# Expected probabilities of misclassification in linear discriminant analysis based on 2-Step monotone missing samples

Nobumichi Shutoh[1,*], Masashi Hyodo[1] and Takashi Seo[2]

[1] *Department of Mathematics, Graduate School of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*

[2] *Department of Mathematical Information Science, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*

## Abstract

As concerns the asymptotic approximation for expected probabilities of misclassification in linear or quadratic discriminant analysis, theoretical results are given under some situations with complete data. However it has not been enough to discuss that under missing samples. In this paper, we derive that with explicit form in linear discriminant function for two groups constructed by 2-Step monotone missing samples. For our purpose, some results of expectations concerning inverted Wishart matrices are also obtained. Finally the numerical evaluations of our result by Monte Carlo simulations are presented.

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*Key Words and Phrases:* linear discriminant function; expected probability of misclassification; asymptotic approximation; monotone missing samples; missing observations.

## 1 Introduction

Linear discriminant analysis is well known as one of statistical procedures to assign $p$ dimensional observation vector $\boldsymbol{x}$ into one of two populations $\Pi^{(1)}$ and $\Pi^{(2)}$. If $\Pi^{(g)}(g = 1, 2)$ has $p$ dimensional normal distribution with mean vector $\boldsymbol{\mu}^{(g)}$ and common covariance matrix $\boldsymbol{\Sigma}$ which are known where $\boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)}$ and $\boldsymbol{\Sigma}$ is positive definite, linear discriminant function (LDF) is constructed as follows:

$$W = (\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})'\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{x} - \frac{1}{2}(\boldsymbol{\mu}^{(1)} + \boldsymbol{\mu}^{(2)})\right]. \tag{1.1}$$

*Corresponding author.
E-mail address: nshutoh1128@yahoo.co.jp

$\boldsymbol{x}$ may be assigned to $\Pi^{(1)}$ if $W > 0$, or $\Pi^{(2)}$ if $W < 0$. The probability of misclassification in (1.1) equals $\Phi(-\frac{1}{2}\Delta)$ exactly, where $\Phi(\cdot)$ denotes the cumulative density function of standard normal distribution and $\Delta^2 \equiv (\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})$ is Mahalanobis squared distance between $\Pi^{(1)}$ and $\Pi^{(2)}$.

However it is usual that both of parameters $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ are unknown. Therefore their own estimators are substituted for $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ respectively in (1.1). If $N_1^{(g)}$ observation vectors $\boldsymbol{x}_j^{(g)}(g = 1, 2,\ j = 1, \ldots, N_1^{(g)})$ which are can be observed completely from $\Pi^{(g)}$, LDF suggested by Wald (1944) is as follows:

$$W_c = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'\boldsymbol{S}^{-1}\left[\boldsymbol{x} - \frac{1}{2}(\overline{\boldsymbol{x}}^{(1)} + \overline{\boldsymbol{x}}^{(2)})\right], \tag{1.2}$$

where $\overline{\boldsymbol{x}}^{(g)}$ and $\boldsymbol{S}$ are sample mean vector from $\Pi^{(g)}$ and pooled sample covariance matrix, and they are the maximum likelihood estimators (MLEs) of $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ respectively. Although it is hard to obtain exact probabilities of misclassification under the situation where parameters are unknown, two types of asymptotic approximations for expected probabilities of misclassification (EPMC) have been considered. Type I approximations are ones of framework such that $N_1^{(1)}$ and $N_1^{(2)}$ are large and $p$ is fixed. Okamoto (1963, 1968) gave the approximation of this type by deriving asymptotic expansion formulas of LDF up to terms of the second order with respect to $(N_1^{(1)^{-1}}, N_1^{(2)^{-1}}, n_1^{-1})$, where $n_1 = N_1^{(1)} + N_1^{(2)} - 2$. Memon and Okamoto (1971) considered it in the case of quadratic discriminant function (QDF). Siotani and Wang (1977) extended Okamoto (1963, 1968) and Memon and Okamoto (1971) up to terms of the third order. It has been known that these approximations are good for small $p$, but are poor as $p$ is large. In the face of this problem, Type II approximations are suggested under the framework such that $N_1^{(1)}$, $N_1^{(2)}$, $p$ and $n_1 - p$ are large. As concerns the asymptotic approximations for EPMC in this type, Raudis (1972) considered one of them in LDF for $N_1^{(1)} = N_1^{(2)}$. Fujikoshi and Seo (1998) extended Raudis (1972) to the case of $N_1^{(1)} \neq N_1^{(2)}$ and also gave it in QDF. Moreover Fujikoshi (2000) gave explicit error bounds for approximation for EPMC proposed by Lachenbruch (1968) and discussed with a method of obtaining asymptotic expansion formulas for EPMC and their error bounds. Besides the asymptotic approximation for EPMC proposed by Lachenbruch (1968) is the one which can fit the situations in both Type I and Type II, so that the asymptotic approximation proposed Lachenbruch (1968) can be used in various situations of data set without missing data. As concerns discussion under elliptical populations, see Wakaki (1994).

On the other hand, as concerns MLEs constructed by observation vectors including missing data, Srivastava (1985) derived MLEs for one and two sample problems without condition of missing patterns by using incidence matrix which can remove missing data from each observation vector. Although they can not be obtained explicitly, Srivastava and Carter (1986) proposed the procedure to obtain them numerically by using Newton-Raphson method. Also MLEs for one sample in the case of 2-Step monotone missing samples from multivariate normal distribution were derived explicitly by Anderson and Olkin (1985). The $k$-Step monotone missing samples from $\Pi^{(g)}$ are defined as data set having following form:

$$
\begin{pmatrix}
x_{11}^{(g)} & \cdots & x_{1\ p_1}^{(g)} & x_{1\ p_1+1}^{(g)} & \cdots & x_{1\ p_{k-1}}^{(g)} & \cdots & x_{1\ p_k}^{(g)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
x_{N_1^{(g)}\ 1}^{(g)} & \cdots & x_{N_1^{(g)}\ p_1}^{(g)} & x_{N_1^{(g)}\ p_1+1}^{(g)} & \cdots & x_{N_1^{(g)}\ p_{k-1}}^{(g)} & \cdots & x_{N_1^{(g)}\ p_k}^{(g)} \\
x_{N_1^{(g)}+1\ 1}^{(g)} & \cdots & x_{N_1^{(g)}+1\ p_1}^{(g)} & x_{N_1^{(g)}+1\ p_1+1}^{(g)} & \cdots & x_{N_1^{(g)}+1\ p_{k-1}}^{(g)} & * \ \cdots & * \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
x_{N_2^{(g)}1}^{(g)} & \cdots & x_{N_2^{(g)}p_1}^{(g)} & x_{N_2^{(g)}\ p_1+1}^{(g)} & \cdots & x_{N_2^{(g)}\ p_{k-1}}^{(g)} & * \ \cdots & * \\
\vdots & & \vdots & \vdots & & * & \vdots & \vdots \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
x_{N^{(g)}\ 1}^{(g)} & \cdots & x_{N^{(g)}\ p_1}^{(g)} & * & \cdots & * & * \ \cdots & *
\end{pmatrix}, \qquad (1.3)
$$

where $p = p_1 + \ldots + p_k, N^{(g)} = N_1^{(g)} + \ldots + N_k^{(g)}$ and each part of " $*$ " denotes missing data. Further Kanda and Fujikoshi (1998) introduced some properties of MLEs in the case of $k = 2, 3$ and general $k$. They also gave asymptotic expansions for the distributions of MLEs in the situation when the sample size is large. In these days, Batsidis, Zografos and Loukas (2006) considered LDF based on 2-Step monotone missing samples and gave the asymptotic expression for probability density function of the probability of misclassification.

In this paper, as one of extension for Lachenbruch (1968), we propose asymptotic approximation for EPMC where training data has 2-Step monotone missing samples. That is, we extend the results of Lachenbruch (1968) in the case of LDF constructed by Batsidis, Zografos and Loukas (2006). Although Kanda and Fujikoshi (2004) derived that under known $\boldsymbol{\Sigma}$, we propose that under unknown case as main result in this paper.

The organization of this paper is as follows. In Section 2, we show method for obtaining MLEs in two groups in the case of 2-Step monotone missing samples and give some Lemmas which can support us to give the asymptotic approximation for EPMC considered in Section 3. In Section 3, we consider asymptotic approximation for EPMC in this case by using the results of Section 2 and derive the unbiased estimators of Mahalanobis squared distances which come in useful for obtaining the approximation of EPMC. Further we evaluate our results in Section 3 numerically by Monte Carlo simulations in Section 4. Finally we present conclusion in this paper and future problems in Section 5.

## 2 MLEs Based on 2-Step Monotone Missing Samples

In this section, we derive the MLEs under assuming that 2-Step monotone missing samples are obtained from each group, i.e., we extend MLEs given by Anderson and Olkin (1985) to the case of two groups where $k = 2$ in (1.3). Therefore we assume each data set from $\Pi^{(g)}$ as follows:

$$\begin{pmatrix} x_{11}^{(g)} & x_{12}^{(g)} & \cdots & x_{1\ p_1}^{(g)} & x_{1\ p_1+1}^{(g)} & \cdots & x_{1p}^{(g)} \\ x_{21}^{(g)} & x_{22}^{(g)} & \cdots & x_{2\ p_1}^{(g)} & x_{2\ p_1+1}^{(g)} & \cdots & x_{2p}^{(g)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{N_1^{(g)}\ 1}^{(g)} & x_{N_1^{(g)}\ 2}^{(g)} & \cdots & x_{N_1^{(g)}\ p_1}^{(g)} & x_{N_1^{(g)}\ p_1+1}^{(g)} & \cdots & x_{N_1^{(g)}\ p}^{(g)} \\ x_{N_1^{(g)}+1\ 1}^{(g)} & x_{N_1^{(g)}+1\ 2}^{(g)} & \cdots & x_{N_1^{(g)}+1\ p_1}^{(g)} & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{N^{(g)}\ 1}^{(g)} & x_{N^{(g)}\ 2}^{(g)} & \cdots & x_{N^{(g)}\ p_1}^{(g)} & * & \cdots & * \end{pmatrix},$$

where $p = p_1 + p_2$ and $N^{(g)} = N_1^{(g)} + N_2^{(g)}$. Then we prepare notations and assumption concerning observation vector. Let $\boldsymbol{x}_j^{(g)'} \equiv \left(\boldsymbol{x}_{1j}^{(g)'}, \boldsymbol{x}_{2j}^{(g)'}\right)$ $(g = 1, 2, j = 1, \ldots, N_1^{(g)})$ be the $p$ dimensional observation vector from $\Pi^{(g)}$ which has complete data, where $\boldsymbol{x}_{1j}^{(g)'} = (x_{j1}^{(g)}, \ldots, x_{jp_1}^{(g)})$ $(j = 1, \ldots, N^{(g)})$ and $\boldsymbol{x}_{2j}^{(g)'} = (x_{j\ p_1+1}^{(g)}, \ldots, x_{jp}^{(g)})$ $(j = 1, \ldots, N_1^{(g)})$. Now we assume distribution of observation vector:

$$\boldsymbol{x}_j^{(g)} \sim N_p(\boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma})\ (j = 1, \ldots, N_1^{(g)}) \ \text{ and }\ \boldsymbol{x}_{1j}^{(g)} \sim N_{p_1}(\boldsymbol{\mu}_1^{(g)}, \boldsymbol{\Sigma}_{11})\ (j = N_1^{(g)} + 1, \ldots, N^{(g)})$$

respectively, where

$$\boldsymbol{\mu}^{(g)} = \begin{pmatrix} \boldsymbol{\mu}_1^{(g)} \\ \boldsymbol{\mu}_2^{(g)} \end{pmatrix} \ \text{ and }\ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

$\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ are partitioned according to blocks of data set. Therefore $\boldsymbol{\mu}_1^{(g)}$ is $p_1$ dimensional vector, $\boldsymbol{\mu}_2^{(g)}$ is $p_2$ dimensional vector, $\boldsymbol{\Sigma}_{11}$ is $p_1 \times p_1$ matrix, $\boldsymbol{\Sigma}_{12} = \boldsymbol{\Sigma}_{21}'$ is $p_1 \times p_2$ matrix and $\boldsymbol{\Sigma}_{22}$ is $p_2 \times p_2$ matrix respectively.

Then we consider transformation of observation vector $\boldsymbol{x}_j^{(g)}$ by multiplying

$$\begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{0} \\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{I}_{p_2} \end{pmatrix}$$

on the left-hand. The mean vector $\boldsymbol{\eta}^{(g)}$ and covariance matrix of transformed observation vector are

$$\boldsymbol{\eta}^{(g)} \equiv \begin{pmatrix} \boldsymbol{\eta}_1^{(g)} \\ \boldsymbol{\eta}_2^{(g)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^{(g)} \\ \boldsymbol{\mu}_2^{(g)} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1^{(g)} \end{pmatrix} \ \text{ and }\ \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{22\cdot 1} \end{pmatrix},$$

where $\boldsymbol{\Sigma}_{22\cdot 1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$ and $\boldsymbol{\Sigma}_{11}$ has nonsingularity. Therefore we consider the new parameters $\{\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \boldsymbol{\Psi}\}$ which are one to one corresponding to $\{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \boldsymbol{\Sigma}\}$, where

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{\Sigma}_{22\cdot 1} \end{pmatrix}.$$

Next the likelihood function to obtain MLEs of $\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}$ and $\boldsymbol{\Psi}$ is as follows:

$$\begin{aligned} \mathrm{L}(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \boldsymbol{\Psi}) &= \text{Const.} \times |\boldsymbol{\Psi}_{11}|^{-\frac{1}{2}(N^{(1)}+N^{(2)})} |\boldsymbol{\Psi}_{22}|^{-\frac{1}{2}(N_1^{(1)}+N_1^{(2)})} \\ &\times \ \mathrm{etr}\left( -\frac{1}{2}\sum_{g=1}^{2}\sum_{j=1}^{N^{(g)}} (\boldsymbol{y}_{1j}^{(g)} - \boldsymbol{\eta}_1^{(g)})' \boldsymbol{\Psi}_{11}^{-1} (\boldsymbol{y}_{1j}^{(g)} - \boldsymbol{\eta}_1^{(g)}) \right) \\ &\times \ \mathrm{etr}\left( -\frac{1}{2}\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}} (\boldsymbol{y}_{2j}^{(g)} - \boldsymbol{\eta}_2^{(g)})' \boldsymbol{\Psi}_{22}^{-1} (\boldsymbol{y}_{2j}^{(g)} - \boldsymbol{\eta}_2^{(g)}) \right), \end{aligned} \tag{2.1}$$

where $\boldsymbol{y}_{1j}^{(g)} = \boldsymbol{x}_{1j}^{(g)}$ and $\boldsymbol{y}_{2j}^{(g)} = \boldsymbol{x}_{2j}^{(g)} - \boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}^{(g)}$. If we define the sample mean vectors

$$\begin{aligned} \overline{\boldsymbol{x}}_{1T}^{(g)} &= \frac{1}{N^{(g)}}\sum_{j=1}^{N^{(g)}} \boldsymbol{x}_{1j}^{(g)}, \qquad & \overline{\boldsymbol{x}}_{1F}^{(g)} &= \frac{1}{N_1^{(g)}}\sum_{j=1}^{N_1^{(g)}} \boldsymbol{x}_{1j}^{(g)}, \\ \overline{\boldsymbol{x}}_{2F}^{(g)} &= \frac{1}{N_1^{(g)}}\sum_{j=1}^{N_1^{(g)}} \boldsymbol{x}_{2j}^{(g)} \ \text{ and } & \overline{\boldsymbol{x}}_{1L}^{(g)} &= \frac{1}{N_2^{(g)}}\sum_{j=N_1^{(g)}+1}^{N^{(g)}} \boldsymbol{x}_{1j}^{(g)}, \end{aligned}$$

then by (2.1) and the well known correction of their own coefficients, we can obtain following MLEs

$$\widehat{\boldsymbol{\eta}}^{(g)} = \begin{pmatrix} \widehat{\boldsymbol{\eta}}_1^{(g)} \\ \widehat{\boldsymbol{\eta}}_2^{(g)} \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_{1T}^{(g)} \\ \overline{\boldsymbol{x}}_{2F}^{(g)} - \widehat{\boldsymbol{\Psi}}_{21}\overline{\boldsymbol{x}}_{1F}^{(g)} \end{pmatrix}$$

and

$$\widehat{\boldsymbol{\Psi}} = \begin{pmatrix} \widehat{\boldsymbol{\Psi}}_{11} & \widehat{\boldsymbol{\Psi}}_{12} \\ \widehat{\boldsymbol{\Psi}}_{21} & \widehat{\boldsymbol{\Psi}}_{22} \end{pmatrix},$$

where

$$\begin{aligned} \widehat{\boldsymbol{\Psi}}_{11} &= \frac{1}{n}\sum_{g=1}^{2}\sum_{j=1}^{N^{(g)}} (\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_{1T}^{(g)})(\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_{1T}^{(g)})', \\ \widehat{\boldsymbol{\Psi}}_{12} &= \left[\sum_{g=1}^{2}\sum_{j=1}^{N^{(g)}} (\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_{1F}^{(g)})(\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_{1F}^{(g)})'\right]^{-1} \left[\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}} (\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_{1F}^{(g)})(\boldsymbol{x}_{2j}^{(g)} - \overline{\boldsymbol{x}}_{2F}^{(g)})'\right], \\ \widehat{\boldsymbol{\Psi}}_{22} &= \frac{1}{n_1}\left\{\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}} (\boldsymbol{x}_{2j}^{(g)} - \overline{\boldsymbol{x}}_{2F}^{(g)})(\boldsymbol{x}_{2j}^{(g)} - \overline{\boldsymbol{x}}_{2F}^{(g)})' \right. \end{aligned}$$

$$
-\left[\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}}(\boldsymbol{x}_{2j}^{(g)}-\overline{\boldsymbol{x}}_{2F}^{(g)})(\boldsymbol{x}_{1j}^{(g)}-\overline{\boldsymbol{x}}_{1F}^{(g)})'\right]
$$

$$
\times\left[\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}}(\boldsymbol{x}_{1j}^{(g)}-\overline{\boldsymbol{x}}_{1F}^{(g)})(\boldsymbol{x}_{1j}^{(g)}-\overline{\boldsymbol{x}}_{1F}^{(g)})'\right]^{-1}\left[\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}}(\boldsymbol{x}_{1j}^{(g)}-\overline{\boldsymbol{x}}_{1F}^{(g)})(\boldsymbol{x}_{2j}^{(g)}-\overline{\boldsymbol{x}}_{2F}^{(g)})'\right]\Bigg\},
$$

$n = N^{(1)} + N^{(2)} - 2$ and $n_1 = N_1^{(1)} + N_1^{(2)} - 2$. Also these matrices are expressed as follows:

$$
\widehat{\boldsymbol{\Psi}}_{11} = \frac{1}{n}(\boldsymbol{\Gamma}_{11}^{(1)} + \boldsymbol{\Gamma}^{(2)}),\ \widehat{\boldsymbol{\Psi}}_{12} = \boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)} \ \text{ and } \ \widehat{\boldsymbol{\Psi}}_{22} = \frac{1}{n_1}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)} \tag{2.2}
$$

respectively. $\boldsymbol{\Gamma}^{(1)}$ and $\boldsymbol{\Gamma}^{(2)}$ are matrices such that

$$
\boldsymbol{\Gamma}^{(1)} = \begin{pmatrix} \boldsymbol{\Gamma}_{11}^{(1)} & \boldsymbol{\Gamma}_{12}^{(1)} \\ \boldsymbol{\Gamma}_{21}^{(1)} & \boldsymbol{\Gamma}_{22}^{(1)} \end{pmatrix} = n_1\boldsymbol{S}^{(1)}
$$

and

$$
\boldsymbol{\Gamma}^{(2)} = n_2\boldsymbol{S}^{(2)} + \sum_{g=1}^{2}\left\{\frac{N_1^{(g)}N_2^{(g)}}{N^{(g)}}(\overline{\boldsymbol{x}}_{1F}^{(g)}-\overline{\boldsymbol{x}}_{1L}^{(g)})(\overline{\boldsymbol{x}}_{1F}^{(g)}-\overline{\boldsymbol{x}}_{1L}^{(g)})'\right\},
$$

where

$$
\boldsymbol{S}^{(1)} = \frac{1}{n_1}\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}}(\boldsymbol{x}_j^{(g)}-\overline{\boldsymbol{x}}^{(g)})(\boldsymbol{x}_j^{(g)}-\overline{\boldsymbol{x}}^{(g)})',
$$

$$
\boldsymbol{S}^{(2)} = \frac{1}{n_2}\sum_{g=1}^{2}\sum_{j=N_1^{(g)}+1}^{N^{(g)}}(\boldsymbol{x}_{1j}^{(g)}-\overline{\boldsymbol{x}}_{1L}^{(g)})(\boldsymbol{x}_{1j}^{(g)}-\overline{\boldsymbol{x}}_{1L}^{(g)})' \ \text{ and } \ n_2 = N_2^{(1)} + N_2^{(2)} - 2.
$$

Clearly we can find that $\boldsymbol{S}^{(1)}$ and $\boldsymbol{S}^{(2)}$ are pooled sample covariance matrices of $\boldsymbol{x}_j^{(g)}(g = 1, 2, j = 1, \ldots, N_1^{(g)})$ and $\boldsymbol{x}_{1j}^{(g)}(g = 1, 2, j = N_1^{(g)} + 1, \ldots, N^{(g)})$ respectively. Also $n_1$ is needed to be larger than $p_1$ by nonsingularity of $\boldsymbol{\Gamma}_{11}^{(1)}$. MLEs derived in this section have natural forms of extension of Anderson and Olkin (1985). Here we have Lemmas concerning the distributions and expectations of (2.2).

Lemma 2.1 $\boldsymbol{\Gamma}_{11}^{(1)}, \boldsymbol{\Gamma}_{11}^{(1)} + \boldsymbol{\Gamma}^{(2)}$, $\boldsymbol{\Gamma}_{12}^{(1)}$ *given* $\boldsymbol{\Gamma}_{11}^{(1)}$ *and* $\boldsymbol{\Gamma}_{22\cdot 1}^{(1)}$ *have following distributions respectively:*

$$
\begin{aligned}
\boldsymbol{\Gamma}_{11}^{(1)} &\sim W_{p_1}(n_1, \boldsymbol{\Sigma}_{11}), \\
\boldsymbol{\Gamma}_{11}^{(1)} + \boldsymbol{\Gamma}^{(2)} = n\widehat{\boldsymbol{\Psi}}_{11} &\sim W_{p_1}(n, \boldsymbol{\Sigma}_{11}), \\
\mathrm{Vec}(\boldsymbol{\Gamma}_{12}^{(1)})|\boldsymbol{\Gamma}_{11}^{(1)} &\sim N_{p_1\times p_2}(\mathrm{Vec}(\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}), \boldsymbol{\Sigma}_{22\cdot 1}\otimes\boldsymbol{\Gamma}_{11}^{(1)}), \\
\boldsymbol{\Gamma}_{22\cdot 1}^{(1)} = n_1\widehat{\boldsymbol{\Psi}}_{22} &\sim W_{p_2}(n_1 - p_1, \boldsymbol{\Sigma}_{22\cdot 1}),
\end{aligned}
$$

*where* $W_p(n, \boldsymbol{\Sigma})$ *denotes Wishart distribution with the parameters* $n$ *and* $\boldsymbol{\Sigma}$.

Proof: Since $\mathbf{\Gamma}^{(1)}$ has $W_p(n_1, \mathbf{\Sigma})$, it has seen that $\mathbf{\Gamma}_{11}^{(1)}$ has $W_{p_1}(n_1, \mathbf{\Sigma}_{11})$ and $\mathbf{\Gamma}_{22\cdot 1}^{(1)}$ has $W_{p_2}(n_1 - p_1, \mathbf{\Sigma}_{22\cdot 1})$. The conditional distribution of $\mathbf{\Gamma}_{12}^{(1)}$ given $\mathbf{\Gamma}_{11}^{(1)}$ is derived by joint probability density function of $(\mathbf{\Gamma}_{11}^{(1)}, \mathbf{\Gamma}_{12}^{(1)})$. For a proof, e.g., see Siotani, Hayakawa and Fujikoshi (1985). Moreover the distribution of $\mathbf{\Gamma}^{(2)}$ can be derived since $n_2\boldsymbol{S}^{(2)}$ has $W_{p_1}(n_2, \mathbf{\Sigma}_{11})$, the other matrix has $W_{p_1}(2, \mathbf{\Sigma}_{11})$ and they are independent. Also $\mathbf{\Gamma}_{11}^{(1)}$ and $\mathbf{\Gamma}^{(2)}$ are independently distributed, which proves the result. □

Lemma 2.2 *Let $\boldsymbol{C}_i$ and $\boldsymbol{D}_i$ be $p_i \times p_i (i = 1, 2.)$ constant matrices respectively, then the expectations as follows can be obtained:*

$$\begin{aligned}
\mathrm{E}\left(\mathbf{\Gamma}_{11}^{(1)^{-1}}\right) &= \frac{1}{n_1 - p_1 - 1}\mathbf{\Sigma}_{11}^{-1}, \\
\mathrm{E}\left((\mathbf{\Gamma}_{11}^{(1)} + \mathbf{\Gamma}^{(2)})^{-1}\right) &= \frac{1}{n - p_1 - 1}\mathbf{\Sigma}_{11}^{-1}, \\
\mathrm{E}\left(\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\right) &= \frac{1}{n_1 - p - 1}\mathbf{\Sigma}_{22\cdot 1}^{-1}, \\
\mathrm{E}\left(\mathbf{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_1\mathbf{\Gamma}_{11}^{(1)^{-1}}\right) &= \frac{n_1 - p_1 - 2}{(n_1 - p_1)(n_1 - p_1 - 1)(n_1 - p_1 - 3)} \\
&\quad \times \mathbf{\Sigma}_{11}^{-1}\boldsymbol{C}_1\mathbf{\Sigma}_{11}^{-1} \\
&\quad + \frac{1}{(n_1 - p_1)(n_1 - p_1 - 1)(n_1 - p_1 - 3)} \\
&\quad \times \left\{\mathbf{\Sigma}_{11}^{-1}\boldsymbol{C}_1'\mathbf{\Sigma}_{11}^{-1} + \mathrm{tr}\left[\boldsymbol{C}_1\mathbf{\Sigma}_{11}^{-1}\right]\mathbf{\Sigma}_{11}^{-1}\right\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left((\mathbf{\Gamma}_{11}^{(1)} + \mathbf{\Gamma}^{(2)})^{-1}\boldsymbol{C}_1(\mathbf{\Gamma}_{11}^{(1)} + \mathbf{\Gamma}^{(2)})^{-1}\right) &= \frac{n - p_1 - 2}{(n - p_1)(n - p_1 - 1)(n - p_1 - 3)} \\
&\quad \times \mathbf{\Sigma}_{11}^{-1}\boldsymbol{C}_1\mathbf{\Sigma}_{11}^{-1} \\
&\quad + \frac{1}{(n - p_1)(n - p_1 - 1)(n - p_1 - 3)} \\
&\quad \times \left\{\mathbf{\Sigma}_{11}^{-1}\boldsymbol{C}_1'\mathbf{\Sigma}_{11}^{-1} + \mathrm{tr}\left[\boldsymbol{C}_1\mathbf{\Sigma}_{11}^{-1}\right]\mathbf{\Sigma}_{11}^{-1}\right\}, \\
\mathrm{E}\left(\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{C}_2\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\right) &= \frac{n_1 - p - 2}{(n_1 - p)(n_1 - p - 1)(n_1 - p - 3)} \\
&\quad \times \mathbf{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{C}_2\mathbf{\Sigma}_{22\cdot 1}^{-1} \\
&\quad + \frac{1}{(n_1 - p)(n_1 - p - 1)(n_1 - p - 3)} \\
&\quad \times \left\{\mathbf{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{C}_2'\mathbf{\Sigma}_{22\cdot 1}^{-1} + \mathrm{tr}\left[\boldsymbol{C}_2\mathbf{\Sigma}_{22\cdot 1}^{-1}\right]\mathbf{\Sigma}_{22\cdot 1}^{-1}\right\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_1\mathbf{\Gamma}_{11}^{(1)^{-1}}\right]\mathrm{tr}\left[\boldsymbol{D}_1\mathbf{\Gamma}_{11}^{(1)^{-1}}\right]\right) &= \frac{n_1 - p_1 - 2}{(n_1 - p_1)(n_1 - p_1 - 1)(n_1 - p_1 - 3)} \\
&\quad \times \mathrm{tr}\left[\boldsymbol{C}_1\mathbf{\Sigma}_{11}^{-1}\right]\mathrm{tr}\left[\boldsymbol{D}_1\mathbf{\Sigma}_{11}^{-1}\right]
\end{aligned}$$

$$+\frac{1}{(n_1-p_1)(n_1-p_1-1)(n_1-p_1-3)}$$
$$\times\left\{\mathrm{tr}\left[\boldsymbol{D}_1\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_1\boldsymbol{\Sigma}_{11}^{-1}\right]+\mathrm{tr}\left[\boldsymbol{D}_1\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_1'\boldsymbol{\Sigma}_{11}^{-1}\right]\right\},$$

*respectively.*

Proof: Lemma 2.2 is derived by the original properties concerning the expectation of inverted Wishart matrix. The results can be derived by Haff (1979) and Marx (1981). □

At the end of this section, we show the other expression of some terms included in $V_m$ defined by Section 3. Here we give the definition of Type I beta distribution.

Definition 2.3 *A $p\times p$ symmetric random matrix $\boldsymbol{Y}$ distributed as Type I beta distribution with the parameters $a$ and $b$ is one which has the following probability density function:*

$$\{\beta_p(a,b)\}^{-1}\det(\boldsymbol{Y})^{a-\frac{1}{2}(p+1)}\det(\boldsymbol{I}_p-\boldsymbol{Y})^{b-\frac{1}{2}(p+1)},\quad \boldsymbol{0}<\boldsymbol{Y}<\boldsymbol{I}_p,$$

*where $a>(p-1)/2$, $b>(p-1)/2$ and $\beta_p(a,b)$ is $p$ dimensional beta function given by*

$$\beta_p(a,b)=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}$$

*for the real parts of $a$ and $b$ are larger than $(p-1)/2$ respectively. Further the function $\Gamma_p(a)$ constructing $\beta_p(a,b)$ is the following function such that*

$$\Gamma_p(a)=\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma\left[a-\frac{1}{2}(i-1)\right],$$

*where $\Gamma(\cdot)$ is Gamma function.*

As concerns the details and properties concerning the above definition and the processes of calculating $\beta_p(a,b)$ and $\Gamma_p(a)$, e.g., see Gupta and Nagar (2000). Besides Type I beta distribution with the parameters $a$ and $b$ is denoted by $B_p^I(a,b)$ in this paper.

Lemma 2.4 *Let $\boldsymbol{A}_1$ be a $p_1\times p_1$ constant and symmetric matrix. Then*

$$\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{A}_1\right]$$

*can be also rewritten with following expression including random matrices:*

$$\mathrm{tr}\left[\boldsymbol{\beta}_{I,p_1}^{-1}(\boldsymbol{W}_1+\boldsymbol{W}_2)^{-\frac{3}{2}}\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\boldsymbol{A}_1\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}(\boldsymbol{W}_1+\boldsymbol{W}_2)^{-\frac{1}{2}}\right],$$

*where $\boldsymbol{\beta}_{I,p_1}=(\boldsymbol{W}_1+\boldsymbol{W}_2)^{\frac{1}{2}}\boldsymbol{W}_1^{-1}(\boldsymbol{W}_1+\boldsymbol{W}_2)^{\frac{1}{2}}$ has $B_{p_1}^I(n_1/2,(n_2+2)/2)$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ have $W_{p_1}(n_1,\boldsymbol{I}_{p_1})$ and $W_{p_1}(n_2+2,\boldsymbol{I}_{p_1})$ respectively.*

Proof: Note that $\boldsymbol{\beta}_{I,p_1}$ has $B_{p_1}^I(n_1/2,(n_2+2)/2)$. For a proof, see, e.g., Gupta and Nagar (2000). Moreover by making use of the properties of trace and Wishart matrices, the proof can be completed. □

## 3 Asymptotic Approximation for EPMC

Now we consider the asymptotic approximation for EPMC when the training data has 2-Step monotone missing samples. The approximation is obtained by making use of Lachenbruch (1968) and results obtained in Section 2. If $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ are unknown, the training data has 2-Step monotone samples and $\boldsymbol{x}$ does not have missing data, then LDF is constructed by the same way as (1.2):

$$W_m = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\widehat{\boldsymbol{\Sigma}}^{-1}\left[\boldsymbol{x} - \frac{1}{2}(\widehat{\boldsymbol{\mu}}^{(1)} + \widehat{\boldsymbol{\mu}}^{(2)})\right], \tag{3.1}$$

where $\widehat{\boldsymbol{\mu}}^{(1)}$, $\widehat{\boldsymbol{\mu}}^{(2)}$ and $\widehat{\boldsymbol{\Sigma}}$ are MLEs obtained by training data. Now we find expectation and variance of $W_m$ given $\widehat{\boldsymbol{\mu}}^{(1)}$, $\widehat{\boldsymbol{\mu}}^{(2)}$ and $\widehat{\boldsymbol{\Sigma}}$ under $\boldsymbol{x} \in \Pi^{(1)}$:

$$\begin{aligned} \mathrm{E}(W_m|\widehat{\boldsymbol{\mu}}^{(1)}, \widehat{\boldsymbol{\mu}}^{(2)}, \widehat{\boldsymbol{\Sigma}}) &= -U_m, \\ \mathrm{Var}(W_m|\widehat{\boldsymbol{\mu}}^{(1)}, \widehat{\boldsymbol{\mu}}^{(2)}, \widehat{\boldsymbol{\Sigma}}) &= V_m, \end{aligned}$$

where

$$U_m = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\widehat{\boldsymbol{\Sigma}}^{-1}(\widehat{\boldsymbol{\mu}}^{(1)} - \boldsymbol{\mu}^{(1)}) - \frac{1}{2}D_m^2, \tag{3.2}$$

$$D_m^2 = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\widehat{\boldsymbol{\Sigma}}^{-1}(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}), \tag{3.3}$$

$$V_m = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}). \tag{3.4}$$

Further if we define

$$Z_m = V_m^{-\frac{1}{2}}(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\widehat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}^{(1)}), \tag{3.5}$$

then $Z_m$ is independent of $(U_m, V_m)$ and distributed as $N(0,1)$ under the situation given $\widehat{\boldsymbol{\mu}}^{(1)}, \widehat{\boldsymbol{\mu}}^{(2)}, \widehat{\boldsymbol{\Sigma}}$. By using (3.2)–(3.5), (3.1) can be also rewritten as

$$W_m = V_m^{\frac{1}{2}} Z_m - U_m. \tag{3.6}$$

Thus $W_m$ can be expressed by $U_m$, $V_m$ and $Z_m$ in the same way as Lachenbruch (1968) in spite of 2-Step monotone missing samples. As concerns the probability of misclassification where $\boldsymbol{x}$ is assigned to $\Pi^{(2)}$ when $\boldsymbol{x}$ comes from $\Pi^{(1)}$, it is expressed by using (3.6) as

$$\Pr\left[W_m \le 0|\boldsymbol{x} \in \Pi^{(1)}\right] = \Pr\left[Z_m \le V_m^{-\frac{1}{2}}U_m|\boldsymbol{x} \in \Pi^{(1)}\right]. \tag{3.7}$$

Since $Z_m$ is distributed as $N(0,1)$ if $\boldsymbol{x}$ comes from $\Pi^{(1)}$, (3.7) can be rewritten easily as $\Phi(V_m^{-\frac{1}{2}}U_m)$. Therefore EPMC of (3.1) where $\boldsymbol{x}$ is assigned to $\Pi^{(2)}$ when $\boldsymbol{x}$ comes from $\Pi^{(1)}$ is obtained as

$$e_m(2|1) = \mathrm{E}_{(U_m,V_m)}\left[\Phi(V_m^{-\frac{1}{2}}U_m)\right]. \tag{3.8}$$

Here we consider the asymptotic approximation for EPMC in (3.1). In actual, $W_m$ is not distributed as normal distribution exactly, but is closely normal asymptotically. Thus as an asymptotic approximation of the right-hand of (3.8), we propose

$$e_m(2|1) \simeq \Phi(\{\mathrm{E}(V_m)\}^{-\frac{1}{2}}\mathrm{E}(U_m)) \tag{3.9}$$

obtained by substituting their expectations of $U_m$ and $V_m$. For the case that $\mathbf{\Sigma}$ is known, Kanda and Fujikoshi (2004) proposed (3.9) where

$$\begin{aligned}\mathrm{E}(U_m) &= -\frac{1}{2}\left[\Delta^2 + \frac{p_1(N^{(1)} - N^{(2)})}{N^{(1)}N^{(2)}} + \frac{p_1(N_1^{(1)} - N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\right],\\ \mathrm{E}(V_m) &= \Delta^2 + \frac{p_1(N^{(1)} + N^{(2)})}{N^{(1)}N^{(2)}} + \frac{p_2(N_1^{(1)} + N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\end{aligned}$$

under the framework such that

$$\frac{N^{(1)}}{N^{(2)}} \to \text{positive const. and } \frac{N_1^{(1)}}{N_1^{(2)}} \to \text{positive const.}$$

as $N_1^{(1)}, N_1^{(2)}, N_2^{(1)}$ and $N_2^{(2)}$ tend to infinity if $p_1$ and $p_2$ are fixed or

$$\frac{N^{(1)}}{N^{(2)}} \to \text{positive const.}, \frac{N_1^{(1)}}{N_1^{(2)}} \to \text{positive const. and } \boldsymbol{\delta}'\boldsymbol{\delta} = O(1)$$

as $N_1^{(1)}, N_1^{(2)}, N_2^{(1)}$, $N_2^{(2)}$, $p_1$ and $p_2$ tend to infinity, where $\boldsymbol{\delta} = \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}$.

If $p_1$ and $p_2$ are fixed, then this approximation corresponds to Type I approximation. If $p_1$ and $p_2$ are also large, that corresponds to Type II approximation. Under unknown $\mathbf{\Sigma}$, for the purpose of obtaining $\mathrm{E}(U_m)$ and $\mathrm{E}(V_m)$, i.e., deriving approximation given in (3.9), we have the following Lemmas.

**Lemma 3.1** *Let* $\mathrm{Vec}(\boldsymbol{X}') \sim N_{p_1 \times p_2}(\mathrm{Vec}(\boldsymbol{M}'), \mathbf{\Sigma} \otimes \mathbf{\Gamma})$, $\boldsymbol{M} = (m_{ij})$, $\mathbf{\Sigma} = (\sigma_{ij})$ *and* $\mathbf{\Gamma} = (\gamma_{ij})$. *Then the expectations with respect to* $\boldsymbol{X}$ *are*

$$\begin{aligned}\mathrm{E}(x_{i_1j_1}x_{i_2j_2}) &= \sigma_{i_1i_2}\gamma_{j_1j_2} + m_{i_1j_1}m_{i_2j_2},\\ \mathrm{E}(x_{i_1j_1}x_{i_2j_2}x_{i_3j_3}) &= m_{i_1j_1}\sigma_{i_2i_3}\gamma_{j_2j_3} + m_{i_2j_2}\sigma_{i_1i_3}\gamma_{j_1j_3}\\ &\quad + m_{i_3j_3}\sigma_{i_1i_2}\gamma_{j_1j_2} + m_{i_1j_1}m_{i_2j_2}m_{i_3j_3},\\ \mathrm{E}(x_{i_1j_1}x_{i_2j_2}x_{i_3j_3}x_{i_4j_4}) &= \sigma_{i_1i_4}\gamma_{j_1j_4}\sigma_{i_2i_3}\gamma_{j_2j_3} + \sigma_{i_1i_2}\gamma_{j_1j_2}\sigma_{i_4i_3}\gamma_{j_4j_3}\\ &\quad + \sigma_{i_1i_3}\gamma_{j_1j_3}\sigma_{i_4i_2}\gamma_{j_4j_2} + m_{i_1j_1}m_{i_2j_2}\sigma_{i_4i_3}\gamma_{j_4j_3}\\ &\quad + m_{i_1j_1}m_{i_3j_3}\sigma_{i_4i_2}\gamma_{j_4j_2} + m_{i_2j_2}m_{i_3j_3}\sigma_{i_1i_4}\gamma_{j_1j_4}\\ &\quad + m_{i_1j_1}m_{i_4j_4}\sigma_{i_2i_3}\gamma_{j_2j_3} + m_{i_4j_4}m_{i_2j_2}\sigma_{i_1i_3}\gamma_{j_1j_3}\\ &\quad + m_{i_4j_4}m_{i_3j_3}\sigma_{i_1i_2}\gamma_{j_1j_2} + m_{i_1j_1}m_{i_2j_2}m_{i_3j_3}m_{i_4j_4}.\end{aligned}$$

Proof: This Lemma was derived by considering the characteristic function of $\boldsymbol{X}$ (e.g., see Gupta and Nagar (2000)). □

Moreover Lemma 2.1 and Lemma 3.1 with putting $\boldsymbol{X} = \boldsymbol{\Gamma}_{12}^{(1)}$ lead the following Lemma concerning conditional expectations of $\widehat{\boldsymbol{\Psi}}_{12}$ given $\boldsymbol{\Gamma}_{11}^{(1)}$.

Lemma 3.2 *Let $\boldsymbol{C}_{ij}(p_i \times p_j)$ and $\boldsymbol{D}_{ij}(p_i \times p_j)$ be constant matrices respectively, then the conditional expectations given $\boldsymbol{\Gamma}_{11}^{(1)}$ as follows can be obtained:*

$$\mathrm{E}(\widehat{\boldsymbol{\Psi}}_{12}|\boldsymbol{\Gamma}_{11}^{(1)}) = \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12},$$

$$\mathrm{E}(\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{21}\widehat{\boldsymbol{\Psi}}_{12}|\boldsymbol{\Gamma}_{11}^{(1)}) = \boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{21}'\boldsymbol{\Sigma}_{22\cdot1} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12},$$

$$\mathrm{E}(\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{22}\widehat{\boldsymbol{\Psi}}_{21}|\boldsymbol{\Gamma}_{11}^{(1)}) = \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\boldsymbol{\Gamma}_{11}^{(1)^{-1}} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1},$$

$$\mathrm{E}(\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{C}_{11}\widehat{\boldsymbol{\Psi}}_{12}|\boldsymbol{\Gamma}_{11}^{(1)}) = \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\right]\boldsymbol{\Sigma}_{22\cdot1} + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12},$$

$$\begin{aligned}&\mathrm{E}(\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{22}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{C}_{11}\widehat{\boldsymbol{\Psi}}_{12}|\boldsymbol{\Gamma}_{11}^{(1)})\\ &= \mathrm{tr}\left[\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\right]\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1} + \boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}'\boldsymbol{\Sigma}_{22\cdot1}\\ &\quad + \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12},\end{aligned}$$

$$\begin{aligned}&\mathrm{E}(\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{21}\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{22}\widehat{\boldsymbol{\Psi}}_{21}|\boldsymbol{\Gamma}_{11}^{(1)})\\ &= \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}} + \mathrm{tr}\left[\boldsymbol{C}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\\ &\quad + \boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{21}'\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1},\end{aligned}$$

$$\begin{aligned}&\mathrm{E}(\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{22}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{C}_{11}\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{D}_{22}\widehat{\boldsymbol{\Psi}}_{21}|\boldsymbol{\Gamma}_{11}^{(1)})\\ &= \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\right]\boldsymbol{\Gamma}_{11}^{(1)^{-1}} + \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}'\boldsymbol{\Sigma}_{22\cdot1}]\,\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\\ &\quad + \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\mathrm{tr}\,[\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}} + \mathrm{tr}\,[\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\\ &\quad + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}'\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}'\boldsymbol{\Gamma}_{11}^{(1)^{-1}} + \mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\\ &\quad + \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\right]\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} + \boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}'\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\\ &\quad + \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\\ &\quad + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1},\end{aligned}$$

$$\begin{aligned}&\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{21}\widehat{\boldsymbol{\Psi}}_{12}\right]\mathrm{tr}\left[\boldsymbol{D}_{21}\widehat{\boldsymbol{\Psi}}_{12}\right]|\boldsymbol{\Gamma}_{11}^{(1)}\right)\\ &= \mathrm{tr}\left[\boldsymbol{C}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{D}_{21}'\boldsymbol{\Sigma}_{22\cdot1}\right] + \mathrm{tr}\left[\boldsymbol{C}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\mathrm{tr}\left[\boldsymbol{D}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right],\end{aligned}$$

$$\begin{aligned}&\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{22}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{C}_{11}\widehat{\boldsymbol{\Psi}}_{12}\right]\mathrm{tr}\left[\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{21}\right]|\boldsymbol{\Gamma}_{11}^{(1)}\right)\\ &= \mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{21}'\boldsymbol{\Sigma}_{22\cdot1}\right] + \mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{C}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\ &\quad + \mathrm{tr}\,[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}]\,\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{21}\right]\\ &\quad + \mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{21}\right],\end{aligned}$$

$$
\begin{aligned}
&\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{22}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{C}_{11}\widehat{\boldsymbol{\Psi}}_{12}\right]\mathrm{tr}\left[\boldsymbol{D}_{22}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{D}_{11}\widehat{\boldsymbol{\Psi}}_{12}|\boldsymbol{\Gamma}_{11}^{(1)}\right]\right)\\
&= \mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{D}_{11}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\right]\mathrm{tr}\left[\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{D}_{11}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}'\boldsymbol{\Sigma}_{22\cdot1}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{D}_{11}'\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\mathrm{tr}\left[\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{D}_{11}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{D}_{11}'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{D}_{22}'\boldsymbol{\Sigma}_{22\cdot1}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{D}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{D}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{D}_{22}'\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{D}_{11}'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot1}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\right]\mathrm{tr}\left[\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{D}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&\quad+\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{C}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\mathrm{tr}\left[\boldsymbol{D}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{D}_{11}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right].
\end{aligned}
$$

Proof: By making use of results of Lemma 2.1 and Lemma 3.1, we can obtain the above results. For some expectations of the above, the concrete methods for calculating them will be given in Appendix A. The other expectations can be also obtained by the methods similarly.

Furthermore we also give the some results of calculating expectations by Lemma 2.4, Konno (1988) and Marx (1981).

Lemma 3.3 *The following expectations can be calculated as*

$$
\begin{aligned}
\mathrm{E}\left(\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\right) &= p_1c_1,\\
\mathrm{E}\left((\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\right) &= c_1\delta_{11}^2,
\end{aligned}
$$

*where* $c_1=(n-1)/\{(n-p_1)(n-p_1-3)(n_1-p_1-1)\}, \delta_{11}^2=(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$.

Proof: The proof will be completed in Appendix B.

By Lemma 2.2, Lemma 3.2 and Lemma 3.3, we can obtain the following theorem.

Theorem 3.4 *If* $\boldsymbol{\mu}^{(g)}$ *and* $\boldsymbol{\Sigma}$ *are unknown, then the expectations of* $U_m$ *and* $V_m$ *defined by (3.2) and (3.4) are*

$$
\begin{aligned}
\mathrm{E}(U_m) &= -\frac{1}{2}u_1\left(\delta_{11}^2+\frac{p_1(N^{(1)}-N^{(2)})}{N^{(1)}N^{(2)}}\right)-\frac{1}{2}p_2u_2\left(\delta_{11}^2+\frac{p_1(N_1^{(1)}-N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\right)\\
&\quad-\frac{1}{2}u_3\left(\Delta^2+\frac{p_1(N^{(1)}-N^{(2)})}{N^{(1)}N^{(2)}}+\frac{p_2(N_1^{(1)}-N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\right),\\
\mathrm{E}(V_m) &= v_1\left(\delta_{11}^2+\frac{p_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\right)+p_2v_2\left(\delta_{11}^2+\frac{p_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\right)
\end{aligned}
$$

$$+v_3\left(\Delta^2+\frac{p_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}+\frac{p_2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\right),$$

*where*

$$\begin{aligned}
u_1 &= \frac{n}{n-p_1-1}-\frac{n_1}{n_1-p-1},\\
u_2 &= \frac{n_1}{(n_1-p-1)(n_1-p_1-1)},\\
u_3 &= \frac{n_1}{n_1-p-1},\\
v_1 &= \frac{n^2(n-1)}{(n-p_1)(n-p_1-1)(n-p_1-3)}+\frac{2p_2nn_1(n-1)}{(n-p_1)(n-p_1-3)(n_1-p_1-1)(n_1-p-1)}\\
&\quad -\frac{n_1^2(n_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)},\\
v_2 &= \frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\quad +\frac{(p_2+1)n_1^2(n_1-1)}{(n_1-p)(n_1-p-3)(n_1-p_1)(n_1-p_1-1)(n_1-p_1-3)}\\
&\quad +\frac{2n_1^2(n_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-1)(n_1-p_1-3)}\\
&\quad +\frac{p_1n_1^2(n_1-p_1-2)}{(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-3)}\\
&\quad +\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-3)},\\
v_3 &= \frac{n_1^2(n_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}.
\end{aligned}$$

Proof: The proof will be completed in Appendix C.

Thus we can derive the asymptotic approximation for EPMC in (3.9) with explicit form. By Theorem 3.4, we derive that under the framework such that

$$\frac{N^{(1)}}{N^{(2)}}\to \text{positive const. and } \frac{N_1^{(1)}}{N_1^{(2)}}\to \text{positive const.}$$

as $N_1^{(1)}, N_1^{(2)}, N_2^{(1)}$ and $N_2^{(2)}$ tend to infinity if $p_1$ and $p_2$ are fixed or

$$\frac{N^{(1)}}{N^{(2)}}\to \text{positive const.}, \ \frac{N_1^{(1)}}{N_1^{(2)}}\to \text{positive const.}, \ \boldsymbol{\delta}'\boldsymbol{\delta}=O(1),$$

$$n-p_1\to\infty \ \text{ and } \ n_1-p \ \to \ \infty$$

as $N_1^{(1)}$, $N_1^{(2)}$, $N_2^{(1)}$, $N_2^{(2)}$, $p_1$ and $p_2$ tend to infinity.

In addition, we can find the following corollary as checking on the results in Theorem 3.4.

**Corollary 3.5** *If we put $N_1^{(g)} = N^{(g)}$ $(g = 1, 2)$ and note that $p = p_1 + p_2$, $\mathrm{E}(U_m)$ and $\mathrm{E}(V_m)$ given by Theorem 3.4 can be reduced to $\mathrm{E}(U)$ and $\mathrm{E}(V)$ derived in Lachenbruch (1968) as follows:*

$$\begin{aligned}\mathrm{E}(U) &= -\frac{n_1}{2(n_1-p)}\left\{\Delta^2 + \frac{(N_1^{(1)} - N_1^{(2)})p}{N_1^{(1)}N_1^{(2)}}\right\},\\ \mathrm{E}(V) &= \frac{n_1^2(n_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\left\{\Delta^2 + \frac{(N_1^{(1)} + N_1^{(2)})p}{N_1^{(1)}N_1^{(2)}}\right\}.\end{aligned}$$

Besides Lachenbruch (1968) proposed the asymptotic approximation for EPMC of LDF in (1.2) by using $\mathrm{E}(U)$ and $\mathrm{E}(V)$:

$$e_c(2|1) \simeq \Phi(\{\mathrm{E}(V)\}^{-\frac{1}{2}}\mathrm{E}(U)). \tag{3.10}$$

As it were, this corollary implies asymptotic approximation proposed in this paper is one of extension for Lachenbruch (1968).

Although we can obtain the asymptotic approximation for EPMC, we can not use $\boldsymbol{\mu}^{(g)}$ and $\boldsymbol{\Sigma}$ actually since they are usually unknown. Then it is natural to consider substituting $d_{m11}^2 \equiv (\widehat{\boldsymbol{\mu}}_1^{(1)} - \widehat{\boldsymbol{\mu}}_1^{(2)})'\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\widehat{\boldsymbol{\mu}}_1^{(1)} - \widehat{\boldsymbol{\mu}}_1^{(2)})$ and $D_m^2$ for $\delta_{11}^2$ and $\Delta^2$ respectively. However under the situation where sample sizes are not very large, the approximation may be affected by the biases of $d_{m11}^2$ and $D_m^2$. Therefore we also propose the unbiased estimators of $\delta_{11}^2$ and $\Delta^2$ respectively at the end of this section.

**Theorem 3.6** *The unbiased estimators $\widehat{\delta}_{11}^2$ and $\widehat{\Delta}^2$ of $\delta_{11}^2$ and $\Delta^2$ which are included in the asymptotic approximation for EPMC given in Theorem 3.4 can be obtained as following forms:*

$$\widehat{\delta}_{11}^2 = \frac{1}{b_1}d_{m11}^2 - \frac{p_1(N^{(1)} + N^{(2)})}{N^{(1)}N^{(2)}}, \tag{3.11}$$

$$\begin{aligned}\widehat{\Delta}^2 &= \frac{1}{b_2}D_m^2 - \frac{1}{b_2}(b_1 - b_3)\widehat{\delta}_{11}^2\\ &\quad -\frac{b_1}{b_2}\frac{p_1(N^{(1)} + N^{(2)})}{N^{(1)}N^{(2)}} - \frac{p_2(N_1^{(1)} + N_1^{(2)})}{N_1^{(1)}N_1^{(2)}} - b_3\frac{p_1p_2(N_1^{(1)} + N_1^{(2)})}{n_1N_1^{(1)}N_1^{(2)}},\end{aligned} \tag{3.12}$$

*where $b_1 = n/(n - p_1 - 1)$, $b_2 = n_1/(n_1 - p - 1)$ and $b_3 = n_1/(n_1 - p_1 - 1)$.*

**Proof**: The proof will be completed in Appendix D.

Besides in the case of complete data, the unbiased estimator of Mahalanobis squared distance has the following form by using $D^2 = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'\boldsymbol{S}^{-1}(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})$:

$$\frac{n_1 - p - 1}{n_1}D^2 - \frac{N_1^{(1)} + N_1^{(2)}}{N_1^{(1)}N_1^{(2)}}p. \tag{3.13}$$

As concerns the asymptotic approximation proposed in this section, it will be evaluated by Monte Carlo simulations in Section 4. Then the above unbiased estimators will be used in stead of Mahalanobis squared distances.

## 4 Simulation Studies

We are interested in the accuracy of the asymptotic approximation for EPMC proposed in this paper. Although we are also interested in comparing it with the other ones, the asymptotic approximations for EPMC in the case of missing samples had not been proposed before. Therefore we compare the accuracy of it with the other ones for the case of complete data, e.g., Lachenbruch (1968) and Fujikoshi and Seo (1998). As concerns the asymptotic approximation for EPMC proposed by Lachenbruch (1968), see Corollary 3.5. On the other hand, Fujikoshi and Seo (1998) proposed the following the asymptotic approximation for EPMC of LDF presented in (1.2):

$$e_c(2|1) \simeq \Phi(\zeta), \tag{4.1}$$

where

$$\zeta = -\frac{1}{2}\left(\frac{N_1^{(1)}+N_1^{(2)}-p}{N_1^{(1)}+N_1^{(2)}}\right)^{1/2}\left\{\Delta^2+\frac{p(N_1^{(1)}-N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\right\}\left\{\Delta^2+\frac{p(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\right\}^{-1/2}.$$

Then we start to compare the accuracy of (3.9), (3.10) and (4.1) by Monte Carlo simulations. As you know, in actual, the exact Mahalanobis squared distances $\delta_{11}^2$ and $\Delta^2$ are unknown respectively. Thus we use the unbiased estimator (3.13) in stead of $\Delta^2$ included in the asymptotic approximations (3.10) and (4.1). In similar, the unbiased estimators (3.11) and (3.12) are used in stead of $\delta_{11}^2$ and $\Delta^2$ in (3.9) respectively.

As concerns the settings of sample sizes concerning complete data, dimension and Mahalanobis distance, we chose the parameters which are similar to Fujikoshi and Seo (1998). The settings of sample sizes and dimensions concerning missing samples are as follows:

$$(N_2^{(1)}, N_2^{(2)}) = (0,0),\ (20,20),\ (20,40),\ (40,40), \quad \text{and} \quad (p_1,p_2) = (1,1),\ (3,2),\ (5,5).$$

The tables in these simulations are separated by the setting of dimensions. Besides the values of asymptotic approximations in tables are average values of the ones calculated 1,000,000 times by Monte Carlo simulations and evaluated the accuracy of ones by them. The results are shown in Tables 1–3 respectively.

## 5 Conclusion and Future Problems

In this paper, we extended the MLEs of 2-Step monotone missing samples to that of two groups in Section 2. By making use of the MLEs, we considered LDF based on 2-Step monotone missing samples. Moreover by giving the many results concerning the expectations which can not be calculated easily(e.g., Lemma 3.3), we proposed the asymptotic approximation for EPMC with an explicit form which is Lachenbruch type of the LDF considered by us. As it were, it is one of theoretical extensions of Lachenbruch (1968). We proposed not only the asymptotic approximation but also the unbiased estimators which are used in calculating the approximation actually in Section 3. Finally we evaluated our result numerically by comparing with the other approximations in Section 4.

By simulations, we find that our result is more accurate than the ones which had been proposed before. Although the asymptotic approximation proposed in this paper is limited only in the case of 2-Step monotone missing samples, constructing LDF by missing samples holds good method for making EPMC lower and giving more accurate asymptotic approximation for EPMC. These results imply that method proposed in this paper can be considered as one of useful methods for experimenters who have put linear discriminant analysis to practical use. In the case of complete data, in general, the results of Fujikoshi and Seo (1998) may be more accurate than Lachenbruch (1968).

As left problems, although it may be easy to be struck with an idea of the extension of the result in this paper to $k$-Step monotone missing samples, the calculations of expectations will be much complicated. If we limit the discussion in the case of 2-Step monotone missing samples, also we will be able to derive theoretical error bounds for the asymptotic approximation proposed in this paper by the method which is similar to Fujikoshi (2000). By the results of simulations in the case of complete data, the extensions of the other types of approximations (e.g., Okamoto (1963, 1968), Fujikoshi and Seo (1998)) to the case of 2-Step monotone missing samples will be needed. Finally we point many discussions (e.g., testing equality procedure, MANOVA) based on monotone missing samples have been left.

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## Appendix A Proof of Lemma 3.2

We show the proof for calculations

$$\mathrm{E}\left(\widehat{\boldsymbol{\Psi}}_{12}\boldsymbol{C}_{22}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{C}_{11}\widehat{\boldsymbol{\Psi}}_{12}|\boldsymbol{\Gamma}_{11}^{(1)}\right) \tag{A.1}$$

and

$$\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{21}\widehat{\boldsymbol{\Psi}}_{12}\right]\mathrm{tr}\left[\boldsymbol{D}_{21}\widehat{\boldsymbol{\Psi}}_{12}\right]|\boldsymbol{\Gamma}_{11}^{(1)}\right). \tag{A.2}$$

At First, we complete the proof of (A.1). (A.1) is also expressed as follows:

$$\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\mathrm{E}\left(\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{C}_{22}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{C}_{11}^{*}\boldsymbol{\Gamma}_{12}^{(1)}|\boldsymbol{\Gamma}_{11}^{(1)}\right), \tag{A.3}$$

where $\boldsymbol{C}_{11}^{*}=\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{C}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}$. Therefore $(i,j)$th element of (A.3) is expressed by

$$\boldsymbol{\Gamma}_{12}^{(1)}=(\Gamma_{12}^{ij}),\ \boldsymbol{\Gamma}_{21}^{(1)}=(\Gamma_{21}^{ij}),\ \boldsymbol{C}_{22}=(C_{22}^{ij})\text{ and }\boldsymbol{C}_{11}^{*}=(C_{11}^{ij})$$

as follows:

$$\begin{aligned}\mathrm{E}\left(\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{C}_{22}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{C}_{11}^{*}\boldsymbol{\Gamma}_{12}^{(1)}|\boldsymbol{\Gamma}_{11}^{(1)}\right)_{ij} &= \sum_{i_1=1}^{p_2}\sum_{i_2=1}^{p_2}\sum_{i_3=1}^{p_1}\sum_{i_4=1}^{p_1}\mathrm{E}\left(\Gamma_{12}^{ii_1}C_{22}^{i_1i_2}\Gamma_{21}^{i_2i_3}C_{11}^{i_3i_4}\Gamma_{12}^{i_4j}|\boldsymbol{\Gamma}_{11}^{(1)}\right)\\ &= \sum_{i_1=1}^{p_2}\sum_{i_2=1}^{p_2}\sum_{i_3=1}^{p_1}\sum_{i_4=1}^{p_1}C_{22}^{i_1i_2}C_{11}^{i_3i_4}\mathrm{E}\left(\Gamma_{21}^{i_1i}\Gamma_{21}^{i_2i_3}\Gamma_{21}^{ji_4}|\boldsymbol{\Gamma}_{11}^{(1)}\right). \end{aligned}\tag{A.4}$$

By Lemma 3.1 with putting

$$\boldsymbol{X}=\boldsymbol{\Gamma}_{12}^{(1)},\ \boldsymbol{M}=\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Gamma}_{11}^{(1)},\ \boldsymbol{\Sigma}=\boldsymbol{\Sigma}_{22\cdot 1}\text{ and }\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_{11}^{(1)},$$

(A.4) is also expressed by

$$\boldsymbol{M}=(m_{ij}),\ \boldsymbol{\Sigma}=(\sigma_{ij}),\ \boldsymbol{\Gamma}=(\gamma_{ij}),\ \boldsymbol{M}'=(m'_{ij}),\ \boldsymbol{\Sigma}'=(\sigma'_{ij})\text{ and }\boldsymbol{\Gamma}'=(\gamma'_{ij})$$

as follows:

$$\begin{aligned}\mathrm{E}\left(\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{C}_{22}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{C}_{11}^{*}\boldsymbol{\Gamma}_{12}^{(1)}|\boldsymbol{\Gamma}_{11}^{(1)}\right)_{ij} &= \sum_{i_1=1}^{p_2}\sum_{i_2=1}^{p_2}\sum_{i_3=1}^{p_1}\sum_{i_4=1}^{p_1}\Big\{m'_{ii_1}C_{22}^{i_1i_2}\sigma_{i_2j}C_{11}^{i_3i_4}\gamma'_{i_4i_3}\\ &\quad+\gamma_{ii_4}C_{11}^{i_4i'_3}m'_{i_3i_2}C_{22}^{i_2i'_1}\sigma_{i_1j}+\gamma_{ii_3}C_{11}^{i_3i_4}m'_{i_4j}C_{22}^{i_1i_2}\sigma'_{i_2i_1}\\ &\quad+m'_{ii_1}C_{22}^{i_1i_2}m_{i_2i_3}C_{11}^{i_3i_4}m'_{i_4j}\Big\}.\end{aligned}$$

Therefore by noting $\boldsymbol{\Sigma}$ and $\boldsymbol{\Gamma}$ are symmetric,

$$\begin{aligned}\mathrm{E}\left(\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{C}_{22}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{C}_{11}^{*}\boldsymbol{\Gamma}_{12}^{(1)}|\boldsymbol{\Gamma}_{11}^{(1)}\right) &= \mathrm{tr}\left[\boldsymbol{C}_{11}^{*}\boldsymbol{\Gamma}_{11}^{(1)}\right]\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot 1}\\ &\quad+\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{C}_{11}^{*\prime}\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}'_{22}\boldsymbol{\Sigma}_{22\cdot 1}\\ &\quad+\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{C}_{11}^{*}\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\mathrm{tr}\left[\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{22\cdot 1}\right]\\ &\quad+\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{C}_{22}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{C}_{11}^{*}\boldsymbol{\Gamma}_{11}^{(1)}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}.\end{aligned}\tag{A.5}$$

By (A.3), multiplying $\mathbf{\Gamma}_{11}^{(1)^{-1}}$ on the left-hand of (A.5) and putting $\boldsymbol{C}_{11}^{*} = \mathbf{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{C}_{11} \mathbf{\Gamma}_{11}^{(1)^{-1}}$ into (A.5) lead the result to prove.

Next we complete the proof of (A.2). (A.2) is also expressed as

$$\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]\mathrm{tr}\left[\boldsymbol{D}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]|\mathbf{\Gamma}_{11}^{(1)}\right), \tag{A.6}$$

where $\boldsymbol{C}_{21}^{*} = \boldsymbol{C}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}$ and $\boldsymbol{D}_{21}^{*} = \boldsymbol{D}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}$. Therefore (A.6) is expressed by

$$\mathbf{\Gamma}_{12}^{(1)} = (\Gamma_{12}^{ij}),\ \mathbf{\Gamma}_{21}^{(1)} = (\Gamma_{21}^{ij}),\ \boldsymbol{C}_{21}^{*} = (C_{21}^{ij}) \text{ and } \boldsymbol{D}_{21}^{*} = (D_{21}^{ij})$$

as follows:

$$\begin{aligned}\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]\mathrm{tr}\left[\boldsymbol{D}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]|\mathbf{\Gamma}_{11}^{(1)}\right) &= \sum_{i_1=1}^{p_2}\sum_{i_2=1}^{p_1}\sum_{i_3=1}^{p_2}\sum_{i_4=1}^{p_1}\mathrm{E}\left(C_{21}^{i_1i_2}\Gamma_{12}^{i_2i_1}D_{21}^{i_3i_4}\Gamma_{12}^{i_4i_3}|\mathbf{\Gamma}_{11}^{(1)}\right)\\ &= \sum_{i_1=1}^{p_2}\sum_{i_2=1}^{p_1}\sum_{i_3=1}^{p_2}\sum_{i_4=1}^{p_1}C_{21}^{i_1i_2}D_{21}^{i_3i_4}\\ &\quad\times\mathrm{E}\left(\Gamma_{21}^{i_1i_2}\Gamma_{21}^{i_3i_4}|\mathbf{\Gamma}_{11}^{(1)}\right).\end{aligned} \tag{A.7}$$

By using Lemma 3.1 with putting

$$\boldsymbol{X} = \mathbf{\Gamma}_{12}^{(1)},\ \boldsymbol{M} = \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Gamma}_{11}^{(1)},\ \mathbf{\Sigma} = \mathbf{\Sigma}_{22\cdot1} \text{ and } \mathbf{\Gamma} = \mathbf{\Gamma}_{11}^{(1)}$$

again, (A.7) is also expressed by

$$\boldsymbol{M} = (m_{ij}),\ \mathbf{\Sigma} = (\sigma_{ij}),\ \mathbf{\Gamma} = (\gamma_{ij}),\ \boldsymbol{M}' = (m'_{ij}) \text{ and } \mathbf{\Sigma}' = (\sigma'_{ij})$$

as follows:

$$\begin{aligned}\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]\mathrm{tr}\left[\boldsymbol{D}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]|\mathbf{\Gamma}_{11}^{(1)}\right) &= \sum_{i_1=1}^{p_2}\sum_{i_2=1}^{p_1}\sum_{i_3=1}^{p_2}\sum_{i_4=1}^{p_1}(C_{21}^{i_1i_2}\gamma_{i_2i_4}D_{21}^{i_4i'_3}\sigma'_{i_3i_1}\\ &\quad+C_{21}^{i_1i_2}m'_{i_2i_1}D_{21}^{i_3i_4}m'_{i_4i_3}).\end{aligned}$$

Therefore by noting that $\mathbf{\Sigma}$ is symmetric,

$$\begin{aligned}\mathrm{E}\left(\mathrm{tr}\left[\boldsymbol{C}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]\mathrm{tr}\left[\boldsymbol{D}_{21}^{*}\mathbf{\Gamma}_{12}^{(1)}\right]|\mathbf{\Gamma}_{11}^{(1)}\right) &= \mathrm{tr}\left[\boldsymbol{C}_{21}^{*}\mathbf{\Gamma}_{11}^{(1)}\boldsymbol{D}_{21}^{*\prime}\mathbf{\Sigma}_{22\cdot1}\right]+\mathrm{tr}\left[\boldsymbol{C}_{21}^{*}\mathbf{\Gamma}_{11}^{(1)}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\right]\\ &\quad\times\mathrm{tr}\left[\boldsymbol{D}_{21}^{*}\mathbf{\Gamma}_{11}^{(1)}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\right].\end{aligned} \tag{A.8}$$

By (A.6), putting $\boldsymbol{C}_{21}^{*} = \boldsymbol{C}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}$ and $\boldsymbol{D}_{21}^{*} = \boldsymbol{D}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}$ into (A.8) leads the result to prove. The other expectations can be obtained in the same way as the above. Thus Lemma 3.2 has been completed. □

## Appendix B Proof of Lemma 3.3

By Lemma 2.4,

$$\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{A}_1\right]=\mathrm{tr}\left[\boldsymbol{\beta}_{I,p_1}^{-1}(\boldsymbol{W}_1+\boldsymbol{W}_2)^{-\frac{3}{2}}\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\boldsymbol{A}_1\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}(\boldsymbol{W}_1+\boldsymbol{W}_2)^{-\frac{1}{2}}\right], \quad \text{(B.1)}$$

where $\boldsymbol{\beta}_{I,p_1} \equiv (\boldsymbol{W}_1+\boldsymbol{W}_2)^{-\frac{1}{2}}\boldsymbol{W}_1(\boldsymbol{W}_1+\boldsymbol{W}_2)^{-\frac{1}{2}}$ has $B_{p_1}^I(n_1/2,(n_2+2)/2)$. Since $\boldsymbol{\beta}_{I,p_1}$ is independent of $\boldsymbol{W}_1+\boldsymbol{W}_2$, we can calculate conditional expectation of (B.1) given $\boldsymbol{W}_1+\boldsymbol{W}_2$. Besides Konno (1988) gave the expectation of $\boldsymbol{\beta}_{I,p_1}^{-1}$ as follows:

$$\mathrm{E}\left(\boldsymbol{\beta}_{I,p_1}^{-1}\right)=\frac{n-p_1-1}{n_1-p_1-1}\boldsymbol{I}_{p_1}.$$

Thus the conditional expectation to obtain can be calculated as

$$\frac{n-p_1-1}{n_1-p_1-1}\mathrm{tr}\left[(\boldsymbol{W}_1+\boldsymbol{W}_2)^{-1}\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\boldsymbol{A}_1\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}(\boldsymbol{W}_1+\boldsymbol{W}_2)^{-1}\right]. \quad \text{(B.2)}$$

Next we also calculate the expectation of (B.2) concerning $\boldsymbol{W}_1+\boldsymbol{W}_2$. Since $\boldsymbol{W}_1+\boldsymbol{W}_2$ is distributed as $W_{p_1}(n,\boldsymbol{I}_{p_1})$, the expectation can be obtained by Marx (1981):

$$\mathrm{E}\left(\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{A}_1\right]\right)=c_1\mathrm{tr}\left[\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\boldsymbol{A}_1\boldsymbol{\Sigma}_{11}^{-\frac{1}{2}}\right], \quad \text{(B.3)}$$

where $c_1=(n-1)/\{(n-p_1)(n-p_1-3)(n_1-p_1-1)\}$. By putting $\boldsymbol{A}_1=\boldsymbol{\Sigma}_{11}$ and $\boldsymbol{A}_1=(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'$ into (B.3), Lemma 3.3 has been completed. □

## Appendix C Proof of Theorem 3.4

Since

$$\widehat{\boldsymbol{\mu}}^{(g)}=\begin{pmatrix}\boldsymbol{I}_{p_1} & \boldsymbol{0}\\ -\widehat{\boldsymbol{\Psi}}_{21} & \boldsymbol{I}_{p_2}\end{pmatrix}^{-1}\widehat{\boldsymbol{\eta}}^{(g)},$$

$$\widehat{\boldsymbol{\Sigma}}=\begin{pmatrix}\boldsymbol{I}_{p_1} & \boldsymbol{0}\\ -\widehat{\boldsymbol{\Psi}}_{21} & \boldsymbol{I}_{p_2}\end{pmatrix}^{-1}\begin{pmatrix}\widehat{\boldsymbol{\Psi}}_{11} & \boldsymbol{0}\\ \boldsymbol{0} & \widehat{\boldsymbol{\Psi}}_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{I}_{p_1} & -\widehat{\boldsymbol{\Psi}}_{12}\\ \boldsymbol{0} & \boldsymbol{I}_{p_2}\end{pmatrix}^{-1}$$

and

$$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{I}_{p_1} & \boldsymbol{0}\\ -\boldsymbol{\Psi}_{21} & \boldsymbol{I}_{p_2}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{\Psi}_{11} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Psi}_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{I}_{p_1} & -\boldsymbol{\Psi}_{12}\\ \boldsymbol{0} & \boldsymbol{I}_{p_2}\end{pmatrix}^{-1},$$

$U_m$ is also expressed as

$$U_m = (\widehat{\boldsymbol{\eta}}_1^{(1)}-\widehat{\boldsymbol{\eta}}_1^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}(\widehat{\boldsymbol{\eta}}_1^{(1)}-\boldsymbol{\mu}_1^{(1)})+(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_2^{(1)})$$

$$
\begin{aligned}
&-\frac{1}{2}(\widehat{\boldsymbol{\eta}}_1^{(1)}-\widehat{\boldsymbol{\eta}}_1^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}(\widehat{\boldsymbol{\eta}}_1^{(1)}-\widehat{\boldsymbol{\eta}}_1^{(2)})-\frac{1}{2}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)})\\
= \;&(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}(\overline{\boldsymbol{x}}_{1T}^{(1)}-\boldsymbol{\mu}_1^{(1)})+(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\boldsymbol{\mu}_2^{(1)})\\
&-(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\boldsymbol{\mu}_1^{(1)})-(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\boldsymbol{\mu}_2^{(1)})\\
&+(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\boldsymbol{\mu}_1^{(1)})-\frac{1}{2}(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})\\
&-\frac{1}{2}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})+(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})\\
&-\frac{1}{2}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)}).
\end{aligned}
$$

Here all sample mean vectors $\overline{\boldsymbol{x}}_{1T}^{(g)}$, $\overline{\boldsymbol{x}}_{1F}^{(g)}$ and $\overline{\boldsymbol{x}}_{2F}^{(g)}$ are independent of $(\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ respectively. Then the expectation of $U_m$ given $\boldsymbol{\Gamma}^{(1)}$ and $\boldsymbol{\Gamma}^{(2)}$ is

$$
\begin{aligned}
\mathrm{E}(U_m|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}) \;=\; &-\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot 1}\right]\\
&-\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&-\frac{n(N^{(1)}-N^{(2)})}{2N^{(1)}N^{(2)}}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\right]\\
&+\frac{n_1(N_1^{(1)}-N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\right]\\
&-\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&-\frac{n}{2}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+n_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{n_1}{2}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{n_1}{2}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}
$$

Now we express $\mathrm{E}(U_m|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ by using $\mathrm{E}(U_{m1}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ and $\mathrm{E}(U_{m2}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ which are separated this expectation given $\boldsymbol{\Gamma}^{(1)}$ and $\boldsymbol{\Gamma}^{(2)}$ into the parts containing $\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}$ and not respectively. Therefore

$$
\mathrm{E}(U_m|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}) \;=\; \mathrm{E}(U_{m1}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})+\mathrm{E}(U_{m2}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}),
$$

where

$$
\mathrm{E}(U_{m1}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}) \;=\; \frac{n_1(N_1^{(1)}-N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\right]
$$

$$-\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$+n_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$-\frac{n_1}{2}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$$

and

$$\begin{aligned}
\mathrm{E}(U_{m2}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}) &= -\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot 1}\right] \\
&\quad -\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&\quad -\frac{n(N^{(1)}-N^{(2)})}{2N^{(1)}N^{(2)}}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\right] \\
&\quad -\frac{n}{2}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad -\frac{n_1}{2}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}).
\end{aligned}$$

Then we consider $\mathrm{E}(U_{m1})$ at first. Since $U_{m1}$ does not include $\boldsymbol{\Gamma}^{(2)}$ and $\boldsymbol{\Gamma}_{22\cdot 1}^{(1)}$ is independent of $(\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)})$, we can obtain $\mathrm{E}(U_{m1}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)})$. By Lemma 2.2, $\mathrm{E}(U_{m1}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)})$ is as follows:

$$\begin{aligned}
\mathrm{E}(U_{m1}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)}) &= \frac{n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&\quad -\frac{n_1}{2(n_1-p-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad +\frac{n_1(N_1^{(1)}-N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right] \\
&\quad -\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right].
\end{aligned}$$

Moreover $\mathrm{E}(U_{m1}|\boldsymbol{\Gamma}_{11}^{(1)})$ can be obtained by Lemma 2.2 and Lemma 3.2.

$$\begin{aligned}
\mathrm{E}(U_{m1}|\boldsymbol{\Gamma}_{11}^{(1)}) &= -\frac{p_2n_1}{2(n_1-p-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad +\frac{n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&\quad -\frac{n_1}{2(n_1-p-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad -\frac{p_2n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right] \\
&\quad +\frac{n_1(N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right].
\end{aligned}$$

Finally by Lemma 2.2,

$$
\begin{aligned}
\mathrm{E}(U_{m1}) &= -\frac{p_2 n_1}{2(n_1-p-1)(n_1-p_1-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad +\frac{n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&\quad -\frac{n_1}{2(n_1-p-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad -\frac{p_1 p_2 n_1 (N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}(n_1-p-1)(n_1-p_1-1)} \\
&\quad +\frac{n_1 (N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]. \qquad \text{(C.1)}
\end{aligned}
$$

On the other hand, we can calculate $\mathrm{E}(U_{m2})$ with leaving $\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}$ out of consideration. By making use of Lemma 2.2,

$$
\begin{aligned}
\mathrm{E}(U_{m2}) &= -\frac{n}{2(n-p_1-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad -\frac{n_1}{2(n_1-p-1)}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&\quad -\frac{n_1 (N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&\quad -\frac{p_1 n (N^{(1)}-N^{(2)})}{2N^{(1)}N^{(2)}(n-p_1-1)} \\
&\quad -\frac{p_2 n_1 (N_1^{(1)}-N_1^{(2)})}{2N_1^{(1)}N_1^{(2)}(n_1-p-1)}. \qquad \text{(C.2)}
\end{aligned}
$$

Thus $\mathrm{E}(U_m)=\mathrm{E}(U_{m1})+\mathrm{E}(U_{m2})$ is derived by (C.1) and (C.2).

On the other hand, $V_m$ is expressed as

$$
\begin{aligned}
V_m &= (\widehat{\boldsymbol{\eta}}_1^{(1)}-\widehat{\boldsymbol{\eta}}_1^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{11}^{-1}(\widehat{\boldsymbol{\eta}}_1^{(1)}-\widehat{\boldsymbol{\eta}}_1^{(2)}) \\
&\quad -2(\widehat{\boldsymbol{\eta}}_1^{(1)}-\widehat{\boldsymbol{\eta}}_1^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)}) \\
&\quad +2(\widehat{\boldsymbol{\eta}}_1^{(1)}-\widehat{\boldsymbol{\eta}}_1^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)}) \\
&\quad +(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)}) \\
&\quad -2(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)}) \\
&\quad +(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)}) \\
&\quad +(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{22}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\widehat{\boldsymbol{\eta}}_2^{(1)}-\widehat{\boldsymbol{\eta}}_2^{(2)}) \\
&= (\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{11}^{-1}(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)}) \\
&\quad -2(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
&+2(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})\\
&+2(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&-2(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'\widehat{\boldsymbol{\Psi}}_{11}^{-1}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})\\
&+(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&-2(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&+(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})\\
&-2(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&+2(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&+2(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&-2(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})\\
&+(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&-2(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&+(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})\\
&+(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{22}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&-2(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{22}\widehat{\boldsymbol{\Psi}}_{22}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})\\
&+(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\widehat{\boldsymbol{\Psi}}_{12}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\boldsymbol{\Psi}_{22}\widehat{\boldsymbol{\Psi}}_{22}^{-1}\widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)}).
\end{aligned}
$$

Also in this calculation, we separate $\mathrm{E}(V_m|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ into the terms containing $\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}$ in the same way as calculating $\mathrm{E}(U_m)$. Thus we express $\mathrm{E}(V_m|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ as follows:

$$
\mathrm{E}(V_m|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})=\mathrm{E}(V_{m1}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})+\mathrm{E}(V_{m2}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}),
$$

where $\mathrm{E}(V_{m1}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ and $\mathrm{E}(V_{m2}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ are the terms which contain $\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}$ and do not respectively.

First $\mathrm{E}(V_{m1}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ is calculated as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m1}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}) \;=\; & -\frac{4nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\right]\\
&-2nn_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]\\
&-2n_1^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-2nn_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot1}\right]
\end{aligned}
$$

$$-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]$$
$$-2n_1^2(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]$$
$$-2n_1^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$+\frac{2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}$$
$$\times\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$+2nn_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}$$
$$\times\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$$
$$+\frac{n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot1}\right]$$
$$+\frac{n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]$$
$$+n_1^2(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$+n_1^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot1}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$$
$$+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]$$
$$+2n_1^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\right]$$
$$+2n_1^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$+n_1^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}$$
$$\times\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$$
$$-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\right.$$
$$\left.\times\,\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\right]$$
$$-2n_1^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}$$

$$
\begin{aligned}
&\times \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}} (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&-2n_1^2 (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})' \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}} \boldsymbol{\Sigma}_{21} \\
&\times \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}} \boldsymbol{\Gamma}_{21}^{(1)} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+ \frac{n_1^2 (N_1^{(1)} + N_1)}{N_1^{(1)} N_1^{(2)}} \mathrm{tr} \left[ \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}} \boldsymbol{\Gamma}_{21}^{(1)} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \right. \\
&\left. \times\, \boldsymbol{\Sigma}_{11} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}} \boldsymbol{\Gamma}_{21}^{(1)} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Sigma}_{11} \right] \\
&n_1^2 (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})' \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}} \boldsymbol{\Gamma}_{21}^{(1)} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \\
&\times \boldsymbol{\Sigma}_{11} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}} \boldsymbol{\Gamma}_{21}^{(1)} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}).
\end{aligned}
$$

Now we consider separating $\mathrm{E}(V_{m1} | \boldsymbol{\Gamma}^{(1)}, \boldsymbol{\Gamma}^{(2)})$ again by the number of $\widehat{\boldsymbol{\Psi}}_{12}$, i.e.,

$$
\begin{aligned}
\mathrm{E}(V_{m1} | \boldsymbol{\Gamma}^{(1)}, \boldsymbol{\Gamma}^{(2)}) \;\; \equiv \;\; & \mathrm{E}(V_{m11} | \boldsymbol{\Gamma}^{(1)}, \boldsymbol{\Gamma}^{(2)}) + \mathrm{E}(V_{m12} | \boldsymbol{\Gamma}^{(1)}, \boldsymbol{\Gamma}^{(2)}) \\
& + \mathrm{E}(V_{m13} | \boldsymbol{\Gamma}^{(1)}, \boldsymbol{\Gamma}^{(2)}) + \mathrm{E}(V_{m14} | \boldsymbol{\Gamma}^{(1)}, \boldsymbol{\Gamma}^{(2)}),
\end{aligned}
$$

where $V_{m1r} (r = 1, \ldots, 4)$ denotes the terms including $r$ $\widehat{\boldsymbol{\Psi}}_{12}$'s in $V_{m1}$. Since we find that $\boldsymbol{\Gamma}_{22\cdot 1}^{(1)}$ is independent of $(\boldsymbol{\Gamma}_{11}^{(1)}, \boldsymbol{\Gamma}_{12}^{(1)}, \boldsymbol{\Gamma}^{(2)})$ and we have the conditional expectation of $\widehat{\boldsymbol{\Psi}}_{12}$ given $\boldsymbol{\Gamma}_{11}^{(1)}$ in Lemma 3.2, we can calculate conditional expectation of $V_{m1r}$ given $(\boldsymbol{\Gamma}_{11}^{(1)}, \boldsymbol{\Gamma}^{(2)})$ for each $r$.

By making use of Lemma 2.2, we can obtain $\mathrm{E}(V_{m11} | \boldsymbol{\Gamma}_{11}^{(1)}, \boldsymbol{\Gamma}_{12}^{(1)}, \boldsymbol{\Gamma}^{(2)})$ as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m11} | \boldsymbol{\Gamma}_{11}^{(1)}, \boldsymbol{\Gamma}_{12}^{(1)}, \boldsymbol{\Gamma}^{(2)}) \;\; = \;\; & -\frac{4 n n_1 (N^{(1)} + N^{(2)})}{N^{(1)} N^{(2)} (n_1 - p - 1)} \\
& \times \mathrm{tr} \left[ (\boldsymbol{\Gamma}_{11}^{(1)} + \boldsymbol{\Gamma}^{(2)})^{-1} \boldsymbol{\Sigma}_{11} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Sigma}_{22\cdot 1}^{-1} \boldsymbol{\Sigma}_{21} \right] \\
& - \frac{2 n n_1}{n_1 - p - 1} \\
& \times (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})' (\boldsymbol{\Gamma}_{11}^{(1)} + \boldsymbol{\Gamma}^{(2)})^{-1} \boldsymbol{\Sigma}_{11} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Sigma}_{22\cdot 1}^{-1} (\boldsymbol{\mu}_2^{(1)} - \boldsymbol{\mu}_2^{(2)}) \\
& - \frac{2 n n_1}{n_1 - p - 1} \\
& \times (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})' (\boldsymbol{\Gamma}_{11}^{(1)} + \boldsymbol{\Gamma}^{(2)})^{-1} \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22\cdot 1}^{-1} \boldsymbol{\Gamma}_{21}^{(1)} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
& - \frac{4 n_1^2 (N_1^{(1)} + N_1^{(2)})}{N_1^{(1)} N_1^{(2)} (n_1 - p)(n_1 - p - 3)} \mathrm{tr} \left[ \boldsymbol{\Sigma}_{22\cdot 1}^{-1} \boldsymbol{\Sigma}_{21} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \right] \\
& - \frac{4 n_1^2 (N_1^{(1)} + N_1^{(2)})}{N_1^{(1)} N_1^{(2)} (n_1 - p)(n_1 - p - 3)} \\
& \times \mathrm{tr} \left[ \boldsymbol{\Sigma}_{22\cdot 1}^{-1} \boldsymbol{\Sigma}_{21} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Gamma}_{12}^{(1)} \boldsymbol{\Sigma}_{22\cdot 1}^{-1} \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12} \right] \\
& - \frac{4 p_2 n_1^2 (N_1^{(1)} + N_1^{(2)})}{N_1^{(1)} N_1^{(2)} (n_1 - p)(n_1 - p - 1)(n_1 - p - 3)} \\
& \times \mathrm{tr} \left[ \boldsymbol{\Sigma}_{22\cdot 1}^{-1} \boldsymbol{\Gamma}_{21}^{(1)} \boldsymbol{\Gamma}_{11}^{(1)^{-1}} \boldsymbol{\Sigma}_{12} \right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}).
\end{aligned}
$$

By making use of Lemma 3.2, we can obtain $\mathrm{E}(V_{m11}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)})$ as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m11}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)}) \;=\;& -\frac{4nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n_1-p-1)}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\right] \\
&-\frac{2nn_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2nn_1}{n_1-p-1} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\right\}^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}).
\end{aligned}
$$

By making use of Lemma 2.2, $\mathrm{E}(V_{m12}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)},\boldsymbol{\Gamma}^{(2)})$ can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m12}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)},\boldsymbol{\Gamma}^{(2)}) \;=\;& \frac{2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n_1-p-1)} \\
&\times\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right] \\
&+\frac{2nn_1}{n_1-p-1} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right] \\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right] \\
&+\frac{n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Sigma}_{11}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right] \\
&\times(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Sigma}_{12}\right] \\
&+\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right]\mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right] \\
&+\frac{2n_1^2(n_1-p-2)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&+\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&+\frac{4n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\right] \\
&+\frac{2n_1^2(n_1-p-2)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&+\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Sigma}_{11}\right] \\
&+\frac{n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}
\end{aligned}
$$

$$
\begin{aligned}
&\times\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}
$$

By making use of Lemma 3.2, $\mathrm{E}(V_{m12}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)})$ can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m12}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)}) \quad = \quad &\frac{2p_2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n_1-p-1)}\\
&\times\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&+\frac{2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n_1-p-1)}\\
&\times\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\right]\\
&+\frac{2p_2nn_1}{n_1-p-1}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{2nn_1}{n_1-p-1}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{2p_2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&+\frac{6n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{6n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\right\}^2\\
&+\frac{n_1^2}{(n_1-p)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{2p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]\\
&+\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]\\
&+\frac{2p_2n_1^2(n_1-p-2)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{4n_1^2}{(n_1-p)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\\
&\times\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{6n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{4n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2n_1^2(n_1-p_1-2)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2}{(n_1-p)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\\
&\times\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{p_2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}
$$

Since $V_{m13}$ does not include $\boldsymbol{\Gamma}^{(2)}$, $\mathrm{E}(V_{m13}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)})$ can be calculated by making use of Lemma 2.2 as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m13}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)}) \;=\;& -\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\right]\\
&-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\right]\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right]\\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\\
&\times\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}
\end{aligned}
$$

$$\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\right]$$
$$\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right]$$
$$\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$-\frac{2n_1^2}{(n_1-p)(n_1-p-3)}$$
$$\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}$$
$$\times\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$$
$$-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right]$$
$$\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).$$

By making use of Lemma 3.2, $\mathrm{E}(V_{m13}|\boldsymbol{\Gamma}_{11}^{(1)})$ can be calculated as follows:

$$\mathrm{E}(V_{m13}|\boldsymbol{\Gamma}_{11}^{(1)}) = -\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]$$
$$-\frac{4p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]$$
$$-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]$$
$$-\frac{2n_1^2}{(n_1-p)(n_1-p-3)}$$
$$\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$
$$-\frac{2p_2n_1^2}{(n_1-p)(n_1-p-3)}$$
$$\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})$$

$$
\begin{aligned}
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{4n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&-\frac{8n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right] \\
&-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&-\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\right\}^2 \\
&-\frac{2p_2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&-\frac{4n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&-\frac{2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&\times (\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}
$$

Since $V_{m14}$ also does not include $\boldsymbol{\Gamma}^{(2)}$, $\mathrm{E}(V_{m14}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)})$ can be calculated by making use of Lemma 2.2 as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m14}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}_{12}^{(1)}) &= \frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right] \\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right] \\
&+\frac{n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times (\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11} \\
&\times \boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&+\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\right] \\
&\times (\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}
$$

By making use of Lemma 3.2, $\mathrm{E}(V_{m14}|\boldsymbol{\Gamma}_{11}^{(1)})$ can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m14}|\boldsymbol{\Gamma}_{11}^{(1)}) &= \frac{p_2^2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right] \\
&+\frac{p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)} \\
&\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right] \\
&+\frac{2p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}
\end{aligned}
$$

$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]$$
$$+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]$$
$$+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]$$
$$+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]$$
$$+\frac{2p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$+\frac{p_2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]$$
$$+\frac{2p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]$$
$$+\frac{4n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}$$
$$\times \mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{12}\right]$$
$$+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}$$
$$\times\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\right\}^2$$
$$+\frac{p_2^2n_1^2}{(n_1-p)(n_1-p-3)}$$
$$\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$$
$$+\frac{p_2n_1^2}{(n_1-p)(n_1-p-3)}$$
$$\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})$$
$$+\frac{2p_2n_1^2}{(n_1-p)(n_1-p-3)}$$

$$
\begin{aligned}
&\times(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{2n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{n_1^2}{(n_1-p)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{2p_2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{p_2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{4n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)} \\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right] \\
&\times(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}).
\end{aligned}
$$

Now that

$$
\mathrm{E}(V_{m1}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)}) = \mathrm{E}(V_{m11}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)}) + \mathrm{E}(V_{m12}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)}) + \mathrm{E}(V_{m13}|\boldsymbol{\Gamma}_{11}^{(1)}) + \mathrm{E}(V_{m14}|\boldsymbol{\Gamma}_{11}^{(1)}),
$$

$$
\begin{aligned}
\mathrm{E}(V_{m1}|\boldsymbol{\Gamma}_{11}^{(1)},\boldsymbol{\Gamma}^{(2)}) \quad = \quad & \frac{p_2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{2p_2nn_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)} + \boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{p_2^2n_1^2}{(n_1-p)(n_1-p-3)}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}) \\
&+\frac{p_2n_1^2}{(n_1-p)(n_1-p-3)}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2p_2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{p_2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&-\frac{2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{2nn_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{2n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{n_1^2}{(n_1-p)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{2p_2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&+\frac{p_2^2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&+\frac{p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&+\frac{2p_2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n_1-p-1)}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&+\frac{p_2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right]\\
&-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&-\frac{2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&-\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\right\}^2\\
&-\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right].
\end{aligned}
$$

Finally $\mathrm{E}(V_{m1})$ can be derived by making use of Lemma 2.2 and Lemma 3.3 as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m1}) \;=\;& \frac{p_2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{2p_2nn_1(n-1)}{(n-p_1)(n-p_1-3)(n_1-p_1-1)(n_1-p-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{p_2(p_2+1)n_1^2(n_1-1)}{(n_1-p)(n_1-p-3)(n_1-p_1)(n_1-p_1-1)(n_1-p_1-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{2p_2n_1^2(n_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-1)(n_1-p_1-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{p_1p_2n_1^2(n_1-p_1-2)}{(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{2p_2n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{n_1^2(n_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&-\frac{2n_1^2(n_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2nn_1}{(n_1-p-1)(n_1-p_1-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{p_1n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&-\frac{n_1^2}{(n_1-p)(n_1-p-3)}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{2p_1p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&+\frac{p_1p_2(p_2+1)n_1^2(n_1-1)(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)(n_1-p_1)(n_1-p_1-1)(n_1-p_1-3)}\\
&+\frac{2p_1p_2n_1^2(n_1-1)(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-1)(n_1-p_1-3)}\\
&+\frac{2p_1p_2nn_1(n-1)(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n-p_1)(n-p_1-3)(n_1-p_1-1)(n_1-p-1)}\\
&+\frac{p_1^2p_2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-2)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-3)}\\
&+\frac{2p_1p_2n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)(n_1-p_1)(n_1-p_1-3)}\\
&-\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&-\frac{2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n_1-p-1)(n-p_1-1)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&-\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\right\}^2\\
&-\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]. \qquad \text{(C.3)}
\end{aligned}
$$

On the other hand, $\mathrm{E}(V_{m2}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$ is calculated as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m2}|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}) \;=\;& \frac{n^2(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\right]\\
&+n^2(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\\
&\times(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+2nn_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{12}\\
&\times\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot 1}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot 1}\right]\\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot 1}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&+n_1^2(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{22\cdot 1}\right]\\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&+n_1^2(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}).
\end{aligned}
$$

By making use of Lemma 2.2 and noting that $\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)}$ and $\boldsymbol{\Gamma}_{22\cdot 1}^{(1)}$ are independent, $\mathrm{E}(V_{m2})$ can be calculated easily as follows:

$$
\begin{aligned}
\mathrm{E}(V_{m2}) \;=\;& \frac{n^2(n-1)}{(n-p_1)(n-p_1-1)(n-p_1-3)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})\\
&+\frac{2nn_1}{(n-p_1-1)(n_1-p-1)}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2(n_1-p_1-1)}{(n_1-p)(n_1-p-1)(n_1-p-3)}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2}{(n_1-p)(n_1-p-1)(n_1-p-3)}\\
&\times\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right](\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{n_1^2}{(n_1-p)(n_1-p-3)}\\
&\times(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})\\
&+\frac{2n_1^2(N_1^{(1)}+N_1^{(2)})(n_1-p_1-1)}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&+\frac{2nn_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n-p_1-1)(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}\left\{\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\right\}^2\\
&+\frac{n_1^2(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-3)}\mathrm{tr}\left[\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right]\\
&+\frac{p_1n^2(n-1)(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n-p_1)(n-p_1-1)(n-p_1-3)}
\end{aligned}
$$

$$+\frac{p_2 n_1^2(n_1-p_1-1)(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p)(n_1-p-1)(n_1-p-3)}. \tag{C.4}$$

Now we have obtained $\mathrm{E}(V_m)=\mathrm{E}(V_{m1})+\mathrm{E}(V_{m2})$ by (C.3) and (C.4). Thus Theorem 3.4 has been completed. □

## Appendix D Proof of Theorem 3.6

First we derive the unbiased estimator of $\delta_{11}^2$. Since

$$\begin{aligned} d_{m11}^2 &= (\widehat{\boldsymbol{\mu}}_1^{(1)}-\widehat{\boldsymbol{\mu}}_1^{(2)})'\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\widehat{\boldsymbol{\mu}}_1^{(1)}-\widehat{\boldsymbol{\mu}}_1^{(2)}) \\ &= n(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)}), \end{aligned}$$

we find the expectation of it given $(\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)})$:

$$\begin{aligned} \mathrm{E}\left(d_{m11}^2|\boldsymbol{\Gamma}^{(1)},\boldsymbol{\Gamma}^{(2)}\right) &= \frac{n(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\right] \\ &\quad +n(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}). \end{aligned}$$

Thus we can obtain the following expectation as follows by Lemma 2.2:

$$\mathrm{E}\left(d_{m11}^2\right) = \frac{n}{n-p_1-1}\left(\delta_{11}^2+\frac{p_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\right).$$

Therefore we derive the following statistic as the unbiased estimator of $\delta_{11}^2$:

$$\widehat{\delta}_{11}^2 = \frac{n-p_1-1}{n}d_{m11}^2-\frac{p_1(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}. \tag{D.1}$$

Next we derive the unbiased estimator of $\Delta^2$. Since

$$\begin{aligned} D_m^2 &= (\widehat{\boldsymbol{\mu}}_1^{(1)}-\widehat{\boldsymbol{\mu}}_1^{(2)})'\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\widehat{\boldsymbol{\mu}}_1^{(1)}-\widehat{\boldsymbol{\mu}}_1^{(2)}) \\ &\quad +(\widehat{\boldsymbol{\mu}}_1^{(1)}-\widehat{\boldsymbol{\mu}}_1^{(2)})'\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22\cdot1}^{-1}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\widehat{\boldsymbol{\mu}}_1^{(1)}-\widehat{\boldsymbol{\mu}}_1^{(2)}) \\ &\quad -2(\widehat{\boldsymbol{\mu}}_1^{(1)}-\widehat{\boldsymbol{\mu}}_1^{(2)})'\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22\cdot1}^{-1}(\widehat{\boldsymbol{\mu}}_2^{(1)}-\widehat{\boldsymbol{\mu}}_2^{(2)}) \\ &\quad +(\widehat{\boldsymbol{\mu}}_2^{(1)}-\widehat{\boldsymbol{\mu}}_2^{(2)})'\widehat{\boldsymbol{\Sigma}}_{22\cdot1}^{-1}(\widehat{\boldsymbol{\mu}}_2^{(1)}-\widehat{\boldsymbol{\mu}}_2^{(2)}) \\ &= n(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}(\overline{\boldsymbol{x}}_{1T}^{(1)}-\overline{\boldsymbol{x}}_{1T}^{(2)}) \\ &\quad +n_1(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})'\boldsymbol{\Gamma}_{22\cdot1}^{-1}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)})' \\ &\quad -2n_1(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}(\overline{\boldsymbol{x}}_{2F}^{(1)}-\overline{\boldsymbol{x}}_{2F}^{(2)}) \\ &\quad +n_1(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Gamma}_{12}^{(1)}\boldsymbol{\Gamma}_{22\cdot1}^{(1)^{-1}}\boldsymbol{\Gamma}_{21}^{(1)}\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\overline{\boldsymbol{x}}_{1F}^{(1)}-\overline{\boldsymbol{x}}_{1F}^{(2)}), \end{aligned}$$

the conditional expectation of $D_m^2$ given $(\mathbf{\Gamma}^{(1)}, \mathbf{\Gamma}^{(2)})$ can be obtained:

$$\begin{aligned}
\mathrm{E}\left(D_m^2|\mathbf{\Gamma}^{(1)}, \mathbf{\Gamma}^{(2)}\right) \;=\; & \frac{n(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\mathrm{tr}\left[(\mathbf{\Gamma}_{11}^{(1)}+\mathbf{\Gamma}^{(2)})^{-1}\mathbf{\Sigma}_{11}\right] \\
& +n(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\mathbf{\Gamma}_{11}^{(1)}+\mathbf{\Gamma}^{(2)})^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
& +\frac{n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\mathbf{\Sigma}_{22\cdot 1}\right] \\
& +\frac{n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\right] \\
& +n_1(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
& -\frac{2n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right] \\
& -2n_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
& +\frac{n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}}\mathrm{tr}\left[\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Sigma}_{11}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right] \\
& +n_1(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Gamma}_{22\cdot 1}^{(1)^{-1}}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}$$

Therefore we find the conditional expectation of $D_m^2$ given $(\mathbf{\Gamma}_{11}^{(1)}, \mathbf{\Gamma}_{12}^{(1)}, \mathbf{\Gamma}^{(2)})$ by Lemma 2.2 as follows:

$$\begin{aligned}
\mathrm{E}\left(D_m^2|\mathbf{\Gamma}_{11}^{(1)}, \mathbf{\Gamma}_{12}^{(1)}, \mathbf{\Gamma}^{(2)}\right) \;=\; & \frac{n(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\mathrm{tr}\left[(\mathbf{\Gamma}_{11}^{(1)}+\mathbf{\Gamma}^{(2)})^{-1}\mathbf{\Sigma}_{11}\right] \\
& +n(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\mathbf{\Gamma}_{11}^{(1)}+\mathbf{\Gamma}^{(2)})^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
& +\frac{p_2 n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)} \\
& +\frac{n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\right] \\
& +\frac{n_1}{n_1-p-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
& -\frac{2n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Sigma}_{21}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right] \\
& -\frac{2n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
& +\frac{n_1(N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Sigma}_{11}\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\right] \\
& +\frac{n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\mathbf{\Gamma}_{11}^{(1)^{-1}}\mathbf{\Gamma}_{12}^{(1)}\mathbf{\Sigma}_{22\cdot 1}^{-1}\mathbf{\Gamma}_{21}^{(1)}\mathbf{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}$$

Moreover we can calculate the conditional expectation of $D_m^2$ given $(\boldsymbol{\Gamma}_{11}^{(1)}, \boldsymbol{\Gamma}^{(2)})$ by Lemma 3.2 as follows:

$$\begin{aligned}
\mathrm{E}\left(D_m^2 | \boldsymbol{\Gamma}_{11}^{(1)}, \boldsymbol{\Gamma}^{(2)}\right) &= \frac{n(N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}}\mathrm{tr}\left[(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}\boldsymbol{\Sigma}_{11}\right] \\
&\quad + n(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'(\boldsymbol{\Gamma}_{11}^{(1)}+\boldsymbol{\Gamma}^{(2)})^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad + \frac{p_2 n_1 (N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)} \\
&\quad + \frac{n_1}{n_1-p-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)})'\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&\quad - \frac{2n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}(\boldsymbol{\mu}_2^{(1)}-\boldsymbol{\mu}_2^{(2)}) \\
&\quad + \frac{p_2 n_1 (N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)}\mathrm{tr}\left[\boldsymbol{\Gamma}_{11}^{(1)^{-1}}\boldsymbol{\Sigma}_{11}\right] \\
&\quad + \frac{p_2 n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Gamma}_{11}^{(1)^{-1}}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}) \\
&\quad + \frac{n_1}{n_1-p-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)})'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22\cdot 1}^{-1}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{\mu}_1^{(1)}-\boldsymbol{\mu}_1^{(2)}).
\end{aligned}$$

By using the result of Lemma 2.2 again, we can obtain $\mathrm{E}(D_m^2)$ as follows:

$$\begin{aligned}
\mathrm{E}(D_m^2) &= \frac{n_1}{n_1-p-1}\Delta^2 + \left(\frac{n}{n-p_1-1}-\frac{n_1}{n_1-p_1-1}\right)\delta_{11}^2 \\
&\quad + \frac{p_1 n (N^{(1)}+N^{(2)})}{N^{(1)}N^{(2)}(n-p_1-1)} + \frac{p_2 n_1 (N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)} \\
&\quad + \frac{p_1 p_2 n_1 (N_1^{(1)}+N_1^{(2)})}{N_1^{(1)}N_1^{(2)}(n_1-p-1)(n_1-p_1-1)}. \qquad \text{(D.2)}
\end{aligned}$$

By the results of (D.1) and (D.2), Theorem 3.6 has been completed. □

Table 1. The Comparisons of the Accuracy of Asymptotic Approximations for EPMC where $p = 2$.

| $p = 2$ $(p_1 = 1,\ p_2 = 1)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(10, 10, 0, 0)$ | 1.05 | 0.3893 | 0.3773 | 0.3291 |
| | 1.68 | 0.2764 | 0.2548 | 0.2217 |
| | 2.56 | 0.1656 | 0.1403 | 0.1164 |
| | 3.29 | 0.1030 | 0.08044 | 0.06144 |
| $(10, 10, 20, 20)$ | 1.05 | 0.3365 | - | 0.3211 |
| | 1.68 | 0.2316 | - | 0.2174 |
| | 2.56 | 0.1249 | - | 0.1129 |
| | 3.29 | 0.06856 | - | 0.05883 |
| $(10, 10, 20, 40)$ | 1.05 | 0.3353 | - | 0.3217 |
| | 1.68 | 0.2299 | - | 0.2177 |
| | 2.56 | 0.1230 | - | 0.1130 |
| | 3.29 | 0.06681 | - | 0.05908 |
| $(10, 10, 40, 40)$ | 1.05 | 0.3306 | - | 0.3201 |
| | 1.68 | 0.2268 | - | 0.2167 |
| | 2.56 | 0.1209 | - | 0.1126 |
| | 3.29 | 0.06529 | - | 0.06190 |
| $(10, 20, 0, 0)$ | 1.05 | 0.3815 | 0.3742 | 0.3307 |
| | 1.68 | 0.2608 | 0.2478 | 0.2210 |
| | 2.56 | 0.1461 | 0.1314 | 0.1141 |
| | 3.29 | 0.08460 | 0.07205 | 0.05927 |
| $(10, 20, 20, 20)$ | 1.05 | 0.3389 | - | 0.3245 |
| | 1.68 | 0.2288 | - | 0.2181 |
| | 2.56 | 0.1201 | - | 0.1116 |
| | 3.29 | 0.06398 | - | 0.05750 |
| $(10, 20, 20, 40)$ | 1.05 | 0.3380 | - | 0.3235 |
| | 1.68 | 0.2276 | - | 0.2169 |
| | 2.56 | 0.1186 | - | 0.1110 |
| | 3.29 | 0.06268 | - | 0.05701 |
| $(10, 20, 40, 40)$ | 1.05 | 0.3239 | - | 0.3228 |
| | 1.68 | 0.2244 | - | 0.2166 |
| | 2.56 | 0.1166 | - | 0.1105 |
| | 3.29 | 0.06126 | - | 0.05658 |

Table 1. Continued.

| $p = 2$ $(p_1 = 1,\ p_2 = 1)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(10, 30, 0, 0)$ | 1.05 | 0.3777 | 0.3724 | 0.3304 |
| | 1.68 | 0.2538 | 0.2445 | 0.2192 |
| | 2.56 | 0.1377 | 0.1274 | 0.1123 |
| | 3.29 | 0.07690 | 0.06828 | 0.05771 |
| $(10, 30, 20, 20)$ | 1.05 | 0.3390 | - | 0.3248 |
| | 1.68 | 0.2272 | - | 0.2167 |
| | 2.56 | 0.1178 | - | 0.1102 |
| | 3.29 | 0.06192 | - | 0.05615 |
| $(10, 30, 20, 40)$ | 1.05 | 0.3383 | - | 0.3241 |
| | 1.68 | 0.2262 | - | 0.2164 |
| | 2.56 | 0.1166 | - | 0.1097 |
| | 3.29 | 0.06090 | - | 0.05591 |
| $(10, 30, 40, 40)$ | 1.05 | 0.3330 | - | 0.3233 |
| | 1.68 | 0.2229 | - | 0.2158 |
| | 2.56 | 0.1146 | - | 0.1090 |
| | 3.29 | 0.05953 | - | 0.05538 |
| $(20, 10, 0, 0)$ | 1.05 | 0.3457 | 0.3363 | 0.3126 |
| | 1.68 | 0.2423 | 0.2285 | 0.2120 |
| | 2.56 | 0.1377 | 0.1230 | 0.1102 |
| | 3.29 | 0.08011 | 0.06780 | 0.05736 |
| $(20, 10, 20, 20)$ | 1.05 | 0.3183 | - | 0.3085 |
| | 1.68 | 0.2181 | - | 0.2092 |
| | 2.56 | 0.1154 | - | 0.1077 |
| | 3.29 | 0.06167 | - | 0.05545 |
| $(20, 10, 20, 40)$ | 1.05 | 0.3174 | - | 0.3079 |
| | 1.68 | 0.2168 | - | 0.2083 |
| | 2.56 | 0.1139 | - | 0.1068 |
| | 3.29 | 0.06034 | - | 0.05492 |
| $(20, 10, 40, 40)$ | 1.05 | 0.3141 | - | 0.3073 |
| | 1.68 | 0.2145 | - | 0.2078 |
| | 2.56 | 0.1122 | - | 0.1065 |
| | 3.29 | 0.05916 | - | 0.05448 |

Table 1. Continued.

| $p = 2$ $(p_1 = 1,\ p_2 = 1)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(20, 20, 0, 0)$ | 1.05 | 0.3428 | 0.3361 | 0.3142 |
| | 1.68 | 0.2357 | 0.2259 | 0.2111 |
| | 2.56 | 0.1292 | 0.1189 | 0.1083 |
| | 3.29 | 0.07232 | 0.06385 | 0.05569 |
| $(20, 20, 20, 20)$ | 1.05 | 0.3225 | - | 0.3117 |
| | 1.68 | 0.2183 | - | 0.2096 |
| | 2.56 | 0.1138 | - | 0.1068 |
| | 3.29 | 0.05998 | - | 0.05447 |
| $(20, 20, 20, 40)$ | 1.05 | 0.3216 | - | 0.3110 |
| | 1.68 | 0.2172 | - | 0.2091 |
| | 2.56 | 0.1126 | - | 0.1067 |
| | 3.29 | 0.05889 | - | 0.05429 |
| $(20, 20, 40, 40)$ | 1.05 | 0.3182 | - | 0.3105 |
| | 1.68 | 0.2149 | - | 0.2089 |
| | 2.56 | 0.1111 | - | 0.1059 |
| | 3.29 | 0.05783 | - | 0.05347 |
| $(20, 40, 0, 0)$ | 1.05 | 0.3391 | 0.3348 | 0.3137 |
| | 1.68 | 0.2292 | 0.2230 | 0.2097 |
| | 2.56 | 0.1215 | 0.1150 | 0.1062 |
| | 3.29 | 0.06544 | 0.06027 | 0.05372 |
| $(20, 40, 20, 20)$ | 1.05 | 0.3233 | - | 0.3122 |
| | 1.68 | 0.2169 | - | 0.2090 |
| | 2.56 | 0.1116 | - | 0.1054 |
| | 3.29 | 0.05793 | - | 0.05306 |
| $(20, 40, 20, 40)$ | 1.05 | 0.3227 | - | 0.3118 |
| | 1.68 | 0.2161 | - | 0.2086 |
| | 2.56 | 0.1108 | - | 0.1054 |
| | 3.29 | 0.05723 | - | 0.05324 |
| $(20, 40, 40, 40)$ | 1.05 | 0.3194 | - | 0.3118 |
| | 1.68 | 0.2142 | - | 0.2082 |
| | 2.56 | 0.1095 | - | 0.1052 |
| | 3.29 | 0.05638 | - | 0.05330 |

Table 1. Continued.

| $p = 2$ $(p_1 = 1,\ p_2 = 1)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(30, 10, 0, 0)$ | 1.05 | 0.3294 | 0.3223 | 0.3060 |
| | 1.68 | 0.2293 | 0.2193 | 0.2079 |
| | 2.56 | 0.1267 | 0.1165 | 0.1070 |
| | 3.29 | 0.07129 | 0.06292 | 0.05524 |
| $(30, 10, 20, 20)$ | 1.05 | 0.3112 | - | 0.3037 |
| | 1.68 | 0.2128 | - | 0.2052 |
| | 2.56 | 0.1116 | - | 0.1051 |
| | 3.29 | 0.05892 | - | 0.05370 |
| $(30, 10, 20, 40)$ | 1.05 | 0.3102 | - | 0.3033 |
| | 1.68 | 0.2115 | - | 0.2047 |
| | 2.56 | 0.1102 | - | 0.1047 |
| | 3.29 | 0.05773 | - | 0.05324 |
| $(30, 10, 40, 40)$ | 1.05 | 0.3080 | - | 0.3022 |
| | 1.68 | 0.2099 | - | 0.2043 |
| | 2.56 | 0.1090 | - | 0.1035 |
| | 3.29 | 0.05686 | - | 0.05260 |
| $(40, 20, 0, 0)$ | 1.05 | 0.3214 | 0.3168 | 0.3051 |
| | 1.68 | 0.2202 | 0.2139 | 0.2057 |
| | 2.56 | 0.1175 | 0.1111 | 0.1044 |
| | 3.29 | 0.06349 | 0.05839 | 0.05290 |
| $(40, 20, 20, 20)$ | 1.05 | 0.3117 | - | 0.3043 |
| | 1.68 | 0.2110 | - | 0.2045 |
| | 2.56 | 0.1091 | - | 0.1035 |
| | 3.29 | 0.05673 | - | 0.05221 |
| $(40, 20, 20, 40)$ | 1.05 | 0.3111 | - | 0.3039 |
| | 1.68 | 0.2102 | - | 0.2040 |
| | 2.56 | 0.1082 | - | 0.1033 |
| | 3.29 | 0.05597 | - | 0.05229 |
| $(40, 20, 40, 40)$ | 1.05 | 0.3094 | - | 0.3036 |
| | 1.68 | 0.2090 | - | 0.2039 |
| | 2.56 | 0.1073 | - | 0.1034 |
| | 3.29 | 0.05533 | - | 0.05225 |

Table 2. The Comparisons of the Accuracy of Asymptotic Approximations for EPMC where $p = 5$.

| $p = 5$ ($p_1 = 3,\ p_2 = 2$) | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(10, 10, 0, 0)$ | 1.05 | 0.4245 | 0.4126 | 0.3704 |
| | 1.68 | 0.3244 | 0.2987 | 0.2640 |
| | 2.56 | 0.2127 | 0.1778 | 0.1505 |
| | 3.29 | 0.1445 | 0.1100 | 0.08746 |
| $(10, 10, 20, 20)$ | 1.05 | 0.3703 | - | 0.3549 |
| | 1.68 | 0.2680 | - | 0.2494 |
| | 2.56 | 0.1572 | - | 0.1383 |
| | 3.29 | 0.09453 | - | 0.07782 |
| $(10, 10, 20, 40)$ | 1.05 | 0.3714 | - | 0.3579 |
| | 1.68 | 0.2675 | - | 0.2507 |
| | 2.56 | 0.1558 | - | 0.1387 |
| | 3.29 | 0.09291 | - | 0.07791 |
| $(10, 10, 40, 40)$ | 1.05 | 0.3643 | - | 0.3525 |
| | 1.68 | 0.2625 | - | 0.2471 |
| | 2.56 | 0.1525 | - | 0.1368 |
| | 3.29 | 0.09061 | - | 0.07667 |
| $(10, 20, 0, 0)$ | 1.05 | 0.4272 | 0.4215 | 0.3828 |
| | 1.68 | 0.3051 | 0.2912 | 0.2626 |
| | 2.56 | 0.1799 | 0.1616 | 0.1423 |
| | 3.29 | 0.1099 | 0.09296 | 0.07811 |
| $(10, 20, 20, 20)$ | 1.05 | 0.3734 | - | 0.3595 |
| | 1.68 | 0.2579 | - | 0.2453 |
| | 2.56 | 0.1410 | - | 0.1304 |
| | 3.29 | 0.07869 | - | 0.07011 |
| $(10, 20, 20, 40)$ | 1.05 | 0.3742 | - | 0.3598 |
| | 1.68 | 0.2573 | - | 0.2446 |
| | 2.56 | 0.1394 | - | 0.1299 |
| | 3.29 | 0.07708 | - | 0.06914 |
| $(10, 20, 40, 40)$ | 1.05 | 0.3653 | - | 0.3542 |
| | 1.68 | 0.2512 | - | 0.2412 |
| | 2.56 | 0.1357 | - | 0.1274 |
| | 3.29 | 0.07467 | - | 0.06816 |

Table 2. Continued.

| $p = 5$ $(p_1 = 3,\ p_2 = 2)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(10, 30, 0, 0)$ | 1.05 | 0.4282 | 0.4245 | 0.3875 |
| | 1.68 | 0.2968 | 0.2873 | 0.2612 |
| | 2.56 | 0.1665 | 0.1543 | 0.1374 |
| | 3.29 | 0.09655 | 0.08562 | 0.07373 |
| $(10, 30, 20, 20)$ | 1.05 | 0.3738 | - | 0.3594 |
| | 1.68 | 0.2536 | - | 0.2424 |
| | 2.56 | 0.1348 | - | 0.1263 |
| | 3.29 | 0.07312 | - | 0.06660 |
| $(10, 30, 20, 40)$ | 1.05 | 0.3744 | - | 0.3593 |
| | 1.68 | 0.2531 | - | 0.2418 |
| | 2.56 | 0.1336 | - | 0.1254 |
| | 3.29 | 0.07186 | - | 0.06549 |
| $(10, 30, 40, 40)$ | 1.05 | 0.3648 | - | 0.3548 |
| | 1.68 | 0.2467 | - | 0.2388 |
| | 2.56 | 0.1299 | - | 0.1237 |
| | 3.29 | 0.06951 | - | 0.06482 |
| $(20, 10, 0, 0)$ | 1.05 | 0.3596 | 0.3490 | 0.3278 |
| | 1.68 | 0.2634 | 0.2472 | 0.2305 |
| | 2.56 | 0.1586 | 0.1402 | 0.1267 |
| | 3.29 | 0.09783 | 0.08144 | 0.07011 |
| $(20, 10, 20, 20)$ | 1.05 | 0.3322 | - | 0.3245 |
| | 1.68 | 0.2342 | - | 0.2249 |
| | 2.56 | 0.1298 | - | 0.1206 |
| | 3.29 | 0.07291 | - | 0.06501 |
| $(20, 10, 20, 40)$ | 1.05 | 0.3331 | - | 0.3253 |
| | 1.68 | 0.2336 | - | 0.2240 |
| | 2.56 | 0.1283 | - | 0.1195 |
| | 3.29 | 0.07139 | - | 0.06392 |
| $(20, 10, 40, 40)$ | 1.05 | 0.3275 | - | 0.3215 |
| | 1.68 | 0.2296 | - | 0.2218 |
| | 2.56 | 0.1257 | - | 0.1181 |
| | 3.29 | 0.06959 | - | 0.06326 |

Table 2. Continued.

| $p = 5$ $(p_1 = 3,\ p_2 = 2)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(20, 20, 0, 0)$ | 1.05 | 0.3674 | 0.3607 | 0.3397 |
| | 1.68 | 0.2585 | 0.2476 | 0.2325 |
| | 2.56 | 0.1470 | 0.1348 | 0.1236 |
| | 3.29 | 0.08575 | 0.07516 | 0.06617 |
| $(20, 20, 20, 20)$ | 1.05 | 0.3411 | - | 0.3315 |
| | 1.68 | 0.2344 | - | 0.2254 |
| | 2.56 | 0.1258 | - | 0.1180 |
| | 3.29 | 0.06851 | - | 0.06213 |
| $(20, 20, 20, 40)$ | 1.05 | 0.3416 | - | 0.3310 |
| | 1.68 | 0.2337 | - | 0.2246 |
| | 2.56 | 0.1245 | - | 0.1169 |
| | 3.29 | 0.06718 | - | 0.06140 |
| $(20, 20, 40, 40)$ | 1.05 | 0.3354 | - | 0.3275 |
| | 1.68 | 0.2296 | - | 0.2225 |
| | 2.56 | 0.1219 | - | 0.1163 |
| | 3.29 | 0.06549 | - | 0.06108 |
| $(20, 40, 0, 0)$ | 1.05 | 0.3692 | 0.3653 | 0.3463 |
| | 1.68 | 0.2515 | 0.2449 | 0.2322 |
| | 2.56 | 0.1359 | 0.1286 | 0.1202 |
| | 3.29 | 0.07508 | 0.06901 | 0.06271 |
| $(20, 40, 20, 20)$ | 1.05 | 0.3444 | - | 0.3349 |
| | 1.68 | 0.2322 | - | 0.2246 |
| | 2.56 | 0.1214 | - | 0.1155 |
| | 3.29 | 0.06430 | - | 0.05983 |
| $(20, 40, 20, 40)$ | 1.05 | 0.3445 | - | 0.3341 |
| | 1.68 | 0.2316 | - | 0.2242 |
| | 2.56 | 0.1204 | - | 0.1148 |
| | 3.29 | 0.06337 | - | 0.05907 |
| $(20, 40, 40, 40)$ | 1.05 | 0.3378 | - | 0.3303 |
| | 1.68 | 0.2274 | - | 0.2219 |
| | 2.56 | 0.1180 | - | 0.1134 |
| | 3.29 | 0.06186 | - | 0.05856 |

Table 2. Continued.

| $p = 5$ $(p_1 = 3,\ p_2 = 2)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(30, 10, 0, 0)$ | 1.05 | 0.3331 | 0.3249 | 0.3099 |
| | 1.68 | 0.2400 | 0.2286 | 0.2167 |
| | 2.56 | 0.1387 | 0.1266 | 0.1172 |
| | 3.29 | 0.08159 | 0.07126 | 0.06333 |
| $(30, 10, 20, 20)$ | 1.05 | 0.3177 | - | 0.3110 |
| | 1.68 | 0.2221 | - | 0.2145 |
| | 2.56 | 0.1205 | - | 0.1134 |
| | 3.29 | 0.06591 | - | 0.05997 |
| $(30, 10, 20, 40)$ | 1.05 | 0.3186 | - | 0.3125 |
| | 1.68 | 0.2215 | - | 0.2141 |
| | 2.56 | 0.1191 | - | 0.1123 |
| | 3.29 | 0.06456 | - | 0.05917 |
| $(30, 10, 40, 40)$ | 1.05 | 0.3143 | - | 0.3103 |
| | 1.68 | 0.2185 | - | 0.2128 |
| | 2.56 | 0.1171 | - | 0.1116 |
| | 3.29 | 0.06316 | - | 0.05846 |
| $(40, 20, 0, 0)$ | 1.05 | 0.3305 | 0.3255 | 0.3155 |
| | 1.68 | 0.2301 | 0.2232 | 0.2156 |
| | 2.56 | 0.1261 | 0.1189 | 0.1126 |
| | 3.29 | 0.07009 | 0.06418 | 0.05927 |
| $(40, 20, 20, 20)$ | 1.05 | 0.3199 | - | 0.3142 |
| | 1.68 | 0.2190 | - | 0.2134 |
| | 2.56 | 0.1155 | - | 0.1104 |
| | 3.29 | 0.06143 | - | 0.05734 |
| $(40, 20, 20, 40)$ | 1.05 | 0.3204 | - | 0.3139 |
| | 1.68 | 0.2185 | - | 0.2131 |
| | 2.56 | 0.1146 | - | 0.1097 |
| | 3.29 | 0.06051 | - | 0.05676 |
| $(40, 20, 40, 40)$ | 1.05 | 0.3170 | - | 0.3120 |
| | 1.68 | 0.2162 | - | 0.2216 |
| | 2.56 | 0.1131 | - | 0.1090 |
| | 3.29 | 0.05949 | - | 0.05615 |

Table 3. The Comparisons of the Accuracy of Asymptotic Approximations for EPMC where $p = 10$.

| $p = 10$ $(p_1 = 5,\ p_2 = 5)$ | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(20, 20, 0, 0)$ | 1.05 | 0.3975 | 0.3905 | 0.3709 |
| | 1.68 | 0.2942 | 0.2814 | 0.2662 |
| | 2.56 | 0.1798 | 0.1637 | 0.1518 |
| | 3.29 | 0.1128 | 0.09761 | 0.08807 |
| $(20, 20, 20, 20)$ | 1.05 | 0.3716 | - | 0.3634 |
| | 1.68 | 0.2675 | - | 0.2578 |
| | 2.56 | 0.1544 | - | 0.1450 |
| | 3.29 | 0.09083 | - | 0.08234 |
| $(20, 20, 20, 40)$ | 1.05 | 0.3735 | - | 0.3659 |
| | 1.68 | 0.2677 | - | 0.2582 |
| | 2.56 | 0.1536 | - | 0.1441 |
| | 3.29 | 0.08972 | - | 0.08143 |
| $(20, 20, 40, 40)$ | 1.05 | 0.3672 | - | 0.3608 |
| | 1.68 | 0.2632 | - | 0.2551 |
| | 2.56 | 0.1507 | - | 0.1421 |
| | 3.29 | 0.08770 | - | 0.08030 |
| $(20, 40, 0, 0)$ | 1.05 | 0.4070 | 0.4036 | 0.3849 |
| | 1.68 | 0.2858 | 0.2788 | 0.2645 |
| | 2.56 | 0.1611 | 0.1523 | 0.1421 |
| | 3.29 | 0.09313 | 0.08532 | 0.07761 |
| $(20, 40, 20, 20)$ | 1.05 | 0.3798 | - | 0.3702 |
| | 1.68 | 0.2623 | - | 0.2529 |
| | 2.56 | 0.1426 | - | 0.1348 |
| | 3.29 | 0.07892 | - | 0.07282 |
| $(20, 40, 20, 40)$ | 1.05 | 0.3809 | - | 0.3720 |
| | 1.68 | 0.2622 | - | 0.2542 |
| | 2.56 | 0.1417 | - | 0.1350 |
| | 3.29 | 0.07792 | - | 0.07290 |
| $(20, 40, 40, 40)$ | 1.05 | 0.3731 | - | 0.3661 |
| | 1.68 | 0.2568 | - | 0.2503 |
| | 2.56 | 0.1384 | - | 0.1333 |
| | 3.29 | 0.07584 | - | 0.07157 |

Table 3. Continued.

| $p = 10$ ($p_1 = 5,\ p_2 = 5$) | | | | |
|---|---|---|---|---|
| $(N_1^{(1)}, N_1^{(2)}, N_2^{(1)}, N_2^{(2)})$ | $\Delta$ | Lachenbruch/Shutoh | Fujikoshi and Seo | Simulation |
| $(30, 10, 0, 0)$ | 1.05 | 0.3393 | 0.3288 | 0.3152 |
| | 1.68 | 0.2583 | 0.2439 | 0.2324 |
| | 2.56 | 0.1615 | 0.1453 | 0.1351 |
| | 3.29 | 0.1024 | 0.08775 | 0.07920 |
| $(30, 10, 20, 20)$ | 1.05 | 0.3250 | - | 0.3216 |
| | 1.68 | 0.2398 | - | 0.2325 |
| | 2.56 | 0.1411 | - | 0.1320 |
| | 3.29 | 0.08372 | - | 0.07552 |
| $(30, 10, 20, 40)$ | 1.05 | 0.3273 | - | 0.3245 |
| | 1.68 | 0.2403 | - | 0.2335 |
| | 2.56 | 0.1404 | - | 0.1319 |
| | 3.29 | 0.08271 | - | 0.07497 |
| $(30, 10, 40, 40)$ | 1.05 | 0.3230 | - | 0.3211 |
| | 1.68 | 0.2371 | - | 0.2310 |
| | 2.56 | 0.1382 | - | 0.1305 |
| | 3.29 | 0.08111 | - | 0.07400 |
| $(40, 20, 0, 0)$ | 1.05 | 0.3418 | 0.3362 | 0.3268 |
| | 1.68 | 0.2457 | 0.2377 | 0.2297 |
| | 2.56 | 0.1412 | 0.1325 | 0.1257 |
| | 3.29 | 0.08236 | 0.07487 | 0.06951 |
| $(40, 20, 20, 20)$ | 1.05 | 0.3311 | - | 0.3265 |
| | 1.68 | 0.2335 | - | 0.2272 |
| | 2.56 | 0.1289 | - | 0.1230 |
| | 3.29 | 0.07184 | - | 0.06668 |
| $(40, 20, 20, 40)$ | 1.05 | 0.3330 | - | 0.3285 |
| | 1.68 | 0.2337 | - | 0.2283 |
| | 2.56 | 0.1282 | - | 0.1230 |
| | 3.29 | 0.07098 | - | 0.06645 |
| $(40, 20, 40, 40)$ | 1.05 | 0.3289 | - | 0.3245 |
| | 1.68 | 0.2308 | - | 0.2255 |
| | 2.56 | 0.1263 | - | 0.1211 |
| | 3.29 | 0.06968 | - | 0.06538 |