An algorithm to compute automorphism groups of $K3$ surfaces

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Let X be an algebraic $K3$ surface.
We denote by S_X the Néron-Severi lattice of X.

Suppose that X is defined over $\mathbb{C}$, or is supersingular in odd characteristic. Then, thanks to the Torelli-type theorem due to Piatetski-Shapiro and Shafarevich (1971) and Ogus (1978, 1983), we can study the automorphism group

$$\mathrm{Aut}(X)$$

of X by S_X.

We denote by $\mathrm{O}(S_X)$ the orthogonal group of S_X. Then we have a natural homomorphism

$$\mathrm{Aut}(X) \to \mathrm{O}(S_X).$$

It is known that this homomorphism has only a finite kernel. We present an algorithm to give a finite set of generators of the image of this homomorphism.

We concentrate on **complex** $K3$ surfaces.

With the cup-product,

$$H := H^2(X, \mathbb{Z})$$

is an even unimodular lattice of signature $(3, 19)$. Let T_X is the orthogonal complement of $S_X = H \cap H^{1,1}$ in H. Let $\omega_X \in T_X \otimes \mathbb{C}$ be a non-zero holomorphic 2-form, and put

$$C_X := \{ \, g \in \mathrm{O}(T_X) \mid \omega_X^g = \lambda\,\omega_X \ \text{ for some } \ \lambda \in \mathbb{C}^\times \, \}.$$

We denote by

$$\mathrm{Nef}(X) := \{ \, x \in S_X \otimes \mathbb{R} \mid \langle x, [C] \rangle \geq 0 \ \text{ for any curve } \ C \, \}$$

the nef cone of X, and put

$$Aut(\mathrm{Nef}(X)) := \{ \, g \in \mathrm{O}(S_X) \mid \mathrm{Nef}(X)^g = \mathrm{Nef}(X) \, \}.$$

Then we have:

Theorem. Via the natural actions of $\mathrm{Aut}(X)$ on S_X and T_X, the group $\mathrm{Aut}(X)$ is identified with the subgroup of

$$Aut(\mathrm{Nef}(X)) \times C_X$$

consisting of pairs

$$(g, h) \in Aut(\mathrm{Nef}(X)) \times C_X$$

such that g and h are restrictions of an element $\gamma \in \mathrm{O}(H)$ to S_X and T_X, respectively.

Remark. There is a simple criterion for the existence of γ by means of discriminant forms.

Hence it is imprttant to calculate $Aut(\mathrm{Nef}(X))$.

The nef cone $\mathrm{Nef}(X)$ is bounded by the hyperplanes $([C])^{\perp}$ perpendicular to the classes of (-2)-curves (that is, smooth rational curves) on X.

The cases where $\mathrm{Nef}(X)$ has only finitely many walls ($\Longleftrightarrow$ $\mathrm{Aut}(X)$ is finite) were classified by Nikulin (1981, 2000) and Vinberg (2007).

The fact that $\mathrm{Aut}(X)$ is finitely generated was proved by Sterk (1985) and Lieblich-Maulik (2011).

Vinberg (1983) gave a set of generators of infinite $\mathrm{Aut}(X)$ for two most algebraic $K3$ surfaces.

Using an idea of Borcherds, Kondo (1998) gave a set of generators of Aut of a generic Jacobian Kummer surface.

Since then, automorphism groups of several $K3$ surfaces have been determined by this method:
Kondo-Keum (2001): Kummer surfaces of product type
Dolgachev-Keum (2002): Hessian quartics
Dolgachev-Kondo (2003): a supersingular $K3$ surface in char 2
Kondo-S. (2012): a supersingular $K3$ surface in char 3

Our method is a generalization of Borcherds-Kondo method.

Example

Let X be a complex $K3$ surface with Picard number 3 (that is, S_X is of rank 3).
Suppose that X admits an elliptic fibration

$$\phi : X \to \mathbb{P}^1$$

with a section $\mathbb{P}^1 \to X$. Considering this section as the origin, we can consider the Mordell-Weil group MW_ϕ.

We assume that MW_ϕ is of rank 1 (that is, ϕ has no reducible fibers).

Then the group $\mathrm{Aut}(X)$ contains the subgroup

$$\mathrm{MW}_\phi \rtimes \{\pm 1\} \cong (\mathbb{Z}/2) * (\mathbb{Z}/2)$$

generated by the translations and the inversion.

We denote by $f_\phi \in S_X$ the class of a fiber of $\phi : X \to \mathbb{P}^1$ and by $z_\phi \in S_X$ the class of the zero section of ϕ.

Then there is $v_3 \in S_X$ such that f_ϕ, z_ϕ, v_3 form a basis of S_X, and that the Gram matrix of S_X with respect to f_ϕ, z_ϕ, v_3 is

$$\mathrm{Gram}_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2k \end{bmatrix},$$

where $-2k := v_3^2$. (The number $2k$ is the discriminant of the Mordell-Weil lattice of $\phi : X \to \mathbb{P}^1$.)

We further assume that the period $\mathbb{C}\omega_X$ of X is sufficiently generic. Then the natural homomorphism

$$\mathrm{Aut}(X) \to \mathrm{O}(S_X) \cong \{\, g \in \mathrm{GL}_3(\mathbb{Z}) \mid g \cdot \mathrm{Gram}_S \cdot {}^t g = \mathrm{Gram}_S \,\}$$

is injective. The image of $\mathrm{MW}_\phi \rtimes \{\pm 1\} \cong (\mathbb{Z}/2) * (\mathbb{Z}/2)$ is generated by

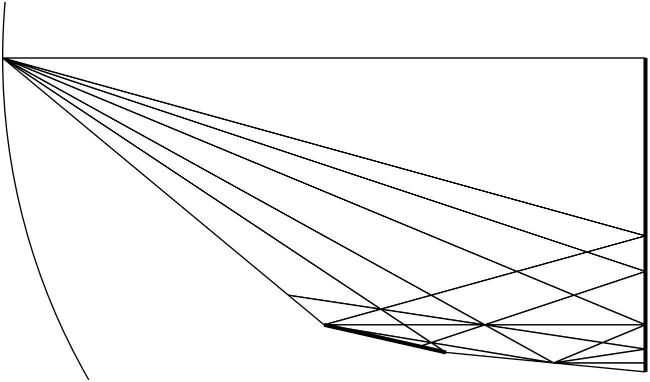
$$h_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad h_2 := \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & -1 \\ 2k & 0 & -1 \end{bmatrix}.$$

Suppose that $k = 11$.
Then we need one more generator for $\mathrm{Aut}(X)$:

$$h_3 := \begin{bmatrix} 20 & 9 & -3 \\ 7 & 2 & -1 \\ 154 & 66 & -23 \end{bmatrix}.$$

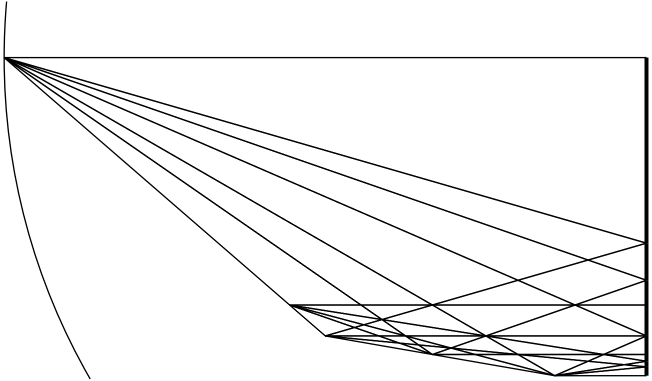
The set of (-2)-curves on X is decomposed into at most two orbits under the action of $\mathrm{Aut}(X)$. There is only one Jacobian fibration on X modulo $\mathrm{Aut}(X)$.

The fundamental domain of $\mathrm{Aut}(X)$ for the case $k = 11$ in $\mathrm{Nef}(X)$ in the projective disc model is as follows.



Suppose that $k = 12$. Then we need two more generators:

$$h_3 := \begin{bmatrix} 37 & 12 & -5 \\ 36 & 13 & -5 \\ 360 & 120 & -49 \end{bmatrix}, \quad h_4 := \begin{bmatrix} 97 & 48 & -14 \\ 0 & 1 & 0 \\ 672 & 336 & -97 \end{bmatrix}.$$



Generalized Borcherds-Kondo method.

Suppose that we have a primitive embedding

$$S_X \hookrightarrow L$$

of S_X into an even **unimodular** hyperbolic lattice

$$L := \mathrm{II}_{1,n-1}$$

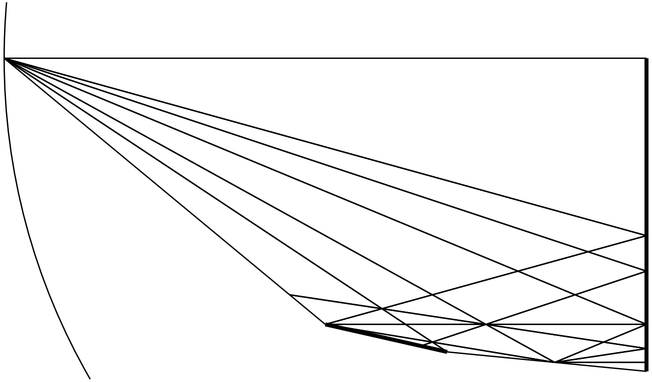
of rank n, where $n = 10, 18$ or 26. Let $\mathcal{P}_L \subset L \otimes \mathbb{R}$ be the positive cone of L containing $\mathrm{Nef}(X)$. Then the hyperplanes $(r)^{\perp}$ in $\mathcal{P}_L$ perpendicular to the vectors $r \in L$ with $r^2 = -2$ decompose $\mathcal{P}_L$ into a union of closed chambers, which we call *Conway chambers*. They are fundamental domain of the reflection group in $\mathrm{O}^+(L)$. Conway (1983) described the shape of Conway chambers.

The tessellation of $\mathcal{P}_L$ by the Conway chambers induces a tessellation of $\mathcal{P}_{S_X} = (S_X \otimes \mathbb{R}) \cap \mathcal{P}_L$. We call the chambers of this induced tessellation *induced chambers*.

Under certain conditions on $S_X \hookrightarrow L$, we have the following:

- The decomposition by induced chambers is $\mathrm{O}^+(S_X)$-invariant.
- The number of $\mathrm{O}^+(S_X)$-orbits on the set of induced chambers is finite.
- The nef cone $\mathrm{Nef}(X)$ is a union of induced chambers.
- Each induced chamber has only finitely many walls, and hence its automorphism group is finite.

Therefore we can find all $\mathrm{O}^+(S_X)$-congruence classes of induced chambers, and hence we obtain the fundamental domain.



Remark.

Borcherds (1987) proved that, if the orthogonal complement of S_X in L is a root lattice, then the induced chambers are $\mathrm{O}^+(S_X)$-congruent to each other.

Remark.

We have applied our algorithm to the Néron-Severi lattice of the complex Fermat quartic. There are too many $\mathrm{O}^+(S_X)$-congruence classes of induced chambers.

Remark.

The preprint is available from
http://arxiv.org/abs/1304.7427