Connected components of the moduli of elliptic $K3$ surfaces

Ichiro Shimada

Hiroshima University

2016 November Chambéry

We work over the complex number field.
All $K3$ surfaces in this talk are algebraic.

Thanks to the Torelli theorem for $K3$ surfaces, we can study the moduli of $K3$ surfaces by lattice theory.

We study connected components of the moduli of elliptic $K3$ surfaces with a fixed combinatorial data.
For this, it is necessary to calculate all the isomorphism classes of lattices in a given genus.

I determine the connected components of elliptic $K3$ surfaces with a fixed combinatorial data, by means of **Miranda-Morrison theory**.

An *elliptic* $K3$ *surface* is a triple (X, f, s), where X is a $K3$ surface, $f\colon X \to \mathbb{P}^1$ is a fibration whose general fiber is a curve of genus 1, and $s\colon \mathbb{P}^1 \to X$ is a section of f.

Let (X, f, s) be an elliptic $K3$ surface. It is well-known that the set of sections of f has a natural structure of the finitely-generated abelian group with the zero element s, which is called the *Mordell-Weil group*. We put

$$A_f := \text{the torsion part of the Mordell-Weil group of } (X, f, s).$$

If an irreducible curve C on X is contained in a singular fiber of f and is disjoint from the zero section s, then C is a smooth rational curve. These curves form an ADE-configuration.

$$\Phi_f := \text{the } ADE\text{-type of the set } \mathcal{R}_f \text{ of these curves.}$$

The *combinatorial type* of (X, f, s) is defined to be (Φ_f, A_f).
The combinatorial type determines a lattice polarization of X.

Theorem (S.- 1999)

There exist exactly 3693 combinatorial types that can be realized as combinatorial types of elliptic $K3$ surfaces.

no.1	A_1	0
	$\dots$	
no.3692	$2A_4 + 2A_3 + 2A_2$	0
no.3693	$6A_3$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

The problem

Determine the connected components of the moduli of elliptic $K3$ surfaces with a fixed combinatorial data (Φ, A).

This work is motivated by the following two works:

[AD] A. Akyol and A. Degtyarev. Geography of irreducible plane sextics. Proc. Lond. Math. Soc. (3), 111(6):1307–1337, 2015.

[G] Ç. Güneș Aktaș. Classification of simple quartics up to equisingular deformation. arXiv:1508.05251.

In [AD], the connected components of the equisingular families of irreducible sextic plane curves with fixed type of ADE-singularities are calculated. In [G], the same calculation was done for non-special singular quartic surfaces with only ADE-singularities.

In both of [AD] and [G], the Miranda-Morrison theory was applied. I developed an algorithm to calculate a spinor norm of an isometry of a p-adic lattice, and made the method fully-automated.

I hope this algorithm is applicable for the moduli of lattice-polarized $K3$ surfaces in general.

Two elliptic $K3$ surfaces (X, f, s) and (X', f', s') are *isomorphic* if there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sim} & X' \\ f \downarrow & & \downarrow f' \\ \mathbb{P}^1 & \xrightarrow{\sim} & \mathbb{P}^1 \end{array}$$

that is compatible with s and s'. A *connected family of elliptic $K3$ surfaces of type* (Φ, A) is a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathbb{P}^1_B \arrow[dl, "\pi_P"] \\ & B & \end{tikzcd}$$

with a section $S\colon \mathbb{P}^1_B \to \mathcal{X}$ of F, where B is a connected analytic variety, $\pi\colon \mathcal{X} \to B$ is a family of $K3$ surfaces, $\pi_P\colon \mathbb{P}^1_B \to B$ is a $\mathbb{P}^1$-fibration, and for any point $t \in B$, the pullback (X_t, f_t, s_t) of $(\mathcal{X}, F, S)$ by $\{t\} \hookrightarrow B$ is an elliptic $K3$ surface of type (Φ, A).

We say (X, f, s) and (X', f', s') are *connected* if there exists a connected family $(\mathcal{X}, F, S)/B$ with two fibers isomorphic to (X, f, s) and (X', f', s').
We define a *connected component of the moduli of elliptic $K3$ surfaces of type* (Φ, A) to be an equivalence class of the relation of connectedness.

Main result

I determined the connected components of the moduli of elliptic $K3$ surfaces of a fixed type for each of the realizable 3693 combinatorial types.

Recall that $\mathcal{R}_f$ is the set of smooth rational curves contained in fibers of f and disjoint from s. We say that (X, f, s) is *extremal* if the cardinality of $\mathcal{R}_f$ attains the possible maximum 18 (in other words, the sum of the indices of ADE-symbols in Φ_f is 18).

List of combinatorial types (Φ, A) with non-connected moduli.

Extremal elliptic $K3$ surfaces

no.	Φ	A	T	$[r, c]$
1	$E_8 + A_9 + A_1$	0	$[2, 0, 10]$	$[2, 0]$
2	$E_8 + A_6 + A_3 + A_1$	0	$[6, 2, 10]$	$[0, 2]$
		$\dots$		
89	$2A_5 + 4A_2$	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$[6, 0, 6]$	$[0, 2]$

Non-extremal elliptic $K3$ surfaces

no.	r	Φ	A	$[c_1, \dots, c_k]$
1	17	$E_7 + D_6 + A_3 + A_1$	$\mathbb{Z}/2\mathbb{Z}$	$[1, 1]$
2	17	$E_7 + 2A_5$	0	$[2]$
		$\dots$		
107	11	$A_3 + 8A_1$	$\mathbb{Z}/2\mathbb{Z}$	$[1, 1]$

The non-connectedness of the moduli comes from three different reasons; one is algebraic, and the other two are transcendental.

For a $K3$ surface X, let

$$S_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

denote the Néron-Severi lattice of X (the $\mathbb{Z}$-module of topological classes of divisors on X with the cup product), and

$$T_X := (S_X \hookrightarrow H^2(X, \mathbb{Z}))^\perp$$

the transcendental lattice of S_X.
For an elliptic $K3$ surface (X, f, s), let

$$L_f = \langle \{ [C] \mid C \in \mathcal{R}_f \} \rangle \subset S_X$$

denote the submodule generated by the set of classes $[C]$ of smooth rational curves $C \in \mathcal{R}_f$.
Note that (X, f, s) is extremal if and only if the rank of L_f attains the possible maximum 18.

(1) The lattice L_f is a root lattice, and its ADE-type is Φ_f.

(2) The Mordell-Weil group of (X, f, s) is isomorphic to $S_X/(U_f \oplus L_f)$, where U_f is the sublattice generated by the classes of a fiber of f and the zero section s. We put

$$M_f := \text{the primitive closure of } L_f \text{ in } S_X,$$

so that $A_f \cong M_f/L_f$.

(3) The Hodge structure of $H^2(X)$ defines a canonical *positive-sign structure* on the transcendental lattice T_X (a choice of one of the two connected components of the manifold parametrizing *oriented* 2-dimensional positive-definite subspace of $T_X \otimes \mathbb{R}$). The complex conjugation switches the positive-sign structures.

Let (X, f, s) and (X', f', s') be *general* members in the connected components $\mathcal{C}$ and $\mathcal{C}'$, respectively. The term "general" means

$$S_X = U_f \oplus M_f, \quad S_{X'} = U_{f'} \oplus M_{f'}.$$

The dimension of $\mathcal{C}$ is

$$20 - \operatorname{rank} S_X = 18 - \operatorname{rank} M_f = 18 - \operatorname{rank} L_f.$$

(a) If there exists no isomorphism $\mathcal{R}_f \xrightarrow{\sim} \mathcal{R}_{f'}$ that induces $M_f \xrightarrow{\sim} M_{f'}$, then $\mathcal{C} \neq \mathcal{C}'$. If there exists such an isomorphism $\mathcal{R}_f \xrightarrow{\sim} \mathcal{R}_{f'}$, we say that $\mathcal{C}$ and $\mathcal{C}'$ are *algebraically equivalent*.

(b) Even if $\mathcal{C}$ and $\mathcal{C}'$ are algebraically equivalent, the primitive embeddings $M_f \hookrightarrow H^2(X, \mathbb{Z})$ and $M_{f'} \hookrightarrow H^2(X', \mathbb{Z})$ may not be isomorphic under any isomorphism $\mathcal{R}_f \xrightarrow{\sim} \mathcal{R}_{f'}$ and $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$. In this case, we have $\mathcal{C} \neq \mathcal{C}'$. In particular, if T_X and $T_{X'}$ are not isomorphic, we have $\mathcal{C} \neq \mathcal{C}'$.

(c) Even if the embeddings are isomorphic, if there exists no isomorphism of the embeddings that is compatible with the positive-sign structures of T_X and $T_{X'}$, we have $\mathcal{C} \neq \mathcal{C}'$. In this case, we say that $\mathcal{C}$ and $\mathcal{C}'$ are *complex conjugate*.

From the list of non-connected moduli, we obtain the following.

Theorem

The moduli of non-extremal elliptic K3 surfaces of type (Φ, A) has more than one connected component that are algebraically equivalent if and only if A is trivial and Φ is one of the following:

$$E_7+2A_5,\ \ E_6+A_{11},\ \ E_6+A_6+A_5,\ \ E_6+2A_5+A_1,$$
$$D_5+2A_6,\ \ D_4+2A_6+A_1,\ \ A_{11}+A_5+A_1,\ \ A_7+2A_5,$$
$$2A_6+A_3+2A_1,\ \ A_6+2A_5+A_1,\ \ E_6+2A_5,\ \ 3A_5+A_1.$$

For each of these types, the moduli has exactly two connected components, and they are complex conjugate to each other.

Corollary

The isomorphim class of T_X of a general member (X, f, s) of a connected component of <u>non-extremal</u> elliptic $K3$ surfaces of type (Φ, A) is determined by the algebraically equivalence class.

This corollary is rather unfortunate, because it shows that there are no phenomena of *arithmetic Zariski pair* type in non-extremal elliptic $K3$ surfaces (that is, with positive dimensional moduli).

Examples of non-extremal elliptic $K3$ surfaces

We investigate the combinatorial type

$$(\Phi, A) = (2D_6 + 4A_1,\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

We have $r = 16$, and hence the moduli is of dimension 2.

The moduli has two connected components I and II with non-isomorphic M_f (that is, they are not algebraically equivalent).

We say that a section $\tau\colon \mathbb{P}^1 \to X$ of an elliptic $K3$ surface (X, f, s) is *narrow at* $P \in \mathbb{P}^1$ if τ and s intersect the same irreducible component of $f^{-1}(P)$.

In the class I, the three non-trivial torsion sections are as follows;

D_6, D_6	A_1, A_1, A_1, A_1
not narrow	narrow at 2 points
not narrow	narrow at other 2 points
not narrow	not narrow at all 4 points.

In the class II, the three non-trivial torsion sections are as follows;

D_6, D_6	A_1, A_1, A_1, A_1
not narrow	narrow at 1 point
not narrow	narrow at 2 points
not narrow	narrow at 1 point.

The fact that the two M_f are non-isomorphic can be shown directly. For this, we need the notion of the *discriminant form* of an even lattice.

Let L be an even lattice; that is, L is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle\colon L\times L\to\mathbb{Z}$$

such that $\langle x,x\rangle\in 2\mathbb{Z}$ for all $x\in L$. Then we have a canonical finite quadratic form

$$q_L\colon D_L:=\mathrm{Hom}(L,\mathbb{Z})/L\to\mathbb{Q}/2\mathbb{Z}$$

which is called the *discriminant form* of L.

For I, the M_f has discriminant form $q\colon(\mathbb{Z}/2\mathbb{Z})^4\to\mathbb{Q}/2\mathbb{Z}$ such that $q(x)\in\{0,1\}$ for all $x\in(\mathbb{Z}/2\mathbb{Z})^4$.
For II, the M_f has discriminant form $q\colon(\mathbb{Z}/2\mathbb{Z})^4\to\mathbb{Q}/2\mathbb{Z}$ such that $q(x)=1/2$ for some $x\in(\mathbb{Z}/2\mathbb{Z})^4$.

Examples of extremal elliptic $K3$ surfaces

A $K3$ surface is *singular* if the rank of S_X attains the possible maximum 20.
In particular, an extremal elliptic $K3$ surface is singular. The moduli of extremal elliptic $K3$ surfaces is of dimension 0.

If X is singular, then T_X is a positive-definite even lattice of rank 2 with a canonical orientation.

Theorem (Shioda-Inose)

The isomorphism class of a singular $K3$ surface X is determined by the isomorphism class of the transcendental lattice T_X with the canonical orientation.

Consider the combinatorial type

$$(\Phi, A) = (E_7 + A_{10} + A_1, 0).$$

The moduli has 3 connected components. They have isomorphic M_f. One has the transcendental lattice

$$T_f \cong \begin{bmatrix} 2 & 0 \\ 0 & 22 \end{bmatrix}.$$

The other two have the transcendental lattice

$$T_f \cong \begin{bmatrix} 6 & 2 \\ 2 & 8 \end{bmatrix},$$

and these two are complex conjugate.

The non-connected moduli whose connected components cannot be distinguished by the algebraic data M_f corresponds to *arithmetic Zariski pairs*.

Example of an arithmetic Zariski pair of plane curves of degree 6

For singular $K3$ surfaces, we have the following:

Theorem (Shioda-Inose)

A singular $K3$ surface is defined over $\overline{\mathbb{Q}}$.

Theorem (Schütt-S.)

Let X and X' be a pair of singular $K3$ surfaces with $S_X \cong S_{X'}$. Then X and X' are conjugate under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Consider the plane curves of degree 6 whose singularities are of type $A_{10} + A_9$. Since the Milnor number 19 is maximal, the moduli is of dimension 0. There are four connected components, that is, there exists four isomorphism classes of such plane curves.
Two of them are irreducible, while the other two are line plus irreducible quintic.
The reducible two curves $C_\pm \subset \mathbb{P}^2$ are defined by

$$z \cdot (\ G(x,y,z) \ \pm \ \sqrt{5} \cdot H(x,y,z)\ ) \ = \ 0,$$

where

$$\begin{aligned} G(x,y,z) &= -9\,x^4z - 14\,x^3yz + 58\,x^3z^2 - 48\,x^2y^2z - 64\,x^2yz^2 + \\ &\quad +10\,x^2z^3 + 108\,xy^3z - 20\,xy^2z^2 - 44\,y^5 + 10\,y^4z, \\ H(x,y,z) &= 5\,x^4z + 10\,x^3yz - 30\,x^3z^2 + 30\,x^2y^2z + 20\,x^2yz^2 - \\ &\quad -40\,xy^3z + 20\,y^5. \end{aligned}$$

Hence C_+ and C_- are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate.
For $C_\pm$, let $X_\pm$ be the singular $K3$ surface obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $C_\pm$. The transcendental lattice of X_+ is

$$\begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix},$$

while the transcendental lattice of X_- is

$$\begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

In particular, the embeddings $C_+ \hookrightarrow \mathbb{P}^2$ and $C_- \hookrightarrow \mathbb{P}^2$ are not homeomorphic; that is, C_+ and C_- form an arithmetic Zariski pair.

Miranda-Morrison theory

To analyze the *transcendental* part of the problem in non-extremal cases, we need a refinement of Miranda-Morrison theory.

Rick Miranda and David R. Morrison.
Embeddings of integral quadratic forms.
http://www.math.ucsb.edu/drm/manuscripts/eiqf.pdf

We say that two even lattices L and L' are *in the same genus* if $L \otimes \mathbb{R}$ and $L' \otimes \mathbb{R}$ have the same signature and their discriminant forms are isomorphic.

Let $\mathcal{G}$ be a genus of isomorphism classes of even indefinite lattices of rank ≥ 3 determined by a signature (s_+, s_-) and a finite quadratic form $q\colon D \to \mathbb{Q}/2\mathbb{Z}$.
Let L be a member of $\mathcal{G}$. We have a natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(q)$.

Miranda and Morrison defined a finite abelian group $\mathcal{M}$ that fits in an exact sequence

$$0 \longrightarrow \mathrm{Coker}(\mathrm{O}(L) \to \mathrm{O}(q)) \longrightarrow \mathcal{M} \longrightarrow \mathcal{G} \longrightarrow 0,$$

and showed how to calculate $\mathcal{M}$.

(1) Their result depends on the strong approximation theorem for the spin group of indefinite lattices of rank ≥ 3.
(2) The group $\mathcal{M}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\ell}$ for some ℓ.
(3) In order to calculate $\mathcal{M}$, we need not to know L. It is enough to know the $\mathbb{Z}_p$-lattices

$$L \otimes \mathbb{Z}_p$$

for each $p | 2\,\mathrm{disc}(L)$. Since we have a complete classification of $\mathbb{Z}_p$-lattices, the $\mathbb{Z}_p$-lattices $L \otimes \mathbb{Z}_p$ can be calculated from the discriminant from q.

We fix the algebraic data M, which is negative-definite of rank r, and compute the connected components such that $M \cong M_f$.

By Nikulin's theorem, if $M_f \cong M$, then the genus $\mathcal{G}_T$ of the T_f is determined by the signature $(2, 18 - r)$ and the discriminant form

$$q_{T_f} \cong -q_M.$$

The embeddings of $U \oplus M$ into the $K3$ lattice (the even unimodular lattice of signature $(3, 19)$) is in one-to-one correspondence with the Miranda-Morrison group $\mathcal{M}$ for $\mathcal{G}_T$.

We need to refine the Miranda-Morrison theory:

- We have to take the positive-sign structures of T_f into account. We enlarge $\mathcal{M}$ to $\widetilde{\mathcal{M}} \subset \mathcal{M} \times \{\pm 1\}$.
- We have to divide $\widetilde{\mathcal{M}}$ by the automorphisms coming from the permutation of the root system $\mathcal{R}_f$ of smooth rational curves.

For the second task, we write the following algorithm.

The input is

- a finite quadratic form

$$q\colon D \to \mathbb{Q}_p/2\mathbb{Z}_p$$

  on a finite abelian p-group D, and,
- an automorphism g of q.

1. Calculate the Gram matrix of an even $\mathbb{Z}_p$-lattice L_p whose discriminant form is q and with minimal rank.
2. Find a lift $\tilde{g} \in \mathrm{O}(L_p)$ of g.
3. Calculate the spinor norm of $\tilde{g}$.

Since $|\mathbb{Z}_p|$ is uncountable, we have to use approximation in the p-adic topology. The estimate of the approximation error is necessary.

The preprint is available from: arXiv:1610.04706

Thank you for your attention