.

Lemma 6.8. *The set* $j(\mathcal{R}(L_{26}, \iota))$ *contains* $\mathcal{R}(S_{3,10})$. *Hence the* $j(\mathcal{R}(L_{26}, \iota))^*$*-chamber* $j(D(X_{3,1}))$ *is contained in the* $\mathcal{R}(S_{3,10})^*$*-chamber* $\mathrm{Nef}^{\circ}(X_{3,10})$.

Proof. It is enough to show that, if $v \in S_{3,1}^{\vee}$ satisfies $v^2 = -2/3$, then there exists $u \in T_{\iota}^{\vee}$ such that $u^2 = -4/3$ and that $u + v$ is contained in $L_{26} \subset S_{3,1}^{\vee} \oplus T_{\iota}^{\vee}$. For this, it suffices to prove that, for any $\bar{u} \in A_{T_{\iota}}$ with $q_{T_{\iota}}(\bar{u}) = -4/3$, there exists $u \in T_{\iota}^{\vee}$ such that $u^2 = -4/3$ and $u \bmod T_{\iota} = \bar{u}$. Since T_{ι} is a root lattice of type $2A_2$, we can confirm this fact by direct computation. □

The set of walls of $j(D(X_{3,1}))$ is equal to

$$\{(j(l_i))^\perp \mid i = 1, \dots, 112\} \;\cup\; \{(j(u_j))^\perp \mid j = 1, \dots, 648\} \;\cup \\ \{(j(w_k))^\perp \mid k = 1, \dots, 5184\}.$$

Note that the vectors $j(l_i)$ are of norm -6, the vectors $j(u_j)$ are of norm -4, and the vectors $j(w_k)$ are of norm -2. Note also that neither $(j(l_i))^\perp$ nor $(j(u_j))^\perp$ are contained in $\mathcal{R}(S_{3,10})^*$. By Proposition 5.3, Theorem 6.7 and Lemma 6.8, we obtain the following:

Theorem 6.9. *The group* $Aut(\mathrm{Nef}(X_{3,10}))$ *is generated by* $\mathrm{PGU}(4, \mathbb{F}_9)'$, Fr', s'_{l_i} *and* $\tau(h_{648})'$.

7. Torelli theorem for supersingular $K3$ surfaces

We review the theory of period mapping and Torelli theorem for supersingular $K3$ surfaces in odd characteristics by Ogus [24], [25]. Throughout this section, we assume that p is odd.

We summarize results on quadratic spaces over finite fields. See, for example, Kitaoka [15, Section 1.3]. Let $\mathbb{F}_q$ be a finite extension of $\mathbb{F}_p$. There exist exactly two isomorphism classes of non-degenerate quadratic forms in 2σ variables $x_1, \dots, x_{2\sigma}$ over $\mathbb{F}_q$. They are represented by

$$f_+ := x_1x_2 + \cdots + x_{2\sigma-1}x_{2\sigma}, \quad \text{and} \tag{7.1}$$

$$f_- := x_1^2 + cx_1x_2 + x_2^2 + x_3x_4 + \cdots + x_{2\sigma-1}x_{2\sigma}, \tag{7.2}$$

where c is an element of $\mathbb{F}_q$ such that $t^2 + ct + 1 \in \mathbb{F}_q[t]$ is irreducible. The quadratic form f_+ (resp. f_-) is called *neutral* (resp. *non-neutral*). The group $\mathrm{O}(\mathbb{F}_q^{2\sigma}, f_\epsilon)$ of the self-isometries of the quadratic space $(\mathbb{F}_q^{2\sigma}, f_\epsilon)$, where $\epsilon = \pm 1$, is of order

$$2\, q^{\sigma(\sigma-1)}(q^\sigma - \epsilon) \prod_{i=1}^{\sigma-1} (q^{2i} - 1).$$

Let N be an even hyperbolic p-elementary lattice of rank 22 with discriminant $-p^{2\sigma}$. We define a quadratic space (N_0, q_0) over $\mathbb{F}_p$ by (1.2). It is known that q_0 is non-degenerate and *non-neutral*. We denote by $\mathrm{O}(N_0, q_0)$ the group of the self-isometries of (N_0, q_0). Note that the scalar multiplications in $\mathrm{O}(N_0, q_0)$ are only ± 1. Let k be a field of characteristic p. We put

$$\varphi := \mathrm{id}_{N_0} \otimes F_k : N_0 \otimes k \to N_0 \otimes k,$$

where F_k is the Frobenius map of k.

Definition 7.1. A subspace K of $N_0 \otimes k$ with $\dim K = \sigma$ is said to be a *characteristic subspace* of (N_0, q_0) if K is totally isotropic with respect to the quadratic form $q_0 \otimes k$ and $\dim(K \cap \varphi(K)) = \sigma - 1$ holds.

Suppose that k is algebraically closed. Let X be a supersingular $K3$ surface with Artin invariant σ defined over k. An isomorphism

$$\eta : N \xrightarrow{\sim} S_X$$

of lattices is called a *marking* of X. We fix a marking η of X. The composite of the marking η and the Chern class map $S_X \to H^2_{\mathrm{DR}}(X/k)$ defines a linear homomorphism

$$\bar{\eta} : N \otimes k \to H^2_{\mathrm{DR}}(X/k).$$

It is known that $\operatorname{Ker} \bar{\eta}$ is contained in $N_0 \otimes k$, and is totally isotropic with respect to $q_0 \otimes k$. We put

$$K_{(X,\eta)} := \varphi^{-1}(\operatorname{Ker} \bar{\eta}),$$

and call $K_{(X,\eta)}$ the *period* of the marked supersingular $K3$ surface (X, η). Then it is proved by Ogus [24], [25] that $K_{(X,\eta)}$ is a characteristic subspace of (N_0, q_0). We denote by $\eta^* : \mathrm{O}(S_X) \xrightarrow{\sim} \mathrm{O}(N)$ the isomorphism induced by the marking η, and let

$$\bar{\eta}^* : \mathrm{O}(S_X) \to \mathrm{O}(N_0, q_0)$$

be the composite of η^* with the natural homomorphism $\mathrm{O}(N) \to \mathrm{O}(N_0, q_0)$. As a corollary of Torelli theorem by Ogus [25, Corollary of Theorem II″], we have the following:

Corollary 7.2. *Let η be a marking of X. Then, as a subgroup of $\mathrm{O}^+(S_X)$, the automorphism group $\mathrm{Aut}(X)$ of X is equal to*

$$\{ \, g \in Aut(\mathrm{Nef}(X)) \;\mid\; K_{(X,\eta)}^{\bar{\eta}^*(g)} = K_{(X,\eta)} \, \}.$$

In particular, the index of $\mathrm{Aut}(X)$ in $Aut(\mathrm{Nef}(X))$ is at most $|\mathrm{O}(N_0, q_0)/\{\pm 1\}|$.

Combining Corollaries 5.8 and 7.2, we obtain the following:

Corollary 7.3. *The automorphism group $\mathrm{Aut}(X)$ is infinite.*

Remark 7.4. When $p = 3$ and $\sigma = 1$, the group $\mathrm{O}(N_0, q_0)$ is of order 8, while the index of $\mathrm{Aut}(X_{3,1})$ in $Aut(\mathrm{Nef}(X_{3,1}))$ is 2 by Theorem 6.7.

Definition 7.5. We say that a supersingular $K3$ surface X with Artin invariant σ is *generic* if there exists a marking η for X such that the subgroup

$$\{ \, \gamma \in \mathrm{O}(N_0, q_0) \;\mid\; K_{(X,\eta)}^{\gamma} = K_{(X,\eta)} \, \} \tag{7.3}$$

of $\mathrm{O}(N_0, q_0)$ consists of only scalar multiplications ± 1.

If X is generic, then the subgroup (7.3) consists of only scalar multiplications for any marking η. The existence of generic supersingular $K3$ surfaces with Artin invariant > 1 (Theorem 1.7) is proved in the next section.

Recall that A_{S_X} is the discriminant group of S_X, and $q_{S_X} : A_{S_X} \to \mathbb{Q}/2\mathbb{Z}$ is the discriminant quadratic form. We will regard A_{S_X} as a 2σ-dimensional vector space over $\mathbb{F}_p$. Note that the image of q_{S_X} is contained in $(2/p)\mathbb{Z}/2\mathbb{Z}$. We define $\bar{q}_{S_X} : A_{S_X} \to \mathbb{F}_p$ by

$$\bar{q}_{S_X}(x \bmod S_X) := p \cdot q_{S_X}(x) \bmod p.$$

Then we obtain a quadratic space $(A_{S_X}, \bar{q}_{S_X})$ over $\mathbb{F}_p$. Note that we can recover q_{S_X} from $\bar{q}_{S_X}$. We have natural homomorphisms

$$\mathrm{O}(S_X) \to \mathrm{O}(q_{S_X}) \cong \mathrm{O}(A_{S_X}, \bar{q}_{S_X}) \twoheadrightarrow \mathrm{PO}(A_{S_X}, \bar{q}_{S_X}) := \mathrm{O}(A_{S_X}, \bar{q}_{S_X})/\{\pm 1\}. \tag{7.4}$$

Let $\eta : N^\vee \xrightarrow{\sim} S_X^\vee$ be the isomorphism induced by a marking η. Then the map

$$px \bmod pN \in N_0 \mapsto \eta(x) \bmod S_X \in A_{S_X} \qquad (x \in N^\vee)$$

induces an isomorphism of quadratic spaces from (N_0, q_0) to $(A_{S_X}, \bar{q}_{S_X})$. By Corollary 7.2, we obtain the following:

Corollary 7.6. *Suppose that X is generic. Then* $\mathrm{Aut}(X)$ *is equal to the kernel of the homomorphism*

$$\Phi : Aut(\mathrm{Nef}(X)) \to \mathrm{PO}(A_{S_X}, \bar{q}_{S_X})$$

obtained by restricting (7.4) *to* $Aut(\mathrm{Nef}(X)) \subset \mathrm{O}(S_X)$.

Remark 7.7. Suppose that X is generic, and that we are given a subset $\{g_1, \dots, g_n\}$ of $Aut(\mathrm{Nef}(X))$ that generate $Aut(\mathrm{Nef}(X))$. Then a finite set of generators of $\mathrm{Aut}(X)$ is obtained by the following procedure. We construct a finite directed graph (V, E) as follows. The set V of vertices is the image of Φ, that is, the subgroup of $\mathrm{PO}(A_{S_X}, \bar{q}_{S_X})$ generated by $\Phi(g_1), \dots, \Phi(g_n)$. The set E of directed edges is the set of triples

$$\alpha = (s_\alpha, g_i, t_\alpha),$$

where $s_\alpha, t_\alpha \in V$ and $s_\alpha \Phi(g_i) = t_\alpha$. The edge α is directed from s_α to t_α and labelled with a generator g_i. We put $\alpha^{-1} := (t_\alpha, g_i^{-1}, s_\alpha)$. We use the identity element $e \in V$ as a base point of the 1-dimensional CW-complex Γ associated with (V, E). Then the fundamental group $\pi_1(\Gamma, e)$ is a free group of finite rank, and its generators are calculated from the graph (V, E). Consider a loop

$$\gamma = \alpha_0^{\varepsilon_0} \dots \alpha_m^{\varepsilon_m}$$

of Γ from e to e, where $\varepsilon_i = \pm 1$ and $\alpha_j^{\varepsilon_j} = (v_j, g_{i_j}^{\varepsilon_j}, v_{j+1})$. Then we have $v_0 = v_{m+1} = e$, and

$$\tilde{\gamma} := g_{i_0}^{\varepsilon_0} \cdots g_{i_m}^{\varepsilon_m} \in Aut(\mathrm{Nef}(X))$$

is mapped to e by Φ. If $\pi_1(\Gamma, e)$ is generated by loops $\gamma_1, \dots, \gamma_l$, then $\mathrm{Aut}(X) = \mathrm{Ker}\,\Phi$ is generated by $\tilde{\gamma}_1, \dots, \tilde{\gamma}_l$.

Remark 7.8. Suppose that $X_{3,10}$ is generic. Applying the procedure in Remark 7.7 to the generators of $Aut(\mathrm{Nef}(X_{3,10}))$ given in Theorem 6.9, we can obtain a finite set of generators of $\mathrm{Aut}(X_{3,10})$. However, a naive application of the procedure would be inexecutable, because, when $p = 3$ and $\sigma = 10$, the order of $\mathrm{O}(N_0, q_0)$ is

$$2^{36} \cdot 3^{90} \cdot 5^{6} \cdot 7^{3} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 37 \cdot 41^{2} \cdot 61 \cdot 73 \cdot 193 \cdot 547 \cdot 757 \cdot 1093 \cdot 1181,$$

which is about 7.886×10^{90}.

For a non-zero vector $v \in S_X \otimes \mathbb{Q}$, we denote by $\langle v \rangle_{\mathbb{Q}}$ the linear subspace of $S_X \otimes \mathbb{Q}$ spanned by v, and put

$$\bar{v} := (\langle v \rangle_{\mathbb{Q}} \cap S_X^{\vee})/(\langle v \rangle_{\mathbb{Q}} \cap S_X),$$

which is a linear subspace of $A_{S_X} \cong \mathbb{F}_p^{2\sigma}$. When $\bar{v} \neq 0$, we denote by

$$[\bar{v}] \in \mathbb{P}(A_{S_X})$$

the corresponding point of the projective space $\mathbb{P}(A_{S_X})$ over $\mathbb{F}_p$. We consider the action of $\mathrm{O}(S_X)$ on $\mathbb{P}(A_{S_X})$.

Remark 7.9. By definition, the reflections s_r with respect to $r \in \mathcal{R}(S_X)$ act on A_{S_X} trivially. Hence the restriction Φ of the homomorphism (7.4) to the subgroup $Aut(\mathrm{Nef}(X))$ of $\mathrm{O}(S_X)$ is also obtained by passing to the quotient $\mathrm{O}(S_X)/(W(S_X) \times \{\pm 1\}) \cong Aut(\mathrm{Nef}(X))$. Thus the orbit of $[\bar{v}]$ under the action of $Aut(\mathrm{Nef}(X))$ is equal to the orbit of $[\bar{v}]$ under the action of $\mathrm{O}(S_X)$.

Corollary 7.10. *Suppose that X is generic. Let $v \in S_X$ be a vector such that $\bar{v} \subset A_{S_X}$ is not zero. Let m be the cardinality of the orbit of $[\bar{v}] \in \mathbb{P}(A_{S_X})$ under the action of $\mathrm{O}(S_X)$. Then the number of $\mathrm{Aut}(X)$-orbits contained in the $\mathrm{O}(S_X)$-orbit of v in S_X is at least m.*

8. Existence of generic supersingular $K3$ surfaces

We prove Theorem 1.7. For the proof, we recall the construction by Ogus [24] of the scheme $\mathcal{M}$ parameterizing characteristic subspaces of the 2σ-dimensional quadratic space (N_0, q_0) over $\mathbb{F}_p$. This scheme $\mathcal{M}$ plays the role of the period domain for supersingular $K3$ surfaces. We continue to assume that p is odd.

Let $\mathrm{Grass}(\nu, N_0)$ denote the Grassmannian variety of ν-dimensional subspaces of N_0, and let $\mathrm{Isot}(\nu, q_0)$ be the subscheme of $\mathrm{Grass}(\nu, N_0)$ parameterizing ν-dimensional totally isotropic subspaces of (N_0, q_0). We put

$$\mathrm{Gen} := \mathrm{Isot}(\sigma, q_0),$$

where Gen is for "generatrix". Note that $\mathrm{Isot}(\nu, q_0)$ is defined over $\mathbb{F}_p$ for any ν. Let k be a field of characteristic p. For a subspace L of $N_0 \otimes k$ with dimension ν, we denote by $[L]$ the k-valued point of $\mathrm{Grass}(\nu, N_0)$ corresponding to L. We then have the following:

(1) If $\nu < \sigma$, then $\mathrm{Isot}(\nu, q_0)$ is geometrically connected.

(2) The scheme $\mathrm{Gen} \otimes \mathbb{F}_{p^2}$ has two connected components Gen_+ and Gen_-, each of which is geometrically connected. Since q_0 is non-neutral, the action of $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.

(3) Let K and K' be two σ-dimensional totally isotropic subspaces of $(N_0, q_0) \otimes k$. Suppose that $\dim(K \cap K') = \sigma - 1$. Then the k-valued points $[K]$ and $[K']$ belong to distinct connected components of Gen.

(4) Suppose that k is algebraically closed. Then, for each k-valued point $[L]$ of the scheme $\mathrm{Isot}(\sigma-1, q_0)$, there exist exactly two σ-dimensional totally isotropic subspaces of $(N_0, q_0)\otimes k$ that contain L.

(5) Let P be the subscheme of $\mathrm{Gen} \times \mathrm{Gen}$ parameterizing pairs (K, K') such that $\dim(K \cap K') = \sigma - 1$. Then the scheme $P \otimes \mathbb{F}_{p^2}$ has two connected components, each of which is isomorphic to $\mathrm{Isot}(\sigma - 1, q_0)$ over $\mathbb{F}_{p^2}$. The action of $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges the two connected components.

Consider the graph

$$\mathrm{id} \times \varphi : \mathrm{Gen} \to \mathrm{Gen} \times \mathrm{Gen}$$

of the Frobenius morphism $\mathrm{Gen} \to \mathrm{Gen}$ given by $K \mapsto \varphi(K)$. The subscheme $\mathcal{M}$ of Gen that parametrizes the characteristic subspaces of (N_0, q_0) is defined by the fiber product

$$\begin{array}{ccc} \mathcal{M} & \hookrightarrow & \mathrm{Gen} \\ \downarrow & \square & \downarrow \mathrm{id} \times \varphi \\ P & \hookrightarrow & \mathrm{Gen} \times \mathrm{Gen}. \end{array}$$

Ogus [24] proved the following:

Theorem 8.1. *The scheme $\mathcal{M}$ defined over $\mathbb{F}_p$ is smooth and projective of dimension $\sigma - 1$. The scheme $\mathcal{M} \otimes \mathbb{F}_{p^2}$ has two connected components $\mathcal{M}_+ = \mathcal{M} \cap \mathrm{Gen}_+$ and $\mathcal{M}_- = \mathcal{M} \cap \mathrm{Gen}_-$, each of which is geometrically connected. The action of* $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ *interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$.*

Proof of Theorem 1.7. Let κ be an algebraic closure of the function field of the scheme $\mathcal{M}_+$ over $\mathbb{F}_{p^2}$, and let $[K_\kappa]$ be the geometric generic point of $\mathcal{M}_+$. By the surjectivity of the period mapping for supersingular $K3$ surfaces (Ogus [25, Theorem III″]), there exist a supersingular $K3$ surface X of Artin invariant σ defined over κ and a marking $\eta : N \xrightarrow{\sim} S_X$ such that $K_{(X,\eta)} = K_\kappa$. We prove that this X is generic, that is,

$$G_\kappa := \{ \gamma \in \mathrm{O}(N_0, q_0) \mid K_\kappa^\gamma = K_\kappa \}$$

is equal to $\{\pm 1\}$. Note that the closure of the point $[K_\kappa]$ coincides with $\mathcal{M}_+$. Therefore we have the following: If a field k contains $\mathbb{F}_{p^2}$, then the action of G_κ leaves K invariant for any k-valued point $[K]$ of $\mathcal{M}_+$.

Suppose that $\sigma \geq 3$. Let u be an arbitrary non-zero isotropic vector of N_0. We will prove that u is an eigenvector of G_κ. Let

$$b_0 : N_0 \times N_0 \to \mathbb{F}_p$$

denote the symmetric bilinear form obtained from q_0. There exists a vector $v \in N_0$ such that $q_0(v) = 0$ and $b_0(u, v) = 1$, and hence (N_0, q_0) has an orthogonal direct-sum decomposition

$$N_0 = U^\perp \oplus U,$$

where U is the subspace spanned by u and v. Repeating this procedure and noting that q_0 is non-neutral, we obtain a basis $a_1, \dots, a_{2\sigma}$ of N_0 with $u = a_{2\sigma}$ such that $q_0(x_1 a_1 + \cdots + x_{2\sigma} a_{2\sigma})$ is equal to the quadratic polynomial f_- in (7.2). Let α and $\bar{\alpha} = \alpha^p$ be the roots in $\mathbb{F}_{p^2}$ of the irreducible polynomial $t^2 + ct + 1 \in \mathbb{F}_p[t]$. We consider the basis

$$\begin{aligned} &b_1^{(-1)} := \alpha a_1 + a_2, \quad b_1^{(1)} := \bar{\alpha} a_1 + a_2, \quad \text{and} \\ &b_i^{(-1)} := a_{2i-1}, \qquad\;\; b_i^{(1)} := a_{2i} \qquad (i = 2, \dots, \sigma) \end{aligned} \tag{8.1}$$

of $N_0 \otimes \mathbb{F}_{p^2}$. Note that each $b_i^{(\pm 1)}$ is isotropic, and that

$$b_0(b_i^{(\alpha)}, b_j^{(\beta)}) = 0 \quad \text{if } i \neq j, \qquad b_0(b_i^{(1)}, b_i^{(-1)}) = \begin{cases} (4 - c^2)/2 & \text{if } i = 1, \\ 1/2 & \text{if } i \geq 2. \end{cases}$$

We put

$$\mathcal{E} := \{1, -1\}^\sigma.$$

For $e = (\varepsilon_1, \dots, \varepsilon_\sigma) \in \mathcal{E}$, we denote by K_e the linear subspace of $N_0 \otimes \mathbb{F}_{p^2}$ spanned by

$$b_1^{(\varepsilon_1)}, \dots, b_\sigma^{(\varepsilon_\sigma)}.$$

It is obvious that K_e is isotropic. Moreover, since

$$\varphi(b_1^{(\varepsilon)}) = b_1^{(-\varepsilon)} \quad \text{and} \quad \varphi(b_i^{(\varepsilon)}) = b_i^{(\varepsilon)} \ \text{ if } i \geq 2,$$

we have $\dim(K_e \cap \varphi(K_e)) = \sigma - 1$. Therefore K_e is a characteristic subspace of (N_0, q_0). Suppose that e and $e' \in \mathcal{E}$ differ only at one component. Then we have $\dim(K_e \cap K_{e'}) = \sigma - 1$, and hence the $\mathbb{F}_{p^2}$-valued points $[K_e]$ and $[K_{e'}]$ of $\mathcal{M}$ belong to distinct connected components. We put

$$\mathcal{E}_+ := \{\, e \in \mathcal{E} \mid \text{the number of } -1 \text{ in } e \text{ is even} \,\}, \quad \mathbf{1} := (1, \dots, 1) \in \mathcal{E}_+.$$

Interchanging α and $\bar{\alpha}$ if necessary, we can assume that $[K_{\mathbf{1}}]$ is an $\mathbb{F}_{p^2}$-valued point of $\mathcal{M}_+$, and hence $[K_e]$ is an $\mathbb{F}_{p^2}$-valued point of $\mathcal{M}_+$ for any $e \in \mathcal{E}_+$. It follows that K_e is invariant under the action of G_κ for any $e \in \mathcal{E}_+$. Let $b_i^{(\alpha)}$ be an arbitrary element among the basis (8.1). Recall that we have assumed $\sigma \geq 3$. Therefore, for each element $b_j^{(\beta)}$ among the basis (8.1) that is distinct from $b_i^{(\alpha)}$, there exists $e(j, \beta) = (\varepsilon_1, \dots, \varepsilon_\sigma) \in \mathcal{E}_+$ such that $\varepsilon_i = \alpha$ and $\varepsilon_j \neq \beta$. Since

$$\bigcap_{(j,\beta) \neq (i,\alpha)} K_{e(j,\beta)} = \langle b_i^{(\alpha)} \rangle$$

is invariant under the action of G_κ, we see that $b_i^{(\alpha)}$ is an eigenvector of G_κ. In particular, the isotropic vector $u = a_{2\sigma} = b_\sigma^{(1)}$ given at the beginning is an eigenvector of G_κ.

Let

$$\lambda_i^{(\alpha)} : G_\kappa \to \mathbb{F}_{p^2}^\times$$

be the character defined by $b_i^{(\alpha)}$. Suppose that $i, j \geq 2$ and $i \neq j$. Then $b_i^{(\alpha)} + b_j^{(\beta)}$ is an isotropic vector of N_0 for any choice of $\alpha, \beta \in \{\pm 1\}$, and hence is an eigenvector of G_κ. Therefore we have

$$\lambda_i^{(\alpha)} = \lambda_j^{(\beta)} \qquad \text{if } i, j \geq 2 \text{ and } i \neq j. \tag{8.2}$$

Since the cardinality of $\{x^2 \mid x \in \mathbb{F}_p\}$ is $(p+1)/2$, there exist $\xi, \eta \in \mathbb{F}_p$ such that

$$(4 - c^2) + \xi^2 + \eta^2 = 0.$$

Then

$$b_1^{(1)} + b_1^{(-1)} + \xi(b_2^{(1)} + b_2^{(-1)}) + \eta(b_3^{(1)} + b_3^{(-1)})$$

is also an isotropic vector of N_0, and hence is an eigenvector of G_κ. Therefore we have

$$\lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_2^{(1)} = \lambda_2^{(-1)} \quad \text{or} \quad \lambda_1^{(1)} = \lambda_1^{(-1)} = \lambda_3^{(1)} = \lambda_3^{(-1)}. \tag{8.3}$$

Combining (8.2) and (8.3), we see that all the characters $\lambda_i^{(\alpha)}$ are equal to each other. Thus G_κ consists of only scalar multiplications.

Suppose that $\sigma = 2$. In this case, the scheme $\mathcal{M}$ coincides with $\mathrm{Isot}(2, q_0)$, which is the scheme parametrizing lines on the smooth quadratic surface $Q_0 = \{q_0 = 0\}$ in the projective space $\mathbb{P}_* N_0 = \mathrm{Grass}(1, N_0)$. Hence $\mathcal{M}_+$ and $\mathcal{M}_-$ correspond to the two rulings of Q_0. Let g be an element of G_κ. Then g leaves every line in the ruling of Q_0 corresponding to $\mathcal{M}_+$ invariant. Since g is defined over $\mathbb{F}_p$ and $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ interchanges $\mathcal{M}_+$ and $\mathcal{M}_-$, we see that g also leaves every line in the other ruling of Q_0 invariant. Therefore g fixes every point of Q_0, and hence every point of $\mathbb{P}_* N_0$. □

9. Lattice equivalence classes versus Aut-equivalence classes on $X_{3,10}$

Suppose that $p > 2$ and $\sigma + \sigma' = 11$. We denote by $A_{p,\sigma'}$ the discriminant group $S_{p,\sigma'}^\vee / S_{p,\sigma'}$ of $S_{p,\sigma'}$, and use the notation in Section 7.

Let $\phi : X_{p,\sigma} \to \mathbb{P}^1$ be a genus one fibration, and let $\phi' : X_{p,\sigma'} \to \mathbb{P}^1$ be a genus one fibration whose lattice equivalence class $[\phi'] \in \mathbb{E}(X_{p,\sigma'})$ corresponds to $[\phi] \in \mathbb{E}(X_{p,\sigma})$ by Theorem 1.3. By Remark 4.6, we have an embedding $j : S_{p,\sigma} \hookrightarrow S_{p,\sigma'}$ inducing $S_{p,\sigma}^\vee(p) \cong S_{p,\sigma'}$ such that $j(f_\phi)$ is a positive scalar multiple of $f_{\phi'}$. Suppose that

$$\overline{f_{\phi'}} = \overline{j(f_\phi)} = (\langle f_{\phi'} \rangle_\mathbb{Q} \cap S_{p,\sigma'}^\vee) / (\langle f_{\phi'} \rangle_\mathbb{Q} \cap S_{p,\sigma'}) \quad \subset \quad A_{p,\sigma'}$$

is not zero. Let m be the cardinality of the orbit of $[\overline{f_{\phi'}}] \in \mathbb{P}(A_{p,\sigma'})$ by the action of $\mathrm{O}(S_{p,\sigma'})$ (or equivalently, in virtue of Remark 7.9, by the action of $Aut(\mathrm{Nef}(X_{p,\sigma'}))$). By Corollary 7.10, the number of Aut-equivalence classes of genus one fibrations contained in the lattice equivalence class $[\phi']$ is at least m, provided that $X_{p,\sigma'}$ is generic.

Remark 9.1. We can regard $S_{p,\sigma'}$ as a submodule of $S_{p,\sigma} \otimes \mathbb{Q}$ by j. Then $S_{p,\sigma'}^{\vee}$ is equal to $(1/p)S_{p,\sigma}$. Hence $(1/p)j(f_{\phi})$ is contained in $S_{p,\sigma'}^{\vee}$.

As a consequence of the fact that $Aut(\mathrm{Nef}(X_{3,10}))$ contains the subgroup $\mathrm{PGU}(4, \mathbb{F}_9)'$ of order 13063680, we obtain the following:

Proposition 9.2. *Suppose that $X_{3,10}$ is generic. Then there exists a genus one fibration on $X_{3,10}$ whose lattice equivalence class contains at least* 6531840 Aut*-equivalence classes.*

Proof. Let (w, x, y) be the affine coordinates of the Fermat quartic surface

$$X_{\mathrm{FQ}} = \{w^4 + x^4 + y^4 + 1 = 0\}$$

in characteristic 3, and let i denote $\sqrt{-1} \in \mathbb{F}_9$. Consider the following ten lines on $X_{\mathrm{FQ}} \cong X_{3,1}$:

$$\begin{array}{ll}
\ell_1 := \{\, w + (1+i) \;=\; x + (1+i)\, y = 0\}, & \ell_2 := \{\, w + (1+i) \;=\; x + (1-i)\, y = 0\}, \\
\ell_3 := \{\, w + i\, y - i \;=\; x + i\, y + i \;= 0\}, & \ell_4 := \{\, w + i\, y + 1 \;=\; x + i\, y - 1 \;= 0\}, \\
\ell_5 := \{\, w - \; y + 1 \;=\; x - \; y - 1 \;= 0\}, & \ell_6 := \{\, w - i\, y - 1 \;=\; x + \; y + i \;= 0\}, \\
\ell_7 := \{\, w + (1-i) \;=\; x - (1+i)\, y = 0\}, & \ell_8 := \{\, w - (1-i)\, y = \; x + (1+i) \;= 0\}, \\
\ell_9 := \{\, w + (1+i)\, x = \; y + (1-i) \;= 0\}, & \ell_{10} := \{\, w + i\, y - 1 \;=\; x - i\, y - 1 \;= 0\}.
\end{array}$$

They form a configuration of (-2)-curves whose dual graph is the affine Dynkin diagram of type $\widetilde{A}_9$. Then the class $f_\phi := \sum_{k=1}^{10} [\ell_k]$ defines a genus one fibration $\phi : X_{3,1} \to \mathbb{P}^1$ in the lattice equivalence class No. 20 of Table 4.2. The line defined by $\{\, w + \; y + 1 \;=\; x + i\, y - i \;= 0\}$ provides us with a section of ϕ that intersects ℓ_{10}.

Let $\phi' : X_{3,10} \to \mathbb{P}^1$ be a genus one fibration corresponding to ϕ by Theorem 1.3. Since the Néron-Severi lattice of X_{FQ} is generated by the classes of lines, we can calculate the action of $\mathrm{PGU}(4, \mathbb{F}_9)$ on $S_{3,1}$ from the permutations of lines induced by $\mathrm{PGU}(4, \mathbb{F}_9)$, and thus we can calculate the action of $\mathrm{PGU}(4, \mathbb{F}_9)'$ on $S_{3,10}$. By computer, we calculate the action of $\mathrm{PGU}(3, \mathbb{F}_4)'$ on the vector space $A_{3,10} \cong \mathbb{F}_3^{20}$. It turns out that the stabilizer subgroup of the non-zero vector

$$(1/3)j(f_\phi) \bmod S_{3,10} \in A_{3,10}$$

is trivial. Hence the orbit of $[\overline{f_{\phi'}}] \in \mathbb{P}(A_{3,10}) \cong \mathbb{P}^{19}(\mathbb{F}_3)$ by the action of $\mathrm{PGU}(4, \mathbb{F}_9)'$ contains at least $|\mathrm{PGU}(4, \mathbb{F}_9)|/|\mathbb{F}_3^{\times}|$ points. □

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