On equisingular families of plane curves of degree 6

Ichiro Shimada

Hiroshima University

2009.01.19 at University of Tokyo

§1 Four equivalence relations of simple sextics

$B \subset \mathbb{P}^2$: a complex reduced projective plane curve of degree 6.

B `is a simple sextic`

$\Leftrightarrow$ B `has only simle singularities (`ADE`-singularities)`

$\Leftrightarrow$ `the minimal resolution` X_B `of the double cover` $Y_B \to \mathbb{P}^2$ `branching along` B `is a` $K3$ `surface`

- μ_B : the total Milnor number of B.
- R_B : the ADE type of $\operatorname{Sing} B$.
- $\mathcal{E}_B$: the set of exceptional (-2)-curves for the minimal resolution $X_B \to Y_B$. We have $|\mathcal{E}_B| = \mu_B$.
- $\Sigma_B \subset H^2(X_B, \mathbb{Z})$: the sublattice generated by the classes $[E]$ of $E \in \mathcal{E}_B$ and the polarization class $h := [\rho^* \mathcal{O}_{\mathbb{P}^2}(1)]$, where $\rho : X_B \to Y_B \to \mathbb{P}^2$. We have $\operatorname{rank} \Sigma_B = 1 + \mu_B$.
- $\overline{\Sigma}_B \subset H^2(X_B, \mathbb{Z})$: the primitive closure of Σ_B in $H^2(X_B, \mathbb{Z})$.

Example by Zariski in 1930's

There exist two irreducible simple sextics with six ordinary cusps (that is, $R_B = 6A_2$)

$B_{trs} = \{f^3 + g^2 = 0\}$ `(torus type)` and B_{ntrs} `(non-torus type)`

that cannot be connected by an equisingular family.
Their differences are described in a several ways:

- $\pi_1(\mathbb{P}^2 \setminus B_{trs}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, while
  $\pi_1(\mathbb{P}^2 \setminus B_{ntrs}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.
- $\exists$ a smooth conic $\Gamma = \{f = 0\}$ passing through the six cusps of B_{trs}, which splits into two curves Γ_+ and Γ_- in $X_{B_{trs}}$, while $\nexists$ a conic passing through the six cusps of B_{ntrs}.
- the finite abelian group $\overline{\Sigma}_{B_{trs}}/\Sigma_{B_{trs}}$ is cyclic of order 3 generated by $[\Gamma_+] \in \overline{\Sigma}_{B_{trs}}$, while $\overline{\Sigma}_{B_{ntrs}}/\Sigma_{B_{ntrs}} = 0$.

$$
\begin{array}{rcl}
B \sim_{\mathrm{eqs}} B' & \Leftrightarrow & B \text{ and } B' \text{ are connected by an equisingular family.} \\
B \sim_{\mathrm{lat}} B' & \Leftrightarrow & \exists \text{ a bijection } \mathcal{E}_B \cong \mathcal{E}_{B'} \text{ that induces, with } \phi(h) = h, \\
& & \text{an isometry of lattices } \phi : \overline{\Sigma}_B \cong \overline{\Sigma}_{B'}. \\
B \sim_{\mathrm{cfg}} B' & \Leftrightarrow & \exists \text{ tubular nbds } T \subset \mathbb{P}^2 \text{ of } B \text{ and } T' \subset \mathbb{P}^2 \text{ of } B' \\
& & \exists \text{ a homeo } \varphi : (T, B) \xrightarrow{\sim} (T', B') \text{ such that} \\
& & \bullet \deg B_i = \deg \varphi(B_i) \text{ for each irred comp } B_i \text{ of } B, \\
& & \bullet\ \varphi \text{ induces a local analytic isomorphism} \\
& & \text{at each singular point of } B \text{ and } B'. \\
B \sim_{\mathrm{top}} B' & \Leftrightarrow & \exists \text{ a homeo } \varphi : (\mathbb{P}^2, B) \xrightarrow{\sim} (\mathbb{P}^2, B') \text{ that induces} \\
& & \text{a local analytic isom at each singular point.}
\end{array}
$$

$$
\boxed{
\begin{array}{ccccc}
 & & B \sim_{\mathrm{lat}} B' & & \\
 & \nearrow\!\!\!\!\nearrow & & \searrow\!\!\!\!\searrow^{\text{(Yang)}} & \\
B \sim_{\mathrm{eqs}} B' & & & & B \sim_{\mathrm{cfg}} B' \\
 & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\nearrow & \\
 & & B \sim_{\mathrm{top}} B' & &
\end{array}
}
$$

Example

For the example by Zariski, we have

$$B_{trs} \sim_{\mathrm{cfg}} B_{ntrs}, \quad \texttt{but}$$

$$B_{trs} \not\sim_{\mathrm{eqs}} B_{ntrs}, \quad B_{trs} \not\sim_{\mathrm{lat}} B_{ntrs}, \quad B_{trs} \not\sim_{\mathrm{top}} B_{ntrs}.$$

Remark

The torus curves $B_{trs} = \{f^3 + g^2 = 0\}$ of Zariski form a connected equisingular family. An explicit defining equation of a non-torus curve B_{ntrs} of Zariski was first given by Oka (1994). The connectedness of the equisingular family of non-torus curves was established by Degtyarev (2008).

Aim: to compare these four equivalence relations.

§2 Comparison of $\sim_{\mathrm{lat}}$ and $\sim_{\mathrm{cfg}}$

Using the surjectivity of the period mapping for complex $K3$ surfaces, Yang (1996) classified all lattice types (the equivalence classes of $\sim_{\mathrm{lat}}$). He has also established an algorithm to determine the configuration type of a given lattice type.

Numbers of lattice types and configuration types:

μ_B	0	1	2	3	4	5	6	7	8	9	10	11
$\sim_{\mathrm{cfg}}$	1	1	2	3	6	10	18	30	53	89	148	246
$\sim_{\mathrm{lat}}$	1	1	2	3	6	10	18	30	53	89	148	246

μ_B	12	13	14	15	16	17	18	19	total
$\sim_{\mathrm{cfg}}$	415	684	1090	1623	2139	2283	1695	623	11159
$\sim_{\mathrm{lat}}$	416	686	1096	1639	2171	2330	1734	629	11308

Definition

A configuration type consisting of k lattice types with $k > 1$ is called a *lattice Zariski k-plet*.

Aim: Desribe all lattice Zariski k-plets.

There are no lattice Zariski k-plets with $k \geq 4$.

Example of a lattice Zariski triple

There are three lattice types $\Lambda_1, \Lambda_2, \Lambda_3$ in the configuration type of $B = C \cup Q$, where C is a smooth conic and Q is a quartic with a tacnode P_1 intersecting C at P_2, P_3 with multiplicity 4, so that $R_B = A_3 + 2A_7$.
These three lattice types are distinguished as follows:

$$\begin{aligned} B \in \Lambda_1 \quad &\Leftrightarrow \quad [\overline{\Sigma}_B : \Sigma_B] = 4, \\ B \in \Lambda_2 \quad &\Leftrightarrow \quad [\overline{\Sigma}_B : \Sigma_B] = 8, \\ B \in \Lambda_3 \quad &\Leftrightarrow \quad [\overline{\Sigma}_B : \Sigma_B] = 2. \end{aligned}$$

Definition

A simple sextic B is said to be *lattice-generic* if $\overline{\Sigma}_B = \mathrm{NS}(X_B)$ holds.

Remark

For any B, there exists a lattice-generic B' such that $B \sim_{\mathrm{eqs}} B'$. In particular, every lattice type contains a lattice-generic member.

The three lattice types above are distinguished geometrically.
Let B be a lattice-generic member of the configuration type above.

- $B \in \Lambda_1$ if and only if $\exists$ a smooth conic Γ passing through P_1, P_2, P_3 such that $\mathrm{mult}_{P_i}(B, \Gamma) = 4$ for $i = 1, 2, 3$.
- $B \in \Lambda_2$ if and only if $\exists$ a line Γ passing through P_1, P_2, P_3.
- $B \in \Lambda_3$ if and only if there are no such conics or lines.

Z-splitting curves (Z stands for "Zariski")

Definition

A reduced irreducible curve $\Gamma \subset \mathbb{P}^2$ is called *splitting for B* if the strict transform of Γ by $X_B \to \mathbb{P}^2$ splits into distinct irred components $\Gamma_+ \subset X_B$ and $\Gamma_- \subset X_B$, which are called the *lifts* of Γ.

A splitting curve $\Gamma \subset \mathbb{P}^2$ is called *Z-splitting* if the class $[\Gamma_+]$ of $\Gamma_+ \subset X_B$ is contained in the primitive closure $\overline{\Sigma}_B \subset H^2(X_B, \mathbb{Z})$.

Remark

(1) A splitting curve is Z-splitting if and only if it is stable under a small equisingular deformation of B.
(2) We have a numerical criterion (involving the intersection multiplicities of B and Γ) to determine whether a splitting curve Γ is Z-splitting or not.

Example

Consider the torus curve $B_{trs} = \{f^3 + g^2 = 0\}$ of Zariski, where f and g are general. Then the conic $\Gamma = \{f = 0\}$ is Z-splitting.

If $f = f_1 f_2$ with $\deg f_1 = \deg f_2 = 1$, then the lines $\Gamma_1 = \{f_1 = 0\}$ and $\Gamma_2 = \{f_2 = 0\}$ are splitting but not Z-splitting. In this case, the simple sextic

$$B' := \{f_1^3 f_2^3 + g^2 = 0\}$$

with $R_{B'} = 6A_2$ is contained in the same lattice type as B_{trs}, but is **not** lattice-generic.

We have written an algorithm to determine all Z-splitting curves of degree ≤ 2 for a lattice-generic member of a given lattice type. In particular, if B and B' are lattice-generic and $B \sim_{\rm lat} B'$, then the numbers of Z-splitting lines (resp. conics) for B and for B' are equal.

By this algorithm, we have obtained the following:

Theorem

The lattice types in any lattice Zariski k-plets $(k > 1)$ are distinguished by the numbers of Z-splitting lines and Z-splitting conics for their lattice-generic members.

We are going to classify all Z-splitting lines and Z-splitting conics for simple sextics.

§3 Classification of Z-splitting lines and conics

For simplicity, we call a pair (B, Γ) of a lattice-generic simple sextic B and a Z-splitting curve Γ a *lattice-generic Z-pair*.

Definition

Let (B, Γ) and (B', Γ') be lattice-generic Z-pairs. We say that (B, Γ) and (B', Γ') are of the same lattice type and write $(B, \Gamma) \sim_{\mathrm{lat}} (B', \Gamma')$ if there exists a bijection $\mathcal{E}_B \cong \mathcal{E}_{B'}$ that induces an isometry of lattices $\phi : \overline{\Sigma}_B \cong \overline{\Sigma}_{B'}$ such that

- ϕ preserves h, and
- ϕ maps the class $[\Gamma_+] \in \overline{\Sigma}_B$ to $[\Gamma'_+] \in \overline{\Sigma}_{B'}$ or $[\Gamma'_-] \in \overline{\Sigma}_{B'}$.

The lattice type containing a lattice-generic Z-pair (B, Γ) is denoted by $\lambda(B, \Gamma)$.

We have $(B, \Gamma) \sim_{\mathrm{lat}} (B', \Gamma') \Longrightarrow B \sim_{\mathrm{lat}} B' \Longrightarrow B \sim_{\mathrm{cfg}} B'$.

Definition

The *order* of a lattice type $\lambda(B, \Gamma)$ is the order of the class $[\Gamma_+] \in \overline{\Sigma}_B$ in the finite abelian group $\overline{\Sigma}_B / \Sigma_B$.

Definition

Let λ and λ_0 be lattice types of lattice-generic Z-pairs. We say that λ_0 is a *specialization* of λ if there exists an analytic family $(B_t, \Gamma_t)_{t \in \Delta}$ of lattice-generic Z-pairs parametrized by a unit disc Δ such that $(B_t, \Gamma_t) \in \lambda$ for $t \neq 0$, and $(B_0, \Gamma_0) \in \lambda_0$.

We are going to classify the lattice types of lattice-generic Z-pairs (B, Γ) with $\deg \Gamma \leq 2$ from which all lattice types are obtained by specializations.

Theorem (Classification of Z-splitting lines)

Let λ be the lattice type of a lattice-generic Z-pair (B, Γ) with $\deg \Gamma = 1$. Then the order d of λ is $6, 8, 10$ or 12, and λ is a specialization of the following lattice type $\lambda_{lin,d} = \lambda(B_d, \Gamma_d)$:

	R_{B_d}	degrees of irreducible components
$\lambda_{lin,6}$	$3A_5$	$[3, 3]$ (the cubics are smooth)
$\lambda_{lin,8}$	$A_3 + 2A_7$	$[2, 4]$ (the quartic has A_3)
$\lambda_{lin,10}$	$2A_4 + A_9$	$[1, 5]$ (the quintic has $2A_4$)
$\lambda_{lin,12}$	$A_3 + A_5 + A_{11}$	$[2, 4]$ (the quartic has A_5).

For each $\lambda_{lin,d} = \lambda(B_d, \Gamma_d)$, the Z-splitting line Γ_d passes through the three singular points of B_d. The finite abelian group $\overline{\Sigma}_{B_d}/\Sigma_{B_d}$ is cyclic of order d, and is generated by the class $[(\Gamma_d)_+]$ of the lift.

Theorem (Classification of Z-splitting conics)

Let λ be the lattice type of a lattice-generic Z-pair (B, Γ) with $\deg \Gamma = 2$. Then the order d of λ is $3, 4, 5, 6, 7$ or 8, and λ is a specialization of the following lattice type $\lambda_{con,d} = \lambda(B_d, \Gamma_d)$:

	R_{B_d}	degs
$\lambda_{con,3}$	$6A_2$	$[6]$
$\lambda_{con,4}$	$2A_1 + 4A_3$	$[2,4]$ (the quartic has $2A_1$)
$\lambda_{con,5}$	$4A_4$	$[6]$
$\lambda_{con,6}$	$2A_1 + 2A_2 + 2A_5$	$[2,4]$ (the quartic has $2A_2$)
$\lambda_{con,7}$	$3A_6$	$[6]$
$\lambda_{con,8}$	$A_1 + A_3 + 2A_7$	$[2,4]$ (the quartic has $A_1 + A_3$).

The finite abelian group $\overline{\Sigma}_{B_d}/\Sigma_{B_d}$ is cyclic of order d, and is generated by the class $[(\Gamma_d)_+]$ of the lift.

Each of the simple sextics in these “generating” lattice types

$$\lambda_{lin,d} \ \ (d = 6, 8, 10, \neq 12) \quad \text{and} \quad \lambda_{con,d} \ \ (d = 3, \ldots, 8)$$

is a member of lattice Zariski k-plets ($k > 1$).

Example

The simple sextic in $\lambda_{con,3}$ ($R_B = 6A_2, \mathrm{degs} = [6]$) is the torus curve B_{trs} of Zariski. It has a non-torus partner B_{ntrs}.

Example

The simple sextic in $\lambda_{lin,8}$ ($R_B = A_3 + 2A_7, \mathrm{degs} = [2, 4]$) is a member of the lattice Zariski triple presented above.

What classes generates the finite abelian group $\overline{\Sigma}_B/\Sigma_B$?

Let B be a lattice-generic simple sextic.

$\Sigma'_B \subset H^2(X_B, \mathbb{Z})$: the sublattice generated by Σ_B and the reduced parts of the strict transforms of the irred components of B.

$$\Sigma_B \subset \Sigma'_B \subset \overline{\Sigma}_B.$$

Theorem (Exceptional simple sextic)

(1) If $\exists$ a Z-splitting curve of degree ≤ 2, then $\overline{\Sigma}_B/\Sigma'_B$ is generated by the classes of the lifts of these Z-splitting curves.

(2) There exists a lattice-generic B_{exc} with $R_{B_{exc}} = 3A_1 + 4A_3$ and $\operatorname{degs} B_{exc} = [2, 4]$ (the quartic has $3A_1$) such that $\Sigma'_{B_{exc}} \neq \overline{\Sigma}_{B_{exc}}$ but there are no Z-splitting curves of degree ≤ 2.

(3) If $\Sigma'_B \neq \overline{\Sigma}_B$ but there are no Z-splitting curves of degree ≤ 2, then the lattice type of B is a specialization of that of B_{exc}.

The group $\overline{\Sigma}_{B_{exc}}/\Sigma_{B_{exc}}$ is cyclic of order 4, and is generated by the classes of the lift $[\Gamma_+]$ of Z-splitting **cubic** curves Γ. Hence we have

Corollary

For a lattice-generic B, $\overline{\Sigma}_B = \mathrm{NS}(X_B)$ is generated over Σ'_B by the classes of the lifts of Z-splitting curves of degree ≤ 3.

Remark

The exceptional simple sextic $B_{exc} = C \cup Q$ is a member of lattice Zariski couple. Let $B'_{exc} = C' \cup Q'$ be a lattice-generic member of the *partner* in this lattice Zariski couple. Then $\overline{\Sigma}_{B'_{exc}}/\Sigma_{B'_{exc}}$ is also cyclic of order 4. There exists a Z-splitting conic Γ for B'_{exc} and $\overline{\Sigma}_{B'_{exc}}/\Sigma_{B'_{exc}}$ is generated by $[\Gamma_+]$.

§4 Comparison of $\sim_{\mathrm{lat}}$ and $\sim_{\mathrm{top}}$

Recall the implications:

$$\begin{array}{ccccc} & & B \sim_{\mathrm{lat}} B' & & \\ & \nearrow\!\!\!\!\nearrow & & \searrow\!\!\!\!\searrow & \\ B \sim_{\mathrm{eqs}} B' & & & & B \sim_{\mathrm{cfg}} B' \\ & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\nearrow & \\ & & B \sim_{\mathrm{top}} B' & & \end{array}$$

For a simple sextic B, we denote by

$$T_B \subset H^2(X_B, \mathbb{Z})$$

the orthogonal complement of $\Sigma_B \subset H^2(X_B, \mathbb{Z})$. (If B is lattice-generic, then T_B is the *transcendental lattice* of X_B.)

Theorem

If $B \sim_{\mathrm{top}} B'$, then the lattices T_B and $T_{B'}$ are isomorphic.

Proof

We consider the open $K3$ surface $U_B := \rho^{-1}(\mathbb{P}^2 \setminus B) \subset X_B$. We put

$$J_\infty(U_B) \ := \ \bigcap_K \mathrm{Im}(H_2(U_B \setminus K, \mathbb{Z}) \to H_2(U_B, \mathbb{Z})),$$

where K runs through the set of compact subsets of U_B, and put $V_2(U_B) := H_2(U_B, \mathbb{Z})/J_\infty(U_B)$. Then the intersection pairing

$$\iota_B : H_2(U_B, \mathbb{Z}) \times H_2(U_B, \mathbb{Z}) \to \mathbb{Z}$$

induces $\bar{\iota}_B : V_2(U_B) \times V_2(U_B) \to \mathbb{Z}$. By construction, we have

$$B \sim_{\mathrm{top}} B' \implies (V_2(U_B), \bar{\iota}_B) \ \cong \ (V_2(U_{B'}), \bar{\iota}_{B'}).$$

Then Theorem follows from

$$(V_2(U_B), \bar{\iota}_B) \cong T_B.$$

□

We have obtained following theorem by means of the Shioda-Inose construction of the singular $K3$ surfaces (the complex $K3$ surfaces with Picard number 20).

Theorem (S.- and Schütt)

Let X and X' be singular $K3$ surfaces defined over $\overline{\mathbb{Q}}$ such that their transcendental lattices are in the same genus. Then X and X' are conjugate under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Corollary

Let B be a simple sextic with $\mu_B = 19$ defined over $\overline{\mathbb{Q}}$. If the genus containing T_B contains more than one isomorphism classes of lattices, then $\exists\ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $B \sim_{\mathrm{lat}} B^{\sigma}$ and $B \not\sim_{\mathrm{top}} B^{\sigma}$. (Remark that the lattice type is determined algebraically.)

Can this corollary be generalized to equisingular families of simple sextics with $\mu_B < 19$?

Example (Arima and S.-)

Consider $\quad B_{\pm} : z \cdot (G(x,y,z) \pm \sqrt{5} \cdot H(x,y,z)) = 0, \quad$ where

$$\begin{aligned} G(x,y,z) &:= -9\,x^4 z - 14\,x^3 yz + 58\,x^3 z^2 - 48\,x^2 y^2 z - 64\,x^2 yz^2 \\ &\quad +10\,x^2 z^3 + 108\,xy^3 z - 20\,xy^2 z^2 - 44\,y^5 + 10\,y^4 z, \\ H(x,y,z) &:= 5\,x^4 z + 10\,x^3 yz - 30\,x^3 z^2 + 30\,x^2 y^2 z + \\ &\quad +20\,x^2 yz^2 - 40\,xy^3 z + 20\,y^5. \end{aligned}$$

We have $\mathrm{degs}\, B_{\pm} = [1,5]$ with the quintic having A_{10} and $R_{B_{\pm}} = A_{10} + A_9$. Their transcendental lattices are

$$T_{B_+} \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T_{B_-} \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

Hence we have $B_+ \sim_{\mathrm{lat}} B_-$ but $B_+ \not\sim_{\mathrm{top}} B_-$.

Does $B \sim_{\mathrm{top}} B'$ imply $B \sim_{\mathrm{lat}} B'$?

Since $H^2(X_B, \mathbb{Z})$ is unimodular, we have $|\operatorname{disc} T_B| = |\operatorname{disc} \overline{\Sigma}_B|$.

Proposition

Suppose that $B \sim_{\mathrm{cfg}} B'$ (and hence $\Sigma_B \cong \Sigma_{B'}$). Then

$$[\overline{\Sigma}_B : \Sigma_B] \neq [\overline{\Sigma}_{B'} : \Sigma_{B'}] \implies B \not\sim_{\mathrm{top}} B'.$$

In many cases (but not in all cases), the lattice types in a configuration type are distinguished by $[\overline{\Sigma}_B : \Sigma_B]$.

Proposition

Let B and B' be simple sextics such that

$$B \sim_{\mathrm{cfg}} B', \quad [\overline{\Sigma}_B : \Sigma_B] = [\overline{\Sigma}_{B'} : \Sigma_{B'}], \quad B \not\sim_{\mathrm{lat}} B'.$$

Then either the lattice types of $[B, B']$ are specializations of those of $[B_{exc}, B'_{exc}]$, or $R_B = A_1 + A_3 + 2A_7$ and $\operatorname{degs} B = [2, 4]$.

§5 Comparison of $\sim_{\mathrm{lat}}$ and $\sim_{\mathrm{eqs}}$

$$
\begin{tikzcd}
& B \sim_{\mathrm{lat}} B' \arrow[dr, Rightarrow] & \\
B \sim_{\mathrm{eqs}} B' \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & B \sim_{\mathrm{cfg}} B' \\
& B \sim_{\mathrm{top}} B' \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

Using the refined version of the surjectivity of the period mapping for complex $K3$ surfaces, Degtyarev (2008) has given an algorithm to determine the connected components of the equisingular family (the equivalence classes of $\sim_{\mathrm{eqs}}$) in a given lattice type.

His algorithm involves a calculation of the orthogonal group $O(T_B)$. Since T_B is indefinite for $\mu_B < 19$, the complete table of the connected components of the equisingular family has not yet obtained *except for the case* $\mu_B = 19$.

λ : a given lattice type.

We want to calculate

$\mathrm{CES}(\lambda) := \lambda/\sim_{\mathrm{eqs}}$: the set of connected components of the equisingular family in λ.

- $\overline{\Sigma}$: the Néron-Severi lattice of λ; $\operatorname{sgn}\overline{\Sigma} = (1, \mu)$.
- $G \subset O(\overline{\Sigma})$: the subgroup of isometries of $\overline{\Sigma}$ preserving the set of exceptional (-2)-curves $[E]$ and the polarization h.
- $\mathcal{T}\mathrm{s}$: the set of isomorphism classes of even lattices T with $\operatorname{sgn} T = (2, 19-\mu)$ whose discriminant form is (-1) times the discriminant form of $\overline{\Sigma}$ (the set of possible transcendental lattices).

We have a natural projection

$$p : \mathrm{CES}(\lambda) \twoheadrightarrow \mathcal{T}\mathrm{s}.$$

For $T \in \mathcal{T}\mathrm{s}$, we put

- $L(\overline{\Sigma}, T)$: the set of even unimodular overlattices L of $\overline{\Sigma} \oplus T$ in which $\overline{\Sigma}$ and T are primitive (these L are the $K3$ lattice).
- $c\Omega(T)$: the set of connected components of the cone $\{x \in T \otimes \mathbb{R} \mid x^2 > 0\}$.

Theorem (Degtyarev)

We have

$$p^{-1}(T) \;\cong\; (G \times O(T))\backslash(L(\overline{\Sigma}, T) \times c\Omega(T)).$$

We say that a connected component is *real* if the corresponding $(G \times O(T))$-orbit $\subset L(\overline{\Sigma}, T) \times c\Omega(T)$ is stable under the interchanging of the two elements of $c\Omega(T)$.

Example

The simple sextics with $R_B = A_{18} + A_1$ form one lattice type. The equisingular family has three connected components; one is real and the other two are non-real. Artal, Carmona and Cogolludo (2002) constructed these simple sextics defined over $\mathbb{Q}(\alpha)$, where α is a root of

$$19x^3 + 50x^2 + 36x + 8 = 0,$$

which has one real root and two non-real roots. Their transcendental lattices are

$$\begin{bmatrix} 38 & 0 \\ 0 & 2 \end{bmatrix} \quad \texttt{(for real)}, \qquad \begin{bmatrix} 8 & \pm 2 \\ \pm 2 & 10 \end{bmatrix} \quad \texttt{(for non-real)}.$$

Example

The simple sextics with $R_B = A_{19}$ form one lattice type. The equisingular family has two connected components, and both are real. Artal et al. showed that they are conjugate by $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. Their transcendental lattices are

$$\begin{bmatrix} 2 & 0 \\ 0 & 20 \end{bmatrix}.$$

We would like to know whether these two are $\sim_{\mathrm{top}}$ or not.

Example

The simple sextics with $R_B = A_{14} + A_4 + A_1$ form one lattice type. The equisingular family has six non-real connected components. Their transcendental lattices are all isomorphic to

$$\begin{bmatrix} 10 & 0 \\ 0 & 30 \end{bmatrix}.$$

Summary

The “zoology” of simple sextics is interesting.