On Jarque-Bera tests for assessing multivariate normality

Kazuyuki Koizumi[1,2], Naoya Okamoto[3] and Takashi Seo[3*]

[1] *Department of Mathematics, Graduate School of Science, Tokyo University of Science*

[2] *Research Fellow of Japan Society for the Promotion of Science*

[3] *Department of Mathematical Information Science, Faculty of Science, Tokyo University of Science*

Abstract

In this paper, we consider some tests for the multivariate normality based on the sample measures of multivariate skewness and kurtosis. Sample measures of multivariate skewness and kurtosis were defined by Mardia (1970), Srivastava (1984) and so on. We derive new multivariate normality tests by using Mardia's and Srivastava's moments. For univariate sample case, Jarque and Bera (1987) proposed bivariate test using skewness and kurtosis. We propose some new test statistics for assessing multivariate normality which are natural extensions of Jarque-Bera test. Finally, the numerical results by Monte Carlo simulation are shown in order to evaluate accuracy of expectations, variances, frequency distributions and upper percentage points for new test statistics.

Key Words and Phrases: Jarque-Bera test; multivariate skewness; multivariate kurtosis; normality test.

1 Introduction

In statistical analysis, the test for normality is an important problem. This problem has been considered by many authors. Shapiro and Wilk's (1965) W-statistic is well known as the univariate normality test. For the multivariate case, some tests based on W-statistic were proposed by Malkovich and Afifi (1973), Royston

*corresponding author. *E-mail addresses:* koizu702@yahoo.co.jp (K. Koizumi), okamoto@ed.kagu.tus.ac.jp (N. Okamoto), seo@rs.kagu.tus.ac.jp (T. Seo)

(1983), Srivastava and Hui (1987) and so on. Mardia (1970) and Srivastava (1984) gave different definitions of the multivariate measures of skewness and kurtosis, and discussed the test statistics using these measures for assessing multivariate normality, respectively. Mardia (1974) derived exact expectations and variances of multivariate sample skewness and kurtosis, and discussed their asymptotic distributions. Srivastava's (1984) sample measures of multivariate skewness and kurtosis have been discussed by many authors. Seo and Ariga (2006) derived a normalizing transformation of test statistic using Srivastava's kurtosis by the asymptotic expansion. Okamoto and Seo (2008) derived the exact expectation and variance of Srivastava's skewness and improved χ^2 statistic defined by Srivastava (1984) for assessing multivariate normality.

In this paper, our purpose is to propose new Jarque-Bera tests for assessing multivariate normality by using Mardia's and Srivastava's measures, respectively. For univariate sample case, Jarque and Bera (1987) proposed an omnibus test using skewness and kurtosis. Improved Jarque-Bera tests have been discussed by many authors. (see, e.g. Urzúa (1996)) But Jarque-Bera test for multivariate sample case has not been considered by any authors. In Section 2 we describe some properties of Mardia's and Srivastava's multivariate skewness and kurtosis. In Section 3 we propose new tests for assessing multivariate normality. New test statistics are asymptotically distributed as χ^2-distribution under the normal population. These tests are extensions of Jarque-Bera test. In Section 4 we investigate accuracy of expectations, variances, frequency distributions and upper percentage points for multivariate Jarque-Bera tests by Monte Carlo simulation.

2 Multivariate measures of skewness and kurtosis

2.1 Mardia's (1970) skewness and kurtosis

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)'$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_p)'$ be random p-vectors distributed identically and independently with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$ and covariance matrix Σ, $\Sigma > 0$. Mardia (1970) has defined the population measures of multivariate skewness and kurtosis as

$$\beta_{M,1} = \mathrm{E}\left[\{(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\}^3\right],$$
$$\beta_{M,2} = \mathrm{E}\left[\{(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\}^2\right],$$

respectively. When $p = 1$, $\beta_{M,1}$ and $\beta_{M,2}$ are reduced to the ordinary univariate measures. It is obvious that for any symmetric distribution about $\boldsymbol{\mu}$, $\beta_{M,1} = 0$. Under the normal distribution $N_p(\boldsymbol{\mu}, \Sigma)$,

$$\beta_{M,1} = 0, \quad \beta_{M,2} = p(p+2).$$

To give the sample counterparts of $\beta_{M,1}$ and $\beta_{M,2}$, let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be samples of size N from a multivariate p-dimensional population. And let $\overline{\boldsymbol{x}}$ and S be the sample mean vector and the sample covariance matrix as follows:

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_j,$$
$$S = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})',$$

respectively.

Then Mardia (1970) has defined the sample measures of skewness and kurtosis by

$$b_{M,1} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{(\boldsymbol{x}_i - \overline{\boldsymbol{x}})'S^{-1}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})\}^3,$$
$$b_{M,2} = \frac{1}{N}\sum_{i=1}^{N}\{(\boldsymbol{x}_i - \overline{\boldsymbol{x}})'S^{-1}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})\}^2,$$

respectively.

Mardia (1970, 1974) have given the following Lemma.

Lemma 1 *Mardia (1970, 1974) have given the exact expectation of $b_{M,1}$, and expectation and variance of $b_{M,2}$ when the population is $N_p(\boldsymbol{\mu}, \Sigma)$.*

$$\begin{aligned} E(b_{M,1}) &= \frac{p(p+2)}{(N+1)(N+3)}\{(N+1)(p+1)-6\}, \\ E(b_{M,2}) &= \frac{p(p+2)(N-1)}{N+1}, \\ Var(b_{M,2}) &= \frac{8p(p+2)(N-3)}{(N+1)^2(N+3)(N+5)}(N-p-1)(N-p+1), \end{aligned}$$

respectively.

Furthermore Mardia (1970) obtained asymptotic distributions of $b_{M,1}$ and $b_{M,2}$ and used them to test the multivariate normality.

Theorem 1 *Let $b_{M,1}$ and $b_{M,2}$ be the sample measures of multivariate skewness and kurtosis, respectively, on the basis of a random sample of size N drawn from $N_p(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$. Then*

$$z_{M,1} = \frac{N}{6} b_{M,1}$$

is asymptotically distributed as χ^2-distribution with $f \equiv p(p+1)(p+2)/6$ degrees of freedom, and

$$z_{M,2} = \sqrt{\frac{N}{8p(p+2)}}(b_{M,2} - p(p+1))$$

is asymptotically distributed as $N(0,1)$.

By making reference to moments of $b_{M,1}$ and $b_{M,2}$, Mardia (1974) considered the following approximate test statistics as competitors of $z_{M,1}$ and $z_{M,2}$:

$$z^*_{M,1} = \frac{N}{6} b_{M,1} \frac{(p+1)(N+1)(N+3)}{N\{(N+1)(p+1)-6\}} \sim \chi^2_f \tag{2.1}$$

asymptotically, and

$$z^*_{M,2} = \frac{\sqrt{(N+3)(N+5)}\{(N+1)b_{M,2} - p(p+2)(N-1)\}}{\sqrt{8p(p+2)(N-3)(N-p-1)(N-p+1)}} \sim N(0,1) \tag{2.2}$$

asymptotically. It is noted that $z^*_{M,1}$ is formed so that $\mathrm{E}(z^*_{M,1}) = f$.

2.2 Srivastava's (1984) skewness and kurtosis

Let $\Gamma = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p)$ be an orthogonal matrix such that $\Gamma'\Sigma\Gamma = D_\lambda$, where $D_\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Note that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of Σ. Then, Srivastava (1984) defined the population measures of multivariate skewness and kurtosis by using the principle component as follows:

$$\beta_{S,1} = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{\mathrm{E}\left[(v_i-\theta_i)^3\right]}{\lambda_i^{\frac{3}{2}}}\right\}^2,$$
$$\beta_{S,2} = \frac{1}{p}\sum_{i=1}^{p}\frac{\mathrm{E}\left[(v_i-\theta_i)^4\right]}{\lambda_i^2},$$

respectively, where $v_i = \boldsymbol{\gamma}_i'\boldsymbol{x}$ and $\theta_i = \boldsymbol{\gamma}_i'\boldsymbol{\mu}$ $(i = 1, 2, \ldots, p)$. We note that $\beta_{S,1} = 0$ and $\beta_{S,2} = 3$ under a multivariate normal population. Let $H = (\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_p)$ be an orthogonal matrix such that $H'SH = D_\omega$, where $D_\omega = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_p)$ and $\omega_1, \omega_2, \ldots, \omega_p$ are the eigenvalues of S. Then, Srivastava (1984) defined the sample measures of multivariate skewness and kurtosis as follows:

$$b_{S,1} = \frac{1}{N^2p}\sum_{i=1}^{p}\left\{\omega_i^{-\frac{3}{2}}\sum_{j=1}^{N}(v_{ij}-\overline{v}_i)^3\right\}^2,$$
$$b_{S,2} = \frac{1}{Np}\sum_{i=1}^{p}\omega_i^{-2}\sum_{j=1}^{N}(v_{ij}-\overline{v}_i)^4,$$

respectively, where $v_{ij} = \boldsymbol{h}_i'\boldsymbol{x}_j$, $\overline{v}_i = (1/N)\sum_{j=1}^{N} v_{ij}$.

Srivastava (1984) obtained the following Lemma:

Lemma 2 *For large N, Srivastava (1984) has given the expectations of $\sqrt{b_{S,1}}$ and $b_{S,1}$ and expectation and variance of $b_{S,2}$ when the population is $N_p(\boldsymbol{\mu}, \Sigma)$.*

$$E(\sqrt{b_{S,1}}) = 0, \;\; E(b_{S,1}) = \frac{6}{N},$$
$$E(b_{S,2}) = 3, \;\; Var(b_{S,2}) = \frac{24}{Np},$$

respectively.

By using Lemma 2, Srivastava (1984) derived the following theorem:

Theorem 2 *Let $b_{S,1}$ and $b_{S,2}$ be the sample measures of multivariate skewness and kurtosis by using the principle component, respectively, on the basis of a random sample of size N drawn from $N_p(\boldsymbol{\mu}, \Sigma)$. Then*

$$z_{S,1} = \frac{Np}{6} b_{S,1}$$

is asymptotically distributed as χ^2-distribution with p degrees of freedom, and

$$z_{S,2} = \sqrt{\frac{Np}{24}} (b_{S,2} - 3)$$

is asymptotically distributed as $N(0, 1)$.

Further Okamoto and Seo (2008) gave the expectation of multivariate sample skewness $b_{S,1}$ without using Taylor expansion. By using the same way as Okamoto and Seo (2008), we can obtain the expectation and variance of multivariate sample kurtosis $b_{S,2}$. Hence we can get the following Lemma:

Lemma 3 *For large N, we give the expectation of $b_{S,1}$ and expectation and variance of $b_{S,2}$ when the population is $N_p(\boldsymbol{\mu}, \Sigma)$.*

$$\begin{aligned} E(b_{S,1}) &= \frac{6(N-2)}{(N+1)(N+3)}, \\ E(b_{S,2}) &= \frac{3(N-1)}{N+1}, \\ Var(b_{S,2}) &= \frac{24}{p} \frac{N(N-2)(N-3)}{(N+1)^2(N+3)(N+5)}, \end{aligned}$$

respectively.

By making reference to moments of $b_{S,1}$ and $b_{S,2}$, we consider following approximate test statistics as competitors of $z_{S,1}$ and $z_{S,2}$:

$$z^*_{S,1} = \frac{(N+1)(N+3)}{6(N-2)} p b_{S,1} \sim \chi^2_p \tag{2.3}$$

asymptotically, and

$$z^*_{S,2} = \frac{\sqrt{p(N+3)(N+5)}\{(N+1)b_{S,2} - 3(N-1)\}}{\sqrt{24N(N-2)(N-3)}} \sim N(0, 1) \tag{2.4}$$

asymptotically.

3 Multivariate Jarque-Bera tests

In this section, we consider new tests for multivariate normality when the population is $N_p(\boldsymbol{\mu}, \Sigma)$. From Theorem 1, we propose a new test statistic using Mardia's measures as follows:

$$MJB_M = N\left\{\frac{b_{M,1}}{6} + \frac{(b_{M,2} - p(p+2))^2}{8p(p+2)}\right\}.$$

MJB_M statistic is asymptotically distributed as χ^2_{f+1}-distribution.

From Theorem 2, we propose a new test statistic using Srivastava's measures as follows:

$$MJB_S = Np\left\{\frac{b_{S,1}}{6} + \frac{(b_{S,2} - 3)^2}{24}\right\}.$$

MJB_S statistic is asymptotically distributed as χ^2_{p+1}-distribution.

Further, by using (2.1) and (2.2), a modified MJB_M is given by

$$MJB^*_M = z^*_{M,1} + {z^*}^2_{M,2}.$$

In the same as MJB_M, this statistic MJB^*_M is distributed as χ^2_{f+1}-distribution asymptotically.

Also, by using (2.3) and (2.4), a modified MJB_S is given by

$$MJB^*_S = z^*_{S,1} + {z^*}^2_{S,2}.$$

In the same as MJB_S, this statistic MJB^*_S is distributed as χ^2_{p+1}-distribution asymptotically.

4 Simulation studies

Accuracy of expectations, variances, frequency distributions and upper percentage points of multivariate Jarque-Bera tests MJB_M, MJB_S, MJB^*_M and MJB^*_S is evaluated by Monte Carlo simulation study. Simulation parameters are as follows: $p = 3,\ 10,\ 20$, $N = 20,\ 50,\ 100,\ 200,\ 400,\ 800$. As a numerical experiment, we carry out 100,000 and 1,000,000 replications for the case of Mardia's measures and Srivastava's measures, respectively.

From Tables 1–2 and Figures 1–6, expectations of approximate χ^2 statistics MJB^*_M and MJB^*_S are invariant for any sample sizes N. That is, MJB^*_M and MJB^*_S are almost close to the exact expectations even for small N. However, accuracy of expectations of MJB_M and MJB_S is not good especially for small N. We note that expectations of MJB_M and MJB_S converge on those of χ^2-distribution for large N. Hence it may be noticed that both MJB^*_M and MJB^*_S are improvements of MJB_M and MJB_S, respectively.

On the other hand, from Tables 1–2 and Figures 7–12, variances of MJB^*_M and MJB^*_S are larger than those of MJB_M and MJB_S. To investigate this cause, we show frequency distributions of multivariate Jarque-Bera tests proposed in this paper. These results are in Figures 13–24. In figures, $f_X(x)$ represents probability density function (p.d.f.) of χ^2-distribution. It may be noticed from these figures that frequencies of MJB^*_M and MJB^*_S are closer to p.d.f. of χ^2-distribution than those of MJB_M and MJB_S, respectively. This tendency appears well when sample size N is small. But the coming off values of MJB^*_M and MJB^*_S are more than those of MJB_M and MJB_S. Therefore there is a tendency for variance to become large.

Finally, in Table 3 and Figures 25–27, we give upper percentage points of MJB_M and MJB^*_M by using Mardia's skewness and kurtosis. MJB_M tends to be conservative. Also MJB^*_M is closer to the upper percentage points of χ^2_{f+1}-distribution even when the sample size N is small. In Table 4 and Figures 28–30, we give upper percentage points of MJB_S and MJB^*_S by using Srivastava's skewness and kurtosis. We note that the tendency is similar to the case using Mardia's moments.

5 Concluding remarks

For univariate sample case, Jarque-Bera test is well known as a simple procedure on practical use. In this paper, we proposed four new test statistics for assessing multivariate normality. MJB_M and MJB_S are natural forms of extensions in the case of multivariate normality tests. But approximations of expectations, frequency distributions and upper percentage points of MJB_M and MJB_S are not good when the sample size N is small. Also we proposed improved multivariate normality test statistics MJB_M^* and MJB_S^*. Hence we improved expectations and upper percentage points of MJB_M and MJB_S. But variances of MJB_M and MJB_S are not improved. This problem still remains. It is an future problem. In order to solve this problem, it may be noted that we have to consider covariance of $z_{M,1}^*$ and ${z^*}^2_{M,2}$ and that of $z_{S,1}^*$ and ${z^*}^2_{S,2}$. We recommend to use MJB_M^* and MJB_S^* from the aspect of aproximate accuracy of upper percentage points of test statistics especially for small N.

References

[1] Jarque, C. M. and Bera, A. K. (1987). A test for normality of observations and regression residuals. *International Statistical Review*, **55**, 163–172.

[2] Malkovich, J. R. and Afifi, A. A. (1973). On tests for multivariate normality. *Journal of the American Statistical Association*, **68**, 176–179.

[3] Mardia, K. V. (1970). Measures of multivariate skewness and kurtosis with applications. *Biometrika*, **57**, 519–530.

[4] Mardia, K. V. (1974). Applications of some measures of multivariate skewness and kurtosis in testing normality and robustness studies. *Sankhyá B*, **36**, 115–128.

[5] Okamoto, N. and Seo, T. (2008). On the Distribution of Multivariate Sample Skewness for Assessing Multivariate Normality. *Technical Report No. 08-01, Statistical Research Group, Hiroshima University.*

[6] Royston, J. P. (1983). Some techniques for assessing multivariate normality based on the Shapiro-Wilk W. *Applied Statistics*, **32**, 121–133.

[7] Seo, T. and Ariga, M. (2006). On the Distribution of Kurtosis Test for Multivariate Normality. *Technical Report No. 06-04, Statistical Research Group, Hiroshima University.*

[8] Shapiro, S. S. and Wilk, M. B. (1965). An analysis of variance test for normality (complete samples). *Biometrika*, **52**, 591–611.

[9] Srivastava, M. S. (1984). A measure of skewness and kurtosis and a graphical method for assessing multivariate normality. *Statistics &Probability Letters*, **2**, 263–267.

[10] Srivastava, M. S. and Hui, T. K. (1987). On assessing multivariate normality based on Shapiro-Wilk W statistic. *Statistics &Probability Letters*, **5**, 15–18.

[11] Urzúa, C. M. (1996). On the correct use of omnibus tests for normality. *Economics Letters*, **90**, 304–309.

Table 1: Expectations and variances of MJB_M and MJB_M^*.

p	N	$E(MJB_M)$	$E(MJB_M^*)$	$f+1$	$Var(MJB_M)$	$Var(MJB_M^*)$	$2(f+1)$
3	20	8.79	10.98	11	14.40	35.78	22
	50	10.00	11.01	11	23.27	36.01	22
	100	10.47	11.01	11	24.95	31.67	22
	200	10.73	11.00	11	24.27	27.55	22
	400	10.83	10.96	11	23.18	24.77	22
	800	10.91	10.98	11	22.51	23.27	22
10	20	189.20	221.02	221	176.65	304.25	442
	50	206.90	220.91	221	427.59	558.73	442
	100	213.73	220.99	221	482.40	562.74	442
	200	217.27	220.96	221	475.28	518.03	442
	400	219.12	220.98	221	468.08	490.30	442
	800	219.93	220.87	221	454.81	465.84	442
20	50	1449.36	1541.18	1541	2447.04	3031.77	3082
	100	1493.80	1541.17	1541	3467.95	3934.50	3082
	200	1516.99	1541.06	1541	3549.40	3818.71	3082
	400	1529.23	1541.37	1541	3406.70	3548.59	3082
	800	1534.60	1540.69	1541	3227.97	3298.83	3082

Table 2: Expectations and variances of MJB_S and MJB_S^*.

p	N	$E(MJB_S)$	$E(MJB_S^*)$	$p+1$	$Var(MJB_S)$	$Var(MJB_S^*)$	$2(p+1)$
3	20	2.93	4.02	4	5.46	18.25	8
	50	3.50	4.01	4	9.06	15.67	8
	100	3.73	4.00	4	9.65	13.03	8
	200	3.86	4.00	4	9.24	10.87	8
	400	3.93	4.00	4	8.74	9.52	8
	800	3.96	4.00	4	8.37	8.74	8
10	20	8.66	11.08	11	12.14	36.89	22
	50	9.91	11.00	11	19.28	31.65	22
	100	10.43	11.00	11	21.42	27.98	22
	200	10.71	11.01	11	22.08	25.44	22
	400	10.86	11.01	11	22.18	23.86	22
	800	10.92	10.99	11	22.06	22.90	22
20	50	19.09	21.01	21	34.82	56.24	42
	100	20.01	21.01	21	39.31	50.83	42
	200	20.49	21.01	21	40.99	46.92	42
	400	20.75	21.01	21	41.72	44.73	42
	800	20.87	21.00	21	41.63	43.13	42

Table 3: The upper 5 percentage points of MJB_M and MJB_M^*.

p	N	MJB_M	MJB_M^*	$\chi^2_{f+1}(0.05)$
3	20	15.80	22.07	19.68
	50	18.67	21.76	19.68
	100	19.43	21.17	19.68
	200	19.71	20.61	19.68
	400	19.64	20.09	19.68
	800	19.63	19.85	19.68
10	20	212.42	252.35	256.68
	50	243.03	262.64	256.68
	100	252.04	262.59	256.68
	200	254.98	260.40	256.68
	400	256.16	258.97	256.68
	800	256.33	257.74	256.68
20	50	1535.31	1637.63	1633.44
	100	1594.99	1649.32	1633.44
	200	1618.15	1646.28	1633.44
	400	1627.42	1641.72	1633.44
	800	1629.50	1636.56	1633.44

Table 4: The upper 5 percentage points of MJB_S and MJB_S^*.

p	N	MJB_S	MJB_S^*	$\chi^2_{p+1}(0.05)$
3	20	6.81	11.24	9.49
	50	8.42	10.58	9.49
	100	8.98	10.16	9.49
	200	9.28	9.90	9.49
	400	9.39	9.71	9.49
	800	9.45	9.60	9.49
10	20	15.03	22.50	19.68
	50	17.86	21.37	19.68
	100	18.87	20.76	19.68
	200	19.34	20.33	19.68
	400	19.54	20.05	19.68
	800	19.60	19.86	19.68
20	50	29.79	34.84	32.67
	100	31.35	34.06	32.67
	200	32.08	33.48	32.67
	400	32.41	33.13	32.67
	800	32.49	32.86	32.67

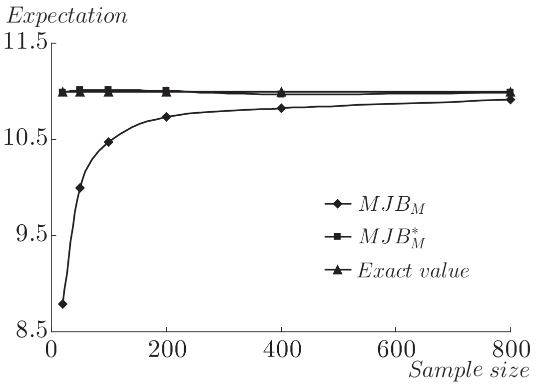

Figure 1: Expectations of MJB_M and MJB_M^* for $p = 3$.

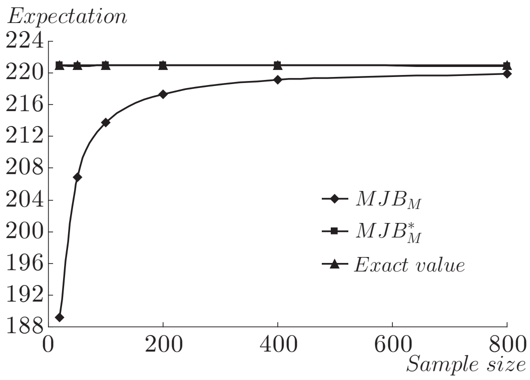

Figure 2: Expectations of MJB_M and MJB_M^* for $p = 10$.

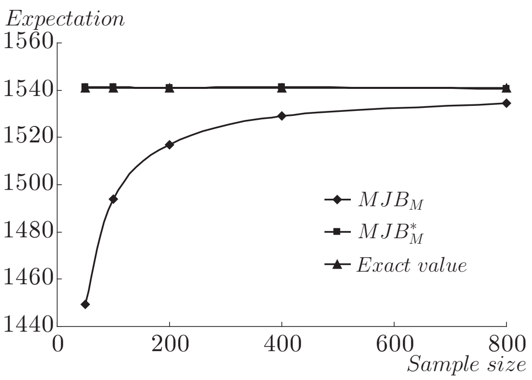

Figure 3: Expectations of MJB_M and MJB_M^* for $p = 20$.

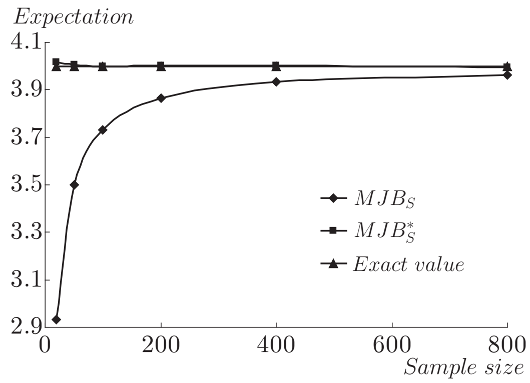

Figure 4: Expectations of MJB_S and MJB_S^* for $p = 3$.

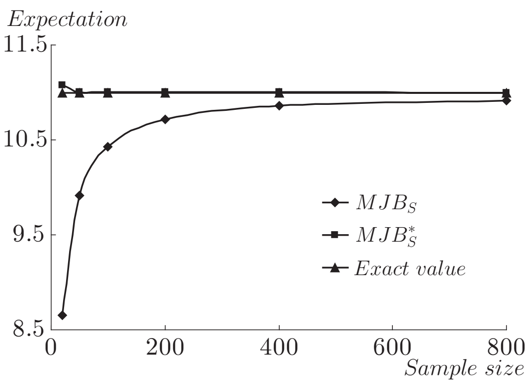

Figure 5: Expectations of MJB_S and MJB_S^* for $p = 10$.

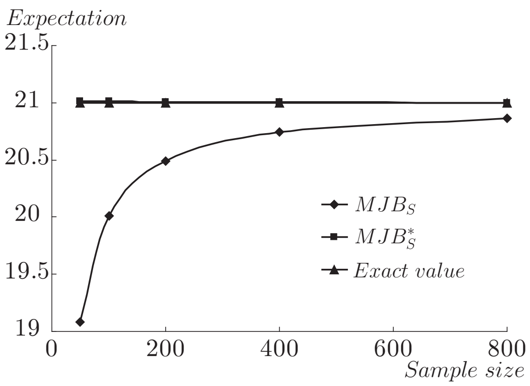

Figure 6: Expectations of MJB_S and MJB_S^* for $p = 20$.

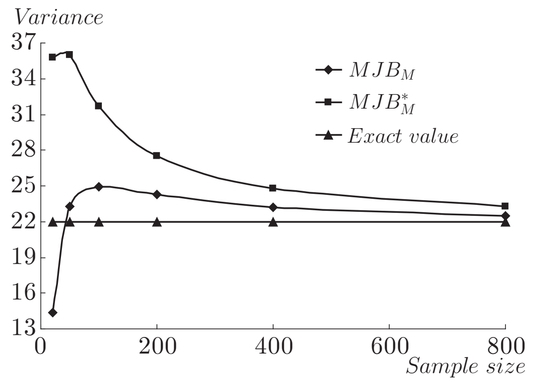

Figure 7: Variances of MJB_M and MJB_M^* for $p = 3$.

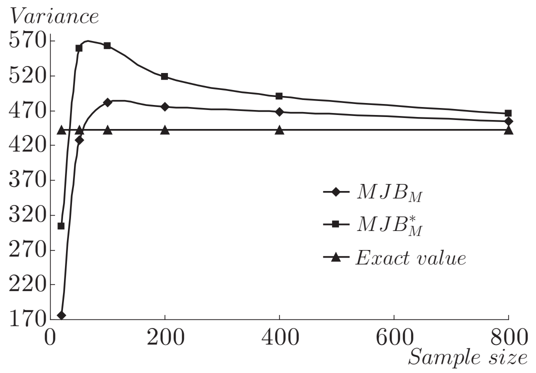

Figure 8: Variances of MJB_M and MJB_M^* for $p = 10$.

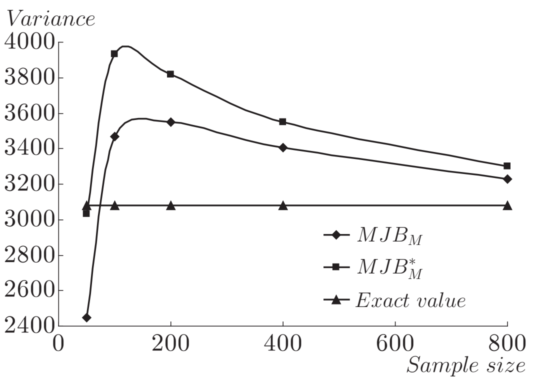

Figure 9: Variances of MJB_M and MJB_M^* for $p = 20$.

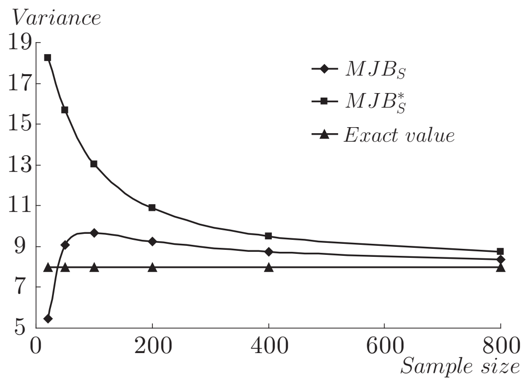

Figure 10: Variances of MJB_S and MJB_S^* for $p = 3$.

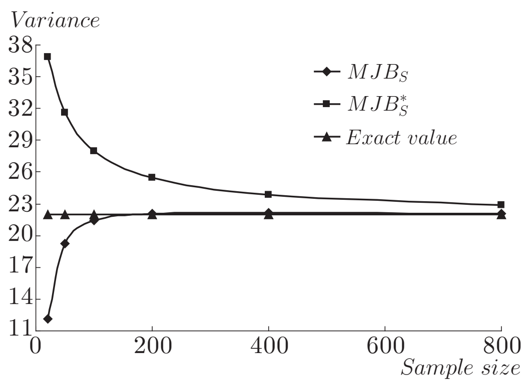

Figure 11: Variances of MJB_S and MJB_S^* for $p = 10$.

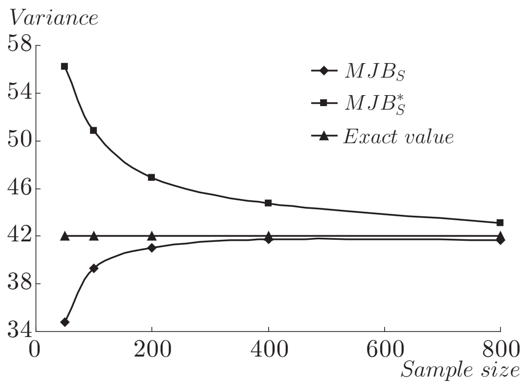

Figure 12: Variances of MJB_S and MJB_S^* for $p = 20$.

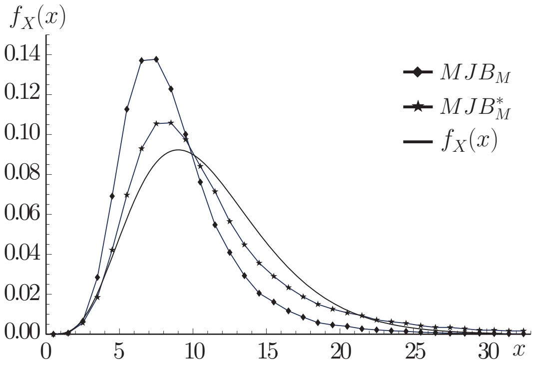

Figure 13: Frequencies of MJB_M and MJB_M^* for $p = 3$, $N = 20$.

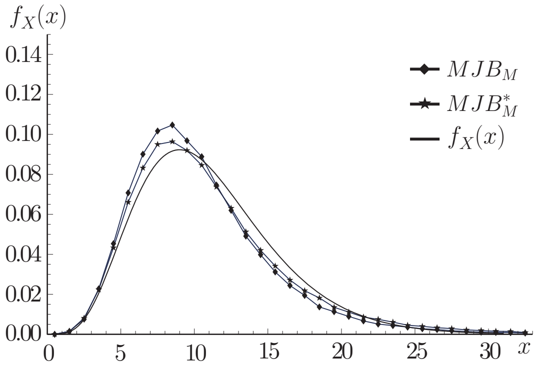

Figure 14: Frequencies of MJB_M and MJB_M^* for $p = 3$, $N = 100$.

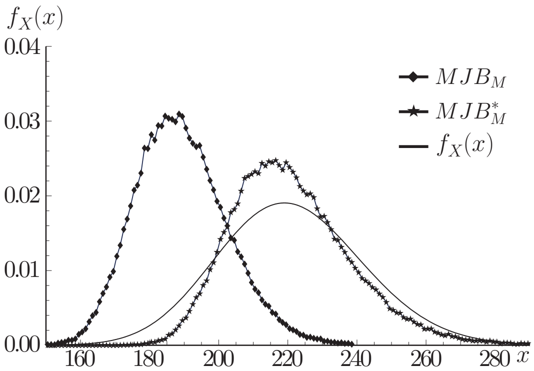

Figure 15: Frequencies of MJB_M and MJB_M^* for $p = 10$, $N = 20$.

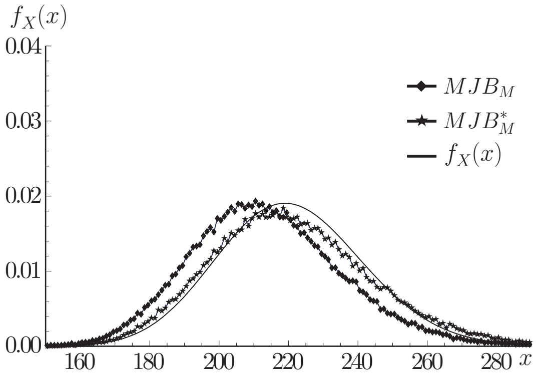

Figure 16: Frequencies of MJB_M and MJB_M^* for $p = 10$, $N = 100$.

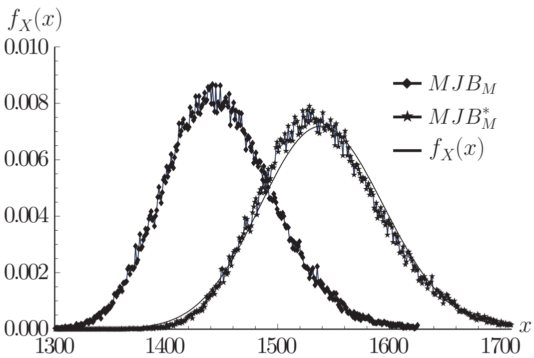

Figure 17: Frequencies of MJB_M and MJB_M^* for $p = 20$, $N = 50$.

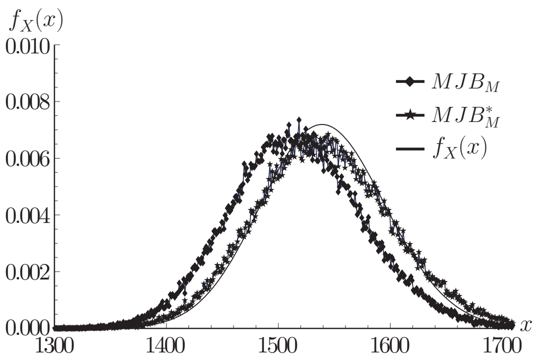

Figure 18: Frequencies of MJB_M and MJB_M^* for $p = 20$, $N = 200$.

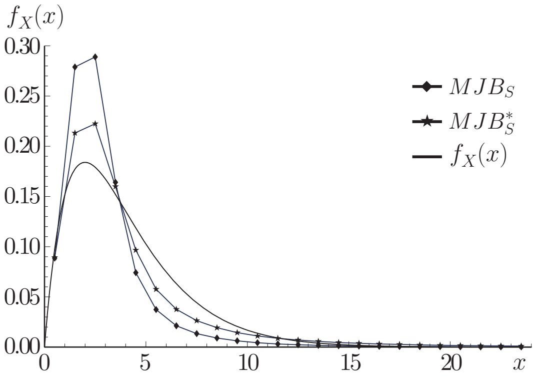

Figure 19: Frequencies of MJB_S and MJB_S^* for $p = 3$, $N = 20$.

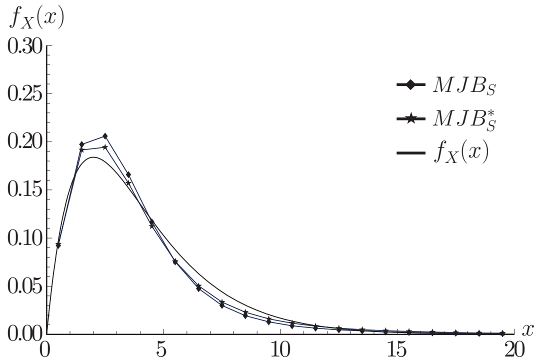

Figure 20: Frequencies of MJB_S and MJB_S^* for $p = 3$, $N = 100$.

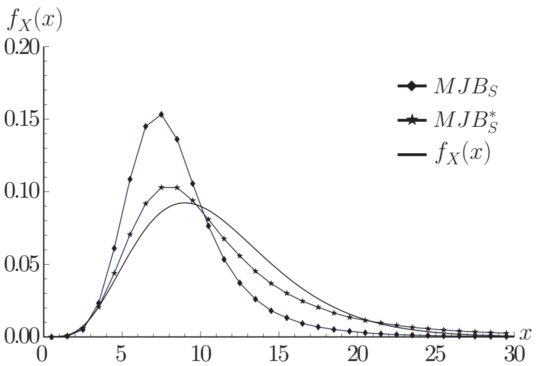

Figure 21: Frequencies of MJB_S and MJB_S^* for $p = 10$, $N = 20$.

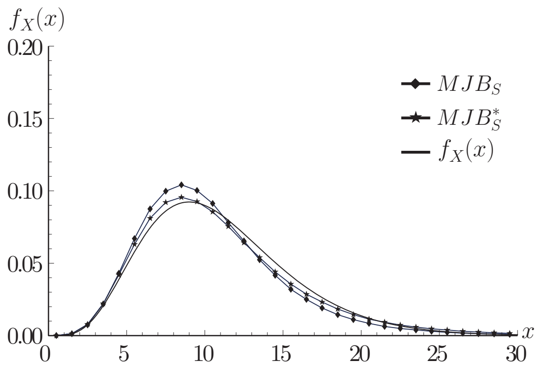

Figure 22: Frequencies of MJB_S and MJB_S^* for $p = 10$, $N = 100$.

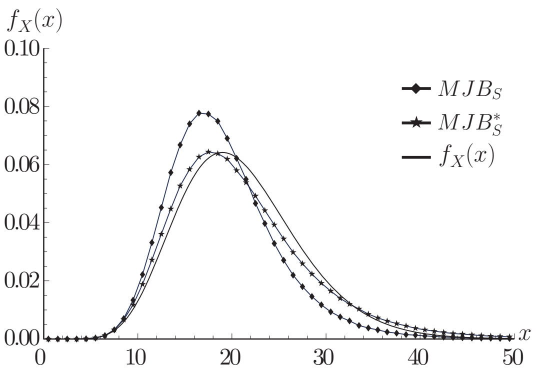

Figure 23: Frequencies of MJB_S and MJB_S^* for $p = 20$, $N = 50$.

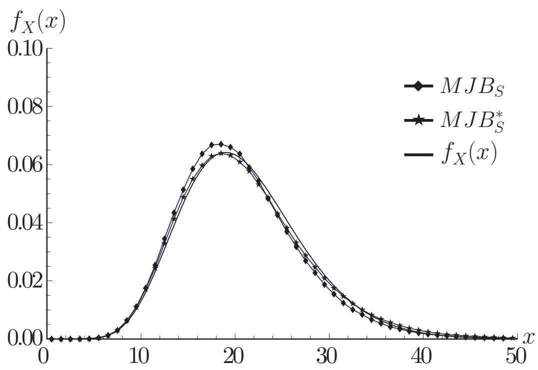

Figure 24: Frequencies of MJB_S and MJB_S^* for $p = 20$, $N = 200$.

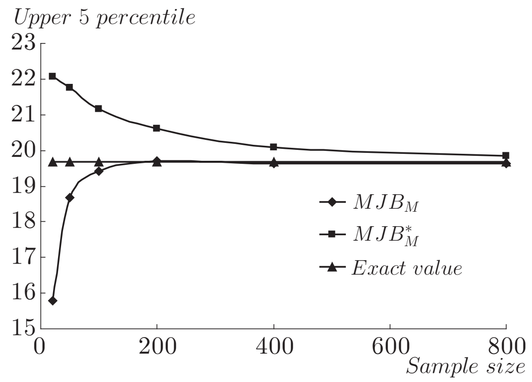

Figure 25: The upper percentiles of MJB_M and MJB_M^* for $p = 3$.

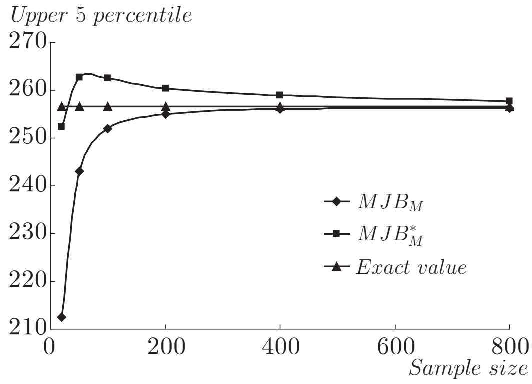

Figure 26: The upper percentiles of MJB_M and MJB_M^* for $p = 10$.

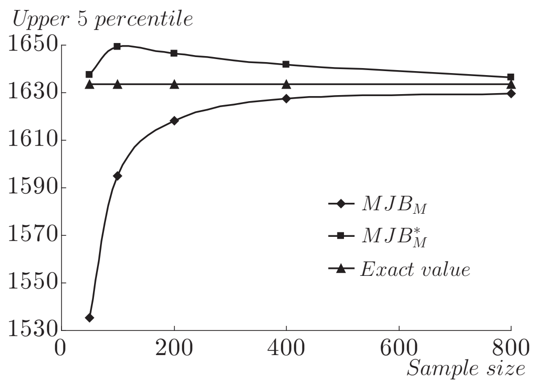

Figure 27: The upper percentiles of MJB_M and MJB_M^* for $p = 20$.

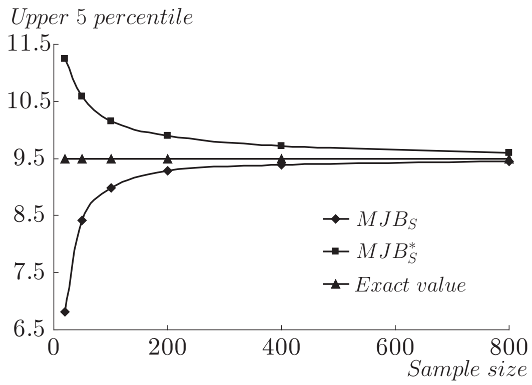

Figure 28: The upper percentiles of MJB_S and MJB_S^* for $p = 3$.

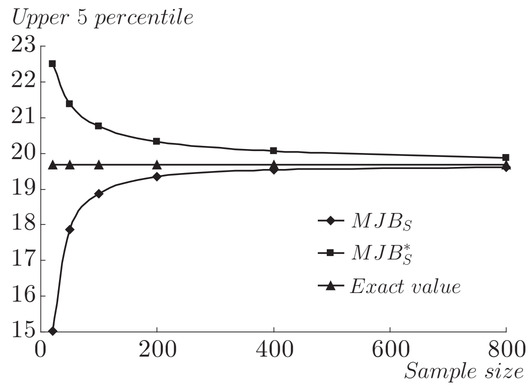

Figure 29: The upper percentiles of MJB_S and MJB_S^* for $p = 10$.

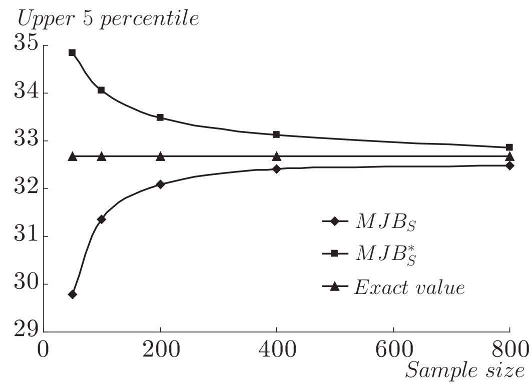

Figure 30: The upper percentiles of MJB_S and MJB_S^* for $p = 20$.