Asymptotic expansions of test statistics for dimensionality and additional information in canonical correlation analysis when the dimension is large

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Abstract

This paper examines asymptotic expansions of test statistics for dimensionality and additional information in canonical correlation analysis based on a sample of size $N = n + 1$ on two sets of variables, i.e., $x_u; p_1 \times 1$ and $x_v; p_2 \times 1$. These problems are related to dimension reduction. The asymptotic approximations of the statistics have been studied extensively when dimensions $p_1$ and $p_2$ are fixed and the sample size $N$ tends to infinity. However, the approximations worsen as $p_1$ and $p_2$ increase. This paper derives asymptotic expansions of the test statistics when both the sample size and dimension are large, assuming that $x_u$ and $x_v$ have a joint $(p_1 + p_2)$-variate normal distribution. Numerical simulations revealed that this approximation is more accurate than the classical approximation as the dimension increases.

Key Words and Phrases: Asymptotic expansion, Tests for dimensionality, Additional information, High-dimensional framework.

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1 Introduction

Let $x_u$ and $x_v$ be two random vectors of $p_1$ and $p_2$ components with a joint $(p_1 + p_2)$-variate normal distribution with a mean vector $\mu = (\mu'_u, \mu'_v)'$ and a covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{uu} & \Sigma_{uv} \\ \Sigma_{vu} & \Sigma_{vv} \end{pmatrix},$$

where $\Sigma_{uv}$ is a $p_1 \times p_2$ matrix. Without loss of generality we may assume $p_1 \leq p_2$. Let $\rho_1 \geq \cdots \geq \rho_{p_1} \geq 0$ be the possible nonzero population canonical correlations between $x_u$ and $x_v$. Note that $\rho_2^2 \geq \cdots \geq \rho_{p_1}^2 \geq 0$ are the characteristic roots of $\Sigma_{uu}^{-1}\Sigma_{uv}\Sigma_{vv}^{-1}\Sigma_{vu}$. The coefficient vectors $\alpha_{ui}$ and $\alpha_{vi}$ of the canonical variables are defined as the solutions of

$$\Sigma_{uv}\Sigma_{vv}^{-1}\Sigma_{vu}\alpha_{ui} = \rho_i^2\Sigma_{uu}\alpha_{ui}, \quad \alpha'_{ui}\Sigma_{uu}\alpha_{uj} = \delta_{ij},$$
$$\Sigma_{vu}\Sigma_{uu}^{-1}\Sigma_{uv}\alpha_{vi} = \rho_i^2\Sigma_{vv}\alpha_{vi}, \quad \alpha'_{vi}\Sigma_{vv}\alpha_{vj} = \delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$, 0 for $i \neq j$. Let $k$ be the number of nonzero canonical correlations $\rho_i$. Then $k = \text{rank}(\Sigma_{uv}) \leq p_1$, and the relationships between $x_u$ and $x_v$ can be summarized in terms of the first $k$ canonical variates $(\alpha'_{ui}x_u, \alpha'_{vi}x_v)$, $i = 1, \ldots, k$.

In canonical correlation analysis, the number of nonzero canonical correlations, defines the dimensionality. Consider the problem of testing the hypothesis that the smaller $p_1 - k$ canonical correlations are zero, i.e.,

$$H_0 : \rho_k > \rho_{k+1} = \cdots = \rho_{p_1} = 0. \quad (1.1)$$

This problem is related to reducing the dimension of the canonical variables. Let $S$ be the sample covariance matrix formed from a sample of size $N = n+1$ of $x = (x'_u, x'_v)'$. Corresponding to a partition of $x$, we partition $S$ as

$$S = \begin{pmatrix} S_{uu} & S_{uw} \\ S_{vw} & S_{vv} \end{pmatrix}.$$ 

The following test statistics have been considered (e.g., see Sitotani, Hayakawa and Fujikoshi (1985)):

$$LR = -\log \prod_{j=k+1}^{p_1} (1 - r_j^2), \quad LH = \sum_{j=k+1}^{p_1} \frac{r_j^2}{1 - r_j^2}, \quad BNP = \sum_{j=k+1}^{p_1} r_j^2, \quad (1.2)$$

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where \( r_j^2 \) is the sample canonical correlation. Note that \( r_1^2 > \cdots > r_p^2 > 0 \) are the characteristic roots of \( S_{uu}^{-1}S_{uv}S_{vu} \). Under a large sample framework,

\[ A_0 : p_1 \text{ and } p_2 \text{ are fixed, } n \to \infty, \quad (1.3) \]

some asymptotic results have been obtained (e.g., see Anderson (2003), Siotani, et al. (1985)). Note that these results will not work well as dimension \( p_1 \) or \( p_2 \) increases. In order to overcome this weakness, we study the asymptotic distributions of these statistics under a high-dimensional framework such that

\[ A_1 : p_1; \text{fixed, } p_2 \to \infty, \quad n \to \infty, \quad m = n - p_2 \to \infty, \quad p_2/n \to c \in (0, 1). \quad (1.4) \]

In this paper we also consider asymptotic distributions of test statistics for a hypothesis concerning the sufficiency of the redundancy of a subset of variables from each of \( x_u \) and \( x_v \). This problem is related to reducing the dimension of the original variables. In order to formulate the hypothesis, we partition \( x_u \) and \( x_v \) as

\[
\begin{align*}
  x_u & : q_1 \times 1, \\
  x_v & : q_2 \times 1,
\end{align*}
\]

\[
\begin{align*}
  x_3 & : q_3 \times 1, \\
  x_4 & : q_4 \times 1
\end{align*}
\]

and \( \alpha_{ui}, \alpha_{vi}, \mu_u, \mu_v, \Sigma \) comfortably:

\[
\begin{pmatrix}
  \alpha_{ui} \\
  \alpha_{vi}
\end{pmatrix} = \begin{pmatrix}
  \alpha_{1i} \\
  \alpha_{2i} \\
  \alpha_{3i} \\
  \alpha_{4i}
\end{pmatrix}, \quad \begin{pmatrix}
  \mu_u \\
  \mu_v
\end{pmatrix} = \begin{pmatrix}
  \mu_1 \\
  \mu_2 \\
  \mu_3 \\
  \mu_4
\end{pmatrix},
\]

\[
\Sigma = \begin{pmatrix}
  \Sigma_{uu} & \Sigma_{uv} \\
  \Sigma_{vu} & \Sigma_{vv}
\end{pmatrix} = \begin{pmatrix}
  \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\
  \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\
  \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} \\
  \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44}
\end{pmatrix}.
\]

Note that \( p_1 = q_1 + q_2 \) and \( p_2 = q_3 + q_4 \). Then, the hypothesis of the sufficiency of \( x_1 \) and \( x_3 \), as the redundancy of \( x_2 \) and \( x_4 \), is formulated as follows:

\[ H_1 : \alpha_{2i} = 0, \quad \alpha_{4i} = 0 \quad (i = 1, \ldots, k). \]

Let \( S \) be the sample covariance matrix formed from a sample of size \( N = n + 1 \) of \( (x'_u, x'_v)' \). Corresponding to a partition of \( \Sigma \), we partition \( S \) as

\[
S = \begin{pmatrix}
  S_{uu} & S_{uv} \\
  S_{vu} & S_{vv}
\end{pmatrix} = \begin{pmatrix}
  S_{11} & S_{12} & S_{13} & S_{14} \\
  S_{21} & S_{22} & S_{23} & S_{24} \\
  S_{31} & S_{32} & S_{33} & S_{34} \\
  S_{41} & S_{42} & S_{43} & S_{44}
\end{pmatrix}.
\]
To test $H_1$, we consider the statistic (Fujikoshi (1982)) defined by

$$T = \left| \begin{array}{cc} S_{22,13} & S_{24,13} \\ S_{42,13} & S_{44,13} \end{array} \right| / \{|S_{22,1}| |S_{44,3}|\}. \tag{1.5}$$

This is a likelihood ratio statistic. Here, $S_{22,1} = S_{22} - S_{21}S_{11}^{-1}S_{12}$, $S_{22,13} = S_{22} - S_{213}S_{131}^{-1}S_{132}$, $S_{2,13} = (S_{21}, S_{23})$, etc.

Under a large sample framework $A_0$, an asymptotic expansion is obtained (see Fujikoshi (1982)). However, the result will not work well as dimensions $p_1$ and $p_2$ increase. In order to overcome this weakness, we study asymptotic expansions of the statistics under a high-dimensional framework such that

$$A_2 \quad p = p_1 + p_2 \to \infty, \quad n \to \infty, \tag{1.6}$$
$$m = n - p \to \infty, \quad p/n \to c \in (0, 1).$$

Numerical simulations revealed that our approximation becomes more accurate than the classical approximation as the dimension increases. Similar approximations have been proposed in the MANOVA model and discriminant analysis. Fujikoshi, Himeno, and Wakaki (2006) derived asymptotic distributions of test statistics for dimensionality under $A_1$. Tonda and Fujikoshi (2004) derived an asymptotic expansion of the distribution of Wilks’ lambda statistic $\Lambda$ under a high-dimensional framework. Wakaki (2006) derived similar results for $\Lambda$ under a different high-dimensional framework. For examples of other distributional results in a high-dimensional framework in which both the dimension and sample size are large, see Bai (1999), Johnstone (2001), Ledoit and Wolf (2002), and Raudys and Young (2004).

2 Distributions of tests for dimensionality

In this section we consider the distribution of the three test statistics (1.2) under framework $A_1$. When we consider the distributions of the statistics in (1.2),

$$\Sigma = \left( \begin{array}{cc} I_{p_1} & \tilde{\mathcal{P}}' \\ \tilde{\mathcal{P}} & I_{p_2} \end{array} \right), \quad \tilde{\mathcal{P}} = (\mathcal{P}, O), \quad \mathcal{P} = \text{diag}(\rho_1, \ldots, \rho_{p_1}),$$

since the statistics are expressed as functions of the characteristic roots of $S_{uu}^{-1}S_{uu} S_{uv}^{-1}S_{vu}$ without loss of generality we may assume. Let $A = n\Sigma$. Corresponding to a partition of $\mathbf{z}$, we partition $A$ as

$$A = \left( \begin{array}{cc} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{array} \right).$$
For our derivation, we use the following properties (see Sugiura and Fujikoshi (1969)):

(a) \( A_{uv} \sim W_{p_1}(m, \Delta) \), where \( \Delta = I_{p_1} - \mathcal{P}^2 \) and \( m = n - p_2 \).

(b) Let \( W \) be the first \( p_1 \times p_1 \) submatrix of \( A_{vv} \). Then, given \( W \), \( A_{uv}A_{vu}^{-1}A_{vv} \sim W_{p_1}(p_2, \Delta; \mathcal{P} \mathcal{W} \mathcal{P}) \), and \( A_{uv}A_{vu}^{-1}A_{vv} \) and \( A_{uu} \) are independent.

(c) \( W \sim W_{p_1}(n, I_{p_1}) \), \( W \) and \( A_{uu} \) are independent.

Let
\[
\ell_i^2 = \frac{r_i^2}{1 - r_i^2}, \quad i = 1, \ldots, p_1.
\]
These are the characteristic roots of \( A_{uv}^{-1}A_{uv}A_{vu}^{-1}A_{vv} \). We will derive an asymptotic distribution of a function of \( \ell_1^2 > \ldots > \ell_{p_1}^2 \) that leads to the asymptotic distribution of a function of \( r_1^2 > \ldots > r_{p_1}^2 \), since \( r_i^2 = \ell_i^2/(1 + \ell_i^2) \). Note that without loss of generality, we may assume

(1) \( A_{uv} \sim W_{p_1}(m, I_{p_1}) \).

(2) Let \( W \) be the first \( p_1 \times p_1 \) submatrix of \( A_{vv} \). Then, given \( W \), \( A_{uv}A_{vu}^{-1}A_{vv} \sim W_{p_1}(p_2, I_{p_2}; \Gamma \mathcal{W} \mathcal{T}) \), where \( \Gamma = \Delta^{-1/2} \mathcal{P} \), and \( A_{uv}A_{vu}^{-1}A_{vv} \) and \( A_{uu} \) are independent.

(3) \( W \sim W_{p_1}(n, I_{p_1}) \), \( W \) and \( A_{uu} \) are independent.

Now we consider the perturbation expansion of
\[
Q = A_{uv}^{-\frac{1}{2}}A_{uv}A_{vu}^{-1}A_{vv}A_{uv}^{-\frac{1}{2}}.
\]
Let \( U \) and \( V \) be the matrices defined by
\[
U = \frac{1}{\sqrt{p_2}} \left( A_{uv}A_{vu}^{-1}A_{vv} - (p_2 I_{p_1} + n \Gamma^2) \right) \quad \text{and} \quad V = \frac{1}{\sqrt{m}} (A_{uv} - m I_{p_1}),
\]
respectively. The characteristic function of \( U \) can be expressed as
\[
C_U(T) = E[\exp(i T U)] = E_W[E[\exp(i T U)|W]],
\]
where
where $T$ is a real symmetric matrix whose $(i, j)$ element is given by $(1 + \delta_{ij})t_{ij}/2$. Here, $\delta_{ij}$ is the Kronecker delta, i.e., $\delta_{ii} = 1$, $\delta_{ij} = 0$ $(i \neq j)$. The conditional characteristic function can be evaluated as
\[
C_U(T|W) = E[\exp(i tr TU)|W] = \exp \left( -\frac{1}{\sqrt{p_2}} i trT(p_2 I_{p_1} + n \Gamma^2) \right) \left| I_{p_1} - \frac{2i}{\sqrt{p_2}} T \right|^{-\frac{p_2}{2}} \times \text{etr} \left[ \frac{i}{\sqrt{p_2}} \Gamma W \Gamma T \left( I_{p_1} - \frac{2i}{\sqrt{p_2}} T \right)^{-1} \right] = \text{etr} \left( -T^2 + i \sqrt{\frac{n}{p_2}} \Gamma \Gamma T - 2 \frac{n}{p_2} \Gamma^2 T^2 \right) \times \{1 + o^*(1)\},
\]
where $G = \frac{1}{\sqrt{n}}(W - n I)$ and the notation $o^*_i$ denotes a term that tends to 0 under a high-dimensional framework (1.4). Therefore,
\[
C_U(T) = \int C_U(T|W)f(W)dW = \text{etr} \left( -T^2 - 2 \frac{n}{p_2} \Gamma^2 T^2 - \frac{n}{p_2} (\Gamma T \Gamma)^2 \right) \times \{1 + o^*(1)\}.
\]
Similarly, the characteristic function of $V$ can be expanded as
\[
C_V(T) = \text{etr}(T^2) \times \{1 + o^*(1)\}.
\]
Using these results we can expand $C_{V,U}(T_1, T_2)$ of the joint characteristic function of $V$ and $U$ as follows:
\[
C_{V,U}(T_1, T_2) = E[\exp(i tr T_1 V + i tr T_2 U)] = C_V(T_1) \times E_W \left[ C_U(T_2|W) \right] = \text{etr}(-T_1^2) \text{etr} \left( -T_2^2 - 2 \frac{n}{p_2} \Gamma^2 T_2^2 - \frac{n}{p_2} (T_2 \Gamma)^2 \right) \times \{1 + o^*(1)\}.
\]
Therefore we obtain the following theorem.

**Theorem 2.1** Under assumption A1, each of the elements of $U$ and $V$ is asymptotically and independently distributed as a normal distribution, more
precisely
\[ v_{ii} \xrightarrow{d} N(0,2), \quad v_{ij} \xrightarrow{d} N(0,1), \quad i \neq j, \]
\[ u_{ii} \xrightarrow{d} N\left(0, 2\left(1 + \frac{n}{p_2} \gamma_i^2 + \frac{n}{p_2} \gamma_i^4\right)\right), \quad u_{ij} \xrightarrow{d} N\left(0, 1 + \frac{n}{p_2} \gamma_i^4\right) \quad i \neq j. \]

where \( \gamma_i^2 = \rho_i^2 / (1 - \rho_i^2) \) and \( \xrightarrow{d} \) denotes convergence in distribution.

We can write \( A_{uv} A_{vv}^{-1} A_{vu} \) and \( A_{uu} \cdot v \) in terms of \( U \) and \( V \) as
\[ A_{uv} A_{vv}^{-1} A_{vu} = p_2 \left(I_{p_1} + \frac{n}{p_2} \Gamma^2\right) + \sqrt{p_2} U, \quad A_{uu} \cdot v = m \left(I_{p_1} + \frac{1}{\sqrt{m}} V\right), \quad (2.1) \]
and hence
\[ Q = A_{uu}^{-1/2} A_{uv} A_{vv}^{-1} A_{vu} A_{uu}^{-1/2} \]
\[ = \frac{1}{m} \left(I_{p_1} + \frac{1}{\sqrt{m}} V\right)^{-1/2} \left(p_2 \left(I_{p_1} + \frac{n}{p_2} \Gamma^2\right) + \sqrt{p_2} U\right) \left(I_{p_1} + \frac{1}{\sqrt{m}} V\right)^{-1/2}. \]

Therefore, \( Q \) can be expanded as
\[ Q = \frac{p_2}{m} \left(I_{p_1} - \frac{1}{\sqrt{m}} V + O_i^*\right) \left(I_{p_1} + \frac{n}{p_2} \Gamma^2\right) \left(I_{p_1} + \frac{1}{\sqrt{m}} V + O_i^*\right) \]
\[ \times \left(I_{p_1} - \frac{1}{\sqrt{m}} V + O_i^*\right) \]
\[ = \frac{p_2}{m} \left(I_{p_1} + \frac{n}{p_2} \Gamma^2\right) \]
\[ + \frac{1}{\sqrt{m}} \left\{ \sqrt{\frac{p_2}{m} - \frac{1}{2} \frac{V p_2}{m} \left(I_{p_1} + \frac{n}{p_2} \Gamma^2\right) - \frac{1}{2} \frac{p_2}{m} \left(I_{p_1} + \frac{n}{p_2} \Gamma^2\right) V} \right\} + O_i^*. \quad (2.2) \]

Here, the notation \( O_i^* \) denotes a term of the \( i \)-th order with respect to \((n^{-1}, p_2^{-1}, m^{-1})\).

### 2.1 Null distributions

In this section we consider the null distribution of the three test statistics under framework A1 and
\[ A1.1: \quad \rho_1^2 > \cdots > \rho_k^2 > \rho_{k+1}^2 = \cdots = \rho_{p_1}^2 = 0. \quad (2.3) \]
Consider the transformed test statistics of LR, LH and BNP in (1.2) defined by

\[
T_{LR} = \sqrt{p_2} \left(1 + \frac{m}{p_2}\right) \left\{ \log \prod_{j=k+1}^{p_1} (1 + \ell_j^2) - (p_1 - k) \log \left(1 + \frac{p_2}{m}\right) \right\},
\]

\[
T_{LH} = \sqrt{p_2} \left\{ \frac{m}{p_2} \sum_{j=k+1}^{p_1} \ell_j^2 - (p_1 - k) \right\}, \quad (2.4)
\]

\[
T_{BNP} = \sqrt{p_2} \left(1 + \frac{m}{p_2}\right) \left\{ \left(1 + \frac{m}{p_2}\right) \sum_{j=k+1}^{p_1} \frac{\ell_j^2}{1 + \ell_j^2} - (p_1 - k) \right\}.
\]

Note that \(\ell_1^2, \ldots, \ell_{p_1}^2\) are the characteristic roots of \(Q\), and the three test statistics are symmetric functions of the last \(p_1 - k\) characteristic roots \(\ell_{k+1}^2, \ldots, \ell_{p_1}^2\). Using the fact that \(Q\) has a perturbation expansion as in (2.2), it can be seen (see Lawley (1956, 1959) and Fujikoshi (1977)) that the last \(p_1 - k\) characteristic roots \(\ell_{k+1}^2, \ldots, \ell_{p_1}^2\) are the characteristic roots of

\[
D = \frac{p_2}{m} I_{p_1-k} + \frac{1}{\sqrt{m}} \left( \sqrt{\frac{p_2}{m}} U_{22} - \frac{p_2}{m} V_{22} \right) + O_1^*, \quad (2.5)
\]

where

\[
U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \quad \text{tr} \left(U_{22} - \sqrt{\frac{p_2}{m}} V_{22}\right) + O_{1/2}^*
\]

Using these results we can expand \(T_{LR}, T_{LH}, \text{and } T_{BNP}\) as follows:

\[
T_{LR} = \sqrt{p_2} \left(1 + \frac{m}{p_2}\right) \left\{ \sum_{j=k+1}^{p_1} \left\{ \log \left(1 + \frac{p_2}{m} + \frac{\ell_j^2}{1 + \ell_j^2} - \frac{p_2}{m} \right) \right\} - (p_1 - k) \log \left(1 + \frac{p_2}{m}\right) \right\}
\]

\[
= \sqrt{p_2} \left(1 + \frac{m}{p_2}\right) \left\{ \sum_{j=k+1}^{p_1} \left\{ \log \left(1 + \frac{p_2}{m}\right) + \frac{1}{1 + \ell_j^2} \left(\frac{\ell_j^2}{1 + \ell_j^2} - \frac{p_2}{m} \right) + O_{1/2}^* \right\} - (p_1 - k) \log \left(1 + \frac{p_2}{m}\right) \right\}
\]

\[
= \text{tr} \left(U_{22} - \sqrt{\frac{p_2}{m}} V_{22}\right) + O_{1/2}^*,
\]
\[
T_{LH} = \sqrt{p_2} \left\{ \frac{m}{p_2} \left( p_2 \left( p_1 - k \right) \right) + \frac{1}{\sqrt{m}} \text{tr} \left( \sqrt{\frac{p_2}{m}} U_{22} - \frac{p_2}{m} V_{22} \right) + O_{1/2}^* \right\} - (p_1 - k) \right\} \\
= \text{tr} \left( U_{22} - \sqrt{\frac{p_2}{m}} V_{22} \right) + O_{1/2}^* ,
\]

\[
T_{BNP} = \sqrt{p_2} \left( 1 + \frac{m}{p_2} \right) \left\{ \left( 1 + \frac{m}{p_2} \right) \left( \frac{p_2}{m} \left( 1 + \frac{m}{p_2} \right) \right)^{-1} (p_1 - k) + \frac{1}{\sqrt{m}} \left( 1 + \frac{m}{p_2} \right)^{-2} \text{tr} \left( \sqrt{\frac{p_2}{m}} U_{22} - \frac{p_2}{m} V_{22} \right) \right\} - (p_1 - k) + O_{1/2}^* \right\} \\
= \text{tr} \left( U_{22} - \sqrt{\frac{p_2}{m}} V_{22} \right) + O_{1/2}^* .
\]

Each of the diagonal elements of \( U_{22} \) and \( V_{22} \) is asymptotically distributed as \( N(0, 2) \). Therefore, we obtain the following theorem.

**Theorem 2.2** Under assumptions (1.4) and (2.3),

\[
\frac{T_G}{\sigma_G} \xrightarrow{d} N(0, 1),
\]

where \( G=LR, \ LH, \ BNP, \) and

\[
\sigma_G = \sqrt{2(p_1 - k) \left( 1 + \frac{p_2}{m} \right)} .
\]

### 2.2 Non-null distribution

In this section we derive the asymptotic non-null distributions of the three test statistics for dimensionality under the alternative hypothesis:

\[
\tilde{H}_0: \rho_b > \rho_{b+1} = \ldots = \rho_{p_1} = 0, \quad k < b \leq p_1.
\]

For simplicity, we assume that the first \( b \) canonical correlations are differ, i.e.,

\[
A1.2: \quad \rho_1^2 > \ldots > \rho_b^2 > \rho_{b+1}^2 = \ldots = \rho_{p_1}^2 = 0. \quad (2.6)
\]
This is equivalent to $\gamma_1^2 > \cdots > \gamma_b^2 > \gamma_{b+1}^2 = \cdots = \gamma_{p_1}^2 = 0$. Note that $\ell_1^2, \ldots, \ell_{p_1}^2$ are the characteristic roots of $A_{uu}^{-1}A_{uv}A_{vu}^{-1}A_{vv}$, which has a perturbation expansion in (2.2).

$$\sqrt{m} \left\{ \ell_j - \frac{p_2}{m} \left( 1 + \frac{n}{p_2} \gamma_j^2 \right) \right\}$$

$$= \sqrt{\frac{p_2}{m}} u_{jj} - \frac{p_2}{m} \left( 1 + \frac{n}{p_2} \gamma_j^2 \right) v_{jj} + O_{1/2}, \quad j = k + 1, \ldots, b. \quad (2.7)$$

Further, from (2.6) the last $p_1 - b$ characteristic roots $\ell_{b+1}^2, \ldots, \ell_{p_1}^2$ are the characteristic roots of

$$\tilde{Q} = \frac{p_2}{m} P_{p_1-b} + \frac{1}{\sqrt{m}} \left( \sqrt{\frac{p_2}{m}} \tilde{U}_{22} - \frac{p_2}{m} \tilde{V}_{22} \right) + O_{1/2}, \quad (2.8)$$

where

$$U = \begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ \tilde{U}_{21} & \tilde{U}_{22} \end{bmatrix}, \quad V = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix},$$

and $\tilde{U}_{22}$ and $\tilde{V}_{22}$ are $(p_1 - b) \times (p_1 - b)$ matrices. Let

$$T_{LR}^* = \sqrt{p_2} \left( 1 + \frac{m}{p_2} \right) \left\{ \log \prod_{j=k+1}^{p_1} (1 + \ell_j^2) \right.\
\left. - \log \prod_{j=k+1}^{p_1} \left( 1 + \frac{p_2}{m} \left( 1 + \frac{n}{p_2} \gamma_j^2 \right) \right) \right\},$$

$$T_{LH}^* = \sqrt{p_2} \left\{ \frac{m}{p_2} \sum_{j=k+1}^{p_1} \ell_j^2 - \frac{m}{p_2} \sum_{j=k+1}^{p_1} \frac{p_2}{m} \left( 1 + \frac{n}{p_2} \gamma_j^2 \right) \right\}, \quad (2.9)$$

$$T_{BNP}^* = \sqrt{p_2} \left( 1 + \frac{m}{p_2} \right) \left\{ \left( 1 + \frac{m}{p_2} \right) \sum_{j=k+1}^{p_1} \frac{\ell_j^2}{1 + \ell_j^2} \right.\
\left. - \left( 1 + \frac{m}{p_2} \right) \sum_{j=k+1}^{p_1} \frac{p_2}{m} \left( 1 + \frac{n}{p_2} \gamma_j^2 \right) \right\}. \right.$$
Using (2.7) and (2.8) we can express $T_{LR}^*$, $T_{LH}^*$ and $T_{BNP}^*$ as follows.

$$T_{LR}^* = \sum_{j=k+1}^{p_1} \left\{ \frac{1 + \frac{p_2}{m}}{1 + \frac{p_2}{m} \left(1 + \frac{n}{p_2 \gamma_j^2}\right)} \right\} \sqrt{\frac{m}{p_2}} \left( \sqrt{\frac{p_2}{m} u_{jj} - \frac{p_2}{m} \left(1 + \frac{n}{p_2 \gamma_j^2}\right) v_{jj}} \right)$$

$$+ \text{tr} \left( \tilde{U}_{22} - \sqrt{\frac{p_2}{m} \tilde{V}_{22}} \right) + O_{1/2}^*;$$

$$T_{LH}^* = \sum_{j=k+1}^{b} \left\{ \sqrt{\frac{m}{p_2}} \left( \sqrt{\frac{p_2}{m} u_{jj} - \frac{p_2}{m} \left(1 + \frac{n}{p_2 \gamma_j^2}\right) v_{jj}} \right) \right\}$$

$$+ \text{tr} \left( \tilde{U}_{22} - \sqrt{\frac{p_2}{m} \tilde{V}_{22}} \right) + O_{1/2}^*;$$

$$T_{BNP}^* = \sum_{j=k+1}^{p_1} \left\{ \left(1 + \frac{p_2}{m}\right)^2 \right\} \frac{1 + \frac{p_2}{m}}{1 + \frac{p_2}{m} \left(1 + \frac{n}{p_2 \gamma_j^2}\right)} \sqrt{\frac{m}{p_2}} \left( \sqrt{\frac{p_2}{m} u_{jj} - \frac{p_2}{m} \left(1 + \frac{n}{p_2 \gamma_j^2}\right) v_{jj}} \right)$$

$$+ \text{tr} \left( \tilde{U}_{22} - \sqrt{\frac{p_2}{m} \tilde{V}_{22}} \right) + O_{1/2}^*.$$

Therefore we can combine the above three expressions as

$$T_G^* = \sum_{j=k+1}^{b} d_j \left( u_{jj} - \sqrt{\frac{p_2}{m} \left(1 + \frac{n}{p_2 \gamma_j^2}\right) v_{jj}} \right)$$

$$+ \text{tr} \left( \tilde{U}_{22} - \sqrt{\frac{p_2}{m} \tilde{V}_{22}} \right) + O_{1/2}^*.$$

where

$$d_j = \frac{1 + \frac{p_2}{m}}{1 + \frac{p_2}{m} \left(1 + \frac{n}{p_2 \gamma_j^2}\right)}.$$

Here, the notation $G$ and $c$ is used such that

$$c = \begin{cases} 
1, & \text{when } G = LR, \\
0, & \text{when } G = LH, \\
2, & \text{when } G = BNP.
\end{cases}$$

Using Theorems 2.1 and 2.2, we obtain the following theorem.
Theorem 2.3 Let $T^*_G$ be the transformed test statistics defined by (2.9), where $G = LR, LH, BNP$. Then, under assumptions (1.4) and (2.6),

$$
\frac{T^*_G}{\sigma^*_G} \xrightarrow{d} N(0,1),
$$

where

$$
\sigma^*_G^2 = 2 \sum_{j=k+1}^b d^2_j \left\{ \left(1 + 2 \frac{n}{p^2_j} \gamma_j^2 + \frac{n}{p^2_j} \gamma_j^4 \right) \right\}
+ 2(p_1 - b) \left(1 + \frac{p_2}{m}\right).
$$

2.3 Asymptotic power

Based on the asymptotic distributions of the three statistics in Theorem 2.3, we obtain their asymptotic powers. Let $\delta_G = T_G - T^*_G$. Then

$$
\delta_{LR} = \sqrt{\frac{m}{p^2}} \left(1 + \frac{p_2}{m} \right) \sum_{j=k+1}^b \log \left(1 + \frac{p_2}{m} \frac{n \gamma_j^2}{(1 + \frac{p_2}{m}) p_2} \right),
$$

$$
\delta_{LH} = \sqrt{p_2} \sum_{j=k+1}^b \frac{n \gamma_j^2}{p^2} ,
$$

$$
\delta_{BNP} = \sqrt{p_2} \left(1 + \frac{p_2}{m} \right) \left\{ \left(1 + \frac{p_2}{m} \right) \sum_{j=k+1}^b \frac{p_2}{m} \frac{1 + \frac{p_2}{m}}{1 + \frac{p_2}{m} \gamma_j^2} - (b - k) \right\}.
$$

We have

$$
P_D = Pr(T_G > \sigma_G z_\alpha) = Pr(T^*_G > \sigma_G z_\alpha - \delta_G),
$$

where $z_\alpha$ is the upper $100\alpha$ % points of the standard normal distribution.

Using Theorem 2.3, the asymptotic power with a level of significance $\alpha$ is expressed as

$$
\lim_{p_2 \to \infty} P_D = \lim_{p_2 \to \infty} \Phi \left( \frac{\delta_G - \sigma_G z_\alpha}{\sigma_G^*} \right),
$$

where $\Phi$ is the distribution function of the standard normal distribution. Under assumption (1.4),

$$
\frac{p_2}{m} = \frac{p_2}{n - p_2} = \frac{1}{n - p_2} \to \frac{1}{c - 1} = \frac{c}{1 - c} > 0.
$$
Therefore, we can obtain $\delta_G \to \infty$ so that the asymptotic power is 1.

3 Distributions of tests for additional information

We are interested in the distribution of $T$ in (1.5). According to Theorem 2 in Fujikoshi (1982), $T$ under $H_1$ is expressed as a product of two independent variables, i.e.,

$$T = T_1 \times T_2, \quad T_1 \sim \Lambda(p_2, q_2, n - p_1), \quad T_2 \sim \Lambda(q_4, q_1, n - r),$$

where $p_1 = q_1 + q_2$, $p_2 = q_3 + q_4$, $r = q_1 + q_3$. Here, we denote the distribution of $\Lambda = |A|/|A + B|$ by $\Lambda(p, q, n)$, where $A$ and $B$ have independent Wishart distributions $W_p(n, \Sigma)$ and $W_p(q, \Sigma)$, respectively. Fujikoshi (1982) derived an asymptotic expansion for the distribution of $T$ under $A_0$. The approximation can be written as

$$P(-m \log T \leq x) = G_f(x) + \frac{\beta}{m^2} \{G_f(x) - G_f(x)\} + O(m^{-2}), \quad (3.2)$$

where $p = q_1 + q_2 + q_3 + q_4$, $r = q_1 + q_3$, $f = (q_1 + q_2)(q_3 + q_4) - q_1q_3$.

$$m = n - \frac{1}{2}(p + 1) - \frac{1}{2}q_1q_3(p - r)/f,$$

$$\beta = \frac{1}{48} \left\{ q_1^2 + q_2^2 + q_3^2 + q_4^2 - 5 \right\} f + 2q_1^2q_2p_2 + 2p_1q_3^2q_4$$

$$+ 2q_2q_4 \left\{ q_1q_2 + q_3q_4 - 3q_1q_3 \right\} - 3(q_1q_3)^2(p - r)^2/f.$$

Let $\Lambda$ be a statistic that is distributed as $\Lambda(p, q, n)$. Tonda and Fujikoshi (2004) derived an asymptotic expansion formula of the distribution of $\Lambda$ when $q$ is fixed, $n \to \infty$, $p \to \infty$ with $p/n \to c \in (0, 1)$. For our derivation, we use their result. Let

$$T_F = -\log \Lambda - m_F/d_F,$$

where

$$m_F = \sum_{j=1}^q \log \frac{n + j}{n - p + j}, \quad d_F = \frac{2}{p} \sum_{j=1}^q \frac{p^2}{(n + j)(n - p + j)}.$$
Then, the characteristic function of $T_F$ can be expanded as

$$C_T(t) = e^{-rac{t^2}{2}} \left\{ 1 + \sum_{\alpha=1}^{2} \kappa_F^{(2\alpha-1)} (it)^{2\alpha-1} + \sum_{\alpha=1}^{3} \kappa_F^{(2\alpha)} (it)^{2\alpha} \right\} + o(p^{-1}),$$

where $\kappa_F^{(\alpha)}$'s are defined by

$$\kappa_F^{(1)} = \frac{1}{\sqrt{p}} \tau_1, \quad \kappa_F^{(3)} = \frac{1}{\sqrt{p}} \tau_3, \quad \kappa_F^{(2)} = \frac{1}{p} \left( \tau_2 + \frac{\tau_1 - \tau_{11}}{2} \right),$$

$$\kappa_F^{(4)} = \frac{1}{p} \left( \tau_4 + \tau_1 \tau_3 - \tau_{13} \right), \quad \kappa_F^{(6)} = \frac{1}{p} \left( \tau_6 + \frac{\tau_3^2 - \tau_{33}}{2} \right),$$

$$\tau_i = \sum_{k=1}^{q} \omega_k^i \tau_{ik}, \quad \tau_{ij} = \sum_{k=1}^{q} \omega_k^{i+j} \tau_{ik} \tau_{jk}.$$
Then, the characteristic function of $T_W$ can be expanded as

$$C_{T_W}(t) = e^{-\frac{1}{2}t^2} \left\{ 1 + \sum_{\alpha=1}^{2} \kappa_W^{(2\alpha-1)}(it)^{2\alpha-1} + \sum_{\alpha=1}^{3} \kappa_W^{(2\alpha)}(it)^{2\alpha} \right\} + o(p^{-1}),$$

where $\kappa_W^{(\alpha)}$s are defined by $\kappa_W^{(1)} = \kappa_W^{(2)} = 0$, 

$$\kappa_W^{(3)} = \frac{\tau^{(3)}}{(\tau^{(2)})^{3/2}}, \quad \kappa_W^{(4)} = \frac{\tau^{(4)}}{(\tau^{(2)})^{2}}, \quad \kappa_W^{(6)} = (\kappa_W^{(3)})^2.$$ 

Using these results, we can obtain the asymptotic distribution of $T$ in (1.5) under various high-dimensional frameworks satisfying A2.

### 3.1 Null distribution under A2

In our framework A2, the conditions "$p \to \infty$" and "one of $q_1, q_2, q_3, q_4 \to \infty$" are equivalent. Under $p_1 \leq p_2$, the condition can be realized as one of the following 12 cases.

(i) $q_1$:fixed, $q_2$:fixed, $q_3$:fixed, $q_4 \to \infty$.  
(ii) $q_1$:fixed, $q_2$:fixed, $q_3 \to \infty$, $q_4$:fixed.  
(iii) $q_1$:fixed, $q_2$:fixed, $q_3 \to \infty$, $q_4 \to \infty$.  
(iv) $q_1$:fixed, $q_2 \to \infty$, $q_3$:fixed, $q_4 \to \infty$.  
(v) $q_1$:fixed, $q_2 \to \infty$, $q_3 \to \infty$, $q_4$:fixed.  
(vi) $q_1$:fixed, $q_2 \to \infty$, $q_3 \to \infty$, $q_4 \to \infty$.  
(vii) $q_1 \to \infty$, $q_2$:fixed, $q_3$:fixed, $q_4 \to \infty$.  
(viii) $q_1 \to \infty$, $q_2$:fixed, $q_3 \to \infty$, $q_4 \to \infty$.  
(ix) $q_1 \to \infty$, $q_2$:fixed, $q_3 \to \infty$, $q_4 \to \infty$.  
(x) $q_1 \to \infty$, $q_2 \to \infty$, $q_3$:fixed, $q_4 \to \infty$.  
(xi) $q_1 \to \infty$, $q_2 \to \infty$, $q_3 \to \infty$, $q_4$:fixed.  
(xii) $q_1 \to \infty$, $q_2 \to \infty$, $q_3 \to \infty$, $q_4 \to \infty$.

We can obtain an asymptotic expansion of the distribution of $T$ in all of the cases except (ii). Note that even in situation (ii), our approximations are good based on the numerical simulation.

We have seen that $T_1$ and $T_2$ have the following asymptotic means and variances:

$$E(-\log T_j) \approx m_j, \quad \text{Var}(-\log T_j) \approx d_j,$$

and hence

$$E(-\log T) \approx m_1 + m_2, \quad \text{Var}(-\log T) \approx d,$$

where $d = \left(d_1^2 + d_2^2\right)^{-1/2}$. Note that $m_j$ and $d_j$ are given by Tonda and Fujikoshi (2004) and Wakaki (2006), respectively. Let $T_H$ be the standardization of $T$ defined by

$$T_H = \frac{-\log T - (m_1 + m_2)}{d}.$$
Noting that \( T_H = w_1 \tilde{T}_1 + w_2 \tilde{T}_2 \) with \( w_j = d_j / d \), the characteristic function \( C_H(t) \) of \( T_H \) is expressed as
\[
C_{T_H}(t) = C_{\tilde{T}_1}(w_1 t) C_{\tilde{T}_2}(w_2 t),
\]
where \( \tilde{T}_j \) are defined by the standardization of \(- \log T_j\). Then the characteristic function of \( T_H \) is expressed by
\[
C_{T_H}(t) = e^{-\frac{1}{2}t^2} \left\{ 1 + \sum_{a=1}^{2} \kappa^{(2a-1)}(it)^{2a-1} + \sum_{a=1}^{3} \kappa^{(2a)}(it)^{2a} \right\} + o(p^{-1}),
\]
where
\[
\kappa^{(1)} = \tilde{\kappa}^{(1)} + \tilde{\kappa}^{(1)}, \quad \kappa^{(3)} = \tilde{\kappa}^{(3)} + \tilde{\kappa}^{(3)}, \quad \kappa^{(2)} = \tilde{\kappa}^{(2)} + \tilde{\kappa}^{(2)} + \tilde{\kappa}^{(1)} \tilde{\kappa}^{(1)}, \\
\kappa^{(4)} = \tilde{\kappa}^{(4)} + \tilde{\kappa}^{(4)} + \tilde{\kappa}^{(3)} \tilde{\kappa}^{(3)} + \tilde{\kappa}^{(3)} \tilde{\kappa}^{(1)} + \tilde{\kappa}^{(1)} \tilde{\kappa}^{(2)} + \tilde{\kappa}^{(3)} \tilde{\kappa}^{(2)}.
\] (3.3)

Here \( \tilde{\kappa}_j^{(a)} \)'s are defined by \( \tilde{\kappa}_j^{(a)} = w^n \kappa_j^{(a)} \), where \( \kappa_j^{(a)} \)'s are given by Tonda and Fujikoshi (2004), Wakaki (2006).

Let \( T_{H1} \) denote the standardization of \( T \) under (i), (iii), and (viii), \( T_{H2} \) under (iv), (vi), and (xii), \( T_{H3} \) under (v), \( T_{H4} \) under (vii), and (ix), and \( T_{H5} \) under (x) and (xii). For convenience, let \((q_{inf}, q_{fix})\) be defined by
\[
(q_{inf}, q_{fix}) = \begin{cases} 
(q_4, q_1) & \text{under } q_4 \to \infty, \; q_1 : \text{fixed}, \\
(q_1, q_4) & \text{under } q_1 \to \infty, \; q_4 : \text{fixed}.
\end{cases}
\]

Then, \( m_j, d_j, \kappa_j^{(a)} \) are obtained from the following table.

<table>
<thead>
<tr>
<th>( T_{H1} )</th>
<th>( m_1, d_1, \kappa_1^{(a)} )</th>
<th>( m_2, d_2, \kappa_2^{(a)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(p_2, q_2, n - p_1) )</td>
<td>( F(q_{inf}, q_{fix}, n - r) )</td>
<td>( F(q_{inf}, q_{fix}, n - r) )</td>
</tr>
<tr>
<td>( W(p_2, q_2, n - p_1) )</td>
<td>( W(q_2, q_3, n - r) )</td>
<td>( W(q_1, q_1, n - r) )</td>
</tr>
<tr>
<td>( F(p_1, q_1, n - p_2) )</td>
<td>( W(q_1, q_1, n - r) )</td>
<td>( W(q_1, q_1, n - r) )</td>
</tr>
<tr>
<td>( W(p_2, q_2, n - p_1) )</td>
<td>( W(q_1, q_1, n - r) )</td>
<td>( W(q_1, q_1, n - r) )</td>
</tr>
</tbody>
</table>

Here, \( F(a, b, c) \) and \( W(a, b, c) \) show that \( (m_j, d_j, \kappa_j^{(a)}) \) are defined from \( (m_F, d_F, \kappa_F^{(a)}) \) or \( (m_W, d_W, \kappa_W^{(a)}) \) by substituting \( (n, p, q) \) for \( (a, b, c) \), respectively. Using \( \Lambda(p, q, n) = \Lambda(q, p, n + q - p) \), cases (viii) and (xii) can be obtained. Note that case (v) is obtained by using the fact that
\[
T = T_1' \times T_2', \quad T_1' \sim \Lambda(p_1, q_4, n - p_1), \quad T_2' \sim \Lambda(q_2, q_3, n - r).
\]

Finally, by inverting the characteristic function of \( T_G \) we have the following theorem.
Theorem 3.1 Let \( T_G \) be the normalization of \( T \) defined by (1.5). Then, the null distribution of \( T_G \) can be expanded as
\[
P(T_G \leq x) = \Phi(x) - \phi(x) \left[ a_1(x) + a_2(x) \right] + o(p^{-1}),
\]
where \( G = H1 \sim H5 \), \( \Phi(x) \) and \( \phi(x) \) are the distribution and density function of the standard normal distribution, respectively and \( a_j(x) \)'s are defined by
\[
a_1(x) = \kappa^{(1)} + \kappa^{(3)} h_2(x),\quad a_2(x) = \kappa^{(2)} h_1(x) + \kappa^{(4)} h_3(x) + \kappa^{(6)} h_5(x).
\]

Here, \( h_j(x) \) is the \( j \)th Hermite polynomial; in particular, \( h_1(x) = x \), \( h_2(x) = x^2 - 1 \), \( h_3(x) = x^3 - x \), \( h_4(x) = x^4 - 6x^2 + 3 \), \( h_5(x) = x^5 - 10x^3 + 15x \), and \( \kappa^{(a)} \)'s are given by (3.3).

Using the coefficients \( a_j(x) \)'s of the asymptotic expansion (3.4), we can obtain the Cornish-Fisher expansion. Let \( x \) and \( \tilde{t}(x) \) denote the percentage point of the limiting distribution of \( T_G \) and the corresponding percentage point of \( T \), respectively, that is
\[
P(T_G \leq \tilde{t}(x)) = \Phi(x), \quad G = H1 \sim H5.
\]
Then from (3.4), \( \tilde{t}_G(x) \) can be expanded as
\[
\tilde{t}(x) = x + a_1(x) + \left\{ a_1(x)a'_1(x) - \frac{1}{2} xa_1(x)^2 + a_2(x) \right\} + o(p^{-1})
= \tilde{t}_G(x) + o(p^{-1})
\]

4 Simulation results

In this section we compare our high-dimensional approximations (denoted as H) with the classical approximations (denoted as C) based on the asymptotic distribution under a large sample framework such that \( p_1 \) and \( p_2 \) are fixed and \( n \) tends to infinity. The numerical accuracy is studied for the upper percentage points and the actual test size.

4.1 Null distributions of tests for dimensionality

It is well-known that under the large sample framework, the three statistics
\[
-n \log \prod_{j=k+1}^{p_1} (1 - r_j^2), \quad n \sum_{j=k+1}^{p_1} \frac{r_j^2}{1 - r_j^2}, \quad n \sum_{j=k+1}^{p_1} r_j^2.
\]
are asymptotically distributed as the $\chi^2$-distribution with $(p_1 - k)(p_2 - k)$ degrees of freedom (e.g., see Siotani et al. (1985)). Under the high-dimensional framework the three statistics $T_G/\sigma_G$ are distributed asymptotically (see Theorem 2.2) as $N(0, 1)$, where $\sigma_G = \sqrt{2(p_1 - k)(1 + p_2/m)}$, $G = LR, LH, BNP$. To facilitate, let $t_C = n^{-1}\chi^2_{(p_1-k)(p_2-k), \alpha}$.

$$t_{LR-H} = (p_1 - k) \log \left( 1 + \frac{p_2}{m} \right) + p_2^{-1/2} \left( 1 + \frac{m}{p_2} \right)^{-1} z_\alpha,$$

$$t_{LH-H} = \frac{p_2}{m} \left[ (p_1 - k) + \sigma_{LH} \times p_2^{-1/2} z_\alpha \right],$$

$$t_{BNP-H} = \left( 1 + \frac{m}{p_2} \right)^{-1} \left\{ (p_1 - k) + \sigma_{BNP} \times p_2^{-1/2} \left( 1 + \frac{m}{p_2} \right)^{-1} z_\alpha \right\},$$

where $\chi^2_{(p_1-k)(p_2-k), \alpha}$ and $z_\alpha$ are $100(1 - \alpha)\%$ point of the $\chi^2$-distribution with $(p_1 - k)(p_2 - k)$ degrees of freedom and the standard normal distribution, respectively.

The values of $p_1, n, p_2$ and $P$ were chosen as follows:

$$(p_1 + p_2, n): (10, 50), (20, 50), (30, 50), (40, 50), (10, 100), (15, 100), (20, 100), (50, 100), (70, 100), (90, 100)(10, 100), (30, 200), (50, 200), (80, 200), (100, 200)(120, 200), (150, 200), (170, 200), (190, 200), (p_1, P) : (3, \text{diag}(0.9, 0.6, 0.0)), (p_1, P) : (4, \text{diag}(0.9, 0.6, 0.0, 0.0)),$$

Table 1 shows the estimated upper 5% points based on a Monte Carlo simulation, the approximated critical points using our method, $t_{LR-H}, t_{LH-H}, t_{BNP-H}$, and the classical approximations $t_C$. Table 2 shows the corresponding actual test sizes. We are interested in the behavior when the dimension is large and close to the sample size.

From Tables 1 and 2, the chi-square type approximation $t_C, \alpha_{LR-C}, \alpha_{LH-C}, \alpha_{BNP-C}$ performs well when $p$ is less than 20. However, the chi-square type approximation is poor when $p$ is greater than 20. When $p$ is large, $\alpha_{LR-C}$ and $\alpha_{LH-C}$ are close to 1 and $\alpha_{BNP-C}$ is close to 0. The normal type approximation $\alpha_{LR-C}, \alpha_{LH-C}$ and $\alpha_{BNP-C}$ performs well when the dimension $p$ is close to half of $N$. When $k = 2$, $\alpha_{LH-C}$ performs well when the dimension $p$ close to $N$. 

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Table 1: Upper 5% point

\( p_1 = 3, \ k = 2, \ \mathcal{P} = \text{diag}(0.9, 0.6, 0.0) \)

<table>
<thead>
<tr>
<th></th>
<th>( p )</th>
<th>( p_2 )</th>
<th>( \text{Simu}_{LR} )</th>
<th>( t_{LR-H} )</th>
<th>( \text{Simu}_{LH} )</th>
<th>( t_{LH-H} )</th>
<th>( \text{Simu}_{BNP} )</th>
<th>( t_{BNP-H} )</th>
<th>( t_C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>10</td>
<td>7</td>
<td>0.24** 0.29*</td>
<td>0.28** 0.32*</td>
<td>0.23** 0.28*</td>
<td>0.23</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>20</td>
<td>17</td>
<td>0.60** 0.65*</td>
<td>0.83 0.88*</td>
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<tr>
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<td>30</td>
<td>27</td>
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<td>1.83 1.91*</td>
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<td>4.67 4.65*</td>
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<tr>
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<td>0.12** 0.15*</td>
<td>0.09** 0.10*</td>
<td>0.11</td>
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<td>0.43 0.48*</td>
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<tr>
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<td>4.10 4.84</td>
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<td>67</td>
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<td>2.82 2.86*</td>
<td>8.82 11.97</td>
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* Denotes \(|\text{Simu}_G - t_{G,H}| \leq 10^{-1}\), where \(G = LR, LH, BNP\).
** Denotes \(|\text{Simu}_G - t_C| \leq 10^{-1}\), where \(G = LR, LH, BNP\).
\( p_1 = 4, \quad k = 2, \quad \mathcal{P} = \text{diag}(0.9, 0.6, 0.0, 0.0) \)

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
N & p & p_2 & \text{Simu}_{LR} & t_{LR-H} & \text{Simu}_{LH} & t_{LH-H} & \text{Simu}_{BNP} & t_{BNP-H} & t_C \\
\hline
50 & 10 & 7 & 0.24** & 0.29* & 0.28** & 0.32* & 0.23** & 0.28* & 0.23 \\
50 & 20 & 17 & 0.60** & 0.65* & 0.83 & 0.88* & 1.30 & 1.69 & 0.51 \\
50 & 30 & 27 & 1.04 & 1.11* & 1.83 & 1.91* & 3.63 & 5.12 & 0.77 \\
50 & 40 & 37 & 1.71 & 1.79* & 4.67 & 4.65* & 7.69 & 11.44 & 1.02 \\
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100 & 15 & 12 & 0.20** & 0.22* & 0.22** & 0.24* & 0.22** & 0.25* & 0.18 \\
100 & 20 & 17 & 0.28** & 0.29* & 0.32** & 0.33* & 0.43 & 0.48* & 0.25 \\
100 & 50 & 47 & 0.82 & 0.84* & 1.28 & 1.28* & 4.10 & 4.84 & 0.62 \\
100 & 70 & 67 & 1.33 & 1.38* & 2.82 & 2.86* & 8.82 & 11.97 & 0.86 \\
100 & 90 & 87 & 2.35 & 2.41* & 9.84 & 9.73* & 15.61 & 23.85 & 1.09 \\
200 & 10 & 7 & 0.06** & 0.07* & 0.06** & 0.07* & 0.04** & 0.04* & 0.06 \\
200 & 30 & 27 & 0.20** & 0.21* & 0.22** & 0.23* & 0.34 & 0.37* & 0.19 \\
200 & 50 & 47 & 0.35** & 0.36* & 0.42 & 0.43* & 1.12 & 1.23 & 0.31 \\
200 & 80 & 77 & 0.61 & 0.61* & 0.83 & 0.83* & 3.65 & 3.99 & 0.48 \\
200 & 100 & 97 & 0.80 & 0.81* & 1.23 & 1.22* & 6.34 & 7.13 & 0.60 \\
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200 & 150 & 147 & 1.51 & 1.53* & 3.56 & 3.54* & 16.97 & 21.03 & 0.87 \\
200 & 170 & 167 & 2.01 & 2.03* & 6.52 & 6.49* & 22.15 & 29.55 & 0.98 \\
200 & 190 & 187 & 2.99 & 3.03* & 19.59 & 19.28* & 26.27 & 40.06 & 1.09 \\
\hline
\end{array}
\]

* Denotes \(|\text{Simu}_G - t_{G.H}| \leq 10^{-1}\), where \(G = LR, LH, BNP\).

** Denotes \(|\text{Simu}_G - t_C| \leq 10^{-1}\), where \(G = LR, LH, BNP\).
Table 2: The corresponding actual test size.

\( p_1 = 3, \ k = 2, \ P = \text{diag}(0.9, 0.6, 0.0) \)

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* Denotes the approximation in \([0.040, 0.060]\)
\( p_1 = 4, \quad k = 2, \quad \mathcal{P} = \text{diag}(0.9, 0.6, 0.0, 0.0) \)

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* Denotes the approximation in [0.040, 0.060]
4.2 Additional information

The Cornish-Fisher expansion of the large sample approximation (3.2) is well known. The expansion is obtained by using an alternative form of (3.2) written as

\[ P(-m \log T \leq x) = G_f(x) + g_f(x) \frac{1}{m^2} \tilde{p}_1(x) + O(m^{-2}), \quad (4.1) \]

where \( g_f(x) \) is the density function of the chi-square variable with \( f \) degrees of freedom and coefficient \( \tilde{p}_1(x) \) is defined by

\[ \tilde{p}_1(x) = \beta \sum_{i=1}^{2} \frac{2x^i}{\prod_{j=1}^{i} f + 2(j - 1)}. \]

Similarly let

\[ P(-m \log T \leq \tilde{t}(x)) = G_f(x). \]

Then the Cornish-Fisher expansion can be written in the same way as in (3.6), that is,

\[ \tilde{t}(x) = x + \frac{1}{m^2} \tilde{p}_1(x) + O(m^{-2}) = \tilde{t}_C + O(m^{-2}). \quad (4.2) \]

For comparison, let \( t_C = m^{-1} \times \tilde{t}_C, t_G = m_1 + m_2 + d \times \tilde{t}_G(x) \) where \( G = H1 \sim H5 \).

Table 3 gives the upper 5% of the points based on a Monte Carlo simulation (Simu), and the approximated critical points of our method, \( t_{H1} \sim t_{H5} \), and the classical approximations \( t_{A0} \). Table 4 gives the corresponding actual test sizes. We are interested in the behavior when the dimension is large and close to the sample size.

From Tables 3 and 4, the chi-square type approximation \( t_C, \alpha_C \) performs well when \( p \) is less than 8. In contrast, the chi-square type approximations are poor when the smallest of \( q_1, q_2, q_3, \) and \( q_4 \) is large. When \( p \) is large, the normal type approximation \( t_{H1} \sim t_{H5}, \alpha_{H1} \sim \alpha_{H5} \) performs better than the chi-square type approximation. Furthermore, when the sample size is much larger than the dimension, the performance of the normal type approximation is similar to that of a large sample approximation. In particular, the approximation \( t_{H5}, \alpha_{H5} \) is the best of these approximation for all cases.
Table 3: Upper 5% point

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* Denotes the approximation in [0.040, 0.060]
Acknowledgments

The author thanks Prof. T. Sugiyama and Prof. Y. Fujikoshi of Chuo University for many useful comments and suggestions.

References


