K3 Surfaces and Lattice Theory

Ichiro Shimada

Hiroshima University

2014 Aug Singapore

Example

Consider two surfaces S_+ and S_- in $\mathbb{C}^3$ defined by

$$w^2(G(x,y) \pm \sqrt{5}\cdot H(x,y)) = 1, \quad \text{where}$$

$$\begin{aligned} G(x,y) &:= -9\,x^4 - 14\,x^3y + 58\,x^3 - 48\,x^2y^2 - 64\,x^2y \\ &\quad +10\,x^2 + 108\,xy^3 - 20\,xy^2 - 44\,y^5 + 10\,y^4, \\ H(x,y) &:= 5\,x^4 + 10\,x^3y - 30\,x^3 + 30\,x^2y^2 + \\ &\quad +20\,x^2y - 40\,xy^3 + 20\,y^5. \end{aligned}$$

Since S_+ and S_- are conjugate by $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$, they can *not* be distinguished algebraically.
But S_+ and S_- are not homeomorphic (in the classical topology).

Many examples of non-homeomorphic conjugate complex varieties are known since Serre (1964).

Introduction

Definition

A smooth projective surface X is called a $K3$ *surface* if

- $\exists$ a nowhere vanishing holomorphic 2-form ω_X on X, and
- $\pi_1(X) = \{1\}$.

We consider the following geometric problems on $K3$ surfaces:

- enumerate elliptic fibrations on a given $K3$ surface,
- enumerate elliptic $K3$ surfaces up to some equivalence relation,
- enumerate projective models of a given $K3$ surface,
- enumerate projective models of $K3$ surfaces,
- determine the automorphism group of a given $K3$ surface,
- $\dots$.

Thanks to the theory of period mapping, some of these problems are reduced to computational problems in lattice theory, and the latter can often be solved by means of *computer*.

In this talk, we explain how to use lattice theory and computer in the study of $K3$ surfaces.

We then demonstrate this method on the problems of constructing Zariski pairs of plane curves of degree 6.

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle \quad \rangle : L \times L \to \mathbb{Z}.$$

Let L be a lattice of rank n. We choose a basis $e_1, \ldots, e_n$ of L. The lattice L is given by the Gram matrix

$$G := \left(\langle e_i, e_j \rangle\right)_{i,j=1,\ldots,n}.$$

- $\mathrm{O}(L)$ is the group of all isometries of L.
- L is *unimodular* if $\det G = \pm 1$.
- The *signature* $\mathrm{sgn}(L)$ is the signature $L \otimes \mathbb{R}$.
- A lattice L is said to be *hyperbolic* if $\mathrm{sgn}(L) = (1, n-1)$, and is *positive-definite* if $\mathrm{sgn}(L) = (n, 0)$.
- A lattice L is *even* if $v^2 \in 2\mathbb{Z}$ for all $v \in L$.
- A sublattice L' of L is *primitive* if L/L' is torsion free.

Lattices associated to a $K3$ surface

$K3$ surfaces are diffeomorphic to each other.
Suppose that X is a $K3$ surface. Then $H^2(X, \mathbb{Z})$ with the cup product is an even unimodular lattice of signature $(3, 19)$, and hence is isomorphic to

$$U^{\oplus 3} \oplus E_8^{-\oplus 2},$$

where U is the hyperbolic plane with a Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and E_8^- is the negative definite root lattice of type E_8.

$$\begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

The Gram matrix of E_8^-

The *Néron-Severi lattice*

$$S_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

of cohomology classes of divisors on X is an even hyperbolic lattice of rank ≤ 20. Moreover the sublattice S_X of $H^2(X, \mathbb{Z})$ is primitive.

Problem

Suppose that an even hyperbolic lattice of rank ≤ 20 is given. Is there a $K3$ surface X such that $S \cong S_X$?

We have the following corollary of *the surjectivity of the period map*:

Theorem

Let S be a primitive hyperbolic sublattice of $U^{\oplus 3} \oplus E_8^{-\oplus 2}$. Then $\exists$ a K3 surface X such that $S \cong S_X$.

Problem

Suppose that an even lattice L and an even unimodular lattice M are given. Can L be embedded into M primitively?

A lattice L is canonically embedded into its *dual lattice*

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z})$$

as a submodule of finite index. The finite abelian group

$$D_L := L^\vee / L$$

is called the *discriminant group* of L.

The symm. bil. form on L extends to a $\mathbb{Q}$-valued symm. bil. form on $L^\vee$, and it defines a finite quadratic form

$$q_L \colon D_L \to \mathbb{Q}/2\mathbb{Z}, \quad \bar{x} \mapsto x^2 \bmod 2\mathbb{Z}.$$

Let M be an even *unimodular* lattice containing L primitively with the orthogonal complement $L^\perp$. Then we have

$$(D_L, q_L) \cong (D_{L^\perp}, -q_{L^\perp}).$$

Conversely, if R is an even lattice such that

$$(D_L, q_L) \cong (D_R, -q_R),$$

then there exist an even unimodular lattice M and a primitive embedding $L \hookrightarrow M$ such that $L^\perp \cong R$.

Problem

Suppose that $s_+, s_- \in \mathbb{Z}_{\geq 0}$ and a finite quadratic form (D, q) are given. Can we determine whether $\exists$ an even lattice L such that $\mathrm{sgn}(L) = (s_+, s_-)$ and $(D_L, q_L) \cong (D, q)$?

Theorem

YES.

Corollary

We can determine whether a given even hyperbolic lattice of rank ≤ 20 is a Néron–Severi lattice of a $K3$ surface X or not.

Polarized $K3$ surfaces

For $v \in S_X$, let $\mathcal{L}_v \to X$ be the corresponding line bundle.

Definition

For $d \in \mathbb{Z}_{>0}$, a vector $h \in S_X$ of $h^2 = d$ is a *polarization of degree d* if $|\mathcal{L}_h| \neq \emptyset$ and has no fixed-components.

Let h be a polarization of degree d. Then $|\mathcal{L}_h|$ defines $\Phi_h : X \to \mathbb{P}^{1+d/2}$. We denote by

$$X \xrightarrow{\phi_h} Y_h \xrightarrow{\psi_h} \mathbb{P}^{1+d/2}$$

the Stein factorization of Φ_h. The normal surface Y_h is the *projective model* of (X, h), and has only rational double points as its singularities.

Example

A plane curve B is a *simple sextic* if B is of degree 6 and has only simple singularities (ADE-singularities; ordinary nodes, ordinary cusps, tacnodes, ...).

Let B be a simple sextic, and $Y_B \to \mathbb{P}^2$ the double covering branched along B. The minimal resolution X_B of Y_B is a $K3$ surface.

We denote by

$$\Phi_B : X_B \to Y_B \to \mathbb{P}^2$$

the composite of the min. resol. and the double covering, and by $h_B \in S_{X_B}$ the class of the pull-back of a line. Then h_B is a polarization of degree 2, and Y_B is its projective model.

Problem

Suppose that $h \in S_X$ with $h^2 > 0$ is given. Is h a polarization? If so, what is the ADE-type of $\mathrm{Sing}(Y_h)$?

We consider the second question first.

Proposition

The ADE-type of $\mathrm{Sing}\, Y_h$ is equal to the ADE-type of the root system $\{r \in S_X \mid \langle h, r\rangle = 0, \langle r, r\rangle = -2\}$.

The sublattice $\{x \in S_X \mid \langle h, x\rangle = 0\}$ is negative-definite.

Problem

Given a positive-definite lattice L and an integer d. Calculate the set $\{r \in L \mid \langle r, r\rangle = d\}$.

Suppose that we are given a triple $[Q, \lambda, c]$, where

- Q is a positive-definite $n \times n$ symmetric matrix with entries in $\mathbb{Q}$,
- λ is a column vector of length n with entries in $\mathbb{Q}$,
- $c \in \mathbb{Q}$.

For $QT := [Q, \lambda, c]$, we define $F_{QT} : \mathbb{R}^n \to \mathbb{R}$ by

$$F_{QT}(v) := v\, Q\, {}^t v + 2\, v\, \lambda + c.$$

We have an algorithm to calculate the finite set

$$E(QT) := \{\, v \in \mathbb{Z}^n \mid F_{QT}(v) \le 0 \,\}$$

by induction on n, and hence we can determine the ADE-type of $\operatorname{Sing} Y_h$.

Criterion for a polarization

Let L be an even hyperbolic lattice. Let $\mathcal{P}_L$ be one of the two connected components of $\{x \in L \otimes \mathbb{R} \mid x^2 > 0\}$.
We put

$$\mathcal{R}_L := \{ \, r \in L \; \mid \; r^2 = -2 \, \}.$$

Each $r \in \mathcal{R}_L$ defines a reflection s_r into the hyperplane $(r)^\perp := \{x \in \mathcal{P}_L \mid \langle x, r\rangle = 0\}$:

$$s_r : x \mapsto x + \langle x, r\rangle r,$$

The closure in $\mathcal{P}_L$ of each connected component of

$$\mathcal{P}_L \setminus \bigcup_{r \in \mathcal{R}_L} (r)^\perp$$

is a standard fundamental domain of the action on $\mathcal{P}_L$ of

$$W(L) := \langle \, s_r \mid r \in \mathcal{R}_L \, \rangle.$$

Let $\mathcal{P}(X) \subset S_X \otimes \mathbb{R}$ be the positive cone that contains an ample class (e.g., the class of a hyperplane section). We put

$$N(X) := \{x \in \mathcal{P}(X) \mid \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } X \}.$$

Proposition

This $N(X)$ is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}(X)$.

It is obvious that, if h is a polarization, then $h \in N(X)$.

Proposition

Let $h \in S_X$ be a vector with $h^2 = 2$. Then h is a polarization if and only if $h \in N(X)$ and $\nexists\ e \in S_X$ with $e^2 = 0$ and $\langle e, h \rangle = 1$.

Problem

Suppose that $h \in S_X$ with $h^2 > 0$ is given. Does h belong to $N(X)$?

Since $N(X)$ is bounded by $(r)^\perp$, this problem is reduced to the following:

Problem

Suppose that we are given vectors $h, h_0 \in \mathcal{P}_L$. Then, for a negative integer d, calculate the set

$$\{ \, r \in S_X \mid \langle r, h \rangle > 0, \ \langle r, h_0 \rangle < 0, \ \langle r, r \rangle = -2 \, \}.$$

There is an algorithm for this task, and hence we can determine whether a given $h \in S_X$ with $h^2 = 2$ is a polarization or not.

Zariski pairs

For a simple sextic B,

- R_B : the ADE-type of $\mathrm{Sing}\, B$ (or of $\mathrm{Sing}\, Y_B$),
- $\mathrm{degs}\, B$ the list of degrees of irreducible components of B.

We say that B and B' are of the same config type and write $B \sim_{\mathrm{cfg}} B'$ if

- $R_B = R_{B'}$, $\mathrm{degs}\, B = \mathrm{degs}\, B'$,
- their intersection patterns of irred. comps are same.

Example

Zariski showed the existence of a pair $[B, B']$ such that

- $R_B = R_{B'} = 6A_2$, $\mathrm{degs}\, B = \mathrm{degs}\, B' = [6]$, and
- $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/(2) * \mathbb{Z}/(3)$,
  $\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/(2) \times \mathbb{Z}/(3)$.

For a simple sextic B with

$$\Phi_B : X_B \to Y_B \to \mathbb{P}^2,$$

let $\mathcal{E}_B$ be the set of exceptional curves of $X_B \to Y_B$, and let

$$\Sigma_B := \langle\, [E] \mid E \in \mathcal{E}_B \,\rangle \oplus \langle h_B \rangle \ \subset \ H^2(X_B, \mathbb{Z}),$$

where h_B is the class of the pull-back of a line. We have

$$B \sim_{\rm cfg} B' \ \Rightarrow \ \Sigma_B \cong \Sigma_{B'}.$$

We denote the primitive closure of Σ_B by

$$\overline{\Sigma}_B \ \subset \ S_{X_B} \ \subset \ H^2(X_B, \mathbb{Z}).$$

After the partial results by Urabe, Yang (1996) made the complete list configuration type of simple sextics by classifying all such $\overline{\Sigma}_B$.

We write $B \sim_{\mathrm{emb}} B'$ if there exists a homeomorphism

$$\psi : (\mathbb{P}^2, B) \xrightarrow{\sim} (\mathbb{P}^2, B').$$

We have $B \sim_{\mathrm{emb}} B' \implies B \sim_{\mathrm{cfg}} B'$.

$$\#\text{ of config types} = 11159 < \#\text{ of emb-top types} = ?$$

Definition

A *Zariski pair* is a pair $[B, B']$ of projective plane curves of the same degree with only simple singularities such that $B \sim_{\mathrm{cfg}} B'$ but $B \not\sim_{\mathrm{emb}} B'$.

We consider the finite abelian group

$$G(B) := \overline{\Sigma}_B / \Sigma_B .$$

We put

$$\Theta_B := (\Sigma_B \subset H^2(X_B, \mathbb{Z}))^{\perp}.$$

Theorem

If $B \sim_{\mathrm{emb}} B'$, then $\Theta_B \cong \Theta_{B'}$.

In fact, Θ_B is a topological invariant of the open surface

$$U_B := \Phi_B^{-1}(\mathbb{P}^2 \setminus B) \quad \subset \quad X_B,$$

because we have $\Theta_B \ \cong \ H^2(U_B, \mathbb{Z}) / \mathrm{Ker}$, where

$$\mathrm{Ker} := \{ \ v \in H^2(U_B) \ \mid \ \langle v, x \rangle = 0 \text{ for all } x \in H^2(U_B) \ \}.$$

Since the discriminant groups of $\overline{\Sigma}_B$ and Θ_B are isomorphic, we have:

Corollary

If $B \sim_{\mathrm{cfg}} B'$ but $|G(B)| \neq |G(B')|$, then $B \not\sim_{\mathrm{emb}} B'$.

This corollary produces many examples of Zariski pairs.

Example

In Zariski's example $[B, B']$ with $R_B = R_{B'} = 6A_2$, $\mathrm{degs}\, B = \mathrm{degs}\, B' = [6]$ and

$$\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/(2) * \mathbb{Z}/(3), \quad \pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/(2) \times \mathbb{Z}/(3),$$

we have $G(B) \cong \mathbb{Z}/3\mathbb{Z}$ and $G(B') = 0$.

Singular $K3$ surfaces

Definition

A $K3$ surface X is called *singular* if $\mathrm{rank}(S_X) = 20$.

Theorem (Shioda and Inose)

The map

$$X \mapsto T(X) := (S_X \subset H^2(X, \mathbb{Z}))^\perp$$

is a bijection from the set of isom. classes of singular $K3$ surfaces to the set of isom. classes of oriented positive-definite even lattices of rank 2.

Theorem (Shioda and Inose)

Every singular $K3$ surface X is defined over $\overline{\mathbb{Q}}$.

Theorem (S. and Schütt)

Let X and X' be singular $K3$ surfaces over $\overline{\mathbb{Q}}$ such that $q_{T(X)} \cong q_{T(X')}$.
Then there $\exists\ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $X' \cong X^\sigma$.

If B is a simple sextic with total Milnor number $\mu(B) = 19$, then X_B is a singular $K3$ surface with $\Theta_B \cong T(X_B)$.

Corollary

Let B be a sextic with $\mu(B) = 19$ defined over $\overline{\mathbb{Q}}$.
If $\exists$ an even pos-def lattice T' of rank 2 with $q_{T'} \cong q_{T(X_B)}$ and $T' \not\cong T(X_B)$,
then $\exists\, \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $B \not\sim_{\mathrm{emb}} B^\sigma$.

The first example revisited

Consider the config type of sextics $B = L + Q$, where

- $\deg L = 1$, $\deg Q = 5$,
- L and Q are tangent at one point with multiplicity 5 (A_9-singularity), and
- Q has one A_{10}-singular point.

Such sextics are projectively isomorphic to

$$z \cdot (G(x, y, z) \pm \sqrt{5} \cdot H(x, y, z)) = 0,$$

where $G(x, y, z)$ and $H(x, y, z)$ are homogenizations of the polynoms in the 1st slide with $L = \{z = 0\}$.
The genus corresponding to $(D_{\overline{\Sigma}_B}, -q_{\overline{\Sigma}_B})$ and signature $(2, 0)$ (that is, the genus containing $T(X_B)$) consists of

$$\begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix} (\text{for } +\sqrt{5}), \qquad \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} (\text{for } -\sqrt{5}).$$