Vanishing cycles of a double plane branching along a real line arrangement

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What is a vanishing cycle?

The classical notion of a *vanishing cycle* of a complex algebraic variety was conceived by S. Lefschetz in his book

L'anysis situs et la géométrie algébrique (1924).

We recall this notion in the simple case of complex surfaces in $\mathbb{P}^3$.

Let $\{X_t\}_{t\in\Delta}$ be a family of surfaces of degree d in $\mathbb{P}^3$ such that X_t is smooth for $t \neq 0$ and that X_0 has an ordinary node $P \in X_0$ as its only singularities. Then there exists a neighborhood U of P in $\mathbb{P}^3$ with local analytic coordinates (x, y, z) such that

$$U \cong \{\, (x, y, z) \in \mathbb{C}^3 \mid |x|^2 + |y|^2 + |z|^2 \leq r^2 \,\}$$

for some $r \in \mathbb{R}_{>0}$ and that, for any $t \in \Delta$, X_t is defined in U by

$$x^2 + y^2 + z^2 = t.$$

Let $\varepsilon \in \Delta$ be a positive real number $\ll r$. Then

$$U \cap X_\varepsilon = \{\, (x, y, z) \in \mathbb{C}^3 \mid \ |x|^2 + |y|^2 + |z|^2 \le r^2,\ \ x^2 + y^2 + z^2 = \varepsilon \ \}$$

is diffeomorphic to a closed tubular neighborhood $\mathcal{T}$ of the zero section of the tangent bundle $T_{S^2} \to S^2$ of a 2-sphere S^2:

$$U \cap X_\varepsilon \ \cong \ \mathcal{T} \hookrightarrow T_{S^2} \to S^2.$$

We give an orientation to S^2. Then the zero section of $T_{S^2} \to S^2$ gives a topological 2-cycle $\Sigma \subset U \cap X_\varepsilon$, which is given by

$$\Sigma = \{\, (x, y, z) \in \mathbb{C}^3 \mid \ x, y, z \text{ are real, and } x^2 + y^2 + z^2 = \varepsilon \ \}.$$

This 2-cycle Σ or its class $[\Sigma] \in H_2(X_\varepsilon, \mathbb{Z})$ is called the *vanishing cycle* on X_ε associated with the ordinary node $P \in X_0$.

Properties of vanishing cycles on the surface X_ε.

- The self-intersection number $\langle [\Sigma], [\Sigma] \rangle$ of $[\Sigma]$ is -2.
- The kernel of the specialization homomorphism $H_2(X_\varepsilon, \mathbb{Z}) \to H_2(X_0, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 1 generated by $[\Sigma]$.
- If $H \subset X_\varepsilon$ is a general hyperplane section, then $\langle [H], [\Sigma] \rangle = 0$.
- The orthogonal complement $[H]^\perp$ of the class $[H]$ in $H_2(X_\varepsilon, \mathbb{Z})$ is generated by vanishing cycles on X_ε for ordinary nodes that appear on members of the total family $\mathcal{X} \to \mathcal{B}$ of surfaces of degree d in $\mathbb{P}^3$.
- The monodromy representation on the space $[H]^\perp \otimes \mathbb{C}$ of vanishing cycles of the family $\mathcal{X} \to \mathcal{B}$ is irreducible.

Introduction

The main theme of this talk is to construct topological cycles $\Delta(C, \gamma_C)$ in a double plane $X \to \mathbb{A}^2(\mathbb{C})$ branching along a real line arrangement. The construction uses the real structure of the arrangement. The cycles $\Delta(C, \gamma_C)$ resemble vanishing cycles for ordinary nodes.

This method is similar to the construction of topological cycles on Fermat varieties due to F. Pham in his paper

Formules de Picard-Lefschetz généralisées et ramification des intégrales

published in 1965. The construction of Pham has many interesting applications; for example, the study of integral Hodge conjecture for Fermat varieties. We hope that our construction (and its higher dimensional analogue) also has some applications.

A double plane branching along a nodal arrangement of real lines

Let $\mathbb{A}^2(\mathbb{R})$ be a real affine plane. A *nodal real line arrangement* is an arrangement of real lines on $\mathbb{A}^2(\mathbb{R})$ such that no three are concurrent.

We consider a nodal arrangement of n real lines

$$\mathcal{A} := \{\ell_1(\mathbb{R}), \dots, \ell_n(\mathbb{R})\},$$

and its complexification

$$\mathcal{A} \otimes \mathbb{C} := \{\ell_1(\mathbb{C}), \dots, \ell_n(\mathbb{C})\},$$

which is an arrangement of complex affine lines in the complex affine plane $\mathbb{A}^2(\mathbb{C})$. We put

$$B(\mathbb{R}) := \bigcup_{i=1}^{n} \ell_i(\mathbb{R}), \quad B(\mathbb{C}) := \bigcup_{i=1}^{n} \ell_i(\mathbb{C}).$$

Then $B(\mathbb{C})$ is a complex nodal affine plane curve of degree n.

We will investigate the topology of a smooth surface X defined by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}), \end{array}$$

where

- $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ is the double covering whose branch locus is equal to the union $B(\mathbb{C})$ of lines in $\mathcal{A} \otimes \mathbb{C}$,
- $\rho\colon X \to W$ is the minimal resolution,
- $\beta\colon Y \to \mathbb{A}^2(\mathbb{C})$ is the blowing up at the singular points of $B(\mathbb{C})$, and
- $\phi\colon X \to Y$ is the double covering whose branch locus is the strict transform of $B(\mathbb{C})$ by β.

For $P \in \operatorname{Sing} B(\mathbb{C})$, let E_P denote the exceptional (-1)-curve of β over P, and D_P the pull-back $\phi^{-1}(E_P)$ of E_P by ϕ. Then D_P is the exceptional (-2)-curve over the singular point of W over P.

Construction of topological 2-cycles in X

A *chamber* is the closure in $\mathbb{A}^2(\mathbb{R})$ of a connected component of $\mathbb{A}^2(\mathbb{R}) \setminus B(\mathbb{R})$. Let $\boldsymbol{Ch}$ be the set of chambers, and let $\boldsymbol{Ch}_{\mathrm{b}}$ be the set of bounded chambers. For $C \in \boldsymbol{Ch}_{\mathrm{b}}$, we put

$$\mathrm{Vert}(C) := C \cap \mathrm{Sing}\, B(\mathbb{C}), \qquad C^{\bullet} := C \setminus \mathrm{Vert}(C),$$
$$\beta^{\natural} C := \text{the closure in } Y \text{ of } \beta^{-1}(C^{\bullet}), \quad \Delta^{\bullet}(C) := \phi^{-1}(\beta^{\natural} C).$$

Suppose that $P \in \mathrm{Vert}(C)$ is the intersection point of $\ell_i(\mathbb{C})$ and $\ell_j(\mathbb{C})$. Let $\tilde{\ell}_i(\mathbb{C})$ and $\tilde{\ell}_j(\mathbb{C})$ be the strict transforms of $\ell_i(\mathbb{C})$ and $\ell_j(\mathbb{C})$ by β, and let $Q_{P,i}$ (resp. $Q_{P,j}$) be the intersection point of the exceptional (-1)-curve E_P and $\tilde{\ell}_i(\mathbb{C})$ (resp. $\tilde{\ell}_j(\mathbb{C})$). Then

$$J_{C,P} := \beta^{\natural} C \cap E_P$$

is a simple path on the 2-sphere E_P connecting the intersection points $Q_{P,i}$ and $Q_{P,j}$.

The points $Q_{P,i}$ and $Q_{P,j}$ are the branch points of the double cover $\phi|D_P\colon D_P \to E_P$. Therefore

$$S_{C,P} := \Delta^\bullet(C) \cap D_P = \phi^{-1}(J_{C,P})$$

is a circle on the 2-sphere D_P.

Note that $S_{C,P}$ divides D_P into two closed hemispheres. If we choose an appropriate affine parameter ζ on D_P, then

$$S_{C,P} = \{\, \zeta \mid \zeta \in \mathbb{R} \cup \{\infty\} \,\}.$$

The complex structures on the hemispheres induce on the boundary $S_{C,P}$ *opposite* orientations.

The space $\Delta^\bullet(C)$ is homeomorphic to a 2-sphere minus a union of disjoint open discs, each of which corresponds to a point of $\mathrm{Vert}(C)$.

We have

$$\partial\,\Delta^\bullet(C) = \bigsqcup_{P\in \mathrm{Vert}(C)} S_{C,P}.$$

Let γ_C be an orientation of $\Delta^\bullet(C)$. Then γ_C induces an orientation $\gamma_{C,P}$ on each $S_{C,P}$. Let $H_{C,P}$ be the hemisphere of D_P such that the orientation on $\partial\,H_{C,P} = S_{C,P}$ induced by the complex structure on $H_{C,P}$ is *opposite* to the orientation $\gamma_{C,P}$ induced by γ_C. Then

$$\Delta(C,\gamma_C) := \Delta^\bullet(C) \;\cup \bigsqcup_{P\in \mathrm{Vert}(C)} H_{C,P}$$

with the orientation γ_C on $\Delta^\bullet(C)$ and the orientations coming from the complex structure on each $H_{C,P}$, where $P \in \mathrm{Vert}(C)$, is a topological 2-cycle.

We call the cycle $\Delta(C, \gamma_C)$, or its homology class

$$[\Delta(C, \gamma_C)] \in H_2(X, \mathbb{Z}),$$

the *vanishing cycle associated with the chamber* C. By definition, we have

$$[\Delta(C, \gamma_C)] + [\Delta(C, -\gamma_C)] = \sum_{P \in \mathrm{Vert}(C)} [D_P],$$

where $[D_P] \in H_2(X, \mathbb{Z})$ is the class of the exceptional (-2)-curve D_P.

Our first main result is as follows:

Theorem

We choose an orientation γ_C *for each bounded chamber* C*. Then* $[\Delta(C, \gamma_C)]$*, where* C *runs through* $\boldsymbol{Ch}_{\mathrm{b}}$*, and* $[D_P]$*, where* P *runs through* $\mathrm{Sing}\, B(\mathbb{C})$*, form a basis of* $H_2(X, \mathbb{Z})$*.*

We investigate $H_2(X, \mathbb{Z})$ by calculating the intersection numbers of $[\Delta(C, \gamma_C)]$ and $[D_P]$.

Intersection numbers

Theorem

Let C and C' be bounded chambers.

- $\langle [\Delta(C, \gamma_C)], [\Delta(C, \gamma_C)] \rangle = -2$.
- *If C and C' are disjoint, then*

$$\langle [\Delta(C, \gamma_C)], [\Delta(C', \gamma_{C'})] \rangle = 0.$$

- $$\langle [\Delta(C, \gamma_C)], [D_P] \rangle = \begin{cases} -1 & \textit{if } P \in \mathrm{Vert}(C), \\ 0 & \textit{if } P \notin \mathrm{Vert}(C). \end{cases}$$

Note that this theorem does not depend on the choice of γ_C.

To calculate $\langle [\Delta(C, \gamma_C)], [\Delta(C', \gamma'_C)] \rangle$ for C, C' with $C \cap C' \neq \emptyset$, we define a *standard orientation* σ_C for each C.

We fix an orientation $\sigma_{\mathbb{A}}$ of the real affine plane $\mathbb{A}^2(\mathbb{R})$.
We also fix, for $i = 1, \dots, n$, an affine linear function $\lambda_i \colon \mathbb{A}^2(\mathbb{R}) \to \mathbb{R}$ such that $\ell_i(\mathbb{R}) = \lambda_i^{-1}(0)$, and put

$$f := \prod_{i=1}^{n} \lambda_i.$$

Then $\pi \colon W \to \mathbb{A}^2(\mathbb{C})$ is given by the first projection from

$$W = \{\, (w, P) \in \mathbb{C} \times \mathbb{A}^2(\mathbb{C}) \mid w^2 = f(P) \,\}.$$

Let C be a bounded chamber, and C° the interior of C in $\mathbb{A}^2(\mathbb{R})$. Then $\pi^{-1}(C^\circ)$ has two connected components Π_a and Π_b. Note that we have

$$\pi^{-1}(C^\circ) = \Pi_a \sqcup \Pi_b \;\subset\; \Delta^\bullet(C) = \phi^{-1}(\beta^\natural C)$$

Let γ_C be an orientation of $\Delta^\bullet(C)$, and let Π_a and Π_b be oriented by γ_C. Then $\pi \colon W \to \mathbb{A}^2(\mathbb{C})$ restricted to one of Π_a and Π_b is an orientation-preserving isomorphism to C° (oriented by the orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$), whereas π restricted to the other is orientation-reversing.

The connected component on which π is orientation-preserving is called the *good sheet* with respect to $\sigma_{\mathbb{A}}$ and γ_C.

Note that we have either

$$f(C^\circ) \subset \mathbb{R}_{>0} \quad \text{or} \quad f(C^\circ) \subset \mathbb{R}_{<0}.$$

In the former case, the two connected components Π_a and Π_b are distinguished by the sign of $w = \pm\sqrt{f} \in \mathbb{R}$ in the equation $w^2 = f$ of W, and in the latter case, the two are distinguished by the sign of $w/\sqrt{-1} = \pm\sqrt{-f} \in \mathbb{R}$.

Definition

A *standard orientation* σ_C of a bounded chamber $C \in \boldsymbol{Ch}_{\mathrm{b}}$ (with respect to $\sigma_{\mathbb{A}}$ and f) is the orientation of $\Delta^\bullet(C)$ such that

- when $f(C^\circ) \subset \mathbb{R}_{>0}$, the connected component with $w > 0$ is the good sheet, and
- when $f(C^\circ) \subset \mathbb{R}_{<0}$, the connected component with $w/\sqrt{-1} > 0$ is the good sheet.

Our third result is as follows:

Theorem

Let C and C' be bounded chambers.

- *If $C \cap C'$ consists of a single point, then*

$$\langle [\Delta(C, \sigma_C)], [\Delta(C', \sigma_{C'})] \rangle = 0.$$

- *If $C \cap C'$ is a line segment on $\mathbb{A}^2(\mathbb{R})$, then*

$$\langle [\Delta(C, \sigma_C)], [\Delta(C', \sigma_{C'})] \rangle = -1.$$

These formulas of intersection numbers give us the complete description of the intersection form $\langle \ , \ \rangle$ on $H_2(X, \mathbb{Z})$.

Projective completion

We explain a method to check these formulas by calculating examples.

We consider a real projective plane $\mathbb{P}^2(\mathbb{R})$ containing the real affine plane $\mathbb{A}^2(\mathbb{R})$ as an affine part. We put

$$\ell_\infty(\mathbb{R}) := \mathbb{P}^2(\mathbb{R}) \setminus \mathbb{A}^2(\mathbb{R}).$$

We take the closure of each $\ell_i(\mathbb{R}) \in \mathcal{A}$ in $\mathbb{P}^2(\mathbb{R})$ and define the arrangement $\widetilde{\mathcal{A}}$ of real projective lines by

$$\widetilde{\mathcal{A}} := \begin{cases} \mathcal{A} \cup \{\ell_\infty(\mathbb{R})\} & \text{if } n \text{ is odd,} \\ \mathcal{A} & \text{if } n \text{ is even.} \end{cases}$$

Let $\widetilde{B}(\mathbb{C})$ be the union of the complex projective lines in the complexification $\widetilde{\mathcal{A}} \otimes \mathbb{C}$ of $\widetilde{\mathcal{A}}$.

Since deg $\widetilde{B}(\mathbb{C}) = |\widetilde{\mathcal{A}}|$ is even, we have a double covering

$$\tilde{\pi}\colon \widetilde{W} \to \mathbb{P}^2(\mathbb{C})$$

whose branch locus is equal to $\widetilde{B}(\mathbb{C})$. Let

$$\tilde{\rho}\colon \widetilde{X} \to \widetilde{W}$$

be the minimal resolution. We put

$$\Lambda_\infty := \tilde{\rho}^{-1}(\tilde{\pi}^{-1}(\ell_\infty(\mathbb{C}))) \quad \subset \quad \widetilde{X}.$$

Then we have

$$X = \widetilde{X} \setminus \Lambda_\infty.$$

The inclusion $\iota\colon X \hookrightarrow \widetilde{X}$ induces a natural homomorphism

$$\iota_*\colon H_2(X, \mathbb{Z}) \to H_2(\widetilde{X}, \mathbb{Z}),$$

which preserves the intersection form.

For simplicity, we assume that $H_2(\widetilde{X}, \mathbb{Z})$ is torsion free. Since $\widetilde{X}$ is compact, the intersection form on $H_2(\widetilde{X}, \mathbb{Z})$ is non-degenerate. Let

$$H_\infty \subset H_2(\widetilde{X}, \mathbb{Z})$$

be the submodule generated by the classes of irreducible components of $\Lambda_\infty := \tilde{\rho}^{-1}(\tilde{\pi}^{-1}(\ell_\infty(\mathbb{C})))$. Then the image of

$$\iota_* \colon H_2(X, \mathbb{Z}) \to H_2(\widetilde{X}, \mathbb{Z})$$

is equal to the orthogonal complement $H_\infty^\perp$ of H_∞ in $H_2(\widetilde{X}, \mathbb{Z})$. The kernel of ι_* is then equal to

$$\mathrm{Ker}\langle\ ,\ \rangle := \{\, x \in H_2(X, \mathbb{Z}) \mid \langle x, y \rangle = 0 \ \text{ for any } \ y \in H_2(X, \mathbb{Z})\,\}.$$

Therefore we can calculate the sublattice $H_\infty^\perp \subset H_2(\widetilde{X}, \mathbb{Z})$ from $\langle\ ,\ \rangle$ on $H_2(X, \mathbb{Z})$.

Comparing the lattice $H_\infty^\perp$ with the lattice $H_2(X, \mathbb{Z})/\mathrm{Ker}\langle\ ,\ \rangle$, we can check the validity of our formulas of intersection numbers on $H_2(X, \mathbb{Z})$.

Examples

We consider the case where $|\mathcal{A}| = n = 6$. We assume that, for any $\ell_i(\mathbb{R})$, at most two other lines in $\mathcal{A}$ is parallel to $\ell_i(\mathbb{R})$. Then $\widetilde{B}(\mathbb{C})$ has only a_1 or d_4 singular points, and hence $\widetilde{X}$ is a $K3$ surface. In particular, $H_2(\widetilde{X}, \mathbb{Z})$ is an even unimodular lattice of rank 22 with signature $(3, 19)$.

Suppose that no pair of lines of $\mathcal{A}$ is parallel. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 10, \quad |\operatorname{Sing} B(\mathbb{C})| = 15,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 25. On the other hand, Λ_∞ is irreducible and H_∞ is of rank 1 with signature $(1, 0)$ and with discriminant group $\mathbb{Z}/2\mathbb{Z}$. Hence $H_\infty^\perp$ is of rank 21 with signature $(2, 19)$ and with discriminant group $\mathbb{Z}/2\mathbb{Z}$.
For randomly generated such arrangements, we checked that $H_2(X, \mathbb{Z})/\operatorname{Ker}\langle\ ,\ \rangle$ is of rank 21 with signature $(2, 19)$ and discriminant group $\mathbb{Z}/2\mathbb{Z}$. Remark that there are several combinatorial structures of nodal arrangements of 6 real lines with no parallel pairs.

Suppose that $\mathcal{A}$ consists of three pairs of parallel lines. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 7, \quad |\operatorname{Sing} B(\mathbb{C})| = 12,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 19. On the other hand, H_∞ is of rank 5 with signature $(1, 4)$ and discriminant group $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$. (Remark that the strict transform of $\ell_\infty(\mathbb{C})$ in $\widetilde{X}$ splits into two irreducible components.) Hence $H_\infty^\perp$ is of rank 17 with signature $(2, 15)$ and with discriminant group $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$.

For randomly generated such arrangements, we checked that $H_2(X, \mathbb{Z})/\operatorname{Ker}\langle\ ,\ \rangle$ is of rank 17 with signature $(2, 15)$ and discriminant group $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$.

Thank you very much for listening!