On the Distribution of Kurtosis Test for Multivariate Normality

Takashi Seo and Mayumi Ariga

Department of Mathematical Information Science
Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 Japan

In this paper, we consider the multivariate normality test based on measure of multivariate kurtosis defined by Srivastava (1984). Under normality, asymptotic expansions of the expectation and the variance for Srivastava's sample measure of multivariate kurtosis are given by perturbation method. An asymptotic expansion for the distribution of the sample measure of multivariate kurtosis are also given. From the result, the normalizing transformations are derived. Finally, the numerical results by Monte Carlo simulations are presented.

Keywords: Asymptotic distribution, Asymptotic expansion, Multivariate normal distribution, Measure of multivariate kurtosis, Normalizing transformation.

1. Introduction

In multivariate statistical analysis, the test for multivariate normality is an important problem and has been studied by many authors. To assess multivariate normality, for example, the multivariate sample measure of skewness and kurtosis and the exact null distributions are given in Mardia (1970, 1974). Srivastava (1984) gave a different definition for the multivariate sample measure of skewness and kurtosis and their asymptotic distributions. Multivariate extensions of the Shapiro-Wilk (1965) test have been proposed by Malkovich and Afifi (1973), Royston (1983), Srivastava and Hui (1987), and so on. Also, Small (1980) gave multivariate extensions of univariate skewness and kurtosis. For a comparison of these methods, see, Looney (1995).

Mardia and Kanazawa (1983) discussed the normal approximation for Mardia's sample measure of multivariate kurtosis by the asymptotic expansions of the third moments. For the asymptotic distributions of Mardia's and Srivastava's measures of multivariate kurtosis under elliptical populations, see, e.g., Berkane and Bentler (1990), Seo and Toyama (1996), Maruyama (2005). The limit distribution of Mardia's multivariate kurtosis under Watson rotational symmetric distributions were discussed by Zhao and Konishi (1997). Henze (1994) discussed with the asymptotic distribution for Mardia's measure of multivariate kurtosis under non-normal populations. For a survey on measure of multivariate kurtosis, see, Schwager (1985).

In this paper, we consider Srivastava's sample measure of multivariate kurtosis under normal population for two cases when the covariance matrix Σ is known and unknown. By using an asymptotic expansion for the distribution of the sample measure of multivariate kurtosis, a normalizing transformation with a considerably good normal approximation for the distribution can be derived. The organization of the paper is as follows. In Section 2, we provide asymptotic expansions up to the order N^{-2} for the first, second and third moments of Srivastava's sample measure of multivariate kurtosis by perturbation method. In Section 3, an asymptotic expansion of the distribution for the sample multivariate kurtosis is given. From this result, a normalizing transformation for the measure of multivariate kurtosis is derived. Finally, in Section 4, we investigate the accuracy of the asymptotic expansion approximation and the normal approximation for the normalizing transformation by Monte Carlo simulation for some selected parameters.

2. Asymptotic expansions for the first, second and third moments

Let $\boldsymbol{x}$ be a random p-vector with the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\Sigma = \Gamma' D_\lambda \Gamma$, where $\Gamma = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p)$ is an orthogonal matrix and $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Note that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the characteristic roots of Σ. Then Srivastava (1984) defined the population measure of multivariate kurtosis as

$$\beta_{2,p} = \frac{1}{p} \sum_{i=1}^{p} \frac{\mathrm{E}[(y_i - \theta_i)^4]}{\lambda_i^2} (\equiv \beta_{\mathrm{S}}),$$

where $y_i = \boldsymbol{\gamma}_i' \boldsymbol{x}$ and $\theta_i = \boldsymbol{\gamma}_i' \boldsymbol{\mu}$, $i = 1, 2, \ldots, p$. We note that $\beta_{\mathrm{S}} = 3$ under multivariate normality. As a remark, the population measure of multivariate kurtosis by Mardia (1970) is defined as

$$\beta_{2,p} = \mathrm{E}[\{(\boldsymbol{x} - \boldsymbol{\mu})' \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\}^2] (\equiv \beta_{\mathrm{M}}).$$

We note that $\beta_{\mathrm{M}} = p(p+2)$ under multivariate normality. For the moments and the approximation for Mardia's sample measure of multivariate kurtosis, see, Mardia and Kanazawa (1983), Siotani, Hayakawa and Fujikoshi(1985).

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be a sample of size N from a multivariate population. Let $\overline{\boldsymbol{x}}$ and $S = H' D_\omega H$ be the sample mean vector and the sample covariance matrix based on a sample of size N, where $H = (\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_p)$ is an orthogonal matrix and $D_\omega =$

$\mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_p)$. We note that $\omega_1, \omega_2, \ldots, \omega_p$ are the characteristic roots of S. Then, by Srivastava (1984), the sample measure of multivariate kurtosis is defined as

$$b_{\mathrm{S}} = \frac{1}{Np}\sum_{i=1}^{p}\frac{1}{\omega_i^2}\sum_{j=1}^{N}(y_{ij} - \overline{y_i})^4, \tag{2.1}$$

where $y_{ij} = \boldsymbol{h}_i'\boldsymbol{x_j}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, N$, $\overline{y}_i = N^{-1}\sum_{j=1}^{N} y_{ij}$, $i = 1, 2, \ldots, p$. We note that Srivastava's measure of multivariate kurtosis b_{S} is defined based on the MLE(maximum likelihood estimator), that is,

$$S = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})'.$$

Without loss of generality, we may assume that $\Sigma = I$ and $\boldsymbol{\mu} = \boldsymbol{0}$ when we consider the sample measure of multivariate kurtosis (2.1). Since the following derivation method can also be used for the latter case of unknown Σ, we first consider asymptotic expansions of the moments for the case when Σ is known under normality.

Assuming that Σ is known, we can write

$$b_{\mathrm{S}} = \frac{1}{p}\sum_{i=1}^{p} m_{4i},$$

where $m_{4i} = N^{-1}\sum_{j=1}^{N}(y_{ij} - \overline{y}_i)^4$. In order to avoid the dependence of y_{ij} and $\overline{y}_i$, let $\overline{y}_i^{(\alpha)}$ be a mean defined on the subset of $y_{i1}, y_{i2}, \ldots, y_{iN}$ by deleting some $y_{i\alpha}$ at random, that is,

$$\overline{y}_i^{(\alpha)} = \frac{1}{N-1}\sum_{j=1, j\neq\alpha}^{N} y_{ij}.$$

Then, we can write

$$m_{4i} = \frac{1}{N}\left(1 - \frac{1}{N}\right)^4 \sum_{\alpha=1}^{N}(y_{i\alpha} - \overline{y}_i^{(\alpha)})^4.$$

Note that $y_{i\alpha}$ is independent of $\overline{y}_i^{(\alpha)}$.

To obtain the expectation of b_{S} by the perturbation method, we put

$$\overline{y}_i^{(\alpha)} = \frac{1}{\sqrt{N-1}}Z.$$

Since and Z is distributed as a standard normal distribution, the odd order moments equal zero and

$$\mathrm{E}[Z^{2k}] = (2k-1)\cdots 5\cdot 3\cdot 1, \quad k = 1, 2, \ldots, 6.$$

We shall also use above result when we calculate second and third moments.

Therefore, we can obtain

$$\mathrm{E}[b_{\mathrm{S}}] = \mathrm{E}[m_{4i}] = 3 - \frac{6}{N} + \frac{3}{N^2} + O(N^{-3}). \tag{2.2}$$

Similarly, we may obtain $\mathrm{E}[b_{\mathrm{S}}^2]$ and $\mathrm{E}[b_{\mathrm{S}}^3]$ as

$$\mathrm{E}[b_{\mathrm{S}}^2] = 9 + \frac{60}{N} - \frac{258}{N^2} + O(N^{-3}), \tag{2.3}$$

$$\mathrm{E}[b_{\mathrm{S}}^3] = 27 + \frac{702}{N} + \frac{5373}{N^2} + O(N^{-3}), \tag{2.4}$$

respectively. Therefore, we obtain the variance for kurtosis b_{S} as

$$\mathrm{Var}[b_{\mathrm{S}}] = \frac{96}{p} \cdot \frac{1}{N} - \frac{312}{p} \cdot \frac{1}{N^2} + O(N^{-3}). \tag{2.5}$$

Next, we consider asymptotic expansions of first, second and third moments for b_{S} when Σ is unknown.

First note that

$$\omega_i = \boldsymbol{h}_i' S \boldsymbol{h}_i = \frac{1}{N}\sum_{j=1}^{N}(y_{ij} - \overline{y}_i)^2, \quad i = 1, 2, \ldots, p.$$

Then, since S is defined as the maximum likelihood estimator, we can write

$$b_{\mathrm{S}} = \frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2}.$$

Under normality, $y_{i1}, y_{i2}, \ldots, y_{iN}$ are independently normally distributed. Hence, by Srivastava(1984), for large N, we note that $\mathrm{E}[m_{\nu i}^k m_{2i}^{-\nu k/2}]\mathrm{E}[m_{2i}^{\nu k/2}] = \mathrm{E}[m_{\nu i}^k]$. Therefore, since $\mathrm{E}[m_{4i}]$ is given by (2.2), we shall calculate

$$\mathrm{E}[m_{2i}^2] = \frac{1}{N^2}\{N\mathrm{E}[C_{i\alpha}^4] + N(N-1)\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2]\},$$

where $C_{i\alpha} = y_{i\alpha} - \overline{y}_i$.

Let $\overline{y}^{(\alpha,\beta)}$ be a mean defined on the subset of $y_{i1}, y_{i2}, \ldots, y_{iN}$ by deleting variable $y_{i\alpha}$ and $y_{i\beta}$ at random, that is,

$$\overline{y}^{(\alpha,\beta)} = \frac{1}{N-2}\sum_{j=1, j\neq\alpha,\beta}^{N} y_{ij},$$

and let

$$\overline{y}^{(\alpha,\beta)} = \frac{1}{\sqrt{N-2}}Z.$$

Calculating the expectation with respect to $y_{i\alpha}$, $y_{i\beta}$ and Z, we have

$$\mathrm{E}[C_{i\alpha}^2 C_{i\beta}^2] = 1 - \frac{2}{N} + \frac{3}{N^2} + O(N^{-3}).$$

Therefore, the expectation for the kurtosis b_{S} is given by

$$\mathrm{E}[b_{\mathrm{S}}] = 3 - \frac{6}{N} + \frac{6}{N^2} - \frac{6}{N^3} + O(N^{-4}). \tag{2.6}$$

Further, we have

$$\begin{aligned}
\mathrm{E}[m_{4i}^2] &= \frac{1}{N^2}\left\{N\mathrm{E}[C_\alpha^8] + N(N-1)\mathrm{E}[C_\alpha^4 C_\beta^4]\right\},\\
\mathrm{E}[m_{2i}^4] &= \frac{1}{N^4}\{N\mathrm{E}[C_\alpha^8] + 4N(N-1)\mathrm{E}[C_\alpha^6 C_\beta^2] + 3N(N-1)\mathrm{E}[C_\alpha^4 C_\beta^4]\\
&\quad +6N(N-1)(N-2)\mathrm{E}[C_\alpha^4 C_\beta^2 C_\gamma^2] + N(N-1)(N-2)(N-3)\mathrm{E}[C_\alpha^2 C_\beta^2 C_\gamma^2 C_\delta^2]\}.
\end{aligned}$$

After a great deal of calculation for the above expectations, we can obtain $\mathrm{E}[b_{\mathrm{S}}^2]$ as

$$\mathrm{E}[b_{\mathrm{S}}^2] = 9 - \frac{12}{N} - \frac{288}{N^2} + O(N^{-3}). \tag{2.7}$$

Hence, when Σ is unknown, we have the variance for kurtosis b_{S} as

$$\mathrm{Var}[b_{\mathrm{S}}] = \frac{24}{p}\cdot\frac{1}{N} - \frac{360}{p}\cdot\frac{1}{N^2} + O(N^{-3}). \tag{2.8}$$

Similarly, we may obtain

$$\mathrm{E}[b_{\mathrm{S}}^3] = 27 + \frac{54}{N} - \frac{1458}{N^2} + O(N^{-3}). \tag{2.9}$$

The expectation (2.6) and variance (2.8) for b_{S} are the same as the asymptotic expansions that we expand the results in Srivastava(1984).

3. Asymptotic expansions for the distribution of b_{S}

In this section, an asymptotic expansion of the distribution function for b_{S} is given. Also, as improved approximation to the distribution of Srivastava's sample measure of kurtosis, we consider the normalizing transformation for the two cases when Σ is known and unknown.

Let $Y = \sqrt{N}(b_{\mathrm{S}} - \beta_{\mathrm{S}})$ for the case when Σ is known. Then, by using (2.2), (2.3) and (2.4), the first three cumulants of Y have the following forms:

$$\begin{aligned}
\kappa_1(Y) &= \frac{1}{\sqrt{N}}a_1 + O(N^{-3/2}),\\
\kappa_2(Y) &= \sigma^2 + O(N^{-1}),\\
\kappa_3(Y) &= \frac{6}{\sqrt{N}}a_3 + O(N^{-3/2}),
\end{aligned}$$

where $a_1 = -6$, $\sigma^2 = 96/p$, $a_3 = 1584/p^2$. By a general theory of asymptotic expansions, the distribution function for b_{S} can be expanded as

$$\Pr\left\{\frac{\sqrt{N}(b_{\mathrm{S}} - \beta_{\mathrm{S}})}{\sigma} \le y\right\} = \Phi(y) - \frac{1}{\sqrt{N}}\left\{\frac{a_1}{\sigma}\Phi^{(1)}(y) + \frac{a_3}{\sigma^3}\Phi^{(3)}(y)\right\} + O(N^{-1}),$$

where $\Phi(y)$ is the cumulative distribution function of $N(0,1)$ and $\Phi^{(j)}(y)$ in the jth derivative of $\Phi(y)$.

The variance stabilizing transformation and the normalizing transformation of some statistics in multivariate analysis are discussed by Konishi(1984) and Konishi(1987), and so on. By using the method in Konishi(1984), if we put

$$f(\beta_{\mathrm{S}}) = -\frac{32}{11}\exp\left[-\frac{11}{32}\beta_{\mathrm{S}}\right],$$

we can obtain

$$\Pr\left[\frac{\sqrt{N}\{f(b_{\mathrm{S}}) - f(\beta_{\mathrm{S}}) - c/N\}}{\sigma f'(\beta_{\mathrm{S}})} \leq y\right] = \Phi(y) + O(N^{-1}).$$

where

$$c = -\left(6 + \frac{33}{2p}\right)\exp\left[-\frac{11}{32}\beta_{\mathrm{S}}\right].$$

Similarly, let $Y^* = \sqrt{N}(b_{\mathrm{S}} - \beta_{\mathrm{S}})$ for the case when Σ is unknown. Then, by using (2.6), (2.7) and (2.9), the first three cumulants of Y^* have the following forms:

$$\begin{aligned}
\kappa_1(Y^*) &= \frac{1}{\sqrt{N}}a_1^* + O(N^{-3/2}),\\
\kappa_2(Y^*) &= \sigma^{*2} + O(N^{-1}),\\
\kappa_3(Y^*) &= \frac{6}{\sqrt{N}}a_3^* + O(N^{-3/2}),
\end{aligned}$$

respectively, where $a_1^* = -6$, $\sigma^{*2} = 24/p$, $a_3^* = 288/p^2$. Therefore, in this case, the distribution function for b_{S} can be expanded as

$$\Pr\left\{\frac{\sqrt{N}(b_{\mathrm{S}} - \beta_{\mathrm{S}})}{\sigma^*} \leq y\right\} = \Phi(y) - \frac{1}{\sqrt{N}}\left\{\frac{a_1^*}{\sigma^*}\Phi^{(1)}(y) + \frac{a_3^*}{\sigma^{*3}}\Phi^{(3)}(y)\right\} + O(N^{-1}).$$

Further, if we put

$$f^*(\beta_{\mathrm{S}}) = -\exp[-\beta_{\mathrm{S}}],$$

we can obtain

$$\Pr\left[\frac{\sqrt{N}\{f^*(b_{\mathrm{S}}) - f^*(\beta_{\mathrm{S}}) - c^*/N\}}{\sigma f'(\beta_{\mathrm{S}})} \leq y\right] = \Phi(y) + O(N^{-1}).$$

where $c^* = -6\exp[-\beta_{\mathrm{S}}](1 + 2/p)$.

4. Accuracy of normal approximation for measure of multivariate kurtosis

We investigate the accuracy of asymptotic approximation for measure of multivariate sample kurtosis by Monte Carlo simulation for some selected parameters. The program for Monte Carlo simulation is written in C, which adopts Mersenne Twister method which makes pseudo-random numbers and the Box-Muller transformation method.

Computations are made for $p = 3, 5, 7, 10$; $N = 20, 50, 100, 200, 400$ for each of cases where Σ is known and unknown for multivariate normal populations. Without any loss of generality, we may assume that $\Sigma = I$. Simulation results based on $1,000,000$ simulations for the cases where Σ is known and unknown are presented in Tables 1 and 2. Further, Tables 1 and 2 give the values of the limiting term (LT), the asymptotic expansion up to the order N^{-1} (AE(N^{-1})), and the one up the order N^{-2} (AE(N^{-2})) for the expectations (2.2), (2.6) and the variances (2.5), (2.8). Note that, in Tables, the notations E and V mean the expectation and the variance, respectively. The results from the numerical examination in Tables 1 and 2 show that the values of the asymptotic expansions are good approximation as N is large. It may be noted from Tables that the values of higher order asymptotic expansions for the expectation and the variance are considerably good approximation even if p is large.

Next, in order to assess the performance of the proposed test statistics in this paper, we investigate the accuracy of normal approximation for four test statistics as follows. These test statistics are asymptotically distributed as $N(0,1)$:

(i)

$$z = \sqrt{\frac{Np}{24}}(b_{\mathrm{S}} - 3),$$

(ii)

$$z^* = \frac{(N+1)\sqrt{p(N+3)(N+5)}\{b_{\mathrm{S}} - 3(1-2/(N+1))\}}{\sqrt{24N(N-2)(N-3)}},$$

(iii)

$$z^{**} = \frac{\sqrt{Np}\{b_{\mathrm{S}} - 3(1-2/N+2/N^2)\}}{\sqrt{24(1-15/N)}},$$

(iv)

$$z_{NT} = \frac{\sqrt{Np}(-e^{-b_{\mathrm{S}}} + e^{-3} + 6e^{-3}(1+\frac{2}{p})/N)}{\sqrt{24}e^{-3}}.$$

The test statistic z is given by Srivastava(1984), z^* can be obtained by making reference to moments in Srivastava(1984). Also, we propose z^{**} by the asymptotic results in Section 2 and z_{NT} by the normalizing transformation in the previous section.

Table 3 gives the values of expectation (E), variance (V) and skewness (S) for z, z^*, z^{**} and z_{NT} when Σ is unknown. Also, simulated values based on $1,000,000$ simulations are presented in Table 3.

Since the values of expectation and skewness are zero and the value of variance is one under standard normal distribution, it may be seen from Table 3 that values of E, V and S converge to zero, one and zero, respectively, as the sample size N is large. Thus, it may be noted from Table 3 that the values for each of statistics give good normal approximations as N is large. Particularly, it may be noted from Table 3 that the value of skewness S for z_{NT} rapidly converges to zero when the sample size N is large. Further, it is seen that the normalizing transformation statistic z_{NT} is considerably good normal approximation even for small sample size. In conclusion, it may be also noted from simulation results that the test statistic z_{NT} proposed in this paper is useful for multivariate normality test.

References

[1] Berkane, M. and Bentler, P. M. (1990). Mardia's coefficient of kurtosis in elliptical populations, *Acta Mathematica Applicatae Sinica*, **6**, 290–294.

[2] Henze, N. (1994). On Mardia's kurtosis test for multivariate normality, *Commun. Statist. -Theor. Meth.*, **23**, 1031–1045.

[3] Konishi, S. (1981). Normalizing transformations of some statistics in multivariate analysis, *Biometrika*, **68**, 647–651.

[4] Konishi, S. (1987). Transformations of statistics in multivariate analysis, *Advances in Multivariate Statistical Analysis*, (A.K. Gupta, ed.), Dordrecht-Holland, D. Reidel Pub. Co, 213–231.

[5] Looney, S. W. (1995). How to use test for univariate normality to assess multivariate normality, *Amer. Statist.*, **49**, 64–70.

[6] Malkovich, J. F. and Afifi, A. A. (1973). On tests for multivariate normality, *J. Amer. Statist. Assoc.*, **68**, 176–179.

[7] Mardia, K. V. (1970). Measures of multivariate skewness and kurtosis with applications, *Biometrika*, **57**, 519–530.

[8] Mardia, K. V. (1974). Applications of some measures of multivariate skewness and kurtosis in testing normality and robustness studies, *Sankhaya B*, **36**, 115–128.

[9] Mardia, K. V. and Kanazawa, M. (1983). The null distribution of multivariate kurtosis, *Commun. Statist. -Simula. Computa.*, **12**, 569–576.

[10] Maruyama, Y. (2005). Asymptotic properties for measures of multivariate kurtosis in elliptical distributions, *Inter. J. Pure and Appl. Math.*, **25**, 407–421.

[11] Royston, J. P. (1983). Some techniques for assessing multivariate normality based on the Shapiro-Wilk W, *Appl. Statist.*, **32**, 121–133.

[12] Schwager, S. J. (1985). Multivariate skewness and kurtosis, *Encyclopedia of Statistical Sciences, Vol. 6*, (S. Kotz and N.L. Johnson, eds.), Wiley, New York, 122–125.

[13] Seo, T and Toyama, T (1996). On the estimation of kurtosis parameter in elliptical distributions, *J. Japan Statist. Soc.*, **26**, 59–68.

[14] Shapiro, S. S. and Wilk, M. B. (1965). An analysis of variance test for normality (complete samples), *Biometrika*, **52**, 591–611.

[15] Siotani, M., Hayakawa, T. and Fujikoshi, Y. (1985). *Modern Multivariate Statistical Analysis: A Graduate Course and Handbook*, American Sciences Press, Inc., Columbus, Ohio.

[16] Small, N. J. H. (1980). Marginal skewness and kurtosis in testing multivariate normality, *Appl. Statist.*, **22**, 260–266.

[17] Srivastava, M. S. (1984). A measure of skewness and kurtosis and a graphical method for assessing multivariate normality, *Statist. & Prob. letters*, **2**, 263–267.

[18] Srivastava, M. S. and Hui, T. K. (1987). On assessing multivariate normality based on Shapiro-Wilk W statistic, *Statist. & Prob. letters*, **5**, 15–18.

[19] Zhao and Konishi (1997). Limit distributions of multivariate kurtosis and moments under Watson rotational symmetric distributions, *Statist. & Prob. letters*, **32**, 291–299.

Table 1. Expectation(E) and variance(V) for multivariate sample kurtosis (Σ is known)

p=3		E				V		
N	LT	AE(N^{-1})	AE(N^{-2})	simulation	LT	AE(N^{-1})	AE(N^{-2})	simulation
20	3.000	2.700	2.708	2.711	0.000	1.600	1.340	1.356
50	3.000	2.880	2.881	2.883	0.000	0.640	0.598	0.618
100	3.000	2.940	2.940	2.940	0.000	0.320	0.310	0.312
200	3.000	2.970	2.970	2.970	0.000	0.160	0.157	0.159
400	3.000	2.985	2.985	2.987	0.000	0.080	0.079	0.080
p=5								
20	3.000	2.700	2.708	2.706	0.000	0.960	0.804	0.826
50	3.000	2.880	2.881	2.879	0.000	0.384	0.359	0.365
100	3.000	2.940	2.940	2.940	0.000	0.192	0.186	0.188
200	3.000	2.970	2.970	2.972	0.000	0.096	0.094	0.096
400	3.000	2.985	2.985	2.987	0.000	0.048	0.048	0.048
p=7								
20	3.000	2.700	2.708	2.709	0.000	0.686	0.574	0.596
50	3.000	2.880	2.881	2.881	0.000	0.274	0.256	0.260
100	3.000	2.940	2.940	2.941	0.000	0.137	0.133	0.134
200	3.000	2.970	2.970	2.971	0.000	0.069	0.067	0.068
400	3.000	2.985	2.985	2.987	0.000	0.034	0.034	0.034
p=10								
20	3.000	2.700	2.708	2.706	0.000	0.480	0.402	0.412
50	3.000	2.880	2.881	2.881	0.000	0.192	0.180	0.181
100	3.000	2.940	2.940	2.942	0.000	0.096	0.093	0.094
200	3.000	2.970	2.970	2.972	0.000	0.048	0.047	0.047
400	3.000	2.985	2.985	2.987	0.000	0.024	0.024	0.024

Table 2. Expectation(E) and variance(V) for multivariate sample kurtosis (Σ is unknown)

p=3		E				V		
N	LT	AE(N^{-1})	AE(N^{-2})	simulation	LT	AE(N^{-1})	AE(N^{-2})	simulation
20	3.000	2.700	2.715	2.716	0.000	0.400	0.100	0.195
50	3.000	2.880	2.882	2.882	0.000	0.160	0.112	0.119
100	3.000	2.940	2.940	2.940	0.000	0.080	0.068	0.069
200	3.000	2.970	2.970	2.971	0.000	0.040	0.037	0.037
400	3.000	2.985	2.985	2.986	0.000	0.020	0.019	0.019
p=5								
20	3.000	2.700	2.715	2.716	0.000	0.240	0.060	0.117
50	3.000	2.880	2.882	2.880	0.000	0.096	0.067	0.071
100	3.000	2.940	2.940	2.941	0.000	0.048	0.041	0.041
200	3.000	2.970	2.970	2.971	0.000	0.024	0.022	0.023
400	3.000	2.985	2.985	2.986	0.000	0.012	0.012	0.012
p=7								
20	3.000	2.700	2.715	2.715	0.000	0.171	0.043	0.083
50	3.000	2.880	2.882	2.882	0.000	0.069	0.048	0.052
100	3.000	2.940	2.940	2.942	0.000	0.034	0.029	0.030
200	3.000	2.970	2.970	2.970	0.000	0.017	0.016	0.016
400	3.000	2.985	2.985	2.985	0.000	0.009	0.008	0.008
p=10								
20	3.000	2.700	2.715	2.713	0.000	0.120	0.030	0.058
50	3.000	2.880	2.882	2.883	0.000	0.048	0.034	0.036
100	3.000	2.940	2.940	2.942	0.000	0.024	0.020	0.021
200	3.000	2.970	2.970	2.971	0.000	0.012	0.011	0.011
400	3.000	2.985	2.985	2.986	0.000	0.006	0.006	0.006

Table 3. Expectation(E), variance(V) and skewness(S) for some test statistics

p=3		E				V				S		
N	z	z^*	z^{**}	z_{NT}	z	z^*	z^{**}	z_{NT}	z	z^*	z^{**}	z_{NT}
20	-0.448	0.005	0.005	-0.231	0.488	1.011	1.952	0.758	0.352	1.049	2.815	-0.193
50	-0.294	0.000	-0.000	-0.169	0.745	1.002	1.065	0.846	0.601	0.936	1.026	-0.154
100	-0.214	-0.004	-0.004	-0.132	0.860	0.998	1.012	0.901	0.589	0.736	0.751	-0.097
200	-0.144	0.005	0.005	-0.090	0.936	1.008	1.012	0.944	0.523	0.585	0.588	-0.048
400	-0.102	0.004	0.004	-0.065	0.974	1.012	1.011	0.977	0.396	0.419	0.419	-0.030
p=5												
20	-0.580	0.004	0.003	-0.208	0.487	1.009	1.948	0.790	0.274	0.817	2.191	-0.158
50	-0.386	-0.007	-0.008	-0.147	0.745	1.001	1.064	0.881	0.463	0.722	0.791	-0.107
100	-0.268	0.004	0.004	-0.099	0.863	1.002	1.015	0.926	0.454	0.568	0.579	-0.077
200	-0.186	0.007	0.007	-0.069	0.940	1.013	1.016	0.963	0.426	0.476	0.479	-0.013
400	-0.130	0.006	0.006	-0.048	0.967	1.005	1.004	0.979	0.303	0.320	0.321	-0.018
p=7												
20	-0.688	0.003	0.001	-0.200	0.486	1.006	1.943	0.812	0.219	0.653	1.752	-0.150
50	-0.449	0.000	0.000	-0.080	0.755	1.015	1.079	0.901	0.424	0.660	0.724	-0.080
100	-0.315	0.006	0.006	-0.083	0.860	0.998	1.011	0.931	0.392	0.490	0.500	-0.050
200	-0.225	0.003	0.003	-0.062	0.929	1.001	1.005	0.966	0.321	0.359	0.361	-0.032
400	-0.158	0.003	0.003	-0.042	0.963	1.000	0.999	0.979	0.263	0.278	0.279	-0.002
p=10												
20	-0.827	-0.003	-0.009	-0.201	0.485	1.005	1.941	0.823	0.187	0.557	1.495	-0.114
50	-0.532	0.006	0.006	-0.107	0.752	1.010	1.074	0.911	0.340	0.529	0.580	-0.069
100	-0.376	0.008	0.008	-0.070	0.866	1.005	1.019	0.947	0.336	0.421	0.429	-0.028
200	-0.268	0.005	0.005	-0.050	0.931	1.003	1.007	0.972	0.286	0.319	0.321	-0.011
400	-0.186	0.007	0.007	-0.031	0.956	0.993	0.993	0.976	0.211	0.223	0.223	-0.001