Lattices of algebraic cycles on varieties of Fermat type (joint work with Nobuyoshi Takahashi)

Ichiro Shimada

Hiroshima University

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Let X be a smooth projective complex surface, and let $D = \sum m_i C_i$ be an effective divisor on X.

We regard

$$H^2(X) := H^2(X, \mathbb{Z})/(\mathrm{torsion})$$

as a unimodular lattice by the cup-product. We consider the submodule

$$\mathcal{L}(X, D) := \langle [C_i] \rangle \subset H^2(X)$$

generated by the classes $[C_i]$ of reduced irreducible components C_i of D, and its primitive closure

$$\bar{\mathcal{L}}(X, D) := (\mathcal{L}(X, D) \otimes \mathbb{Q}) \cap H^2(X) \ \subset \ H^2(X).$$

Problem

How to calculate the finite abelian group

$$A(X, D) := \bar{\mathcal{L}}(X, D)/\mathcal{L}(X, D)?$$

Motivation 1.

Let X_m be the Fermat surface

$$x_0^m + x_1^m + x_2^m + x_3^m = 0,$$

and let D be the union of the $3m^2$ lines on X_m. For simplicity, we assume $m \geq 5$. Shioda showed that

$$(m, 6) = 1 \iff \mathrm{NS}(X_m) = \bar{\mathcal{L}}(X_m, D),$$

and posed the problem

$$(m, 6) = 1 \iff \mathrm{NS}(X_m) = \mathcal{L}(X_m, D)?$$

Recently, Schütt, Shioda and van Luijk showed the following by modulo p reduction technique and computer-aided calculation:

Theorem

Let m be ≤ 100 and prime to 6. Then $\mathrm{NS}(X_m) = \mathcal{L}(X_m, D)$. In particular, $A(X_m, D) = 0$.

Motivation 2.

In 1930's, Coble discovered a pair $[S_0, S_1]$ of quartic surfaces in $\mathbb{P}^3$ with 8 nodes that can *not* be connected by equising deformation: S_0 is called *azygetic*, and S_1 is called *syzygetic*.

They are distinguished by

$$h^0(\mathbb{P}^3, \mathcal{I}_Q(2)) = \begin{cases} 2 & \text{if } Q = \operatorname{Sing} S_0, \\ 3 & \text{if } Q = \operatorname{Sing} S_1, \end{cases}$$

where $\mathcal{I}_Q \subset \mathcal{O}_{\mathbb{P}^3}$ is the ideal sheaf of $Q \subset \mathbb{P}^3$.

Let X_0 and X_1 be the minimal resolutions of S_0 and S_1, respectively, and let D_0 and D_1 be the exceptional divisors. Then we have

$$\begin{cases} A(X_0, D_0) = \bar{\mathcal{L}}(X_0, D_0)/\mathcal{L}(X_0, D_0) & = & 0 \\ A(X_1, D_1) = \bar{\mathcal{L}}(X_1, D_1)/\mathcal{L}(X_1, D_1) & \cong & \mathbb{Z}/2\mathbb{Z}. \end{cases}$$

Using the Torelli theorem for complex $K3$ surfaces, we have found a *quartet* $[S_0, S_1, S_2, S_3]$ of quartic surfaces with RDPs of type

$$2A_1 + 2A_2 + 2A_5$$

such that, for the minimal resolution X_i of S_i and the exceptional divisor D_i on X_i, we have

$$\begin{aligned} A(X_0, D_0) &= 0, \\ A(X_1, D_1) &\cong \mathbb{Z}/2\mathbb{Z}, \\ A(X_2, D_2) &\cong \mathbb{Z}/3\mathbb{Z}, \\ A(X_3, D_3) &\cong \mathbb{Z}/6\mathbb{Z}. \end{aligned}$$

Motivation 3.

Let C_1 and C_2 be smooth conics on $\mathbb{P}^2$ in general position, and let $L_1, \dots, L_4$ be their common tangents. Consider the double covering $S \to \mathbb{P}^2$ branching along

$$T := C_1 + C_2 + L_1 + L_2 + L_3 + L_4.$$

Then S has RDPs of type $8A_3 + 10A_1$. Let $X \to S$ be the minimal resolution of S, and let D be the total transform of T. Then $A(X, D)$ is *non*-trivial.

We have the following classical theorem due to Salmon:

Theorem

There is a conic passing through the eight tacnodes of T.

Let X be a smooth projective complex surface, and $D = \sum m_i C_i$ an effective divisor on X.

For a submodule $M \subset H^2(X)$, we put

$$\operatorname{disc} M := |\det(S)|,$$

where S is the symmetric matrix expressing the cup-product restricted to M. Then

$$M \text{ is a sublattice of } H^2(X) \iff \operatorname{disc} M \neq 0.$$

If $\mathcal{L}(X, D) = \langle [C_i] \rangle$ is a sublattice, then so is $\bar{\mathcal{L}}(X, D)$ and

$$|A(X, D)| = \sqrt{\frac{\operatorname{disc} \mathcal{L}(X, D)}{\operatorname{disc} \bar{\mathcal{L}}(X, D)}}.$$

In particular, if $\operatorname{disc} \mathcal{L}(X, D)$ is square-free, then $A(X, D)$ is trivial.

If we know the configuration of irreducible components C_i of D, then we can calculate $\mathcal{L}(X, D)$ algebro-geometrically.

We present an algorithm to calculate $\operatorname{disc} \bar{\mathcal{L}}(X, D)$.

Remark that

$$\operatorname{disc} \mathcal{L}(X, D), \quad \operatorname{disc} \bar{\mathcal{L}}(X, D), \quad \text{and} \quad A(X, D)$$

depend only on the open surface

$$X \setminus D\,;$$

namely, if X' is another smooth projective surface containing $X \setminus D$ such that $D' := X' \setminus (X \setminus D)$ is a union of curves, then we have

$$\begin{aligned} \operatorname{disc} \mathcal{L}(X, D) &= \operatorname{disc} \mathcal{L}(X', D'), \\ \operatorname{disc} \bar{\mathcal{L}}(X, D) &= \operatorname{disc} \bar{\mathcal{L}}(X', D'), \\ A(X, D) &\cong A(X', D'). \end{aligned}$$

We show that, under certain assumptions, $\operatorname{disc} \bar{\mathcal{L}}(X, D)$ can be calculated *topologically* from $X \setminus D$.

Suppose that $\operatorname{disc} \mathcal{L}(X, D) \neq 0$. Then we have

$$\bar{\mathcal{L}}(X, D) = (\mathcal{L}(X, D)^{\perp})^{\perp},$$

and, since $H^2(X)$ is unimodular, we have

$$\operatorname{disc} \bar{\mathcal{L}}(X, D) = \operatorname{disc} \mathcal{L}(X, D)^{\perp}.$$

Thus it is enough to calculate the orthogonal complement $\mathcal{L}(X, D)^{\perp}$.

Proposition

By the Poincaré duality $H^2(X) \cong H_2(X)$, the orthogonal complement $\mathcal{L}(X, D)^\perp \subset H^2(X)$ is equal to the image of the homomorphism

$$j_* : H_2(X \setminus D) \to H_2(X)$$

induced by the inclusion $j : X \setminus D \hookrightarrow X$.

The proof follows from the following commutative diagram:

$$\begin{array}{ccccc} H_2(X \setminus D) & \xrightarrow{j_*} & H_2(X) & & \\ | \wr & & | \wr & & \\ H^2(X, D) & \longrightarrow & H^2(X) & \longrightarrow & H^2(D) = \bigoplus H^2(C_i). \end{array}$$

Remark

If $\mathcal{L}(X, D)^\perp \cong \operatorname{Im} j_*$ is of rank 0, then $\bar{\mathcal{L}}(X, D) = H^2(X)$ and hence $|A(X, D)| = \sqrt{\operatorname{disc} \mathcal{L}(X, D)}$.

Since $j_* : H_2(X \setminus D) \to H_2(X)$ preserves the *intersection pairing* $(\ ,\ )$ of topological cycles, we have the following:

Proposition

Suppose that $\operatorname{disc} \mathcal{L}(X, D) \neq 0$. Then the lattice $\mathcal{L}(X, D)^{\perp} \;\cong\; \operatorname{Im} j_*$ is isomorphic to the lattice

$$H_2(X \setminus D) / \ker(H_2(X \setminus D)),$$

where $\ker(H_2(X \setminus D))$ denotes the submodule

$$\{\, x \in H_2(X \setminus D) \;|\; (x, y) = 0 \ \text{ for all } \ y \in H_2(X \setminus D) \,\}.$$

Therefore, to calculate $\operatorname{disc} \bar{\mathcal{L}}(X, D)$, it is enough to calculate $H_2(X \setminus D)$ and the intersection pairing on $H_2(X \setminus D)$.

We apply our method to coverings of $\mathbb{P}^2$ branching along 4 lines

$$B_0, \quad B_1, \quad B_2, \quad B_3$$

in general position. Since

$$\pi_1(\mathbb{P}^2 \setminus \bigcup B_i) = (\mathbb{Z}\,\gamma_0 \oplus \cdots \oplus \mathbb{Z}\,\gamma_3)/\langle \gamma_0 + \cdots + \gamma_3 \rangle$$

is abelian, where $\gamma_0, \dots, \gamma_3$ are simple loops around $B_0, \dots, B_3$, these coverings are necessarily abelian.
For a surjective homomorphism

$$\rho : \pi_1(\mathbb{P}^2 \setminus \cup B_i) \to H$$

to a finite abelian group H, we denote by

$$\phi_\rho : Y_\rho \to \mathbb{P}^2$$

the finite covering associated to ρ, and by

$$\varphi_\rho : X_\rho \to Y_\rho \to \mathbb{P}^2$$

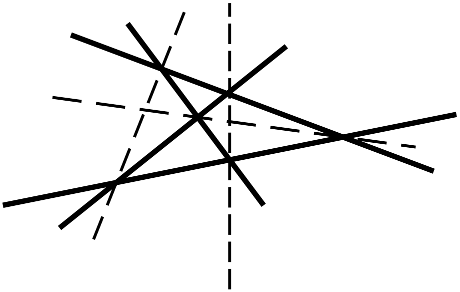
the composite of the resolution $X_\rho \to Y_\rho$ and the covering ϕ_ρ.

As the divisor D, we consider the pull-back of the three lines

$$\Lambda_1 + \Lambda_2 + \Lambda_3$$

passing through two of the six intersection points of $B_0, \dots, B_3$:

$$D_\rho := \varphi_\rho^*(\Lambda_1 + \Lambda_2 + \Lambda_3) \quad \subset \quad X_\rho.$$



Thick lines are B_i, and dash-lines are Λ_ν

Note that D_ρ contains the exceptional divisor of $X_\rho \to Y_\rho$.

When ρ is the maximal homomorphism

$$\begin{aligned}\pi_1(\mathbb{P}^2 \setminus \bigcup B_i) = \;& (\mathbb{Z}\,\gamma_0 \oplus \cdots \oplus \mathbb{Z}\,\gamma_3)/\langle \gamma_0 + \cdots + \gamma_3 \rangle \\ \to \;& (\mathbb{Z}/m\mathbb{Z})^3 = (\mathbb{Z}/m\mathbb{Z})e_0 \oplus (\mathbb{Z}/m\mathbb{Z})e_1 \oplus (\mathbb{Z}/m\mathbb{Z})e_2\end{aligned}$$

to the abelian group of exponent m given by

$$\rho(\gamma_i) = e_i \quad (i = 0, 1, 2) \quad \text{and} \quad \rho(\gamma_3) = -e_0 - e_1 - e_2,$$

then X_ρ is the Fermat surface of degree m, and D_ρ is the union of the $3m^2$ lines.

In general, X_ρ is a resolution of a quotient Y_ρ of the Fermat surface.

The divisor D_ρ is the union of the images of the $3m^2$ lines and the exceptional divisors of the resolution.

Since the singular points on Y_ρ are cyclic quotient singularities, we can resolve them by *Hirzebruch-Jung* method. Thus we can calculate the lattice $\mathcal{L}(X_\rho, D_\rho)$.

(Since $\mathcal{L}(X_\rho, D_\rho)$ contains a vector h with $h^2 > 0$, we have $\operatorname{disc} \mathcal{L}(X_\rho, D_\rho) \neq 0$ by Hodge index theorem.)

On the other hand, we obtain $\operatorname{disc} \bar{\mathcal{L}}(X_\rho, D_\rho)$ by calculating the intersection pairing on $H_2(X_\rho \setminus D_\rho)$.

We have carried out these calculations for all coverings associated to homomorphisms

$$\rho : \pi_1(\mathbb{P}^2 \setminus \bigcup B_i) \to \mathbb{Z}/m\mathbb{Z}$$

to cyclic groups of order $m \leq 40$.

In this case, the open surface $X_\rho \setminus D_\rho$ is a quotient of

$$X_m \setminus (\text{union of the } 3m^2 \text{ lines})$$

by the group $(\mathbb{Z}/m\mathbb{Z})^2$.

It turns out that the finite abelian groups

$$A(X_\rho, D_\rho) = \bar{\mathcal{L}}(X_\rho, D_\rho)/\mathcal{L}(X_\rho, D_\rho)$$

are non-trivial for many cases.

Let

$$R_0, \quad R_1, \quad R_2, \quad R_3$$

be the reduced irreducible curves on X_ρ that are mapped to the branching lines B_0, B_1, B_2, B_3, respectively. It is easy to see that

$$[R_i] \in \bar{\mathcal{L}}(X_\rho, D_\rho).$$

Theorem

Let $\rho : \pi_1(\mathbb{P}^2 \setminus \bigcup B_i) \to \mathbb{Z}/m\mathbb{Z}$ be a surjective homomorphism to a cyclic group of order m with $4 \leq m \leq 40$. Then

$$\bar{\mathcal{L}}(X_\rho, D_\rho) = \mathcal{L}(X_\rho, D_\rho) + \langle [R_0], [R_1], [R_2], [R_3] \rangle.$$

We put

$$\begin{aligned}\mathcal{L}'(X_\rho, D_\rho) &:= \mathcal{L}(X_\rho, D_\rho) + \langle [R_0], [R_1], [R_2], [R_3] \rangle \\ &= \mathcal{L}(X_\rho, D_\rho + R_0 + R_1 + R_2 + R_3).\end{aligned}$$

We can calculate $\operatorname{disc} \mathcal{L}'(X_\rho, D_\rho)$ algebro-geometrically.

The statement of Theorem is equivalent to say that $\mathcal{L}'(X_\rho, D_\rho)$ is primitive in $H^2(X_\rho)$ for $4 \le m \le 40$.

All we have to do is to calculate $\operatorname{disc}(\mathcal{L}(X_\rho, D_\rho)^\perp)$ and to show

$$\operatorname{disc} \mathcal{L}'(X_\rho, D_\rho) = \operatorname{disc}(\mathcal{L}(X_\rho, D_\rho)^\perp).$$

Problem

Is $\mathcal{L}'(X_\rho, D_\rho)$ primitive for all m and ρ?

The homomorphism $\rho : \pi_1(\mathbb{P}^2 \setminus \bigcup B_i) \to \mathbb{Z}/m\mathbb{Z}$ is given by

$$[a_0, a_1, a_2, a_3] := [\rho(\gamma_0), \rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3)].$$

Example

The following is the table of $\operatorname{disc} \mathcal{L}'(X_\rho, D_\rho)$ and $\operatorname{disc} \mathcal{L}(X_\rho, D_\rho)$ for $m = 12$:

$[a_0, a_1, a_2, a_3]$	$\operatorname{disc} \mathcal{L}'(X_\rho, D_\rho)$	$\operatorname{disc} \mathcal{L}(X_\rho, D_\rho)$	rank
$[0, 0, 1, 11]$	1	1	62
$[0, 1, 1, 10]$	1	$(2)^4(3)^4$	62
$[0, 1, 2, 9]$	1	$(2)^4$	50
$[0, 1, 3, 8]$	1	1	46
$[0, 1, 4, 7]$	1	$(3)^4$	50
$[0, 1, 5, 6]$	1	$(2)^4$	50
$[1, 1, 1, 9]$	$(2)^2(3)$	$(2)^{10}(3)^5$	44
$[1, 1, 2, 8]$	$(2)^4(3)$	$(2)^8(3)^5$	36

Example

$[a_0, a_1, a_2, a_3]$	$\operatorname{disc} \mathcal{L}'(X_\rho, D_\rho)$	$\operatorname{disc} \mathcal{L}(X_\rho, D_\rho)$	rank
$[1, 1, 3, 7]$	$(3)^3$	$(2)^6(3)^7$	30
$[1, 1, 4, 6]$	1	$(2)^4(3)^4$	38
$[1, 1, 5, 5]$	$(2)^6$	$(2)^{14}(3)^4$	40
$[1, 1, 11, 11]$	1	$(2)^6(3)^4$	38
$[1, 2, 2, 7]$	$(2)^4(3)^3$	$(2)^{10}(3)^7$	38
$[1, 2, 3, 6]$	$(2)^2(3)^2$	$(2)^8(3)^2$	34
$[1, 2, 4, 5]$	$(2)^4$	$(2)^8(3)^4$	31
$[1, 2, 10, 11]$	1	$(2)^6(3)^4$	38
$[1, 3, 3, 5]$	$(2)^4(3)^2$	$(2)^{10}(3)^2$	30
$[1, 3, 4, 4]$	$(3)^3$	$(3)^7$	34
$[1, 3, 9, 11]$	1	$(2)^6$	26
$[1, 3, 10, 10]$	$(3)^3$	$(2)^6(3)^7$	40

Example

$[a_0, a_1, a_2, a_3]$	$\operatorname{disc} \mathcal{L}'(X_\rho, D_\rho)$	$\operatorname{disc} \mathcal{L}(X_\rho, D_\rho)$	rank
$[1, 4, 8, 11]$	1	$(3)^4$	28
$[1, 4, 9, 10]$	(3)	$(2)^4(3)^5$	33
$[1, 5, 7, 11]$	1	$(2)^6(3)^4$	26
$[1, 5, 9, 9]$	$(2)^6$	$(2)^{14}$	28
$[1, 6, 6, 11]$	1	$(2)^6$	34
$[1, 6, 7, 10]$	$(3)^2$	$(2)^6(3)^6$	34
$[1, 6, 8, 9]$	$(2)^2$	$(2)^6$	30
$[1, 7, 8, 8]$	$(2)^4(3)^3$	$(2)^4(3)^7$	26
$[2, 3, 3, 4]$	$(2)^4$	$(2)^8$	34
$[2, 3, 9, 10]$	1	$(2)^6$	34
$[3, 4, 8, 9]$	1	1	30

We explain the method to calculate $H_2(X_\rho \setminus D_\rho)$ in detail.

First remark that, if Γ is an arbitrary line on $\mathbb{P}^2$, then

$$\mathcal{L}(X_\rho, D_\rho) \;\subset\; \mathcal{L}(X_\rho, D_\rho + \varphi_\rho^*(\Gamma)) \;\subset\; \bar{\mathcal{L}}(X_\rho, D_\rho),$$

and therefore we have

$$\mathcal{L}(X_\rho, D_\rho)^\perp = \mathcal{L}(X_\rho, D_\rho + \varphi_\rho^*(\Gamma))^\perp.$$

Hence it is enough to take suitable lines $\Gamma_1, \dots, \Gamma_k$, put

$$U \;:=\; \mathbb{P}^2 \setminus (\bigcup B_i \cup \bigcup \Lambda_\nu \cup \bigcup \Gamma_q),$$

and calculate the intersection form on $H_2(X^U)$, where

$$X^U := \varphi_\rho^{-1}(U).$$

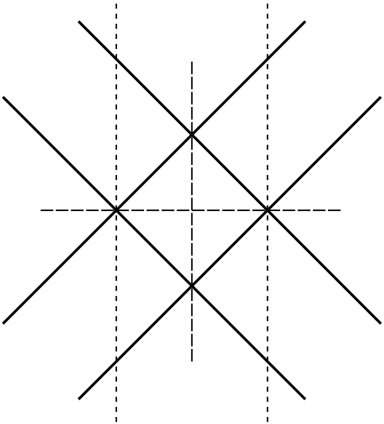
We choose U in such a way that U admits a morphism

$$f : U \to \mathbb{C} \setminus \{P_1, \dots, P_N\}$$

such that the composite

$$f \circ \varphi_\rho \quad : \quad X^U \to U \to \mathbb{C} \setminus \{P_1, \dots, P_N\}$$

is a locally trivial fibration (in the classical topology) with fibers being open Riemann surfaces.



The thick lines are $B_0, \dots, B_3$: The x-axis and the y-axis are Λ_1 and Λ_2 (Λ_3 is the line at infinity):

The fibration f is given by $(x, y) \mapsto x$, and hence we have to remove two extra vertical dash-lines Γ_1 and Γ_2.
In this case, we have $N = 3$.

We choose a base point

$$b \in \mathbb{C} \setminus \{P_1, P_2, P_3\},$$

and consider the open Riemann surface

$$R_b := (f \circ \varphi_\rho)^{-1}(b) \quad \subset \quad X^U.$$

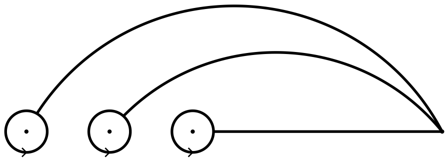
Then R_b is an étale cover of the punctured affine line

$$f^{-1}(b) \subset U.$$

Thus we can calculate $H_1(R_b, \mathbb{Z})$ and the intersection pairing

$$Q : H_1(R_b) \times H_1(R_b) \to \mathbb{Z}.$$

We choose a system of simple loops $\{\sigma_1, \sigma_2, \sigma_3\}$ with the base point b on $\mathbb{C} \setminus \{P_1, P_2, P_3\}$ as follows:



(The loops are $\sigma_1, \sigma_2, \sigma_3$ from left to right.)

When $t \in \mathbb{C}$ moves along σ_i, the punctured and branching points of

$$(f \circ \varphi_\rho)^{-1}(t) \to f^{-1}(t)$$

undergo the *braid monodromies*. Looking at them, we obtain the monodromies along σ_i:

$$\mu_i \ : \ H_1(R_b) \ \to \ H_1(R_b).$$

Since $\mathbb{C} \setminus \{P_1, P_2, P_3\}$ is homotopically equivalent to the union of σ_i, the open surface X^U is homotopically equivalent to the union of the fibers $(f \circ \varphi_\rho)^{-1}(t)$ over these loops.

Let an element

$$([\gamma_1], [\gamma_2], [\gamma_3]) \in \bigoplus_{i=1}^{3} H_1(R_b)$$

represent a topological *chain* on X^U that is the union of tubes drawn by the topological cycle $\gamma_i \subset R_b$ moving over σ_i.

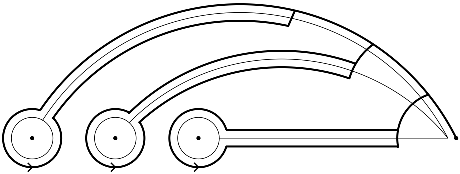
Its boundary is in R_b, and the homology class of the boundary is

$$w([\gamma_1], [\gamma_2], [\gamma_3]) := \sum (1 - \mu_i)([\gamma_i]) \quad \in \quad H_1(R_b).$$

Hence $H_2(X^U)$ is equal to the kernel of

$$w \quad : \quad \bigoplus_{i=1}^{3} H_1(R_b) \ \to \ H_1(R_b).$$

The intersection pairing on $H_2(X^U)$ is calculated by *perturbing* the system of simple loops $\{\sigma_1, \sigma_2, \sigma_3\}$:



For the perturbation above, we have

$$
\begin{aligned}
& -([\gamma_1],[\gamma_2],[\gamma_3]) \cdot (([\gamma_1'],[\gamma_2'],[\gamma_3']) \\
= \quad & Q((1-\mu_1)([\gamma_1]),(1-\mu_2)([\gamma_2'])) \\
& + Q((1-\mu_1)([\gamma_1]),(1-\mu_3)([\gamma_3'])) \\
& + Q((1-\mu_2)([\gamma_2]),(1-\mu_3)([\gamma_3'])) \\
& + Q((1-\mu_1)([\gamma_1]),-\mu_1([\gamma_1'])) + Q((1-\mu_2)([\gamma_2]),-\mu_2([\gamma_2'])) \\
& \qquad + Q((1-\mu_3)([\gamma_3]),-\mu_3([\gamma_3'])).
\end{aligned}
$$

Then we can calculate

$$\ker(H_2(X^U)) := \{ x \in H_2(X^U) \mid (x, y) = 0 \text{ for all } y \in H_2(X^U) \},$$

and the lattice

$$H_2(X^U)/\ker(H_2(X^U)) \;\cong\; \mathcal{L}(X_\rho, D_\rho)^\perp.$$

If $H_2(X^U) \neq \ker(H_2(X^U))$, then we confirm

$$\operatorname{disc}(H_2(X^U)/\ker(H_2(X^U))) = \operatorname{disc} \mathcal{L}'(X_\rho, D_\rho).$$

If $H_2(X^U) = \ker(H_2(X^U))$, then we confirm $\operatorname{disc} \mathcal{L}'(X_\rho, D_\rho) = 1$.

Thus we can conclude that $\mathcal{L}'(X_\rho, D_\rho)$ is primitive.

Remark

For Shioda's original problem of Fermat surface X_m of degree m, we have to consider the covering $X^U \to U$ of mapping degree m^3.

Maple has run out of memory even when $m = 6$ ($m^3 = 216$).

Thank you!