Dessins d'enfants and transcendental lattices of singular $K3$ surfaces

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Dessins d'enfants and transcendental lattices of extremal elliptic surfaces

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- **By a lattice, we mean a finitely generated free $\mathbb{Z}$-module Λ equipped with a non-degenerate symmetric bilinear form**

$$\Lambda \times \Lambda \;\to\; \mathbb{Z}.$$

§1. Introduction of the theory of dessins

Definition. A *dessin d'enfant* (a *dessin*, for short) is a connected graph that is bi-colored (i.e., each vertex is colored by black or while, and every edge connects a black vertex and a white vertex) and oriented (i.e., for each vertex, a cyclic ordering is given to the set of edges emitting from the vertex). Two dessins are *isomorphic* if there exists an isomorphism of graphs between them that preserves the coloring and the orientation.

We denote by $\mathcal{D}(n)$ the set of isomorphism classes of dessins with n edges.

Definition. A *permutation pair* is a pair (σ_0, σ_1) of elements of the symmetric group S_n such that the subgroup $\langle \sigma_0, \sigma_1 \rangle \subset S_n$ is a transitive permutation group.
Two permutation pairs (σ_0, σ_1) and (σ_0', σ_1') are *isomorphic* if there exists $g \in S_n$ such that $\sigma_0' = g^{-1}\sigma_0 g$ and $\sigma_1' = g^{-1}\sigma_1 g$ hold.

We denote by $\mathcal{P}(n)$ the set of isomorphism classes $[\sigma_0, \sigma_1]$ of permutation pairs (σ_0, σ_1) of elements of S_n.

Definition. A *Belyĭ pair* is a pair (C, β) of a compact connected Riemann surface C and a finite morphism $C \longrightarrow \mathbb{P}^1$ that is étale over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
Two Belyĭ pairs (C, β) and (C', β') are *isomorphic* if there exists an isomorphism $\phi : C \cong C'$ such that $\phi \circ \beta' = \beta$.

We denote by $\mathcal{B}(n)$ the set of isomorphism classes of Belyĭ pairs of degree n.

Proposition. For each n, there exist canonical bijections

$$\mathcal{D}(n) \xrightarrow{\sim} \mathcal{P}(n) \xrightarrow{\sim} \mathcal{B}(n).$$

Proof. First we define $f_{\mathcal{DP}} : \mathcal{D}(n) \to \mathcal{P}(n)$. Let $D \in \mathcal{D}(n)$ be given. We number the edges of D by $1, \ldots, n$, and let $\sigma_0 \in S_n$ (resp. $\sigma_1 \in S_n$) be the product of the cyclic permutations of the edges at the black (resp. while) vertices coming from the cyclic ordering. Since D is connected, $\langle \sigma_0, \sigma_1 \rangle$ is transitive. The isomorphism class $[\sigma_0, \sigma_1]$ does not depend on the choice of the numbering of edges. Hence $f_{\mathcal{DP}}(D) := [\sigma_0, \sigma_1]$ is well-defined.

Next, we define $f_{\mathcal{PB}} : \mathcal{P}(n) \to \mathcal{B}(n)$. We choose a base point $b_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ on the real open segment $(0, 1) \subset \mathbb{R}$, and consider the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_0)$, which is a free group generated by the homotopy classes γ_0 and γ_1 of the loops depicted below:

Let $[\sigma_0, \sigma_1] \in \mathcal{P}(n)$ be given. Then we have an étale covering of degree n

$$\beta^0 : C^0 \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

corresponding the homomorphim $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, b_0) \to S_n$ defined by $\gamma_0 \mapsto \sigma_0$ and $\gamma_1 \mapsto \sigma_1$. Compactifying (C^0, β^0), we obtain a Belyĭ pair $f_{\mathcal{PB}}([\sigma_0, \sigma_1]) := (C, \beta)$.

Finally, we define $f_{\mathcal{BD}} : \mathcal{B}(n) \to \mathcal{D}(n)$. Suppose that a Belyĭ pair $(C, \beta) \in \mathcal{B}(n)$ be given. Let D be the bi-colored graph such that the black vertices are $\beta^{-1}(0)$, the white vertices are $\beta^{-1}(1)$, and the edges are $\beta^{-1}(I)$, where $I := [0, 1] \subset \mathbb{R}$ is the closed interval. Then D is connected, since C is connected. We then give a cyclic ordering on the set of edges emitting from each vertex by means of the orientation of C induced by the complex structure of C.

These three maps $f_{\mathcal{DP}}$, $f_{\mathcal{PB}}$ and $f_{\mathcal{BD}}$ yield the bijections

$$\mathcal{D}(n) \xrightarrow{\sim} \mathcal{P}(n) \xrightarrow{\sim} \mathcal{B}(n).$$

□

Proposition. (1) If (C, β) is a Belyĭ pair, then (C, β) can be defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$.
(2) If Belyĭ pairs (C, β) and (C', β') over $\overline{\mathbb{Q}}$ are isomorphic, then the isomorphism is defined over $\overline{\mathbb{Q}}$.

Corollary. For each n, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{D}(n) \cong \mathcal{P}(n) \cong \mathcal{B}(n)$.

Theorem (Belyĭ). A non-singular curve C over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if there exists a finite morphism $\beta : C \longrightarrow \mathbb{P}^1$ such that (C, β) is a Belyĭ pair.

Corollary. We put $\mathcal{B} := \cup_n \mathcal{B}(n)$. Then the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathcal{B}$ is faithful.

Indeed, considering the j-invariants of elliptic curves over $\overline{\mathbb{Q}}$, we see that the action is faithful on a subset $\mathcal{B}_1 \subset \mathcal{B}$ of Belyĭ pairs of genus 1. In fact, the action is faithful on a subset $\mathcal{B}_{0,\mathrm{tree}} \subset \mathcal{B}$ of Belyĭ pairs of genus 0 whose dessins are trees (L. Schneps, H. W. Lenstra, Jr).

§2. Elliptic surfaces of Belyĭ type

The goal is to introduce an invariant of dessins by means of elliptic surfaces.

By an elliptic surface, we mean a non-singular compact complex relatively-minimal elliptic surface $\varphi : X \to C$ with a section $O_\varphi : C \to X$. We denote by

$$\Sigma_\varphi \subset C$$

the finite set of points $v \in C$ such that $\varphi^{-1}(v)$ is singular, by

$$J_\varphi : C \to \mathbb{P}^1$$

the functional invariant of $\varphi : X \to C$, and by

$$h_\varphi : \pi_1(C \setminus \Sigma_\varphi, b) \to \mathrm{Aut}(H_1(E_b)) \cong SL_2(\mathbb{Z})$$

the homological invariant of $\varphi : X \to C$, where $b \in C \setminus \Sigma_\varphi$ is a base point, and $H_1(E_b)$ is the first homology group $H_1(E_b, \mathbb{Z})$ of $E_b := \varphi^{-1}(b)$ with the intersection pairing.

Definition. An elliptic surface $\varphi : X \to C$ is *of Belyĭ type* if (C, J_φ) is a Belyĭ pair and $\Sigma_\varphi \subset J_\varphi^{-1}(\{0, 1, \infty\})$.

Consider the homomorphim

$$\bar{h} \;:\; \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, b_0) = \langle \gamma_0, \gamma_1 \rangle \to PSL_2(\mathbb{Z})$$

given by

$$\bar{h}(\gamma_0) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \bmod \pm I_2, \qquad \bar{h}(\gamma_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bmod \pm I_2.$$

Let (C, β) be a Belyĭ pair, and let $b \in C$ be a point such that $\beta(b) = b_0$. Then the elliptic surfaces $\varphi : X \to C$ of Belyĭ type with $J_\varphi = \beta$ are in one-to-one correspondence with the homomorphisms

$$h \;:\; \pi_1(C \setminus \beta^{-1}(\{0,1,\infty\}), b) \;\to\; SL_2(\mathbb{Z})$$

that make the following diagram commutative:

$$\begin{array}{ccc} \pi_1(C \setminus \beta^{-1}(\{0,1,\infty\}), b) & \xrightarrow{h} & SL_2(\mathbb{Z}) \\ \beta_* \downarrow & & \downarrow \\ \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, b_0) & \xrightarrow[\bar{h}]{} & PSL_2(\mathbb{Z}). \end{array}$$

We denote by

$$\mathrm{NS}(X) := (H^2(X, \mathbb{Z})/\text{torsion}) \cap H^{1,1}(X)$$

the Néron-Severi lattice of X, and by P_φ the sublattice of NS(X) generated by the classes of the section O_φ and the irreducible components of singular fibers.

Definition. **An elliptic surface $\varphi : X \to C$ is *extremal* if**

$$P_\varphi \otimes \mathbb{C} = \mathrm{NS}(X) \otimes \mathbb{C} = H^{1,1}(X);$$

(that is, the Picard number of X is equal to $h^{1,1}(X)$, and the Mordell-Weil rank is 0.)

Theorem (Mangala Nori). Let $\varphi : X \to C$ be an elliptic surface. Suppose that J_φ is non-constant. Then $\varphi : X \to C$ is extremal if and only if the following hold:

- $\varphi : X \to C$ is of Belyĭ type,
- the dessin of (C, J_φ) has valencies ≤ 3 at the black vertices, and valencies ≤ 2 at the white vertices, and
- there are no singular fibers of type I_0^*, II, III or IV.

Example. A $K3$ surface of Picard number 20 with the transcendental lattice

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

has a structure of the extremal elliptic surface with singular fibers of the type I_0^*, II^*, IV^*. The J-invariant of this elliptic $K3$ surface is therefore constant 0.

We define a *topological* invariant Q_φ of an elliptic surface $\varphi : X \to C$. We put

$$X_\varphi^0 := X \setminus (\varphi^{-1}(\Sigma_\varphi) \cup O_\varphi(C)),$$

and let

$$H_2(X_\varphi^0) := H_2(X_\varphi^0, \mathbb{Z})/\text{torsion}$$

be the second homology group modulo the torsion with the intersection pairing

$$(\ \ ,\ \ ) : H_2(X_\varphi^0) \times H_2(X_\varphi^0) \to \mathbb{Z}.$$

We then put

$$I(X_\varphi^0) := \{\, x \in H_2(X_\varphi^0) \ \mid \ (x, y) = 0 \ \text{ for all } \ y \,\},$$

and

$$Q_\varphi := H_2(X_\varphi^0)/I(X_\varphi^0).$$

Then Q_φ is torsion-free, and $(\ \ ,\ \ )$ induces a non-degenerate symmetric bilinear form on Q_φ. Thus Q_φ is a lattice.

Proposition. The invariant Q_φ is isomorphic to the orthogonal complement of

$$P_\varphi = \langle O_\varphi, \text{the irred. components in fibers}\rangle \subset H^2(X)$$

in $H^2(X)$.

Corollary. If $\varphi : X \to C$ is an extremal elliptic surface, then Q_φ is isomorphic to the transcendental lattice of X.

We can calculate Q_φ from the homological invariant

$$h_\varphi : \pi_1(C \setminus \Sigma_\varphi, b) \to \mathrm{Aut}(H_1(E_b)).$$

For simplicity, we assume that $r := |\Sigma_\varphi| > 0$. We choose loops

$$\lambda_i : I \to C \setminus \Sigma_\varphi \quad (i = 1, \ldots, N := 2g(C) + r - 1)$$

with the base point b such that their union is a strong deformation retract of $C \setminus \Sigma_\varphi$. Then $\pi_1(C \setminus \Sigma_\varphi, b)$ is a free group generated by $[\lambda_1], \ldots, [\lambda_N]$. Then X^0_φ is homotopically equivalent to a topological space obtained from

$$E_b \setminus \{O_\varphi(b)\} \quad \sim \quad \mathbb{S}^1 \vee \mathbb{S}^1$$

by attaching $2N$ tubes $\mathbb{S}^1 \times I$, two of which lying over each loop λ^i.

We prepare N copies of $H_1(E_b) \cong \mathbb{Z}^2$, and consider the homomorphism

$$\partial \ : \ \bigoplus_{i=1}^{N} H_1(E_b) \ \to \ H_1(E_b)$$

defined by

$$\partial(x_1, \ldots, x_N) := \sum_{i=1}^{N} (h_\varphi([\lambda_i])x_i - x_i).$$

Then $H_2(X_\varphi^0)$ is isomorphic to $\operatorname{Ker} \partial$. The intersection pairing on $H_2(X_\varphi^0)$ is calculated by perturbing the loops λ_i to the loops λ_i' with the base point $b' \neq b$.

§3. An invariant of dessins

Let (C, β) be a Belyĭ pair. We put

$$\begin{aligned}
\beta^{-1}(0) &= \beta^{-1}(0)_{0(3)} \sqcup \beta^{-1}(0)_{1(3)} \sqcup \beta^{-1}(0)_{2(3)}, \\
\beta^{-1}(1) &= \beta^{-1}(1)_{0(2)} \sqcup \beta^{-1}(1)_{1(2)}, \\
\beta^{-1}(\infty) &= \beta^{-1}(\infty)_1 \sqcup \beta^{-1}(\infty)_2 \sqcup \beta^{-1}(\infty)_3 \sqcup \dots,
\end{aligned}$$

where

$$\begin{aligned}
\beta^{-1}(p)_{a(m)} &= \left\{ x \in \beta^{-1}(p) \;\middle|\; \begin{array}{l} \text{the ramification index} \\ \text{of } \beta \text{ at } x \text{ is } \equiv a \bmod m \end{array} \right\}, \\
\beta^{-1}(\infty)_b &= \left\{ x \in \beta^{-1}(\infty) \;\middle|\; \begin{array}{l} \beta \text{ has a pole of order } b \\ \text{at } x \end{array} \right\}.
\end{aligned}$$

A *type-specification* is a list

$$s = [s_{00}, s_{01}, s_{02}, s_{10}, s_{11}, s_{\infty b}\ (b = 1, 2, \dots)]$$

of maps, where

$$\begin{aligned}
s_{00} &: \beta^{-1}(0)_{0(3)} \longrightarrow \{I_0, I_0^*\},\\
s_{01} &: \beta^{-1}(0)_{1(3)} \longrightarrow \{II, IV^*\},\\
s_{02} &: \beta^{-1}(0)_{2(3)} \longrightarrow \{II^*, IV\},\\
s_{10} &: \beta^{-1}(1)_{0(2)} \longrightarrow \{I_0, I_0^*\},\\
s_{11} &: \beta^{-1}(1)_{1(2)} \longrightarrow \{III, III^*\},\\
s_{\infty b} &: \beta^{-1}(\infty)_b \longrightarrow \{I_b, I_b^*\}.
\end{aligned}$$

If $g(C) > 0$, then, for each type-specification s, there exist exactly $2^{2g(C)-1}$ elliptic surfaces $\varphi : X \to C$ of Belyĭ type such that $J_\varphi = \beta$ and that the types of singular fibers are s.

If $g(C) = 0$, then, for exactly half of all the type-specifications s, there exists an elliptic surface $\varphi : X \to C$ of Belyĭ type (unique up to isomorphism) such that $J_\varphi = \beta$ and that the types of singular fibers are s.

The set of pairs of the type-specification s and the invariant Q_φ of the corresponding elliptic surface $\varphi : X \to C$ of Belyĭ type is an invariant of the Belyĭ pair (C, β).

Example. Consider the simplest dessin

$\beta^{-1}(0)$	$\beta^{-1}(1)$	$\beta^{-1}(\infty)$	X	Q_φ
II	III	I_1	none	–
		I_1^*	rational	0
	III^*	I_1	rational	0
		I_1^*	none	–
IV^*	III	I_1	rational	0
		I_1^*	none	–
	III^*	I_1	none	–
		I_1^*	$K3$	$\begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}$

If the dessin has valencies $\leq$ 3 at black vertices and $\leq$ 2 at white vertices, then we can restrict ourselves to type-specifications *s* that yields extremal elliptic surfaces.

Example. **Consider the dessins of degree 2:**

For each of them, there exists a unique type-specification that yields an extremal non-rational elliptic surface.

	$\beta^{-1}(0)$	$\beta^{-1}(1)$	$\beta^{-1}(\infty)$	X	Q_φ
D_1	II^*	I_0	$I_1^* \times 2$	$K3$	$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
D_2	$IV^* \times 2$	I_0	I_2^*	$K3$	$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$
D_3	II^*	$III^* \times 2$	I_2^*	$\chi = 36$	$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 4 \end{bmatrix}$

§4. Examples

In 1989, Miranda and Persson classified all semi-stable extremal elliptic $K3$ surfaces.

In 2001, S.- and Zhang classified all (not necessarily semi-stable) extremal elliptic $K3$ surfaces, and calculated their transcendental lattices.

In 2007, Beukers and Montanus determined the Belyĭ pairs (or dessins) $(\mathbb{P}^1, J_\varphi)$ associated with the semi-stable extremal elliptic $K3$ surfaces $\varphi : X \to \mathbb{P}^1$.

We will use the dessins by Beukers and Montanus as examples for our invariant.

Definition. A dessin is said to be of MPBM-type if it is of genus 0, of degree 24 and has valency 3 at every black vertices and valency 2 at the white vertices.

Proposition. Let $\varphi : X \to \mathbb{P}^1$ be a semi-stable extremal elliptic $K3$ surface. Then the Belyĭ pair $(\mathbb{P}^1, J_\varphi)$ is of MPBM-type.

By the valencies at ∞ of the Belyĭ pair (C, β), we mean the orders of poles of β.

When we write a dessin of MPBM-type, we omit the white vertices. For example, we write the dessin of MPBM-type with valency $[8, 8, 2, 2, 2, 2]$ at ∞

by

Example. The dessins of MPBM-type with valency $[11, 5, 3, 3, 1, 1]$ at ∞:

Beukers and Montanus showed that they are defined over $\mathbb{Q}(\sqrt{5})$, and are conjugate by $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. By the semi-stable extremal type-specification, the invariants Q_φ of the associated elliptic $K3$ surfaces are

$$\begin{bmatrix} 6 & 3 \\ 3 & 84 \end{bmatrix} \quad \text{for } D_1, \qquad \begin{bmatrix} 24 & 9 \\ 9 & 24 \end{bmatrix} \quad \text{for } D_2.$$

They are not isomorphic.

Example (continued). By the calculation of the transcendental lattices, we have known that the transcendental lattice of an extremal elliptic $K3$ surface of type

$$I_{11} + I_5 + I_3 + I_3 + I_1 + I_1$$

is either

$$\begin{bmatrix} 6 & 3 \\ 3 & 84 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 24 & 9 \\ 9 & 24 \end{bmatrix},$$

and both lattices actually occur.

However, we have not known which lattice corresponds to which dessin.

Example. The dessins of MPBM-type with valency $[6, 6, 5, 5, 1, 1]$ at ∞:

Beukers and Montanus showed that they are defined over $\mathbb{Q}(\sqrt{3})$, and are conjugate by $\mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$. By the semi-stable extremal type-specification, the invariants Q_φ of D_1 and D_2 are both

$$\begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix}.$$

Example (continued). By the extremal type-specification

$$I_6 + I_6 + I_5 + I_5 + I_1^* + I_1^* \quad \text{over } \beta^{-1}(\infty),$$

the invariants Q_φ of D_1 and D_2 are

$$Q_1 = \begin{bmatrix} 580 & -3944 & -7196 & -1440 \\ -3944 & 26846 & 48964 & 9800 \\ -7196 & 48964 & 89326 & 17880 \\ -1440 & 9800 & 17880 & 3580 \end{bmatrix}, \quad \text{and}$$

$$Q_2 = \begin{bmatrix} 260 & 456 & 2232 & 1748 \\ 456 & 876 & 4092 & 3048 \\ 2232 & 4092 & 19574 & 14966 \\ 1748 & 3048 & 14966 & 11764 \end{bmatrix},$$

respectively.

Both are even, positive-definite, and of discriminant 14400. They are not isomorphic, because there are four vectors v such that $Q_1(v, v) = 6$, while there are no vectors v such that $Q_2(v, v) = 6$.

§5. Galois-invariant part

For a lattice Λ, we put

$$\Lambda^{\vee} := \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

Then there is a canonical embedding $\Lambda \hookrightarrow \Lambda^{\vee}$. Then there is a canonical embedding $\Lambda \hookrightarrow \Lambda^{\vee}$ with the finite cokernel

$$D(\Lambda) := \Lambda^{\vee}/\Lambda$$

of order $|\operatorname{disc}\Lambda|$. We have a symmetric bilinear form

$$\Lambda^{\vee} \times \Lambda^{\vee} \longrightarrow \mathbb{Q}$$

that extends the symmetric bilinear form on Λ. We consider the natural non-degenerate quadratic form

$$q(\Lambda) : D(\Lambda) \times D(\Lambda) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

The finite quadratic form $(D(\Lambda), q(\Lambda))$ is called the *discriminant form* of Λ.

Since $H^2(X)$ is a unimodular lattice, we obtain the following:

Proposition. Let $\varphi : X \to C$ be an elliptic surface. Then the finite quadratic forms $(D(Q_\varphi), q(Q_\varphi))$ and $(D(P_\varphi), -q(P_\varphi))$ are isomorphic.

For an embedding $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$, we denote by $\varphi^\sigma : X^\sigma \to C^\sigma$ the pull-back of

$$\begin{array}{ccc} X & \to & C \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}\mathbb{C} & \end{array}$$

by $\sigma^* : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$. Since the lattice P_φ is defined algebraically, we see that P_φ and P_{φ^σ} are isomorphic.

Corollary. For any $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$, the finite quadratic forms $(D(Q_\varphi), q(Q_\varphi))$ and $(D(Q_{\varphi^\sigma}), q(Q_{\varphi^\sigma}))$ are isomorphic.

Thus the finite quadratic form $(D(Q_\varphi), q(Q_\varphi))$ is Galois-invariant. Therefore we can use $(D(Q_\varphi), q(Q_\varphi))$ to distinguish distinct Galois orbits in the set of dessins $\mathcal{D}(n)$.

Remark. If Λ is an even lattice, then we can refine the discriminant form $q(\Lambda) : D(\Lambda) \times D(\Lambda) \to \mathbb{Q}/\mathbb{Z}$ to

$$q(\Lambda) : D(\Lambda) \times D(\Lambda) \to \mathbb{Q}/2\mathbb{Z}.$$

Example. Consider again the two dessins of MPBM-type with valency $[11, 5, 3, 3, 1, 1]$ at ∞, which are conjugate by $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$:

Their invariants under the semi-stable extremal type-specification

$$\begin{bmatrix} 6 & 3 \\ 3 & 84 \end{bmatrix} \quad \text{for } D_1 \text{ and } \quad \begin{bmatrix} 24 & 9 \\ 9 & 24 \end{bmatrix} \quad \text{for } D_2$$

are not isomorphic, but in the same genus, and hence they have isomorphic ($\mathbb{Q}/2\mathbb{Z}$-valued) discriminant forms.

Example. The dessins of MPBM-type with valency $[8, 8, 3, 3, 1, 1]$ at ∞:

By the semi-stable extremal type-specification, the invariants Q_φ are

$$Q_1 = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} \quad \text{for } D_1, \qquad Q_2 = \begin{bmatrix} 24 & 0 \\ 0 & 24 \end{bmatrix} \quad \text{for } D_2.$$

Since $|D(Q_1)| = 144$ and $|D(Q_2)| = 576$, we see that these dessins are not Galois conjugate.

In fact, the Mordell-Weil group of the semi-stable extremal elliptic $K3$ surface over the dessin D_1 has a torsion of order 2, while that of D_2 is torsion-free.

Problem. Let $\varphi : X \to C$ be an extremal elliptic surface defined over $\overline{\mathbb{Q}}$. Let Q' be a lattice that is in the same genus as Q_φ. Is there $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $Q_{\varphi^\sigma} \cong Q'$?

Remark. YES, if X is a $K3$ surface.

Remark. By using prescribed type-specifications, we can use $(D(Q_\varphi), q(Q_\varphi))$ to distinguish the Galois orbits in the set of *marked* dessins.