# Singular K3 surfaces and non-homeomorphic conjugate varieties

**Tokyo, 2007 December**

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- By a lattice, we mean a finitely generated free $\mathbb{Z}$-module $\Lambda$ equipped with a non-degenerate symmetric bilinear form
$$\Lambda \times \Lambda \quad \to \quad \mathbb{Z}.$$
- A lattice $\Lambda$ is said to be *even* if $(v, v) \in 2\mathbb{Z}$ for any $v \in \Lambda$.

## §1. Introduction

For a $K3$ surface $X$ defined over a field $k$, we denote by $\mathrm{NS}(X)$ the Néron-Severi lattice of $X \otimes \bar{k}$.

**Definition.** A $K3$ surface $X$ defined over a field of characteristic 0 is said to be *singular* if $\mathrm{rank}(\mathrm{NS}(X))$ attains the possible maximum 20. For a singular $K3$ surface $X$, we put

$$d(X) := \mathrm{disc}(\mathrm{NS}(X)),$$

which is a negative integer.

Shioda and Inose showed that every singular $K3$ surface $X$ is defined over a number field $F$. We denote by $\mathrm{Emb}(F, \mathbb{C})$ the set of embeddings of $F$ into $\mathbb{C}$, and investigate the the transcendental lattice

$$T(X^\sigma) := (\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X^\sigma, \mathbb{Z}))^\perp$$

for each embedding $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, where $X^\sigma$ is the complex $K3$ surface $X \otimes_{F,\sigma} \mathbb{C}$. Note that each $T(X^\sigma)$ is a positive-definite even lattice of rank 2 with discriminant $-d(X)$.

# §2. Shioda-Mitani-Inose theory

For a negative integer $d$, we put

$$\mathcal{M}_d := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z},\ a > 0,\ c > 0, \\ b^2 - 4ac = d \end{array} \right\},$$

on which $GL_2(\mathbb{Z})$ acts by $M \mapsto {}^t g M g$, where $M \in \mathcal{M}_d$ and $g \in GL_2(\mathbb{Z})$. We denote the set of isomorphism classes of even, positive-definite lattices (resp. *oriented* lattices) of rank 2 with discriminant $-d$ by

$$\mathcal{L}_d := \mathcal{M}_d / \, GL_2(\mathbb{Z}) \quad (\text{resp. } \widetilde{\mathcal{L}}_d := \mathcal{M}_d / \, SL_2(\mathbb{Z}) \ ).$$

Let $S$ be a *complex* singular $K3$ surface. By the Hodge decomposition

$$T(S) \otimes \mathbb{C} = \mathrm{H}^{2,0}(S) \oplus \mathrm{H}^{0,2}(S),$$

we can define a canonical orientation on $T(S)$. We denote by $\widetilde{T}(S)$ the *oriented* transcendental lattice of $S$, and by $[\widetilde{T}(S)] \in \widetilde{\mathcal{L}}_{d(S)}$ the isomorphism class of the oriented transcendental lattice.

**Theorem (Shioda and Inose).** The map $S \mapsto [\widetilde{T}(S)]$ induces a bijection from the set of isomorphism classes of complex singular $K3$ surfaces to the set $\bigcup_d \widetilde{\mathcal{L}}_d$ of isomorphism classes of even, positive-definite oriented lattices of rank 2.

In fact, Shioda and Inose gave an explicit construction of a complex singular $K3$ surface with a given oriented transcendental lattice. Suppose that

$$\widetilde{T}_0 = \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \quad \text{with} \quad d := b^2 - 4ac < 0$$

is given. We put

$E' := \mathbb{C}/(\mathbb{Z} + \tau'\mathbb{Z})$, where $\tau' = (-b + \sqrt{d})/(2a)$, and
$E := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ , where $\tau = (b + \sqrt{d})/2$.

**Theorem (Shioda and Mitani).** The oriented transcendental lattice $\widetilde{T}(E' \times E)$ of the abelian surface $E' \times E$ is isomorphic to $\widetilde{T}_0$.

Shioda and Inose showed that, on the Kummer surface $\mathrm{Km}(E' \times E)$, there are effective divisors $C$ and $\Theta$ such that

(1) $C = C_1 + \cdots + C_8$ and $\Theta = \Theta_1 + \cdots + \Theta_8$ are disjoint,

(2) $C$ is an $ADE$-configuration of $(-2)$-curves of type $\mathbb{E}_8$,

(3) $\Theta$ is an $ADE$-configuration of $(-2)$-curves of type $8\mathbb{A}_1$,

(4) there is $[\mathcal{L}] \in \mathrm{NS}(\mathrm{Km}(E' \times E))$ such that $2[\mathcal{L}] = [\Theta]$.

We make the diagram

$$Y \leftarrow \widetilde{Y} \rightarrow \mathrm{Km}(E' \times E),$$

where $\widetilde{Y} \rightarrow \mathrm{Km}(E' \times E)$ is the double covering branching exactly along $\Theta$, and $Y \leftarrow \widetilde{Y}$ is the contraction of the $(-1)$-curves on $\widetilde{Y}$ (that is, the inverse images of $\Theta_1, \ldots, \Theta_8$).

**Theorem (Shioda and Inose).** The surface $Y$ is a singular $K3$ surface, and the diagram

$$Y \longleftarrow \widetilde{Y} \longrightarrow \mathrm{Km}(E' \times E) \longleftarrow \widetilde{E' \times E} \longrightarrow E' \times E$$

induces an isomorphism

$$\widetilde{T}(Y) \;\cong\; \widetilde{T}(E' \times E) \;(\cong\; \widetilde{T}_0\;)$$

of the oriented transcendental lattices.

## §3. Genera of lattices

**Definition.** Two lattices

$$\boldsymbol{\lambda} : \boldsymbol{\Lambda} \times \boldsymbol{\Lambda} \to \mathbb{Z} \quad \text{and} \quad \boldsymbol{\lambda}' : \boldsymbol{\Lambda}' \times \boldsymbol{\Lambda}' \to \mathbb{Z}$$

are said to be *in the same genus* if

$$\boldsymbol{\lambda} \otimes \mathbb{Z}_p \ : \ \boldsymbol{\Lambda} \otimes \mathbb{Z}_p \times \boldsymbol{\Lambda} \otimes \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{and}$$

$$\boldsymbol{\lambda}' \otimes \mathbb{Z}_p \ : \ \boldsymbol{\Lambda}' \otimes \mathbb{Z}_p \times \boldsymbol{\Lambda}' \otimes \mathbb{Z}_p \to \mathbb{Z}_p$$

are isomorphic for any $p$ including $p = \infty$, where $\mathbb{Z}_\infty = \mathbb{R}$.

We have the following:

**Theorem (Nikulin).** Two even lattices of the same rank are in the same genus if and only if they have the same signature and their discriminant forms are isomorphic.

**Definition.** Let $\Lambda$ be an even lattice. Then $\Lambda$ is canonically embedded into

$$\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z})$$

as a subgroup of finite index, and we have a natural symmetric bilinear form

$$\Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}$$

that extends the symmetric bilinear form on $\Lambda$. The finite abelian group

$$D_\Lambda := \Lambda^\vee / \Lambda,$$

together with the natural quadratic form

$$q_\Lambda : D_\Lambda \to \mathbb{Q}/2\mathbb{Z}$$

is called the *discriminant form of* $\Lambda$.

**Proposition.** Suppose that an even lattice $M$ is embedded into an even unimodular lattice $L$ primitively. Let $N$ denote the orthogonal complement of $M$ in $L$. Then we have

$$(D_M, q_M) \cong (D_N, -q_N)$$

**Proposition.** Let $X$ be a singular $K3$ surface defined over a number field $F$. For $\sigma, \sigma' \in \mathrm{Emb}(F, \mathbb{C})$, the lattices $T(X^\sigma)$ and $T(X^{\sigma'})$ are in the same genus.

This follows from Nikulin's theorem. We have

$$\mathrm{NS}(X) \cong \mathrm{NS}(X^\sigma) \cong \mathrm{NS}(X^{\sigma'}).$$

Since $\mathrm{H}^2(X^\sigma, \mathbb{Z})$ is unimodular, the discriminant form of $T(X^\sigma)$ is isomorphic to $(-1)$ times the discriminant form of $\mathrm{NS}(X^\sigma)$:

$$(D_{T(X^\sigma)}, q_{T(X^\sigma)}) \cong (D_{\mathrm{NS}(X^\sigma)}, -q_{\mathrm{NS}(X^\sigma)}).$$

The same holds for $T(X^{\sigma'})$. Hence $T(X^\sigma)$ and $T(X^{\sigma'})$ have the isomorphic discriminant forms.

**Theorem (S.- and Schütt).** Let $\mathcal{G} \subset \mathcal{L}_d$ be a genus of even positive-definite lattices of rank 2, and let $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{L}}_d$ be the pull-back of $\mathcal{G}$ by the natural projection $\widetilde{\mathcal{L}}_d \to \mathcal{L}_d$. Then there exists a singular $K3$ surface $X$ defined over a number field $F$ such that the set

$$\{\ [\widetilde{T}(X^\sigma)]\ \mid\ \sigma \in \mathrm{Emb}(F, \mathbb{C})\ \} \ \subset\ \widetilde{\mathcal{L}}_d$$

coincides with the oriented genus $\widetilde{\mathcal{G}}$.

Let $\widetilde{T}_0$ be an element of the oriented genus $\widetilde{\mathcal{G}}$, and let $Y$ be a singular $K3$ surface such that $\widetilde{T}(Y) \cong \widetilde{T}_0$. We consider the Shioda-Inose-Kummer diagram

$$Y \longleftarrow \widetilde{Y} \longrightarrow \mathrm{Km}(E' \times E) \longleftarrow \widetilde{E' \times E} \longrightarrow E' \times E,$$

which we assume to be defined over a number field $F$. Then, for each $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, the diagram

$$Y^\sigma \longleftarrow \widetilde{Y}^\sigma \longrightarrow \mathrm{Km}(E' \times E)^\sigma \longleftarrow \widetilde{E'^\sigma \times E^\sigma} \longrightarrow E'^\sigma \times E^\sigma$$

induces $\widetilde{T}(Y^\sigma) \cong \widetilde{T}(E'^\sigma \times E^\sigma)$. The lattice $\widetilde{T}(E'^\sigma \times E^\sigma)$ can be calculated from $\widetilde{T}(E' \times E)$ by the classical class field theory of imaginary quadratic fields.

## §4. Non-homeomorphic conjugate varieties

We denote by $\mathrm{Emb}(\mathbb{C}, \mathbb{C})$ the set of embeddings $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ of the complex number field $\mathbb{C}$ into itself.

**Definition.** For a complex variety $X$ and $\sigma \in \mathrm{Emb}(\mathbb{C}, \mathbb{C})$, we define a complex variety $X^\sigma$ by the following diagram of the fiber product:

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\sigma^*} & \operatorname{Spec}\mathbb{C}. \end{array}$$

Two complex varieties $X$ and $X'$ are said to be *conjugate* if there exists $\sigma \in \mathrm{Emb}(\mathbb{C}, \mathbb{C})$ such that $X'$ is isomorphic to $X^\sigma$ over $\mathbb{C}$.

It is obvious from the definition that conjugate varieties are homeomorphic in Zariski topology.

**Problem.** How about in the classical complex topology?

**We have the following:**

**Example (Serre (1964)).** There exist conjugate smooth projective varieties $X$ and $X^\sigma$ such that their topological fundamental groups are *not* isomorphic:

$$\pi_1(X) \not\cong \pi_1(X^\sigma).$$

In particular, $X$ and $X^\sigma$ are not homotopically equivalent.

**Grothendieck's dessins d'enfant (1984).**
Let $f : C \longrightarrow \mathbb{P}^1$ be a finite covering defined over $\overline{\mathbb{Q}}$ branching only at the three points $0, 1, \infty \in \mathbb{P}^1$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consider the conjugate covering

$$f^\sigma : C^\sigma \longrightarrow \mathbb{P}^1.$$

Then $f$ and $f^\sigma$ are topologically distinct in general.

Belyi's theorem asserts that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of topological types of the covering of $\mathbb{P}^1$ branching only at $0, 1, \infty$ is faithful.

Other examples of non-homeomorphic conjugate varieties.

- Abelson: Topologically distinct conjugate varieties with finite fundamental group.
  Topology 13 (1974).
- Artal Bartolo, Carmona Ruber, Cogolludo Agustín: Effective invariants of braid monodromy.
  Trans. Amer. Math. Soc. 359 (2007).
- S.-: On arithmetic Zariski pairs in degree 6.
  arXiv:math/0611596
- S.-: Non-homeomorphic conjugate complex varieties.
  arXiv:math/0701115
- Easton, Vakil: Absolute Galois acts faithfully on the components of the moduli space of surfaces: A Belyi-type theorem in higher dimension. arXiv:0704.3231
- Bauer, Catanese, Grunewald: The absolute Galois group acts faithfully on the connected components of the moduli space of surfaces of general type. arXiv:0706.1466
- F. Charles: Conjugate varieties with distinct real cohomology algebras. arXiv:0706.3674

Let $V$ be an oriented topological manifold of real dimension 4. We put

$$\mathrm{H}_2(V) := \mathrm{H}_2(V, \mathbb{Z})/\text{torsion},$$

and let

$$\iota_V \,:\, \mathrm{H}_2(V) \times \mathrm{H}_2(V) \;\to\; \mathbb{Z}$$

be the intersection pairing. We then put

$$J_\infty(V) \;:=\; \bigcap_K \mathrm{Im}(\mathrm{H}_2(V \setminus K) \to \mathrm{H}_2(V)),$$

where $K$ runs through the set of compact subsets of $V$, and set

$$\widetilde{B}_V := \mathrm{H}_2(V)/J_\infty(V) \quad \text{and} \quad B_V := (\widetilde{B}_V)/\text{torsion}.$$

Since any topological cycle is compact, the intersection pairing $\iota_V$ induces a symmetric bilinear form

$$\beta_V \,:\, B_V \times B_V \;\to\; \mathbb{Z}.$$

It is obvious that the isomorphism class of $(B_V, \beta_V)$ is a topological invariant of $V$.

**Theorem.** Let $X$ be a complex smooth projective surface, and let $C_1, \ldots, C_n$ be irreducible curves on $X$. We put

$$V := X \setminus \bigcup C_i.$$

Suppose that the classes $[C_1], \ldots, [C_n]$ span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Then $(B_V, \beta_V)$ is isomorphic to the transcendental lattice

$$T(X) := (\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X))^{\perp}/\text{torsion}.$$

Hence $T(X)$ is a topological invariant of the open surface $V \subset X$.

**Construction of examples.**

Let $T_1$ and $T_2$ be even positive-definite lattices of rank 2 that are in the same genus but not isomorphic. We have a singular $K3$ surface $X$ defined over a number field $F$, and embeddings $\sigma_1, \sigma_2 \in \mathrm{Emb}(F, \mathbb{C})$ such that

$$T(X^{\sigma_1}) \cong T_1 \quad \text{and} \quad T(X^{\sigma_2}) \cong T_2.$$

Let $C_1, \dots, C_n$ be irreducible curves on $X$ whose classes span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Enlarging $F$, we can assume that

$$V := X \setminus \bigcup C_i.$$

is defined over $F$. Then the conjugate open varieties

$$V^{\sigma_1} \quad \text{and} \quad V^{\sigma_2}$$

are not homeomorphic.

# §5. Maximizing sextic

**Definition.** (1) A complex plane curve $C \subset \mathbb{P}^2$ of degree 6 is called a *maximizing sextic* if $C$ has only simple singularities and its total Milnor number is 19.
(2) Two complex projective plane curves $C$ and $C'$ are said to be *conjugate* if there exists $\sigma \in \mathrm{Emb}(\mathbb{C}, \mathbb{C})$ such that $C^\sigma \subset \mathbb{P}^2$ is projectively equivalent to $C' \subset \mathbb{P}^2$.

If $C$ is a maximizing sextic, the minimal resolution $X_C \to Y_C$ of the double cover $Y_C \to \mathbb{P}^2$ of $\mathbb{P}^2$ branching exactly along $C$ is a complex singular $K3$ surface. We put

$$T[C] := T(X_C), \quad \text{and} \quad \widetilde{T}[C] := \widetilde{T}(X_C).$$

**Theorem.** Let $C$ be a maximizing sextic, and let $\widetilde{T}'$ be an oriented lattice such that its underlying (non-oriented) lattice is in the same genus, but not isomorphic, with $T[C]$. Then there is a maximizing sextic $C'$ such that $\widetilde{T}[C'] \cong \widetilde{T}'$, and that $C$ and $C'$ are conjugate, but $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic.

**Definition.** A pair of complex projective plane curves $C$ and $C'$ is called an *arithmetic Zariski pair* if they are conjugate but $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic.

**Remark.** The first example of an arithmetic Zariski pair was discovered by Artal, Carmona and Cogolludo in degree 12 by means of completely different method.

Using Torelli theorem, we can make a complete list of the $ADE$-types of maximizing sextics and their oriented transcendental lattices. Thus we obtain the following complete list of arithmetic Zariski pairs of maximizing sextics.

In the table below, $L[2a, b, 2c]$ denotes the lattice of rank 2 given by the matrix

$$\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

| | | | |
|---|---|---|---|
| 1 | $E_8 + A_{10} + A_1$ | $L[6,2,8]$, | $L[2,0,22]$ |
| 2 | $E_8 + A_6 + A_4 + A_1$ | $L[8,2,18]$, | $L[2,0,70]$ |
| 3 | $E_6 + D_5 + A_6 + A_2$ | $L[12,0,42]$, | $L[6,0,84]$ |
| 4 | $E_6 + A_{10} + A_3$ | $L[12,0,22]$, | $L[4,0,66]$ |
| 5 | $E_6 + A_{10} + A_2 + A_1$ | $L[18,6,24]$, | $L[6,0,66]$ |
| 6 | $E_6 + A_7 + A_4 + A_2$ | $L[24,0,30]$, | $L[6,0,120]$ |
| 7 | $E_6 + A_6 + A_4 + A_2 + A_1$ | $L[30,0,42]$, | $L[18,6,72]$ |
| 8 | $D_8 + A_{10} + A_1$ | $L[6,2,8]$, | $L[2,0,22]$ |
| 9 | $D_8 + A_6 + A_4 + A_1$ | $L[8,2,18]$, | $L[2,0,70]$ |
| 10 | $D_7 + A_{12}$ | $L[6,2,18]$, | $L[2,0,52]$ |
| 11 | $D_7 + A_8 + A_4$ | $L[18,0,20]$, | $L[2,0,180]$ |
| 12 | $D_5 + A_{10} + A_4$ | $L[20,0,22]$, | $L[12,4,38]$ |
| 13 | $D_5 + A_6 + A_5 + A_2 + A_1$ | $L[12,0,42]$, | $L[6,0,84]$ |
| 14 | $D_5 + A_6 + 2A_4$ | $L[20,0,70]$, | $L[10,0,140]$ |
| 15 | $A_{18} + A_1$ | $L[8,2,10]$, | $L[2,0,38]$ |
| 16 | $A_{16} + A_3$ | $L[4,0,34]$, | $L[2,0,68]$ |
| 17 | $A_{16} + A_2 + A_1$ | $L[10,4,22]$, | $L[6,0,34]$ |
| 18 | $A_{13} + A_4 + 2A_1$ | $L[8,2,18]$, | $L[2,0,70]$ |
| 19 | $A_{12} + A_6 + A_1$ | $L[8,2,46]$, | $L[2,0,182]$ |
| 20 | $A_{12} + A_5 + 2A_1$ | $L[12,6,16]$, | $L[4,2,40]$ |
| 21 | $A_{12} + A_4 + A_2 + A_1$ | $L[24,6,34]$, | $L[6,0,130]$ |
| 22 | $A_{10} + A_9$ | $L[10,0,22]$, | $L[2,0,110]$ |
| 23 | $A_{10} + A_9$ | $L[8,3,8]$, | $L[2,1,28]$ |
| 24 | $A_{10} + A_8 + A_1$ | $L[18,0,22]$, | $L[10,2,40]$ |
| 25 | $A_{10} + A_7 + A_2$ | $L[22,0,24]$, | $L[6,0,88]$ |
| 26 | $A_{10} + A_7 + 2A_1$ | $L[10,2,18]$, | $L[2,0,88]$ |
| 27 | $A_{10} + A_6 + A_2 + A_1$ | $L[22,0,42]$, | $L[16,2,58]$ |
| 28 | $A_{10} + A_5 + A_3 + A_1$ | $L[12,0,22]$, | $L[4,0,66]$ |
| 29 | $A_{10} + 2A_4 + A_1$ | $L[30,10,40]$, | $L[10,0,110]$ |
| 30 | $A_{10} + A_4 + 2A_2 + A_1$ | $L[30,0,66]$, | $L[6,0,330]$ |
| 31 | $A_8 + A_6 + A_4 + A_1$ | $L[22,4,58]$, | $L[18,0,70]$ |
| 32 | $A_7 + A_6 + A_4 + A_2$ | $L[24,0,70]$, | $L[6,0,280]$ |
| 33 | $A_7 + A_6 + A_4 + 2A_1$ | $L[18,4,32]$, | $L[2,0,280]$ |
| 34 | $A_7 + A_5 + A_4 + A_2 + A_1$ | $L[24,0,30]$, | $L[6,0,120]$ |

**Definition.** We say that an oriented lattice $\widetilde{T}$ of rank 2 is *real* if it is isomorphic to its reverse; that is, if $\widetilde{T}$ is represented by a $2 \times 2$-matrix $M$, then there exists $P \in GL_2(\mathbb{Z})$ with $\det P = -1$ such that $M = {}^tPMP$.

For a singular $K3$ surface $X$ defined over $\mathbb{C}$, we put

$$\overline{X} := X \otimes_{\mathbb{C},-} \mathbb{C},$$

where $^- : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ is the complex conjugate. Then $\widetilde{T}(\overline{X})$ is the reverse of $\widetilde{T}(X)$. Therefore, if $X$ is defined over $\mathbb{R}$, then $\widetilde{T}(X)$ is real in the sense above.

Let $C$ be a maximizing sextic. If $\widetilde{T}[C]$ is non-real, then the complex singular $K3$ surfaces $X_C$ and $X_{\overline{C}}$ are not isomorphic (since $\widetilde{T}[C]$ is not isomorphic to $\widetilde{T}[\overline{C}]$), and hence $C$ is not projectively equivalent to $\overline{C}$, but $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, \overline{C})$ are obviously homeomorphic.

# §6. Example by Artal, Carmona and Cogolludo

We consider the following cubic extension of $\mathbb{Q}$:

$$K := \mathbb{Q}[t]/(\varphi), \quad \text{where} \quad \varphi = 17t^3 - 18t^2 - 228t + 556.$$

The roots of $\varphi = 0$ are $\alpha, \bar{\alpha}, \beta$, where

$$\alpha = 2.590 \cdots + 1.108 \cdots \sqrt{-1}, \qquad \beta = -4.121 \cdots .$$

There are three corresponding embeddings

$$\sigma_\alpha : K \hookrightarrow \mathbb{C}, \quad \sigma_{\bar{\alpha}} : K \hookrightarrow \mathbb{C} \quad \text{and} \quad \sigma_\beta : K \hookrightarrow \mathbb{C}.$$

There exists a homogeneous polynomial

$$\Phi(x_0, x_1, x_2) \in K[x_0, x_1, x_2]$$

of degree 6 with coefficients in $K$ such that the plane curve

$$C = \{\Phi = 0\}$$

has three simple singular points of type

$$A_{16} + A_2 + A_1$$

as its only singularities.

**Consider the conjugate complex plane curves**

$$C_\alpha = \{\Phi^{\sigma_\alpha} = 0\}, \quad C_{\bar{\alpha}} = \{\Phi^{\sigma_{\bar{\alpha}}} = 0\} \quad \text{and} \quad C_\beta = \{\Phi^{\sigma_\beta} = 0\}.$$

**They show that, if $C'$ is a maximizing sextic of type $A_{16} + A_2 + A_1$, then $C'$ is projectively equivalent to $C_\alpha$, $C_{\bar{\alpha}}$ or $C_\beta$.**

**Using Torelli theorem, we see that their oriented transcendental lattices are**

$$\begin{bmatrix} 10 & \pm 4 \\ \pm 4 & 22 \end{bmatrix} \quad \text{(non-real)}$$

**and**

$$\begin{bmatrix} 6 & 0 \\ 0 & 34 \end{bmatrix} \quad \text{(real)}.$$

**Therefore we have**

$$T[C_\alpha] \cong \begin{bmatrix} 10 & \pm 4 \\ \pm 4 & 22 \end{bmatrix} \quad \text{and} \quad T[C_\beta] \cong \begin{bmatrix} 6 & 0 \\ 0 & 34 \end{bmatrix}.$$

**Hence $(\mathbb{P}^2, C_\alpha)$ and $(\mathbb{P}^2, C_\beta)$ are not homeomorphic.**

# §7. Example by Arima and S.-

There are 4 connected components in the moduli space of maximizing sextics of type

$$A_{10} + A_9.$$

Two of them have irreducible members, and their oriented transcendental lattices are

$$\begin{bmatrix} 10 & 0 \\ 0 & 22 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 110 \end{bmatrix} \quad \text{(both are real).}$$

The other two have reducible members (a line and an irreducible quintic), and their oriented transcendental lattices are

$$\begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix} \quad \text{(both are real).}$$

We will consider these reducible members.

The reducible members are defined over $\mathbb{Q}(\sqrt{5})$. The defining equation is

$$C_{\pm} \quad : \quad z\cdot(G(x,y,z)\pm\sqrt{5}\cdot H(x,y,z))=0,$$

where

$$\begin{aligned} G(x,y,z) &:= -9\,x^4z-14\,x^3yz+58\,x^3z^2-48\,x^2y^2z-\\ &\quad -64\,x^2yz^2+10\,x^2z^3++108\,xy^3z-\\ &\quad -20\,xy^2z^2-44\,y^5+10\,y^4z,\\ H(x,y,z) &:= 5\,x^4z+10\,x^3yz-30\,x^3z^2+30\,x^2y^2z+\\ &\quad +20\,x^2yz^2-40\,xy^3z+20\,y^5. \end{aligned}$$

The singular points are

$$[0:0:1]\ \ (A_{10}) \quad \text{and} \quad [1:0:0]\ \ (A_9).$$

We have two possibilities:

$$T[C_+]\cong\begin{bmatrix}8&3\\3&8\end{bmatrix} \quad \text{and} \quad T[C_-]\cong\begin{bmatrix}2&1\\1&28\end{bmatrix},$$

or

$$T[C_+]\cong\begin{bmatrix}2&1\\1&28\end{bmatrix} \quad \text{and} \quad T[C_-]\cong\begin{bmatrix}8&3\\3&8\end{bmatrix}.$$

**Problem.** Which is the case?

**Remark.** This problem cannot be solved by any algebraic methods.

More generally, we can raise the following:

**Problem.** How to calculate the transcendental lattice of $X_C$ from the defining equation of a maximizing sextic $C$?

This task can be done by a *Zariski-van Kampen method* for homology cycles.

The Zariski-van Kampen method is a method to calculate $\pi_1(\mathbb{P}^2 \setminus C)$ for a projective plane curve $C \subset \mathbb{P}^2$. Tradittionally, the invariant $\pi_1(\mathbb{P}^2 \setminus C)$ has been used to distinguish embedding topologies of plane curves with the same type of singularities.

**Remark.** We have $\pi_1(\mathbb{P}^2 \setminus C_+) \cong \pi_1(\mathbb{P}^2 \setminus C_-) \cong \mathbb{Z}$.

For simplicity, we put $X_\pm := X_{C_\pm}$. Let $D \subset X_\pm$ be the total transform of the union of the lines

$$\{z = 0\} \cup \{x = 0\},$$

on which the two singular points of $C_\pm$ locate, and let $X^0_\pm$ be the complement of $D$. Since the irreducible components of $D$ span $\mathrm{NS}(X_\pm)$, the inclusion $X^0_\pm \hookrightarrow X_\pm$ induces a surjection

$$\mathrm{H}_2(X^0_\pm, \mathbb{Z}) \twoheadrightarrow T(X_\pm).$$

We will describe the generators of $\mathrm{H}_2(X^0_\pm, \mathbb{Z})$ and the intersection numbers among them.

We put

$$f_\pm(y, z) := G(1, y, z) \pm \sqrt{5} \cdot H(1, y, z),$$

and set

$$Q_\pm := \{f_\pm(y, z) = 0\}.$$

Then $Q_\pm$ is a smooth affine quintic curve, and it intersects the line

$$L := \{z = 0\}$$

at the origin with the multiplicity 5. The open surface $X^0_\pm$ is a double covering of $\mathbb{A}^2 \setminus L$ branching along $Q_\pm$.

Let

$$\pi_{\pm} : X^0_{\pm} \longrightarrow \mathbb{A}^2 \setminus L$$

be the double covering. We consider the projection

$$p : \mathbb{A}^2 \longrightarrow \mathbb{A}^1_z \qquad p(y,z) := z$$

and the composite

$$q_{\pm} : X^0_{\pm} \longrightarrow \mathbb{A}^2 \setminus L \longrightarrow \mathbb{A}^1_z.$$

There are four critical points of the finite covering

$$p|Q_{\pm} : Q_{\pm} \longrightarrow \mathbb{A}^1_z.$$

Three of them $R_{\pm}, S_{\pm}, \overline{S}_{\pm}$ are simple critical values, while the critical point over 0 is of multiplicity 5. Their positions are

$$R_+ = 0.42193..., \qquad S_+ = 0.23780... + 0.24431... \cdot \sqrt{-1},$$

and

$$R_- = 0.12593..., \qquad S_- = 27.542... + 45.819... \cdot \sqrt{-1}.$$

We choose a base point $b$ on $\mathbb{A}^1_z$ as a sufficiently small positive real number (say $b = 10^{-3}$), and define the loop $\lambda$ and the paths $\rho_{\pm}$, $\sigma_{\pm}$, $\bar{\sigma}_{\pm}$ on the $z$-line $\mathbb{A}^1_z$ as in the figure:

**on** $\mathbb{A}^1_z$

We put

$$\mathbb{A}^1_y := p^{-1}(b), \quad F_\pm := q_\pm^{-1}(b) = \pi_\pm^{-1}(\mathbb{A}^1_y).$$

Then the morphism

$$\pi_\pm | F_\pm : F_\pm \longrightarrow \mathbb{A}^1_y$$

is the double covering branching exactly at the five points $\mathbb{A}^1_y \cap Q_\pm$. Hence $F_\pm$ is a genus 2 curve minus one point.

We choose a system of oriented simple closed curves $a_1, \ldots, a_5$ on $F_\pm$ in such a way that their images by the double covering

$$\pi_\pm | F_\pm : F_\pm \longrightarrow \mathbb{A}^1_y$$

are given in the figure and that the orientations are given so that

$$a_i a_{i+1} = -a_{i+1} a_i = 1$$

holds for $i = 1, \ldots, 5$, where $a_6 := a_1$. Then $\mathrm{H}_1(F_\pm, \mathbb{Z})$ is generated by $[a_1], \ldots, [a_4]$, and we have

$$[a_5] = -[a_1] - [a_2] - [a_3] - [a_4].$$

$$\textbf{on } \boldsymbol{p}^{-1}(\boldsymbol{b}) = \mathbb{A}^1_{\boldsymbol{y}}$$

The monodromy along the loop $\lambda$ around $z = 0$ is given by

$$a_i \mapsto a_{i+1}.$$

Hence the open surface $X^0_\pm$ is homotopically equivalent to the 2-dimensional $CW$-complex obtained from $F_\pm$ by attaching

- four tubes

$$T_i := S^1 \times I \quad (i = 1, \ldots, 4)$$

  with $\partial T_i = a_{i+1} - a_i$ and zipping up $\{0\} \times I$ together appropriately,

  and

- three thimbles

$$\Theta(\rho_\pm), \quad \Theta(\sigma_\pm), \quad \Theta(\bar{\sigma}_\pm)$$

  corresponding to the vanishing cycles on $F_\pm$ for the simple critical values $R_\pm$, $S_\pm$ and $\overline{S}_\pm$.

Hence the homology group $\mathrm{H}_2(X^0_{\pm}, \mathbb{Z})$ is equal to the kernel of the homomorphism

$$\bigoplus_{i=1}^{4} \mathbb{Z}[T_i] \oplus \mathbb{Z}[\Theta(\rho_{\pm})] \oplus \mathbb{Z}[\Theta(\sigma_{\pm})] \oplus \mathbb{Z}[\Theta(\bar{\sigma}_{\pm})] \longrightarrow \bigoplus_{i=1}^{4} \mathbb{Z}[a_i]$$

given by $[M] \mapsto [\partial(M)]$. Therefore the problem is reduced to the calculation of the vanishing cycles $\partial\Theta(\rho_{\pm})$, $\partial\Theta(\sigma_{\pm})$ and $\partial\Theta(\bar{\sigma}_{\pm})$.

**When $z$ moves from $b$ to $R_\pm$ along the path $\rho_\pm$, the branch points $p^{-1}(z) \cap Q_\pm$ moves as follows:**

**Therefore, putting an orientation on the thimble, we have**

$$[\partial\Theta(\rho_+)] = [a_1] - [a_2] + [a_3] - [a_4],$$

**while**

$$[\partial\Theta(\rho_-)] = [a_2] + [a_3].$$

**When $z$ moves from $b$ to $S_\pm$ along the path $\sigma_\pm$, the branch points $p^{-1}(z) \cap Q_\pm$ moves as follows:**

**Therefore, putting an orientation on the thimble, we have**

$$[\partial\Theta(\sigma_+)] = [a_1] - [a_2] - [a_3],$$

**while**

$$[\partial\Theta(\sigma_-)] = 2[a_1] - [a_2] - [a_3] - [a_4].$$

**Thus we can describe the generators of $H_2(X^0_\pm, \mathbb{Z})$ explicitly. It is a free $\mathbb{Z}$-module of rank 3. The intersection numbers between them are calculated by perturbing these cycles:**

**By this calculation, we obtain the following:**

**Proposition.**

$$T[C_+] \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T[C_-] \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$