On lattice-theoretic invariants of curves on a surface

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By a *lattice*, we mean a free $\mathbb{Z}$-module L of finite rank with a nondegenerate symmetric bilinear form

$$L \times L \to \mathbb{Z}.$$

A lattice L is naturally embedded into $\mathrm{Hom}(L, \mathbb{Z})$.
The *discriminant group* of L is the finite abelian group

$$\operatorname{disc} L := \mathrm{Hom}(L, \mathbb{Z})/L.$$

We fix a simply connected complex projective surface S, and consider reduced (possibly reducible) curves B on S.

We define several invariants of (S, B) by means of abelian coverings of S branching along B, and apply them for the constrution of examples of *Zariski couples*.

Let B and B' be curves on S.

Definition

We say that a homeomorphism $f : B \xrightarrow{\sim} B'$ *preserves the classes of irreducible components* if we have $[B_i] = [f(B_i)]$ in $H^2(S, \mathbb{Z})$ for any irreducible component B_i of B.

Definition

We say that B and B' have the *same embedding topology* and write

$$B \sim_{\rm top} B'$$

if there is a homeomorphism $h : (S, B) \cong (S, B')$ such that $h|B : B \xrightarrow{\sim} B'$ preserves the classes of irreducible components.

Definition

We say that B and B' are *of the same configuration type* and write

$$B \sim_{\rm cfg} B'$$

if there are

- tubular neighborhoods $\mathcal{T} \subset S$ of B and $\mathcal{T}' \subset S$ of B',
- a homeomorphism $\tau : (\mathcal{T}, B) \cong (\mathcal{T}', B')$

such that $\tau|B : B \xrightarrow{\sim} B'$ preserves the classes of irred components.

It is obvious that

$$B \sim_{\text{top}} B' \implies B \sim_{\text{cfg}} B'$$

According to Artal-Bartolo (1994), we define as follows:

Definition

A couple $[B, B']$ of curves on S is said to be a *Zariski couple* if $B \sim_{\text{cfg}} B'$ but $B \not\sim_{\text{top}} B'$.

The first example was discovered by Zariski in 1930's.

Example

There are irreducible curves B and B' of degree 6 on $\mathbb{P}^2$ with six ordinary cusps as their only singularities such that

- $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, and
- $\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Hence $[B, B']$ is a Zariski couple of *simple sextics*.

Definition

A plane curve $B \subset \mathbb{P}^2$ of degree 6 is a *simple sextic* if it has only simple singularities:

$$\begin{array}{lll} A_n & x^{n+1} + y^2 = 0 & (n \geq 1) \\ D_n & x^{n-1} + xy^2 = 0 & (n \geq 4) \\ E_6 & x^4 + y^3 = 0 & \\ E_7 & x^3y + y^3 = 0 & \\ E_8 & x^5 + y^3 = 0 & \end{array}$$

For simple sextics, we have

$$\begin{array}{ll} & \# \text{ of config types} = 11159 \\ < & \# \text{ of emb-top types} = ? \\ < & \# \text{ of connected componets of equi-sing families} = ? \end{array}$$

Example

We have three plane curves of degree 6

$$B_1 = C_1 + Q_1, \quad B_2 = C_2 + Q_2, \quad B_4 = C_4 + Q_4,$$

where Q_i is a quartic with one tacnode (A_3 point) and C_i is a smooth conic tangent to Q_i at two points with multiplicity 4 (two A_7-points).
Let $E_i \to Q_i$ be the normalization of Q_i. Then E_i is of genus 1 and has four special points

$$p, q \text{ the pull-back of } A_3, \quad s, t \text{ the pull-back of } 2A_7.$$

Then the order of $[p + q - s - t]$ in $\mathrm{Pic}^0(E_i)$ is 1, 2 and 4 according to $i = 1, 2, 4$. Their emb-top types are different, and hence $[B_1, B_2, B_4]$ is a Zariski *triple*.

Example

Consider two simple sextics

$$B_{\pm} \quad : \quad z \cdot (G(x,y,z) \pm \sqrt{5} H(x,y,z)) = 0, \quad \text{where}$$

$$\begin{aligned} G &= -9\,x^4 z - 14\,x^3 yz + 58\,x^3 z^2 - 48\,x^2 y^2 z - 64\,x^2 yz^2 + \\ &\quad 10\,x^2 z^3 + 108\,xy^3 z - 20\,xy^2 z^2 - 44\,y^5 + 10\,y^4 z, \\ H &= 5\,x^4 z + 10\,x^3 yz - 30\,x^3 z^2 + 30\,x^2 y^2 z + \\ &\quad 20\,x^2 yz^2 - 40\,xy^3 z + 20\,y^5. \end{aligned}$$

The quintic $G \pm \sqrt{5} H = 0$ has a A_{10}- singular point and intersects the line $z = 0$ at only one point that is a A_9- singular point.

They are *conjugate* under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, but have the different embedding topology. Hence $[B_+, B_-]$ is an *arithmetic Zariski couple*.

We fix a finite abelian group A once and for all.

Let B be a curve on S with the irred components $B_1, \dots, B_m$. Since S is simply connected, all étale Galois coverings of $S \setminus B$ with the Galois group A are in one-to-one correspondence with the set

$$\mathcal{C}_A(S, B) := \left\{ \gamma \;\middle|\; \begin{array}{l} \gamma \text{ is a surjective homomorphism} \\ \qquad H^2(B) = \bigoplus \mathbb{Z}[B_i] \twoheadrightarrow A \\ \text{such that the image of the restriction} \\ \text{map } r : H^2(S) \to H^2(B) = \bigoplus \mathbb{Z}[B_i] \\ \text{is contained in } \operatorname{Ker} \gamma \end{array} \right\}.$$

For $\gamma \in \mathcal{C}_A(S, B)$, we denote the corresponding covering by

$$\varphi_\gamma : W_\gamma \to S \setminus B.$$

Definition

A *smooth projective completion* of $\varphi_\gamma : W_\gamma \to S \setminus B$ is a morphism

$$\phi : X \to S$$

from a smooth projective surface X such that

- X contains W_γ as a Zariski open dense subset, and
- ϕ extends $\varphi_\gamma : W_\gamma \to S \setminus B$.

We choose a smooth projective completion $\phi : X \to S$ of φ_γ (not necessarily A-equivariant), and put

$$\mathcal{E}(X) := \left\{ \; E \subset X \;\middle|\; \begin{array}{l} E \text{ is an irreducible curve on } X \text{ such} \\ \text{that } \phi(E) \text{ is a point on } S \end{array} \right\}.$$

We consider

$$H^2(X)' := H^2(X)/(\text{the torsion part})$$

as a lattice by the cup-product. In this lattice, we have two submodules

$$\begin{aligned} \phi^*\mathrm{NS}(S) &= \langle\, [\phi^* C] \mid C \text{ is a curve on } S \,\rangle, \quad \text{and} \\ \langle \mathcal{E}(X) \rangle &= \langle\, [E] \mid E \in \mathcal{E}(X) \,\rangle. \end{aligned}$$

The cup-product is non-degenerate on each of them. Moreover we have

$$\phi^*\mathrm{NS}(S) \perp \langle \mathcal{E}(X) \rangle.$$

Hence the cup-product is non-degenerate on

$$\Sigma(X) := \phi^*\mathrm{NS}(S) \oplus \langle \mathcal{E}(X) \rangle,$$

that is, $\Sigma(X)$ is a sublattice of $H^2(X)'$.

We denote by

$$\overline{\Sigma}(X) := (\Sigma(X) \otimes \mathbb{Q}) \cap H^2(X)'$$

the *primitive closure* of $\Sigma(X) = \phi^* \mathrm{NS}(S) \oplus \langle \mathcal{E}(X) \rangle$ in $H^2(X)'$.

Definition

We put

$$\begin{aligned} F_A(S, B, \gamma) &:= \overline{\Sigma}(X)/\Sigma(X), \\ \sigma_A(S, B, \gamma) &:= \operatorname{disc} \Sigma(X) = \mathrm{Hom}(\Sigma(X), \mathbb{Z})/\Sigma(X), \end{aligned}$$

which are finite abelian groups, and

$$T_A(S, B, \gamma) := \Sigma(X)^{\perp} = \overline{\Sigma}(X)^{\perp} \subset H^2(X)',$$

which is a primitive sublattice of $H^2(X)'$.

Proposition

Neither $F_A(S, B, \gamma)$ nor $\sigma_A(S, B, \gamma)$ nor $T_A(S, B, \gamma)$ does depend on the choice of the smooth projective completion $\phi : X \to S$ of $\varphi_\gamma : W_\gamma \to S \setminus B$.

The proof is very easy. Suppose that $\phi' : X' \to S$ is another smooth projective completion. Then we have a commutative diagram

$$
\begin{array}{ccccc}
 & & X'' & & \\
 & \swarrow & & \searrow & \\
X & & & & X' \\
 & \searrow & & \swarrow & \\
 & & S & & ,
\end{array}
$$

where X'' is a smooth projective surface, and $X'' \to X$ and $X'' \to X'$ are birational morphisms that are isom over $S \setminus B$.

Since a birational morphism between smooth surfaces are composite of blowing-ups at points, we obtain orthogonal direct-sum decompositions

$$\begin{aligned} \Sigma(X'') &= \Sigma(X) \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle \quad \text{and} \\ H^2(X'')' &= H^2(X)' \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle, \end{aligned}$$

where $e_1, \ldots, e_N$ are classes with $e_i^2 = -1$. Hence we obtain

$$\begin{aligned} \overline{\Sigma}(X)/\Sigma(X) &\cong \overline{\Sigma}(X'')/\Sigma(X''), \\ \mathrm{Hom}(\Sigma(X), \mathbb{Z})/\Sigma(X) &\cong \mathrm{Hom}(\Sigma(X''), \mathbb{Z})/\Sigma(X''), \\ \Sigma(X)^{\perp} &\cong \Sigma(X'')^{\perp}. \end{aligned}$$

The same isomorphisms hold between X' and X''. □

By means of the celebrated theorem of Villamayor on the existence of the equivariant resolution of singularities, we can prove the following:

Proposition

There exists an A-equivariant smooth projective completion $\phi : X \to S$ for $\varphi_\gamma : W_\gamma \to S \setminus B$.

Hence the finite abelian groups $F_A(S, B, \gamma)$, $\sigma_A(S, B, \gamma)$ and the lattice $T_A(S, B, \gamma)$ are A-modules.

Consider the case $S = \mathbb{P}^2$ and $A = \mathbb{Z}/2\mathbb{Z}$.

Example

The example of Zariski revisited:
Let B be a plane curve of degree 6 defined by

$$f^3 + g^2 = 0, \quad \deg f = 2, \ \deg g = 3, \ \text{general}.$$

Then B is irred and $\operatorname{Sing} B$ consists of six cusps. The conic $Q : f = 0$ passes through $\operatorname{Sing} B$. On the other hand, there is an irred sextic B' with $\operatorname{Sing} B'$ consisting of 6 cusps such that there are no conics passing through them. (Del Pezzo, Segre, Zariski.)

- $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}, \quad \pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$,
- $F_A(\mathbb{P}^2, B', \gamma) = 0, \qquad \pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

In fact, $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by the class of the lift of the conic Q. (The conic Q splits in X into $\tilde{Q}^+ \cup \tilde{Q}^-$.)

Consider again the case $S = \mathbb{P}^2$ and $A = \mathbb{Z}/2\mathbb{Z}$.

Example

Recall the conjugate plane sextics B_+ and B_- with $A_9 + A_{10}$-singular points:

$$B_\pm \quad : \quad z \cdot (G(x,y,z) \pm \sqrt{5}\, H(x,y,z)) = 0.$$

We have

$$T_A(\mathbb{P}^2, B_+, \gamma) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T_A(\mathbb{P}^2, B_-, \gamma) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

The invariant $T_A(S, B, \gamma)$ is a *topological* invariant. Let

$$h : (S, B) \xrightarrow{\sim} (S, B')$$

be a homeomorphism. Since h induces a homeomorphism $S \setminus B \cong S \setminus B'$, we obtain a bijection

$$h^* : \mathcal{C}_A(S, B') \xrightarrow{\sim} \mathcal{C}_A(S, B).$$

For $\gamma \in \mathcal{C}_A(S, B')$, the covering $\varphi_{h^*\gamma} : W_{h^*\gamma} \to S \setminus B$ is obtained by pulling back $\varphi_\gamma : W_\gamma \to S \setminus B'$ by $h : S \setminus B \cong S \setminus B'$.

Theorem

Suppose that the classes $[B_i]$ span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$. Then

$$T_A(S, B, h^*\gamma) \cong T_A(S, B', \gamma).$$

Proof Since $W_{h^*\gamma}$ is homeomorphic to W_γ, it is enough to show that $T_A(S, B, \gamma)$ is determined by the homeo-type of W_γ. Let

$$\iota_W : H_2(W_\gamma) \times H_2(W_\gamma) \to \mathbb{Z}$$

be the intersection pairing. We put

$$\mathrm{Ker}(\iota_W) := \{\, x \in H_2(W_\gamma) \mid \iota_W(x, y) = 0 \ \text{ for all } \ y \in H_2(W_\gamma) \,\}.$$

Then ι_W induces a non-degenerate symmetric bilinear form

$$\bar{\iota}_W \ : \ H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W) \ \times \ H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W) \ \to \ \mathbb{Z}.$$

The lattice $(H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W), \ \bar{\iota}_W)$ is determined by the homeomorphism type of W_γ. Hence the proof is completed by showing that the homomorphism

$$H_2(W_\gamma) \to H_2(X)$$

induced by the inclusion $W_\gamma \hookrightarrow X$ induces

$$H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W) \cong T_A(S, B, \gamma) \subset H_2(X)'.$$

Definition

A *map of equi-configuration* is a homeomorphism $\tau : (\mathcal{T}, B) \xrightarrow{\simeq} (\mathcal{T}', B')$, where $\mathcal{T}, \mathcal{T}' \subset S$ are tubular nbds of B, B' respectively, such that the induced homeomorphic $B \xrightarrow{\simeq} B'$ preserves the classes of irreducible components.

If $\tau : (\mathcal{T}, B) \xrightarrow{\simeq} (\mathcal{T}', B')$ is a map of equi-configuration, then we have the following commutative diagram:

$$\begin{array}{ccc} H^2(S) & \xrightarrow{r} & H^2(B') \\ \| & & \downarrow \tau^* \\ H^2(S) & \xrightarrow{r} & H^2(B). \end{array}$$

Therefore τ induces a bijection

$$\tau_* : \mathcal{C}_A(S, B) \xrightarrow{\simeq} \mathcal{C}_A(S, B').$$

Corollary

Let $\tau : (\mathcal{T}, B) \xrightarrow{\simeq} (\mathcal{T}', B')$ be a map of equi-configuration. If $T_A(S, B, \gamma) \ncong T_A(S, B', \tau_*\gamma)$, then $[B, B']$ is a Zariski couple.

By this corollary, we can obtain many examples of Zariski couples.

We can also prove the following for

$$\sigma_A(S, B, \gamma) := \operatorname{disc} \Sigma(X)$$

by means of the minimal good embedded resolution of $B \subset S$.

Proposition

Let $\tau : (\mathcal{T}, B) \xrightarrow{\simeq} (\mathcal{T}', B')$ be a map of equi-configuration. Then $\sigma_A(S, B, \gamma) \cong \sigma_A(S, B', \tau_*\gamma)$.

Since the lattice $H^2(X)'$ is unimodular, the three invariants

$$\begin{aligned} F_A(S,B,\gamma) &= \overline{\Sigma}(X)/\Sigma(X), \\ \sigma_A(S,B,\gamma) &= \operatorname{disc}\Sigma(X) = \operatorname{Hom}(\Sigma(X),\mathbb{Z})/\Sigma(X) \\ T_A(S,B,\gamma) &= \Sigma(X)^{\perp} \end{aligned}$$

satisfy

$$|\operatorname{disc}(T_A(S,B,\gamma))| = \frac{|\sigma_A(S,B,\gamma)|}{|F_A(S,B,\gamma)|^2}.$$

Corollary

Let $\tau : (\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$ be a map of equi-configuration, so that $\sigma_A(S,B,\gamma) \cong \sigma_A(S,B',\tau_*\gamma)$. If $|F_A(S,B,\gamma)| \neq |F_A(S,B',\tau_*\gamma)|$, then $[B, B']$ is a Zariski couple.

Example

For the Zariski triple

$$B_1 = C_1 + Q_1, \quad B_2 = C_2 + Q_2, \quad B_4 = C_4 + Q_4$$

described above, we have

$$F_A(\mathbb{P}^2, B_4) \cong \mathbb{Z}/2\mathbb{Z}, \;\; F_A(\mathbb{P}^2, B_2) \cong \mathbb{Z}/4\mathbb{Z}, \;\; F_A(\mathbb{P}^2, B_1) \cong \mathbb{Z}/8\mathbb{Z}.$$

These cyclic groups are generated by the classes of

$$\begin{array}{l} \text{the reduced part of the proper transform of } C_4, \\ \text{the lift of a conic passing through } \operatorname{Sing} B_2, \\ \text{the lift of a line passing through } \operatorname{Sing} B_1, \end{array}$$

respectively.

Definition

We say that a complex $K3$ surface X is *singular* if $\mathrm{rank}(\mathrm{NS}(X))$ attains the possible maximum 20.

Shioda and Inose showed that the isomorphism class of a singular $K3$ surface is determined by its transcendental lattice

$$T(X) := \mathrm{NS}(X)^{\perp}$$

with its Hodge decomposition $T(X) \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$.
We denote by

$$\tilde{T}(X)$$

the oriented transcendental lattice of X, which is a positive definite lattice of rank 2.

Shioda and Inose also gave an explicit way of constructing a singular $K3$ surface X from the oriented transcendental lattice $\tilde{T}(X)$. The singular $K3$ surface X is obtained as a certain double cover of the Kummer surface

$$\mathrm{Km}(E \times E'),$$

where E and E' are elliptic curves with CM by some orders of $\mathbb{Q}(\sqrt{-|\operatorname{disc} T(X)|})$. In particular, we have

Theorem

Every singular $K3$ surface is defined over a number field.

S.- and Schütt (2007) proved the following by means of the class field theory of imaginary quadratic fields:

Theorem

Let X and X' be singular $K3$ surfaces defined over $\overline{\mathbb{Q}}$. If their transcendental lattices $T(X)$ and $T(X')$ are contained in the same genus, then there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $X' \cong X^{\sigma}$.

Definition

A plane curve $B \subset \mathbb{P}^2$ of degree 6 is called a *maximizing sextic* if B has only simple singularities and the sum of the Milnor numbers attains the possible maximum 19.

Let B be a maximizing sextic. Consider the double covering $W_\gamma \to \mathbb{P}^2 \setminus B$ branching along every irreducible component of B. Then we obtain a singular $K3$ surface X as a smooth projective completion $X \to \mathbb{P}^2$ of $W_\gamma \to \mathbb{P}^2 \setminus B$, and the lattice invariant $T_A(\mathbb{P}^2, B, \gamma)$ is the transcendental lattice $T(X)$.

By searching for maximizing sextics B such that the genus of the lattice $T_A(\mathbb{P}^2, B, \gamma)$ contains more than one isomorphism classes, we obtain many examples of arithmetic Zariski couples of maximizing sextics:

No.	$sing - type$	$T_A(\mathbb{P}^2, B, \gamma)$,	$T_A(\mathbb{P}^2, B^\sigma, \gamma)$
1	$E_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
2	$E_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
3	$E_6 + D_5 + A_6 + A_2$	$L[12,0,42]$,	$L[6,0,84]$
4	$E_6 + A_{10} + A_3$	$L[12,0,22]$,	$L[4,0,66]$
5	$E_6 + A_{10} + A_2 + A_1$	$L[18,6,24]$,	$L[6,0,66]$
6	$E_6 + A_7 + A_4 + A_2$	$L[24,0,30]$,	$L[6,0,120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$L[30,0,42]$,	$L[18,6,72]$
8	$D_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
9	$D_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
	$\dots$		
	$\dots$		
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$L[24,0,30]$,	$L[6,0,120]$

where

$$L[a,b,c] := \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Question:

Are there any arithmetic Zariski couple $[B, B']$ such that $\pi_1(\mathbb{P}^2 \setminus B)$ is not isomorphic to $\pi_1(\mathbb{P}^2 \setminus B')$?

Note that $\pi_1(\mathbb{P}^2 \setminus B)$ and $\pi_1(\mathbb{P}^2 \setminus B')$ have isomorphic pro-finite completions.

Digression:

Recall the arithmetic Zariski couple $[B_+, B_-]$ with

$$T_A(\mathbb{P}^2, B_+, \gamma) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T_A(\mathbb{P}^2, B_-, \gamma) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

Consider the double coverings of the complements:

$$W_\pm \quad : \quad w^2 \cdot (G(x, y, 1) \pm \sqrt{5} \cdot H(x, y, 1)) = 1.$$

Both of them are smooth affine surfaces in $\mathbb{C}^3$. They are not homeomorphic.

Remark

Many examples of conjugate but non-homeomorphic complex varieties have been constructed since Serre (1960).

The invariant $F_A(S, B, \gamma)$ is related to $\pi_1(S \setminus B)$.

We denote by $N_\gamma \subset \pi_1(S \setminus B)$ the kernel of the homomorphism

$$\pi_1(S \setminus B) \twoheadrightarrow A$$

corresponding to $\varphi_\gamma : W_\gamma \to S \setminus B$. Then the action of A on $H_1(W_\gamma) = N_\gamma/[N_\gamma, N_\gamma]$ is associated with

$$0 \to H_1(W_\gamma) \to \pi_1(S \setminus B)/[N_\gamma, N_\gamma] \to A \to 0 :$$

that is, $a \in A$ acts on $x \in H_1(W_\gamma)$ by

$$a(x) = \tilde{a} \cdot x \cdot \tilde{a}^{-1},$$

where $\tilde{a} \in \pi_1(S \setminus B)/[N_\gamma, N_\gamma]$ is the pre-image of $a \in A$, and we regard $H_1(W_\gamma)$ as a normal subgroup of $\pi_1(S \setminus B)/[N_\gamma, N_\gamma]$. Hence, if A acts on $H_1(W_\gamma)$ non-trivially, $\pi_1(S \setminus B)$ is non-abelian.

We can prove the following:

Theorem

Assume the following:

(a) *A is a cyclic group of prime order l,*

(b) *the classes $[B_1], \dots, [B_m]$ span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$, and*

(c) *$\gamma([B_i]) \neq 0$ for $i = 1, \dots, m$.*

If the p-part $F_A(S, B, \gamma)_p$ of $F_A(S, B, \gamma)$ is non-trivial for some $p \neq l$, then $\pi_1(S \setminus B)$ acts on $H_1(W_\gamma)$ non-trivially and hence is non-abelian.

The invariant $\sigma_A(S, B, \gamma)$ can be computed from the configuration data of B.

We have developed a general method of Zariski-van Kampen type to calculate the lattice $T_A(S, B, \gamma)$.

Hence $|F_A(S, B, \gamma)|$ can be also calculated.

However, there seems to be no general method to determine $F_A(S, B, \gamma)$ so far, and it would be more difficult to find algebraic cycles that generate $F_A(S, B, \gamma)$ explicitly.

Thank you!