Error bounds for high–dimensional Edgeworth expansions for some tests on covariance matrices

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Abstract

Problems of testing three hypotheses : (i) equality of covariance matrices of several multivariate normal populations, (ii) sphericity, and (iii) that a covariance matrix is equal to a specified one, are treated. High–dimensional Edgeworth expansions of the null distributions of the modified likelihood ratio test statistics are derived. Computable error bounds of the expansions are derived for each expansions. The Edgeworth expansion and its error bound for non–null distribution of the test statistic for (iii) are also derived.

1 Introduction

This paper is concerned with problems of testing hypotheses on covariance matrices of multivariate normal populations. The null hypotheses considered are

$$\begin{aligned}&H_0 : \Sigma_1 = \Sigma_2 = \cdots = \Sigma_r,\\ &H_0 : \Sigma = \lambda I_p \quad \text{and}\\ &H_0 : \Sigma = \Sigma_0 \quad \text{(a specified matrix).}\end{aligned}$$

Let $\boldsymbol{x}_{i1}, \cdots, \boldsymbol{x}_{iN_i}$ be a random sample from p–variate normal population $N_p(\boldsymbol{\mu}_i, \Sigma_i)$, where $\boldsymbol{\mu}_i$ and Σ are the mean vector and the covariance matrix, respectively ($i = 1, \cdots, r$). The modified likelihood ratio criterion, suggested by Bartlett [2] for testing the hypothesis

$$H_0 : \Sigma_1 = \cdots = \Sigma_r$$

against the alternatives that H_0 is not true, is given by

$$\Lambda = \frac{\prod_{i=1}^r (\det A_i)^{n_i/2}}{(\det A)^{n/2}} \frac{n^{pn/2}}{\prod_{i=1}^r n_i^{pn_i/2}}, \tag{1.1}$$

where

$$\begin{aligned}&A_i = \sum_{j=1}^{N_j} (\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)', \quad \bar{\boldsymbol{x}}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} \boldsymbol{x}_{ij}\\ &n_i = N_i - 1 \quad (i = 1, \cdots, r),\end{aligned}$$

and

$$A=\sum_{i=1}^{r}A_i, \quad n=\sum_{i=1}^{r}n_i.$$

The unbiasedness of the case $r = 2$ was proved by Sugiura and Nagao [12] and Perlman [11] proved for general r (See Muirhead [9, chapter 8] for details). An asymptotic expansion of the null distribution of $-2\rho\log\Lambda$ was given by Box [3] for large $N = \rho n$ as

$$\Pr\{-2\rho\log\Lambda \le x\} = G_f(x) + \frac{\gamma}{M^2}\{G_{f+4}(x) - G_f(x)\} + O(M^{-3}), \tag{1.2}$$

where G_l is the distribution function of χ^2 distribution with degree of freedom l,

$$\begin{aligned}
M &= \rho n = n - \frac{2p^2+3p-1}{6(p+1)(r-1)}\left(\sum_{i=1}^{r}\frac{n}{n_i}-1\right),\\
f &= \frac{1}{2}p(p+1)(r-1) \quad \text{and}\\
\gamma &= \frac{p(p+1)}{48}\left\{(p-1)(p+2)\left(\sum_{i=1}^{r}\frac{n^2}{n_i^2}-1\right)-6(r-1)n^2(1-\rho)^2\right\}.
\end{aligned}$$

It is known that (1.2) does not give very good approximation if p is large. So we consider to use the Edgeworth expansion of the null distribution in such cases.

Let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N$ be a random sample from $N_p(\boldsymbol{\mu}, \Sigma)$. The likelihood ratio criterion for testing

$$H_0 : \Sigma = \lambda I_p, \quad \lambda : \text{unknown},$$

against the alternatives that H_0 is not true, derived by Mauchly [8] is given by

$$V = \frac{\det A}{\left(\frac{1}{p}\mathrm{tr}A\right)^p}, \tag{1.3}$$

where

$$A=\sum_{j=1}^{N}(\boldsymbol{x}_j-\bar{\boldsymbol{x}})(\boldsymbol{x}_j-\bar{\boldsymbol{x}})', \quad \bar{\boldsymbol{x}}=\frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_j. \tag{1.4}$$

The unbiasedness of (1.3) was first proved by Gleser [6] (see Muirhead [9]). An asymptotic expansion of the null distribution for large n, first given by Anderson (1958) (see [1, sectin 10.7]) is

$$\Pr\{-n\rho\log V \le x\} = G_f(x) + \frac{\gamma}{M^2}\{G_{f+4}(x) - G_f(x)\} + O(M^{-3}), \tag{1.5}$$

where

$$\begin{aligned}
M &= \rho n = n - \frac{2p^2+p+2}{6p}, \quad f = \frac{1}{2}(p+2)(p-1) \quad \text{and}\\
\gamma &= \frac{(p-1)(p-2)(p+2)(2p^3+6p^2+3p+2)}{288p^2}.
\end{aligned}$$

We also treat the problem of testing

$$H_0 : \Sigma = \Sigma_0,$$

where Σ_0 is a specified positive definite matrix, against $H_1 : \Sigma \neq \Sigma_0$. The modified likelihood ratio statistic is given by

$$\Lambda = \left(\frac{e}{n}\right)^{pn/2} \mathrm{etr}\left(-\frac{1}{2}\Sigma_0^{-1}A\right)(\det \Sigma_0^{-1}A)^{n/2}, \tag{1.6}$$

where A is given by (1.4) and $n = N - 1$. The unbiasedness was proved by Nagao [10] and Das Gupta [4]. An asymptotic expansion of the null distribution for large n, given by Davis [5] is

$$\Pr\{-2\rho \log \Lambda \leq x\} = G_f(x) + \frac{\gamma}{M^2}\{G_{f+4}(x) - G_f(x)\} + O(M^{-3}), \tag{1.7}$$

where

$$M = \rho n = n - \frac{2p^2 + 3p - 1}{6(p+1)}, \quad f = \frac{1}{2}p(p+1) \quad \text{and}$$
$$\gamma = \frac{p(2p^4 + 6p^3 + p^2 - 12p - 13)}{288(p+1)}.$$

It is known that if p is large, the approximation formulas given by (1.5) and (1.7) are not very good.

The purpose of this paper is to derive the Edgeworth expansion for the test statistic (1.1) under a framework

$$p \to \infty,\ n_i \to \infty,\ \frac{p}{n_i} \to c_i \in (0,1)\ (i = 1, \cdots, r), \tag{1.8}$$

and the Edgeworth expansions for the test statistics (1.3) and (1.6) under a framework

$$p \to \infty,\ n \to \infty,\ \frac{p}{n} \to c \in (0,1), \tag{1.9}$$

and to give computable error bounds of the derived approximation formulas.

2 Testing Equality of the covariance matrices

In this section we derive the Edgeworth expansion of the null distribution of the test statistic given by (1.1) and give computable error bounds of the derived approximation formula.

2.1 Edgeworth expansion

Let

$$V = (\det A)^{-1} \prod_{i=1}^{r} (\det A_i).$$

The hth moment of V is given by Box [3] as

$$\mathrm{E}[V^h] = \prod_{j=1}^{p} \left\{ \frac{\Gamma[\frac{n-p+j}{2}]}{\Gamma[\frac{n-p+j}{2}+h]} \prod_{i=1}^{r} \frac{\Gamma[\frac{n_i-p+j}{2}+h]}{\Gamma[\frac{n_i-p+j}{2}]} \right\}.$$

Hence the characteristic function and the sth cumulant of $-\log V$ are given by

$$\varphi_V(t) = \mathrm{E}[V^{-it}] = \prod_{j=1}^{p} \left\{ \frac{\Gamma[\frac{n-p+j}{2}]}{\Gamma[\frac{n-p+j}{2}-it]} \prod_{i=1}^{r} \frac{\Gamma[\frac{n_i-p+j}{2}-it]}{\Gamma[\frac{n_i-p+j}{2}]} \right\}$$

and

$$\begin{aligned} \kappa_V^{(s)} &= (-i)^{-s} \frac{\partial^s}{\partial t^s} \log \varphi_V(t) \bigg|_{t=0} \\ &= (-1)^s \sum_{j=1}^{p} \left\{ -\psi^{(s-1)}\Big(\frac{n-p+j}{2}\Big) + \sum_{i=1}^{r} \psi^{(s-1)}\Big(\frac{n_i-p+j}{2}\Big) \right\}, \end{aligned} \tag{2.1}$$

respectively, where $\psi^{(s)}$ is the polygamma function defined by

$$\psi^{(s)}(a) = \Big(\frac{d}{da}\Big)^{s+1} \log \Gamma[a] = \begin{cases} -C + \displaystyle\sum_{k=0}^{\infty} \Big(\frac{1}{1+k} - \frac{1}{k+a}\Big) & (s=0) \\ \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^{s+1} s!}{(k+a)^{s+1}} & (s=1,2,\cdots) \end{cases}, \tag{2.2}$$

and C is the Euler constant. Let

$$T = \frac{-\log V - \kappa_V^{(1)}}{(\kappa_V^{(2)})^{1/2}}, \tag{2.3}$$

and denote the standardized cumulant as

$$\kappa^{(s)} = \frac{\kappa_V^{(s)}}{(\kappa_V^{(2)})^{s/2}} \quad (s=3,4,\cdots).$$

Then upper bounds for the standardized cumulants are given by the following lemma.

Lemma 2.1 *Assume that $n_1 \le n_2 \le \cdots \le n_r$ without loss of generality. Let*

$$\begin{aligned} m &= \frac{n_1 - p - \frac{1}{2}}{2} (\kappa_V^{(2)})^{1/2}, \\ b_s &= \frac{2}{\kappa_V^{(2)}(s+1)(s+2)(s+3)} \left[\sum_{i=1}^{r} \left\{ \Big(\frac{n_1-p-\frac{1}{2}}{n_i-p-\frac{1}{2}}\Big)^{s+1} - (\frac{n_1-p-\frac{1}{2}}{n_i-\frac{1}{2}}\Big)^{s+1} \right\} \right. \\ &\qquad \left. - (\frac{n_1-p-\frac{1}{2}}{n-p-\frac{1}{2}}\Big)^{s+1} + (\frac{n_1-p-\frac{1}{2}}{n-\frac{1}{2}}\Big)^{s+1} \right]. \end{aligned} \tag{2.4}$$

Then it holds that

$$0 < \frac{\kappa^{(s)}}{s!} < m^{-(s-2)} b_{s-3} \quad (s=3,4,\cdots). \tag{2.5}$$

Proof From (2.1) and (2.2) the sth cumulant $\kappa_V^{(s)}$ $(s \geq 2)$ can be expressed as

$$\kappa_V^{(s)} = \sum_{k=0}^{\infty} \sum_{j=1}^{p} \left\{ \sum_{i=1}^{r} \frac{(s-1)!}{(\frac{n_i-p+j}{2}+k)^s} - \frac{(s-1)!}{(\frac{n-p+j}{2}+k)^s} \right\}.$$

So apparently $\kappa_V^{(s)}$ is positive for $s \geq 2$. Since

$$f(x,y) = \sum_{i=1}^{r} \frac{(s-1)!}{(\frac{n_i-p+y}{2}+x)^s} - \frac{(s-1)!}{(\frac{n-p+y}{2}+x)^s}$$

is decreasing and convex as a function of x and y,

$$\begin{aligned}
\kappa_V^{(s)} &\leq \int_{-1/2}^{\infty} \left\{ \int_{1/2}^{p+1/2} f(x,y) dy \right\} dx \\
&= \sum_{i=1}^{r} \left\{ \frac{2(s-3)!}{(\frac{n_i-p-\frac{1}{2}}{2})^{s-2}} - \frac{2(s-3)!}{(\frac{n_i-\frac{1}{2}}{2})^{s-2}} \right\} - \frac{2(s-3)!}{(\frac{n-p-\frac{1}{2}}{2})^{s-2}} + \frac{2(s-3)!}{(\frac{n-\frac{1}{2}}{2})^{s-2}} \Bigg\},
\end{aligned}$$

which gives the bound (2.5) immediately. ∎

Next lemma gives a lower bound of $\kappa_V^{(2)}$.

Lemma 2.2 *It holds that*

$$\kappa_V^{(2)} \geq \sum_{i=1}^{r} 2 \log \frac{n_i}{n_i - p} - 2 \log \frac{n}{n-p}. \tag{2.6}$$

Proof It can be easily checked that

$$f(a) = \frac{1}{a^2} - \int_{-1}^{0} \left\{ \int_{0}^{1} \frac{1}{(a+x+\frac{y}{2})^2} dx \right\} dy$$

is decreasing and positive in $((1+\sqrt{13})/6, \infty)$. Hence

$$\begin{aligned}
\kappa_V^{(2)} &= \sum_{k=0}^{\infty} \sum_{j=1}^{p} \left\{ \sum_{i=1}^{r} \frac{1}{(\frac{n_i-p+j}{2}+k)^2} - \frac{1}{(\frac{n-p+j}{2}+k)^2} \right\} \\
&\geq \int_{0}^{\infty} \left[\int_{0}^{p} \left\{ \sum_{i=1}^{r} \frac{1}{(\frac{n_i-p+y}{2}+x)^2} - \frac{1}{(\frac{n-p+y}{2}+x)^2} \right\} dy \right] dx,
\end{aligned}$$

which leads (2.6). ∎

Lemma 2.2 assures b_s in (2.4) is bounded and $m \to \infty$ under the framework (1.8). The characteristic function of T given by (2.3) can be expanded as

$$\begin{aligned}\varphi(t) &:= \exp\Big\{-\frac{t^2}{2} + \sum_{s=3}^{\infty} \frac{\kappa^{(s)}}{s!}(it)^s\Big\} \\ &= \exp\Big(-\frac{t^2}{2}\Big)\Big\{1 + \sum_{k=1}^{\infty} \frac{(it)^{3k}}{k!}\Big(\sum_{s=0}^{\infty} \frac{\kappa^{(s+3)}}{(s+3)!}(it)^s\Big)^k\Big\} \\ &= \exp\Big(-\frac{t^2}{2}\Big)\Big\{1 + \sum_{k=1}^{\infty} \frac{(it)^{3k}}{k!} \sum_{j=0}^{\infty} \gamma_{k,j}(it)^j\Big\},\end{aligned} \tag{2.7}$$

where

$$\gamma_{k,j} = \sum_{s_1+\cdots+s_k=j} \frac{\kappa^{(s_1+3)}\cdots\kappa^{(s_k+3)}}{(s_1+3)!\cdots(s_k+3)!}. \tag{2.8}$$

Lemma 2.1 leads that $\gamma_{k,j} = O(m^{-(j+k)})$. Therefore let

$$\varphi_s(t) = \exp\Big(-\frac{t^2}{2}\Big)\Big\{1 + \sum_{k=1}^{s} \frac{(it)^{3k}}{k!} \sum_{j=0}^{s-k} \gamma_{k,j}(it)^j\Big\}. \tag{2.9}$$

Then it holds that

$$\varphi(t) = \varphi_s(t) + O(m^{-(s+1)}).$$

Inverting (2.9), we obtain the Edgeworth expansion of the null distribution of the standardized test statistic T up to the order $O(m^{-s})$ as

$$Q_s(x) = \Phi(x) - \phi(x)\Big\{\sum_{k=1}^{s} \frac{1}{k!} \sum_{j=0}^{s-k} \gamma_{k,j} h_{3k+j-1}(x)\Big\}, \tag{2.10}$$

where Φ and ϕ are the distribution function and the probability density function of the standard normal distribution, respectively, $\gamma_{k,j}$ is given by (2.6), and $h_r(x)$ is the r–th order Hermite polynomial defined by

$$\Big(\frac{d}{dx}\Big)^r \exp\Big(-\frac{x^2}{2}\Big) = (-1)^r h_r(x) \exp\Big(-\frac{x^2}{2}\Big).$$

2.2 Error bound

Using the inverse Fourier transformation we obtain a uniform bound for the error of the above Edgeworth expansion as

$$\begin{aligned}\sup_x |\mathrm{P}(T \le x) - Q_s(x)| &\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|t|} |\varphi(t) - \varphi_s(t)|\, dt \\ &= \frac{1}{2\pi}(I_1[v] + I_2[v] + I_3[v]),\end{aligned} \tag{2.11}$$

where

$$I_1[v] = \int_{-mv}^{mv} \frac{1}{|t|} \left|\varphi(t) - \varphi_s(t)\right| dt,$$

$$I_2[v] = \int_{|t|>mv} \frac{1}{|t|} \left|\varphi_s(t)\right| dt \quad \text{and} \quad I_3[v] = \int_{|t|>mv} \frac{1}{|t|} \left|\varphi(t)\right| dt$$

with some positive constant $v < 1$. In order to find a bound for each integral I_1, I_2 and I_3, we prepare some lemmas.

Lemma 2.3 *Let T be a random variable such that*

$$\mathrm{E}[|T|^s] < \infty \text{ for any } s > 0, \text{ and}$$
$$\log \mathrm{E}[\exp(itT)] = -\frac{t^2}{2} + \sum_{s=3}^{\infty} \frac{\kappa^{(s)}}{s!}(it)^s$$

in some neighborhood of $t = 0$. Assume that there are a sequence $\{b_s\}_{s=0,1,2,\cdots}$ of positive numbers and positive numbers v and m such that

$$\frac{|\kappa^{(s)}|}{s!} \le m^{-(s-2)} b_{s-3} \ (s \ge 3) \text{ and } B[v] := \sum_{s=0}^{\infty} b_s v^s < \infty. \tag{2.12}$$

Define $R_{k,l}[v]$ and $\varphi_s(t)$ as

$$R_{k,l}[v] = v^{-l}\left\{(B[v])^k - \sum_{j=0}^{l-1}\Big(\sum_{s_1+\cdots+s_k=j} b_{s_1}\cdots b_{s_k}\Big)v^j\right\} \text{ and}$$

$$\varphi_s(t) = \exp\Big(-\frac{t^2}{2}\Big)\left\{1 + \sum_{k=1}^{s} \frac{(it)^{3k}}{k!} \sum_{j=0}^{s-k} \sum_{s_1+\cdots+s_k=j} \frac{\kappa^{(s_1+3)}\cdots\kappa^{(s_k+3)}}{(s_1+3)!\cdots(s_k+3)!}(it)^j\right\}.$$

If $|t| \le mv$, then

$$\begin{aligned}\frac{1}{|t|}|\mathrm{E}[\exp(itT)] - \varphi_s(t)| \le \frac{1}{m^{s+1}} \exp\Big(-\frac{t^2}{2}\Big)&\left\{\sum_{k=1}^{s} \frac{1}{k!}|t|^{s+2k} R_{k,s-k+1}[v]\right.\\ &\left.+ \frac{1}{(s+1)!}|t|^{3s+2}(B[v])^{s+1}\exp\Big(t^2 vB[v]\Big)\right\},\end{aligned} \tag{2.13}$$

and hence

$$\int_{-mv}^{mv} \frac{1}{|t|}|\mathrm{E}[\exp(itT)] - \varphi_s(t)|dt \le U_1[v; m, B],$$

where

$$\begin{aligned}U_1[v; m, B] = \frac{2}{m^{s+1}}&\left\{\sum_{k=1}^{s} \frac{1}{k!} R_{k,s-k+1}[v] \int_0^{mv} t^{s+2k} \exp\Big(-\frac{t^2}{2}\Big)dt\right.\\ &\left.+ \frac{1}{(s+1)!}(B[v])^{s+1} \int_0^{mv} t^{3s+2} \exp\Big(-\frac{t^2}{2}c_v\Big)dt\right\}\end{aligned} \tag{2.14}$$

and $c_v = 1 - 2vB[v]$.

Proof The difference between the characteristic function of T and φ_s is

$$\mathrm{E}[\exp(itT)] - \varphi_s(t) = \exp\Big(-\frac{t^2}{2}\Big)\Big\{1 + \sum_{k=1}^{s} \frac{(it)^{3k}}{k!} \sum_{j=s-k+1}^{\infty} \gamma_{k,j}(it)^j + \sum_{k=s+1}^{\infty} \frac{(it)^{3k}}{k!}\Big(\sum_{j=0}^{\infty} \frac{\kappa^{(j+3)}}{(j+3)!}(it)^j\Big)^k\Big\},$$

where $\gamma_{k,j}$'s have the same definition as (2.9) with $\kappa^{(s)}$'s in the lemma. Using (2.12) we have

$$\Big|\sum_{k=s+1}^{\infty} \frac{(it)^{3k}}{k!}\Big(\sum_{j=0}^{\infty} \frac{\kappa^{(j+3)}}{(j+3)!}(it)^j\Big)^k\Big| \le \sum_{k=s+1}^{\infty} \frac{|t|^{3k}}{k!}\Big(\sum_{j=0}^{\infty} m^{-(j+1)} b_j (mv)^j\Big)^k$$
$$\le \frac{|t|^{3(s+1)}}{(s+1)!} m^{-(s+1)} (B[v])^{s+1} \sum_{k=0}^{\infty} \frac{1}{k!}(t^2 v B[v])^k$$

and

$$\Big|(it)^{3k} \sum_{j=s-k+1}^{\infty} \gamma_{k,j}(it)^j\Big|$$
$$\le |t|^{2k+1}(mv)^{k-1} \sum_{j=s-k+1}^{\infty} m^{-(k+j)} \sum_{s_1+\cdots+s_k} (b_{s_1}\cdots b_{s_k})|t|^s (mv)^{j-s}$$
$$= \frac{1}{m^{s+1}}|t|^{s+2k+1} v^{-(s-k+1)}\Big\{(B[v])^k - \sum_{j=0}^{s-k} \sum_{s_1+\cdots+s_k} (b_{s_1}\cdots b_{s_k}) v^j\Big\}.$$

Hence (2.4) holds. ■

Lemma 2.4 *If $p < n_1 < n_2$, then*

$$\log\Big|\prod_{j=1}^{p} \frac{\Gamma[\frac{n_2-p+j}{2}]\Gamma[\frac{n_1-p+j}{2} - it]}{\Gamma[\frac{n_2-p+j}{2} - it]\Gamma[\frac{n_1-p+j}{2}]}\Big| < -\frac{1}{4}\sum_{j=1}^{p} \int_{n_1-p}^{n_2-p} \log\Big\{1 + \frac{4t^2}{(j+x)^2}\Big\} dx$$
$$< -\frac{(n_2-n_1)p}{4} \log\Big\{1 + \frac{16t^2}{(n_1+n_2-p+1)^2}\Big\}, \tag{2.15}$$

and

$$\log\Big|\prod_{j=1}^{p} \frac{\Gamma[\frac{n_1-p+j}{2} - it]}{\Gamma[\frac{n_1-p+j}{2}]}\Big| < -\frac{1}{2}\sum_{j=1}^{p} \int_{(n_1-p+j)/2}^{\infty} \log\Big(1 + \frac{t^2}{x^2}\Big) dx$$
$$< -\frac{p|t|}{2} c\Big[\frac{2n_1-p+1}{4|t|}\Big] \tag{2.16}$$

where

$$c[z] = \int_z^{\infty} \log\Big(1 + \frac{1}{y^2}\Big) dy = 2\arctan\Big(\frac{1}{z}\Big) - z\log\Big(1 + \frac{1}{z^2}\Big). \tag{2.17}$$

Proof It is known that

$$\left|\frac{\Gamma[x+yi]}{\Gamma[x]}\right|^2 = \prod_{k=0}^{\infty}\left\{1+\frac{y^2}{(x+k)^2}\right\}^{-1}$$

for any real number $x > 0$ and y. Since

$$\log\left\{1+\frac{t^2}{(a+x)^2}\right\} - \log\left\{1+\frac{t^2}{(b+x)^2}\right\}$$

is a decreasing function of x for $a < b$,

$$\begin{aligned}
&\log\left|\prod_{j=1}^{p}\frac{\Gamma[\frac{n_2-p+j}{2}]\Gamma[\frac{n_1-p+j}{2}-it]}{\Gamma[\frac{n_2-p+j}{2}-it]\Gamma[\frac{n_1-p+j}{2}]}\right| \\
&= -\frac{1}{2}\sum_{j=1}^{p}\sum_{k=0}^{\infty}\left[\log\left\{1+\frac{t^2}{(\frac{n_1-p+j}{2}+k)^2}\right\} - \log\left\{1+\frac{t^2}{(\frac{n_2-p+j}{2}+k)^2}\right\}\right] \\
&< -\frac{1}{2}\sum_{j=1}^{p}\int_0^{\infty}\left[\log\left\{1+\frac{t^2}{(\frac{n_1-p+j}{2}+x)^2}\right\} - \log\left(1+\frac{t^2}{(\frac{n_2-p+j}{2}+x)^2}\right)\right]dx \\
&= -\frac{1}{4}\sum_{j=1}^{p}\int_{n_1-p}^{n_2-p}\log\left\{1+\frac{4t^2}{(j+x)^2}\right\}dx \\
&< -\frac{(n_2-n_1)p}{4}\log\left\{1+4t^2\Big(\frac{p+1}{2}+\frac{n_2+n_1-2p}{2}\Big)^{-2}\right\},
\end{aligned}$$

which leads (2.15). Here we used the fact that if $f(x,y)$ is convex and $c < d$, then

$$\sum_{j=1}^{p}\int_c^d f(x,j)dx > p(d-c)f\Big(\frac{c+d}{2},\frac{p+1}{2}\Big)$$

for the last inequality.

Similarly,

$$\begin{aligned}
\log\left|\prod_{j=1}^{p}\frac{\Gamma[\frac{n_1-p+j}{2}-it]}{\Gamma[\frac{n_1-p+j}{2}]}\right| &= -\frac{1}{2}\sum_{j=1}^{p}\sum_{k=0}^{\infty}\log\left\{1+\frac{t^2}{(\frac{n_1-p+j}{2}+k)^2}\right\} \\
&< -\frac{1}{2}\sum_{j=1}^{p}\int_{(n_1-p+j)/2}^{\infty}\log\Big(1+\frac{t^2}{x^2}\Big)dx = -\frac{|t|}{2}\sum_{j=1}^{p}c\Big[\frac{n_1-p+j}{2|t|}\Big] \\
&< -\frac{p|t|}{2}c\Big[\frac{n_1-\frac{p-1}{2}}{2|t|}\Big]
\end{aligned}$$

because $c[z]$ is a convex function of z. ■

Now we can give upper bounds of the integrals in (2.11). Let

$$L_3[v] = \begin{cases} \dfrac{3v-2}{4v} - \dfrac{(1-v)^2}{2v^2}\log(1-v) & (0 < |v| < 1) \\ 0 & (v = 0) \end{cases} \tag{2.18}$$

Then it is easily checked that $L_3[v]$ can be expanded as

$$L_3[v] = \sum_{s=1}^{\infty} \frac{1}{s(s+1)(s+2)} v^s.$$

So a bound of $I_1[v]$ is immediately given by lemma 2.3, that is

$$I_1[v] \le U_1[v; m, B],$$

where U_1 is given by (2.14) with

$$\begin{aligned} B[v] = \sum_{s=0}^{\infty} b_s v^s = \frac{2}{v\kappa_V^{(2)}} \Bigg\{ \sum_{i=1}^{r} \Bigg(L_3\left[\frac{n_1 - p - \frac{1}{2}}{n_i - p - \frac{1}{2}} v\right] - L_3\left[\frac{n_1 - p - \frac{1}{2}}{n_i - \frac{1}{2}} v\right]\Bigg) \\ - L_3\left[\frac{n_1 - p - \frac{1}{2}}{n - p - \frac{1}{2}} v\right] + L_3\left[\frac{n_1 - p - \frac{1}{2}}{n - \frac{1}{2}} v\right]\Bigg\}. \end{aligned} \tag{2.19}$$

The calculation of integral I_2 is not difficult. From (2.9)

$$I_2[v] = \int_{mv}^{\infty} \frac{2}{t} \exp(-\frac{t^2}{2}) dt + \sum_{k=1}^{s} \frac{2}{k!} \sum_{j=0}^{s-k} \gamma_{k,j} \int_{mv}^{\infty} t^{3k+j-1} \exp(-\frac{t^2}{2}) dt. \tag{2.20}$$

A bound for I_3 is obtained by using lemma 2.4. Let

$$\begin{aligned} F[z] &= \int_0^z \Big\{ \int_0^y \log\Big(1 + \frac{1}{(x+y)^2}\Big) dx \Big\} dy \\ &= \frac{z^2}{2} \log\Big(1 + \frac{1}{z^2}\Big) + 2z \arctan(z) - \frac{1}{2} \log(1 + z^2). \end{aligned} \tag{2.21}$$

Then from (2.15) and (2.16)

$$\begin{aligned} \log|\varphi_V(t)| &< -\frac{1}{4} \int_1^{p+1} \Big\{ \int_{n_1-p}^{n-p} \log\Big(1 + \frac{4t^2}{(y+x)^2}\Big) dx \Big\} dy \\ &\quad - \frac{1}{4} \sum_{i=2}^{r} \int_1^{p+1} \Big\{ \int_{n_i-p}^{\infty} \log\Big(1 + \frac{4t^2}{(y+x)^2}\Big) dx \Big\} dy \\ &= -t^2 G(t; n_1, \dots, n_r, p), \end{aligned}$$

where

$$\begin{aligned} &G(t; n_1, \dots, n_r, p) \\ &\quad = \frac{p(r-1)\pi}{2|t|} + F\Big(\frac{n+1}{2|t|}\Big) - F\Big(\frac{n-p+1}{2|t|}\Big) - \sum_{i=1}^{r} \Big\{ F\Big(\frac{n_i+1}{2|t|}\Big) - F\Big(\frac{n_i - p + 1}{2|t|}\Big) \Big\}. \end{aligned}$$

Hence

$$\begin{aligned} I_3[v] \le U_3[v] &:= \int_{mv}^{\infty} \frac{2}{t} \exp\Big\{ -\frac{t^2}{\kappa_V^{(2)}} G\big[t(\kappa_V^{(2)})^{-1/2}; n_1, \dots, n_r, p\big] \Big\} dt \\ &= \int_{m_0 v}^{\infty} \frac{2}{t} \exp\{-t^2 G(t; n_1, \dots, n_r, p)\} dt, \end{aligned} \tag{2.22}$$

where

$$m_0 = \frac{n_1 - p - \frac{1}{2}}{2}.$$

The result obtained here is summarized in the following theorem.

Theorem 2.1 *Let T be the standardized test statistic given by* (2.3) *and Q_s be the Edgeworth expansion of the null distribution function of T given by* (2.10)*. Then*

$$\sup_x |\mathrm{P}(T \le x) - Q_s(x)| < \frac{1}{2\pi}(U_1[v; m, B]) + I_2[v] + U_3[v]),$$

where U_1 is given by (2.14) *with B given by* (2.19)*, I_2 is given by* (2.20) *and U_3 is given by* (2.22)*.*

The calculation of the integral $U_3[v]$ may be difficult. Using lemma 2.4 we can obtain a simpler bound for $I_3[v]$. Since $c[z]$ defined by (2.17) is a decreasing function, if $t \ge m_0 v$, then

$$|\varphi_V(t)| < C_r \left\{ 1 + \frac{16t^2}{(n+n_1-p+1)^2} \right\}^{-(n-n_1)p/4},$$

where

$$C_r = \exp\left\{ -\frac{pm_0 v}{2} \sum_{i=2}^{r} c\left[\frac{2n_i - p + 1}{4m_0 v} \right] \right\}.$$

Hence

$$\begin{aligned} I_3[v] &< C_r \int_{m_0 v}^{\infty} \frac{2}{t} \left\{ 1 + \frac{16t^2}{(n+n_1-p+1)^2} \right\}^{-(n-n_1)p/4} dt \\ &= C_r \int_{1+\alpha}^{\infty} (s-1)^{-1} s^{-(n-n_1)p/4} ds = C_r \sum_{j=0}^{\infty} \frac{1}{\frac{(n-n_1)p}{4} + j} (1+\alpha)^{-(n-n_1)p/4-j} \qquad (2.23) \\ &< C_r \frac{4}{(n-n_1)p} (1+\alpha)^{-(n-n_1)p/4} \frac{1+\alpha}{\alpha} = \tilde{U}_3[v] \quad \text{(say)}, \end{aligned}$$

where

$$\alpha = \frac{16 m_0^2 v^2}{(n+n_1-p+1)^2}.$$

3 The sphericity test

In this section we derive an error bound of the Edgeworth expansion of the null distribution of the test statistic given by (1.3). The hth moment of V is given by Khatri and Srivastava [7] as

$$\mathrm{E}[V^h] = p^{ph} \frac{\Gamma[\frac{pn}{2}]}{\Gamma[\frac{pn}{2} + ph]} \prod_{j=1}^{p} \frac{\Gamma[\frac{n-p+j}{2} + h]}{\Gamma[\frac{n-p+j}{2}]}$$

(see also Muirhead [9, chapter 8.3.2]). Hence the characteristic function and the sth cumulant of $-\log V$ are obtained similarly to (2.1) as

$$\varphi_V(t) = p^{-itp} \frac{\Gamma[\frac{pn}{2}]}{\Gamma[\frac{pn}{2} - pit]} \prod_{j=1}^{p} \frac{\Gamma[\frac{n-p+j}{2} - it]}{\Gamma[\frac{n-p+j}{2}]}, \quad \text{and}$$

$$\kappa_V^{(s)} = \begin{cases} -p\log p + p\psi\Big(\dfrac{np}{2}\Big) - \displaystyle\sum_{j=1}^{p} \psi\Big(\dfrac{n-p+j}{2}\Big) & (s=1) \\ (-1)^{s-1}\Big\{p^s \psi^{(s-1)}\Big(\dfrac{np}{2}\Big) - \displaystyle\sum_{j=1}^{p} \psi^{(s-1)}\Big(\dfrac{n-p+j}{2}\Big)\Big\} & (s=2,3,\cdots) \end{cases}, \tag{3.1}$$

respectively. Let

$$T = \frac{-\log V - \kappa_V^{(1)}}{(\kappa_V^{(2)})^{1/2}}, \tag{3.2}$$

and

$$\kappa^{(s)} = \frac{\kappa_V^{(s)}}{(\kappa_V^{(2)})^{s/2}} \quad (s = 3, 4, \cdots). \tag{3.3}$$

Then upper bounds for the standardized cumulants are given by the following lemma.

Lemma 3.1 *Let*

$$\begin{aligned} m &= \frac{n-p-\frac{1}{2}}{2} (\kappa_V^{(2)})^{1/2}, \\ b_s &= \frac{2}{\kappa_V^{(2)}(s+1)(s+2)(s+3)} \Big\{1 - \Big(\frac{n-p-\frac{1}{2}}{n-\frac{1}{2}}\Big)^{s+1}\Big\} \\ &\quad - \Big\{\frac{2}{\kappa_V^{(2)}(s+3)n^2} + \frac{2p}{\kappa_V^{(2)}(s+2)(s+3)n}\Big\}\Big(\frac{n-p-\frac{1}{2}}{n}\Big)^{s+1}. \end{aligned} \tag{3.4}$$

Then

$$0 < \frac{\kappa^{(s)}}{s!} < m^{-(s-2)} b_{s-3} \quad (s = 3, 4, \cdots). \tag{3.5}$$

Proof Using multiplication formula

$$\Gamma[pz] = \frac{p^{pz-\frac{1}{2}}}{(2\pi)^{(p-1)/2}} \prod_{j=0}^{p-1} \Gamma\big[z + \frac{j}{p}\big]$$

the characteristic function of $-\log V$ can be written as

$$\varphi_V(t) = \mathrm{E}[V^{-it}] = \prod_{j=1}^{p-1} \Big\{ \frac{\Gamma[\frac{n-p+j}{2} - it]}{\Gamma[\frac{n-p+j}{2}]} \frac{\Gamma[\frac{n}{2} + \frac{j}{p}]}{\Gamma[\frac{n}{2} + \frac{j}{p} - it]} \Big\}. \tag{3.6}$$

Hence the sth cumulant can be represented as

$$\kappa_V^{(s)} = (-1)^s \sum_{j=1}^{p-1} \Big\{ \psi^{(s-1)}\Big(\frac{n-p+j}{2}\Big) - \psi^{(s-1)}\Big(\frac{n}{2}+\frac{j}{p}\Big)\Big\},$$

which leads that $\kappa_V^{(s)} > 0$ since $(-1)^s\psi^{(s-1)}(a)$ is a decreasing function of a.

Since $(a+x)^{-s}$ is a decreasing and convex function of x $(x > -a)$, if $a > \frac{1}{2}$ then

$$(-1)^s\psi^{(s-1)}(a) = \sum_{k=0}^{\infty} \frac{(s-1)!}{(a+x)^s} < \int_{-\frac{1}{2}}^{\infty} \frac{(s-1)!}{(a+x)^s}dx = \frac{(s-2)!}{(a-\frac{1}{2})^{s-1}}, \tag{3.7}$$

and if $a > 0$ then

$$(-1)^s\psi^{(s-1)}(a) > \frac{(s-1)!}{2a} + \int_0^{\infty} \frac{(s-1)!}{(a+x)^s}dx = \frac{(s-1)!}{2a^s} + \frac{(s-2)!}{a^{s-1}}. \tag{3.8}$$

Hence

$$\begin{aligned} 0 < (-1)^s \sum_{j=1}^{p} \psi^{(s-1)}\Big(\frac{n-p+j}{2}\Big) &< \sum_{j=1}^{p} \frac{(s-2)!2^{s-1}}{(n-p+j-1)^{s-1}} \\ < \int_{\frac{1}{2}}^{p+\frac{1}{2}} \frac{(s-2)!2^{s-1}}{(n-p-1+x)^{s-1}}dx &= \frac{(s-3)!2^{s-1}}{(n-p-\frac{1}{2})^{s-2}}\Big\{1-\Big(\frac{n-p-\frac{1}{2}}{n-\frac{1}{2}}\Big)^{s-2}\Big\}, \end{aligned} \tag{3.9}$$

and

$$(-1)^{s-1}p^s\psi^{(s-1)}\Big(\frac{np}{2}\Big) < -\frac{(s-1)!2^{s-1}}{n^s} - \frac{(s-2)!2^{s-1}p}{n^{s-1}},$$

which lead (3.5). ∎

Let L_1 and L_2 be defined by

$$\begin{aligned} L_1[v] &= \sum_{s=0}^{\infty} \frac{v^{s+1}}{s+3} = \begin{cases} -\frac{1}{v^2}\log(1-v) - \frac{2+v}{2v} & (0 < |v| < 1) \\ 0 & (v=0) \end{cases} \quad \text{and} \\ L_2[v] &= \sum_{s=0}^{\infty} \frac{v^{s+1}}{(s+2)(s+3)} = \begin{cases} \frac{1-v}{v^2}\log(1-v) + \frac{2-v}{2v} & (0 < |v| < 1) \\ 0 & (v=0) \end{cases}, \end{aligned} \tag{3.10}$$

respectively. Then $B[v] = \sum_{s=0}^{\infty} b_s v^s$ for (3.4) can be represented as

$$\begin{aligned} B[v] = \frac{2}{v\kappa_V^{(2)}}\Big\{ & L_3[v] - L_3\Big[\frac{n-p-\frac{1}{2}}{n-\frac{1}{2}}v\Big] \\ & - \frac{1}{n^2}L_1\Big[\frac{n-p-\frac{1}{2}}{n}v\Big] - \frac{p}{n}L_2\Big[\frac{n-p-\frac{1}{2}}{n}v\Big]\Big\}, \end{aligned} \tag{3.11}$$

where L_3 is given by (2.18).

Let $Q_s(x)$ be the Edgeworth expansion of the null distribution function of T given by the same formula as (2.10) but the cumulants $\kappa_V^{(s)}$'s are given by (3.1) and m is given in lemma 3.1. Then a uniform bound for the error is given by the same formula as (2.11) with the characteristic function φ of T given by (3.2). An upper bound for $I_1[v]$ is given by lemma 2.3 with B given above. I_2 the same form as in (3.5).

Similarly to the proof of (2.15) in lemma 2.4, (3.6) leads that

$$\begin{aligned}\log|\varphi_V(t)| &< -\frac{1}{2}\sum_{j=1}^{p-1}\int_{(n-p+j)/2}^{n/2+j/p}\log\Big(1+\frac{t^2}{x^2}\Big)dx\\ &< -\frac{1}{2}\int_1^p\Big\{\int_{(n-p+y)/2}^{n/2+y/p}\log\Big(1+\frac{t^2}{x^2}\Big)dx\Big\}dy = -t^2G(t;n,p),\end{aligned}$$

where

$$G(t;n,p) = \frac{p}{2}\Big\{F\Big(\frac{n+2}{2|t|}\Big) - F\Big(\frac{n}{2|t|}+\frac{1}{p|t|}\Big)\Big\} - F\Big(\frac{n}{2|t|}\Big) + F\Big(\frac{n-p+1}{2|t|}\Big) \quad (3.12)$$

and F is given by (2.21). Hence an upper bound of $I_3[v]$ is obtained by the same formula as (2.22) with above $G(t;n,p)$ instead of $G(t;n_1,\cdots,n_r,p)$.

Thus we obtained the uniform error bound for the Edgeworth expansion.

Theorem 3.1 *Let T be the standardized test statistic given by* (3.2) *and Q_s be the Edgeworth expansion of the null distribution function of T given by* (2.10)*, but the cumulants are given by* (3.3) *with* (3.1)*. Then*

$$\sup_x|\mathrm{P}(T\le x) - Q_s(x)| < \frac{1}{2\pi}(U_1[v;m,B]) + I_2[v] + U_3[v]),$$

where U_1 is given by (2.14) *with B given by* (3.11)*, I_2 is given by* (2.20) *and U_3 is given by* (2.22)*, with $G(t;n,p)$ given by* (3.12) *instead of $G(n;n_1,\cdots,n_r,p)$.*

A simple bound for $I_3[v]$ is also given. Since

$$\int_{(n-p+y)/2}^{n/2+y/p}\log\Big(1+\frac{t^2}{x^2}\Big)dx$$

is a convex function of y,

$$\begin{aligned}\sum_{j=1}^{p-1}\int_{(n-p+j)/2}^{n/2+j/p}\log\Big(1+\frac{t^2}{x^2}\Big)dx &< (p-1)\int_{n/2-p/4}^{(n+1)/2}\log\Big(1+\frac{t^2}{x^2}\Big)dx\\ &< \frac{(p-1)(p+2)}{4}\log\Big\{1+\frac{4t^2}{(n+\frac{1}{2}-\frac{p}{4})^2}\Big\}.\end{aligned}$$

Hence

$$I_3[v] < \int_{m_0v}^{\infty}\frac{2}{t}\Big\{1+\frac{4t^2}{(n+\frac{1}{2}-\frac{p}{4})^2}\Big\}^{-(p-1)(p+2)/8} < \tilde{U}_3[v],$$

where

$$\tilde{U}_3[v] = \frac{8}{(p-1)(p+2)}(1+\alpha)^{-(p-1)(p+2)/8}\frac{1+\alpha}{\alpha}, \quad \text{and}$$
$$\alpha = \Big(\frac{2m_0 v}{n+\frac{1}{2}-\frac{p}{4}}\Big)^2. \tag{3.13}$$

4 Equality of covariance matrix to a specified matrix

In this section we derive Edgeworth expansions of the null and non–null distributions of the modified test statistic given by (1.6). We can assume that $\Sigma_0 = I_p$, the identity matrix, without any loss of generality. Let

$$V = \mathrm{etr}\Big(-\frac{1}{2}A\Big)(\det A)^{n/2}, \tag{4.1}$$

where $\mathrm{etr}(A)$ means $\exp(\mathrm{tr}A)$. Then the modified likelihood ratio test rejects H_0 : $\Sigma = I_p$ for small values of V. The hth moment of V is given by Anderson (1958) (see [1, sectin 10.8]) as

$$\mathrm{E}[V^h] = 2^{nph/2}\frac{\det\Sigma^{nh/2}}{\det(I_p+h\Sigma)^{(1+h)n/2}}\prod_{j=1}^{p}\frac{\Gamma[\frac{n-p+j+nh}{2}]}{\Gamma[\frac{n-p+j}{2}]}.$$

The characteristic function of $-\frac{2}{n}\log V$ is given by

$$\varphi_V(t) = \frac{2^{-pit}(\det\Sigma)^{-it}}{\{\det(I_p-\frac{2it}{n}\Sigma)\}^{\frac{n}{2}-it}}\prod_{j=1}^{p}\frac{\Gamma[\frac{n-p+j}{2}-it]}{\Gamma[\frac{n-p+j}{2}]},$$

where z^w for complex number z means the principal branch, that is, $z^w = \exp(w\log z)$ $= \exp\{w(\log|z| + i\arg z)\}$. Hence the sth cumulants are given by

$$\kappa_V^{(1)} = -p\log 2 - \log\det\Sigma + \mathrm{tr}\Sigma - \sum_{j=1}^{p}\psi\Big(\frac{n-p+j}{2}\Big) \quad \text{and}$$
$$\kappa_V^{(s)} = (s-1)!\Big(\frac{2}{n}\Big)^{s-1}\mathrm{tr}\Big(\Sigma^s - \frac{s}{s-1}\Sigma^{s-1}\Big) \tag{4.2}$$
$$+ (-1)^s\sum_{j=1}^{p}\psi^{(s-1)}\Big(\frac{n-p+j}{2}\Big) \quad (s=2,3,\cdots).$$

Let

$$T = \frac{\frac{-2}{n}\log V - \kappa_V^{(1)}}{(\kappa_V^{(2)})^{1/2}},$$
$$\kappa^{(s)} = \frac{\kappa_V^{(s)}}{(\kappa_V^{(2)})^{s/2}} \quad (s=3,4,\cdots). \tag{4.3}$$

A bound for the standardized cumulant $\kappa^{(s)}$ is given by the following lemma.

Lemma 4.1 *Let*

$$m = \frac{n-p-\frac{1}{2}}{2}(\kappa_V^{(2)})^{1/2} \quad \textit{and}$$

$$\begin{aligned} b_s &= \frac{2}{\kappa_V^{(2)}(s+1)(s+2)(s+3)}\Big\{1-\Big(\frac{n-p-\frac{1}{2}}{n-\frac{1}{2}}\Big)^{s+1}\Big\} \\ &+ \frac{2}{\kappa_V^{(2)}(s+3)}\Big\{\frac{1}{n}\Big(\frac{n-p-\frac{1}{2}}{n}\Big)^{s+1}\mathrm{tr}\Big(\Sigma^{s+3}-\frac{s+3}{s+2}\Sigma^{s+2}\Big)\Big\}. \end{aligned} \tag{4.4}$$

Then

$$0 < \frac{\kappa^{(s)}}{s!} < m^{-(s-2)}b_{s-3} \quad (s=3,4,\cdots)$$

and

$$\begin{aligned} B[v] = \sum_{s=0}^{\infty} b_s v^s &= \frac{2}{v\kappa_V^{(2)}}\Big\{L_3[v] - L_3\Big[\frac{n-p-\frac{1}{2}}{n-\frac{1}{2}}v\Big]\Big\} \\ &+ \frac{2}{v\kappa_V^{(2)}n}\sum_{j=1}^{p}\Big\{(\lambda_j^2-\lambda_j)L_1\Big[\frac{n-p-\frac{1}{2}}{n}\lambda_j v\Big] - \lambda_j L_2\Big[\frac{n-p-\frac{1}{2}}{n}\lambda_j v\Big]\Big\}, \end{aligned} \tag{4.5}$$

where L_1, L_2 *are given by* (3.10), L_3 *is given by* (2.18), *and* $\lambda_1, \cdots, \lambda_p$ *are the eigenvalues of* Σ.

Proof It is easily checked that

$$\mathrm{tr}\Big(\Sigma - \frac{s}{s-1}\Sigma^{s-1}\Big) = \sum_{j=1}^{p}\Big(\lambda_j^s - \frac{s}{s-1}\lambda_j^{s-1}\Big)$$

has the minimum at $\lambda_j = 1$ $(j=1,\cdots,p)$. Hence from (3.8)

$$\kappa_V^{(s)} > \sum_{j=1}^{p}\Big\{\frac{(s-2)!}{(\frac{n-p+j}{2})^{s-1}} - (s-2)!\Big(\frac{2}{n}\Big)^{s-1}\Big\} > 0.$$

From (3.9)

$$\begin{aligned} \frac{\kappa_V^{(s)}}{s!} &< m^{-(s-2)}\frac{2}{s(s-1)(s-2)}\Big\{1-\Big(\frac{n-p-\frac{1}{2}}{n-\frac{1}{2}}\Big)^{s-2}\Big\} \\ &+ \Big(\frac{2}{ns}\Big)^{s-1}\mathrm{tr}\Big(\Sigma^s - \frac{s}{s-1}\Sigma^{s-1}\Big) = m^{-(s-2)}b_{s-3}. \end{aligned}$$

(4.5) can be checked by using the series expressions of L_1, L_2 and L_3. ■

The next lemma is used for evaluating $|\varphi_V(t)|$.

Lemma 4.2 *Let λ be positive. Then*

$$\log\left|\left(1-\frac{2it\lambda}{n}\right)^{-\frac{n}{2}+it}\right| = -\frac{n}{4}\log\left(1+\frac{4t^2\lambda^2}{n^2}\right)+t\arctan\left(\frac{2t\lambda}{n}\right)$$
$$\leq \frac{1}{2}\int_{n/2}^{\infty}\log\left(1+\frac{t^2}{x^2}\right)dx.$$

Proof It is easily checked that

$$\left|\left(1-\frac{2it\lambda}{n}\right)^{-\frac{n}{2}+it}\right| = \exp\left\{-\frac{n}{4}\log\left(1+\frac{4t^2\lambda^2}{n^2}\right)+t\arctan\left(\frac{2t\lambda}{n}\right)\right\}$$

has the maximum at $\lambda = 1$. Hence

$$\log\left|\left(1-\frac{2it\lambda}{n}\right)^{-\frac{n}{2}+it}\right| \leq \frac{t}{2}\left\{2t\arctan\left(\frac{2t}{n}\right)-\frac{n}{2t}\log\left(1+\frac{4t^2}{n^2}\right)\right\}$$
$$= \frac{1}{2}\int_{n/2}^{\infty}\log\left(1+\frac{t^2}{x^2}\right)dx.$$

■

From (2.16) and lemma 4.2

$$\log|\varphi_V(t)| < -\frac{|t|}{2}G(t;n,\lambda_1,\cdots,\lambda_p),$$

where

$$G(t;n,\lambda_1,\cdots,\lambda_p) = \sum_{j=1}^{p}\left\{c\left[\frac{n-p+j}{2|t|}\right]-2\arctan\left(\frac{2|t|\lambda_j}{n}\right)+\frac{n}{2|t|}\log\left(1+\frac{4t^2\lambda_j}{n^2}\right)\right\} \tag{4.6}$$

and $c[\cdot]$ is given by (2.17). Hence

$$I_3[v] \leq U_3[v] := \int_{m_0 v}^{\infty}\frac{2}{t}\exp\left\{-\frac{t}{2}G(t;n,\lambda_1,\cdots,\lambda_p)\right\}dt, \tag{4.7}$$

where

$$m_0 = \frac{n-p-\frac{1}{2}}{2}.$$

Now we can state the last theorem which gives an uniform bound for the Edgeworth expansion of the distribution function of the standardized test statistic.

Theorem 4.1 *Let T be the standardized test statistic given by* (4.3) *and Q_s be the Edgeworth expansion of the distribution function of T given by* (2.10)*, but the cumulants are given by* (4.3) *with* (4.2)*. Then*

$$\sup_x|\mathrm{P}(T\leq x)-Q_s(x)| < \frac{1}{2\pi}(U_1[v;m,B])+I_2[v]+U_3[v]),$$

where U_1 is given by (2.14) *with B given by* (4.5)*, I_2 is given by* (2.20) *and U_3 is given by* (4.7)*.*

A simple bound for $I_3[v]$ is given as follows. From (2.16) and lemma 4.2

$$\begin{aligned}\log|\varphi_V(t)| &< -\frac{1}{2}\sum_{j=1}^{p}\int_{(n-p+j)/2}^{\infty}\log\Big(1+\frac{t^2}{x^2}\Big)dx+\frac{p}{2}\int_{n/2}^{\infty}\log\Big(1+\frac{t^2}{x^2}\Big)dx\\ &= -\frac{1}{2}\sum_{j=1}^{p-1}\int_{(n-p+j)/2}^{n/2}\log\Big(1+\frac{t^2}{x^2}\Big)dx < -\frac{p-1}{2}\int_{(n-p+p/2)/2}^{n}\log\Big(1+\frac{t^2}{x^2}\Big)dx\\ &< -\frac{p(p-1)}{8}\log\Big(1+\frac{t^2}{(\frac{n}{2}-\frac{p}{8})}\Big).\end{aligned}$$

Hence

$$\begin{aligned}I_3[v] &= \int_{m_0}^{\infty}\frac{2}{t}|\varphi_V(t)|\,dt < \int_{m_0}^{\infty}\frac{2}{t}\Big(1+\frac{t^2}{(\frac{n}{2}-\frac{p}{8})^2}\Big)^{-p(p-1)/8}dt\\ &< \frac{8}{p(p-1)}(1+\alpha)^{-p(p-1)/8}\frac{1+\alpha}{\alpha} = \tilde{U}_3[v] \quad \text{(say)},\end{aligned} \tag{4.8}$$

where

$$\alpha = \Big(\frac{2m_0 v}{n-\frac{p}{4}}\Big)^2.$$

Remark In order to evaluate the order of the error term of the Edgeworth expansion for the non–null distribution function under the framework (1.8), we need some assumption on how Σ changes as $p\to\infty$. Let $\lambda_1^{(p)}$ be the maximal eigenvalue of Σ. Then if $\lambda_1^{(p)}$ is bounded under (1.8), we can show that

$$\sup_x |P(T\le x)-Q_s(x)| = O\Big(\frac{1}{m^{s+1}}\Big).$$

5 Examples

In this section we show some tables of the error bounds derived in the previous sections.

In the case of $s=2$, $\varphi_s(t)$ in (2.9) becomes

$$\varphi_2(t) = \exp\Big(-\frac{t^2}{2}\Big)\Big\{1+\frac{1}{6}\kappa^{(3)}(it)^3+\frac{1}{72}(\kappa^{(3)})^2(it)^6+\frac{1}{24}\kappa^{(4)}(it)^4\Big\}.$$

Hence $I_2[v]$ in (2.20) is

$$I_2[v] = 2\int_{mv}^{\infty}\Big\{\frac{1}{t}+\frac{1}{6}\kappa^{(3)}t^2+\frac{1}{72}(\kappa^{(3)})^2t^5+\frac{1}{24}\kappa^{(4)}t^3\Big\}.$$

$U_1[v;m,B]$ in (2.14) becomes

$$U_1[v] = \frac{2}{m^3}\int_0^{mv}\Big\{\Big(R_{1,2}[v]t^4+\frac{1}{2}R_{2,1}[v]t^6\Big)\exp\Big(-\frac{t^2}{2}\Big)+\frac{1}{6}(B[v])^3t^8\exp\Big(-\frac{t^2}{2}c_v\Big)\Big\}dt,$$

where

$$R_{1,2}[v] = \frac{1}{v^2}\{B[v] - b_0 - b_1 v\} \text{ and } R_{2,1}[v] = \frac{1}{v}\{(B[v])^2 - b_0^2\}.$$

$U_3[v]$ and $\tilde{U}_3[v]$ for each test statistic do not depend on s.

Although we can minimize the bounds obtained in the previous sections with respect to v numerically, it is sufficient for actual use to calculate the bounds at

$$v = 0.05, 0.10, \cdots, 0.95$$

and choose the minimum.

Tables 1, 2 and 3 give the two kinds of bounds,

$$\text{BOUND–1} = \min_{v=0.05,\cdots,0.95} \frac{1}{2\pi}\big(U_1[v] + I_2[v] + U_3[v]\big) \quad \text{and}$$

$$\text{BOUND–2} = \min_{v=0.05,\cdots,0.95} \frac{1}{2\pi}\big(U_1[v] + I_2[v] + \tilde{U}_3[v]\big),$$

the sharp and the simple ones of the second order ($s = 2$) Edgeworth expansions of the null distribution functions of the test statistics testing $H_0 : \Sigma_1 = \Sigma_2$, $H_0 : \Sigma = \lambda I_p$ and $H_0 : \Sigma = I_p$, where the values in the parentheses are the values of v at the minimum.

Table 1: Error bounds for testing $H_0 : \Sigma_1 = \Sigma_2$ in the case of $s = 2$

$n_1 = n_2 = 18$			$n_1 = 18, n_2 = 36$		
p	BOUND–1	BOUND–2	p	BOUND–1	BOUND–2
3	0.0037 (0.55)	0.0046 (0.60)	3	0.0035 (0.65)	0.0046 (0.70)
6	0.0013 (0.55)	0.0016 (0.60)	6	0.0013 (0.65)	0.0016 (0.65)
9	0.0009 (0.60)	0.0011 (0.60)	9	0.0010 (0.65)	0.0012 (0.75)
12	0.0010 (0.70)	0.0013 (0.75)	12	0.0012 (0.80)	0.0017 (0.90)
15	0.0037 (0.95)	0.0125 (0.95)	15	0.0097 (0.95)	0.0400 (0.95)

$n_1 = n_2 = 36$					
p	BOUND–1	BOUND–2	p	BOUND–1	BOUND–2
3	0.0009 (0.45)	0.0010 (0.45)	21	$***$	$***$
6	0.0003 (0.35)	0.0003 (0.40)	24	$***$	$***$
9	0.0001 (0.35)	0.0002 (0.35)	27	$***$	0.0001 (0.55)
12	$***$	0.0001 (0.35)	30	0.0002 (0.60)	0.0002 (0.75)
15	$***$	$***$	33	0.0010 (0.95)	0.0051 (0.95)

The notation "$***$" means the value smaller than 0.00001.

We can see that both p and $n - p$ (or $n_1 - p$) are moderately large, the error bounds are sufficiently small and the simple version of bounds suffices for actual use.

Table 4 gives the error bounds of the second order Edgeworth expansions of the non–null distribution functions of the test statistic testing $H_0 : \Sigma = I_p$ in the case that $n = 36$ and $\Sigma = \lambda I_p$, where $\lambda = 0.5$ and 2. BOUND–2 are not good in the case of $\Sigma = 2I_p$. One of the reason is that $\tilde{U}_3[v]$ in (4.8) gives a uniform bound for $I_3[v]$ with respect to Σ.

Table 2: Error bounds for testing $H_0 : \Sigma = \lambda I_p$ in the case of $s = 2$

$n = 30$			$n = 60$		
p	BOUND–1	BOUND–2	p	BOUND–1	BOUND–2
5	0.1417 (0.75)	0.1506 (0.75)	10	0.0255 (0.55)	0.0256 (0.55)
10	0.0276 (0.60)	0.0291 (0.60)	20	0.0031 (0.40)	0.0033 (0.45)
15	0.0093 (0.60)	0.0103 (0.60)	30	0.0008 (0.40)	0.0009 (0.40)
20	0.0052 (0.65)	0.0062 (0.70)	40	0.0004 (0.45)	0.0005 (0.45)
25	0.0067 (0.90)	0.0115 (0.95)	50	0.0004 (0.60)	0.0006 (0.70)

Table 3: Error bounds for testing $H_0 : \Sigma = I_p$ in the case of $s = 2$ under the null hypothesis

$n = 30$			$n = 60$		
p	BOUND–1	BOUND–2	p	BOUND–1	BOUND–2
5	0.0886 (0.70)	0.1889 (0.75)	10	0.0186 (0.50)	0.0289 (0.55)
10	0.0196 (0.60)	0.0325 (0.65)	20	0.0026 (0.40)	0.0034 (0.45)
15	0.0074 (0.60)	0.0113 (0.60)	30	0.0008 (0.40)	0.0009 (0.40)
20	0.0041 (0.65)	0.0066 (0.70)	40	0.0004 (0.40)	0.0005 (0.45)
25	0.0050 (0.85)	0.0121 (0.95)	50	0.0004 (0.55)	0.0006 (0.70)

Table 4: Error bounds for testing $H_0 : \Sigma = I_p$ in the case of $s = 2$ and $n = 36$ under the non–null hypotheses

$\lambda_1 = \cdots = \lambda_p = 0.5$			$\lambda_1 = \cdots = \lambda_p = 2.0$		
p	BOUND–1	BOUND–2	p	BOUND–1	BOUND–2
6	0.0112 (0.50)	0.1072 (0.60)	6	0.0236 (0.25)	0.5866 (0.30)
12	0.0030 (0.45)	0.0119 (0.55)	12	0.0068 (0.25)	0.0937 (0.35)
18	0.0014 (0.45)	0.0032 (0.60)	18	0.0026 (0.30)	0.0209 (0.40)
24	0.0009 (0.55)	0.0019 (0.65)	24	0.0011 (0.40)	0.0053 (0.60)
30	0.0015 (0.75)	0.0039 (0.95)	30	0.0008 (0.65)	0.0034 (0.95)

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