# Improvement of the Quality of the Chi-square Approximation for the ADF Test on a Covariance Matrix with a Linear Structure

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Abstract

The asymptotically distribution-free (ADF) test statistic was proposed by Browne (1984). It is known that the null distribution of the ADF test statistic is asymptotically distributed according to the chi-square distribution. This asymptotic property is always satisfied, even under nonnormality, although the null distributions of other famous test statistics, e.g., the maximum likelihood test statistic and the generalized least square test statistic, do not converge to the chi-square distribution under nonnormality. However, many authors have reported numerical results which indicate that the quality of the chi-square approximation for the ADF test is very poor, even when the sample size is large and the population distribution is normal. In this paper, we try to improve the quality of the chi-square approximation to the ADF test for a covariance matrix with a linear structure by using the Bartlett correction applicable under the assumption of normality. By conducting numerical studies, we verify that the obtained Bartlett correction can perform well even when the assumption of normality is violated.

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## 1. Introduction

Let $\boldsymbol{y}$ be a $p$-dimensional random vector with mean $\mathrm{E}[\boldsymbol{y}] = \boldsymbol{\mu}$ and covariance matrix $\mathrm{Cov}[\boldsymbol{y}] = \boldsymbol{\Sigma}$, and let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ be $n$ independent samples from $\boldsymbol{y}$. Then, we consider the following hypothesis test on the covariance structure:

$$H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta}), \quad H_1 : \text{not } H_0, \tag{1.1}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)'$ is the $q$-dimensional unknown parameter vector,

In order to test the hypothesis (1.1), we will generally assume that $\boldsymbol{y}$ is distributed according to the multivariate normal distribution. Under the assumption of normality, two famous test statistics, the maximum likelihood (ML) test statistic and the generalized least square (GLS) test statistic, have been utilized. It is well known that the null distributions of both test statistics converge to the chi-square distribution with $p^* - q$ degrees of freedom as $n \to \infty$ when $\boldsymbol{y} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $p^* = p(p+1)/2$. However, it is also known that the null distributions of the two test statistics do not converge to the chi-square distribution if the population distribution, i.e., the true distribution of $\boldsymbol{y}$, is not normal. In fact, when the assumption of normality is violated, the null distributions of the ML and GLS statistics converge to a weighted sum of chi-square distributions with one degree of freedom, whose weights depend on the fourth moments of the true distribution (e.g., Browne, 1984; Yuan & Bentler, 1999a; Yanagihara, Tonda & Matsumoto, 2005). Nevertheless, many researchers will take the chi-square distribution as the null distribution of the ML and GLS test statistics because nobody knows the true distribution. The chi-square approximation may therefore lead us to a false conclusion when the true distribution is not normal, because the actual test size (or significance level) becomes different from the nominal test size.

On the other hand, Brown (1984) adjusted the GLS test statistic so that the asymptotic null distribution is distributed according to the chi-square distribution. The adjusted test statistic was called the asymptotically distribution-free (ADF) test statistic. Let $\boldsymbol{S}$ be an unbiased estimator of $\boldsymbol{\Sigma}$, defined by

$$\boldsymbol{S} = \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(\boldsymbol{y}_i - \bar{\boldsymbol{y}})', \tag{1.2}$$

where $\bar{\boldsymbol{y}}$ is the sample mean of $\boldsymbol{y}$ defined by $\bar{\boldsymbol{y}} = n^{-1}\sum_{i=1}^{n}\boldsymbol{y}_i$, and let $F(\boldsymbol{\theta})$ be the discrepancy function given by

$$F(\boldsymbol{\theta}) = \mathrm{vech}\left\{\boldsymbol{S} - \boldsymbol{\Sigma}(\boldsymbol{\theta})\right\}' \boldsymbol{S}_Y^{-1} \mathrm{vech}\left\{\boldsymbol{S} - \boldsymbol{\Sigma}(\boldsymbol{\theta})\right\}, \tag{1.3}$$

where $\boldsymbol{S}_Y$ is the sample covariance matrix of $\mathrm{vech}\{(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(\boldsymbol{y}_i - \bar{\boldsymbol{y}})'\}$ defined as

$$\boldsymbol{S}_Y = \frac{1}{n}\sum_{i=1}^{n}\mathrm{vech}\left\{(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(\boldsymbol{y}_i - \bar{\boldsymbol{y}})' - \boldsymbol{S}\right\}\mathrm{vech}\left\{(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(\boldsymbol{y}_i - \bar{\boldsymbol{y}})' - \boldsymbol{S}\right\}'. \tag{1.4}$$

Here the vech operator is used to transform the lower triangular matrix of a symmetric matrix to a vector by stacking the columns of the matrix in turn (see e.g., Harville, 1997, Chapter 16): i.e.,

$$\mathrm{vech}(\boldsymbol{U}) = (u_{11}, \ldots, u_{p1}, u_{22}, \ldots, u_{2p}, u_{33}, \ldots, u_{pp})',$$

where $u_{ij}$ is the $(i, j)$th element of a $p \times p$ matrix $\boldsymbol{U}$. Then, the ADF test statistic is defined by

$$T = nF(\hat{\boldsymbol{\theta}}), \tag{1.5}$$

where $\hat{\boldsymbol{\theta}}$ is an estimator of $\boldsymbol{\theta}$, which is obtained by minimizing the discrepancy function $F(\boldsymbol{\theta})$ in (1.3). Note that $\boldsymbol{S}_Y$ is an estimator of the asymptotic covariance matrix of $\mathrm{vech}(\boldsymbol{S})$. Thus, the null distribution of $T$ in (1.5) converges to the chi-square distribution with $p^* - q$ degrees of freedom as $n \to \infty$, even if the true distribution is not normal. However, many authors have reported numerical results which show that the quality of the chi-square approximation for the ADF test statistic is very poor, even when the sample size is large and the assumption of normality is correct (see e.g., Hu, Bentler & Kano, 1992; Yuan & Bentler, 1998). On the other hand, it is well known that the Bartlett correction (Bartlett, 1937) is available to improve the quality of the chi-square approximation (see e.g., Fujikoshi, 2000; Yanagihara & Yuan, 2005; Yanagihara, 2007a). Therefore, we try here to improve the quality of the chi-square approximation for the ADF test by applying the Bartlett correction, at least when $\boldsymbol{y} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

The main aim of this paper is to obtain the Bartlett correction to the ADF test statistic under normality. In particular, we specify that the covariance structure is linear, i.e., the covariance structure considered is of the form

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \theta_1 \boldsymbol{G}_1 + \cdots + \theta_q \boldsymbol{G}_q, \tag{1.6}$$

where $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_q$ are known $p \times p$ matrices and linearly independent (see e.g., Anderson, 1969; Siotani, Hayakawa & Fujikoshi, 1985, Chapter 8.6.3). Although the Bartlett correction is derived under normality, we verify that our Bartlett correction performs well, even under nonnormality, by conducting numerical studies.

The paper is organized in follows: In Section 2, we derive the Bartlett correction to the ADF test statistic for a covariance matrix with linear structure, under normality. In Section 3, by conducting numerical simulations, we examine the performance of the ADF test statistic adjusted by the Bartlett correction. Section 4 contains a discussion and our conclusions. Technical details are provided in the Appendix.

## 2. The Bartlett Correction for the ADF Test Statistic

Let $\boldsymbol{s} = \mathrm{vech}(\boldsymbol{S})$ and $\boldsymbol{C} = (\mathrm{vech}(\boldsymbol{G}_1), \ldots, \mathrm{vech}(\boldsymbol{G}_q))$. When we are dealing with a covariance matrix as in (1.6), the discrepancy function $F(\boldsymbol{\theta})$ in (1.3) is rewritten as

$$F(\boldsymbol{\theta}) = (\boldsymbol{s} - \boldsymbol{C}\boldsymbol{\theta})' \boldsymbol{S}_Y^{-1} (\boldsymbol{s} - \boldsymbol{C}\boldsymbol{\theta}). \tag{2.1}$$

By minimizing $F(\boldsymbol{\theta})$ in (2.1), we derive an estimator of $\boldsymbol{\theta}$ as

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\boldsymbol{s}. \tag{2.2}$$

Substituting (2.2) into (2.1) yields the ADF test statistic for (1.6) as

$$T = n\boldsymbol{s}'\left\{\boldsymbol{S}_Y^{-1} - \boldsymbol{S}_Y^{-1}\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\right\}\boldsymbol{s}. \tag{2.3}$$

Note that the test statistic $T$ in (2.3) is invariant with respect to a linear transformation of $\boldsymbol{y}$ to $\boldsymbol{\varepsilon} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{y} - \boldsymbol{\mu})$ when the null hypothesis is true. Therefore, we use $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$ instead of $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ in the derivation of the Bartlett correction.

In order to expand the test statistic $T$ in (2.3), we use the following random vector $\boldsymbol{z}$ and random matrices $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{R}$, which are asymptotically normal;

$$\begin{aligned} \boldsymbol{z} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\boldsymbol{\varepsilon}_j = \sqrt{n}\bar{\boldsymbol{\varepsilon}}, & \boldsymbol{V} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(\boldsymbol{\varepsilon}_j\boldsymbol{\varepsilon}_j' - \boldsymbol{I}_p), \\ \boldsymbol{R} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')\boldsymbol{\varepsilon}_i' - \boldsymbol{\Gamma}\right\}, & \boldsymbol{W} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')' - \boldsymbol{\Omega}\right\}, \end{aligned} \tag{2.4}$$

where the matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ are given by

$$\boldsymbol{\Omega} = \mathrm{E}\left[\mathrm{vech}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\mathrm{vech}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')'\right], \quad \boldsymbol{\Gamma} = \mathrm{E}\left[\mathrm{vech}(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')\boldsymbol{\varepsilon}'\right]. \tag{2.5}$$

Let

$$\boldsymbol{a} = \mathrm{vech}(\boldsymbol{I}_p),\ \boldsymbol{\Psi} = \boldsymbol{\Omega} - \boldsymbol{a}\boldsymbol{a}',\ \boldsymbol{\Xi} = \boldsymbol{\Psi}^{-1} - \boldsymbol{\Psi}^{-1}\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{\Psi}^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{\Psi}^{-1}. \tag{2.6}$$

and

$$\boldsymbol{v} = \mathrm{vech}(\boldsymbol{V}),\ \boldsymbol{h} = \mathrm{vech}(\boldsymbol{z}\boldsymbol{z}'),\ \boldsymbol{Z} = (\boldsymbol{z} \otimes \boldsymbol{I}_p) + (\boldsymbol{I}_p \otimes \boldsymbol{z}). \tag{2.7}$$

From the Appendix, we see that the test statistic $T$ in (2.3) can be expanded as

$$T = T_0 + \frac{1}{\sqrt{n}}T_1 + \frac{1}{n}T_2 + O_p(n^{-3/2}), \tag{2.8}$$

where $T_0$, $T_1$ and $T_2$ are defined by

$$\begin{aligned} T_0 &= \boldsymbol{v}'\boldsymbol{\Xi}\boldsymbol{v} \\ T_1 &= \boldsymbol{v}'\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi}\boldsymbol{v} - 2\boldsymbol{v}'\boldsymbol{\Xi}\boldsymbol{h}, \\ T_2 &= \boldsymbol{v}'\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi}\boldsymbol{v} - \boldsymbol{v}'\boldsymbol{\Xi}\boldsymbol{Q}_2\boldsymbol{\Xi}\boldsymbol{v} + 2\boldsymbol{v}'\boldsymbol{\Xi}\boldsymbol{v} - 2\boldsymbol{v}'\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi}\boldsymbol{h} + \boldsymbol{h}'\boldsymbol{\Xi}\boldsymbol{h}. \end{aligned} \tag{2.9}$$

Here, the random matrices $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$ and $\boldsymbol{Z}$ are defined by

$$\begin{aligned} \boldsymbol{Q}_1 =& \boldsymbol{W} - \boldsymbol{\Gamma}\boldsymbol{Z}'\boldsymbol{D}_p^{+'} - \boldsymbol{D}_p^{+}\boldsymbol{Z}\boldsymbol{\Gamma}' - \boldsymbol{v}\boldsymbol{a}' - \boldsymbol{a}\boldsymbol{v}', \\ \boldsymbol{Q}_2 =& \boldsymbol{D}_p^{+}\boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{D}_p^{+'} - \boldsymbol{R}\boldsymbol{Z}'\boldsymbol{D}_p^{+'} - \boldsymbol{D}_p^{+}\boldsymbol{Z}\boldsymbol{R}' + 2\left(\boldsymbol{a}\boldsymbol{h}' + \boldsymbol{h}\boldsymbol{a}'\right) - \boldsymbol{v}\boldsymbol{v}', \end{aligned} \tag{2.10}$$

where $\boldsymbol{D}_p$ is a $p^2 \times p^*$ duplication matrix and $\boldsymbol{D}_p^+$ is the Moore-Penrose inverse of $\boldsymbol{D}_p$, defined by

$$\boldsymbol{D}_p \mathrm{vech}(\boldsymbol{U}) = \mathrm{vec}(\boldsymbol{U}), \quad \boldsymbol{D}_p^+ \mathrm{vec}(\boldsymbol{U}) = \mathrm{vech}(\boldsymbol{U}), \tag{2.11}$$

(see e.g., Harville, 1997, Chapter 16).

Note that $\boldsymbol{\Gamma}$ is zero when $\boldsymbol{\varepsilon} \sim \mathrm{N}_p(\boldsymbol{0}_p, \boldsymbol{I}_p)$, where $\boldsymbol{0}_p$ is a $p$-dimensional vector, all of whose elements are 0. If the null hypothesis (1.6) is true, $\boldsymbol{\Xi a} = \boldsymbol{0}_{p^*}$ is satisfied. Moreover, it is easy to obtain the equation $\mathrm{tr}(\boldsymbol{\Xi \Psi}) = p^* - q$. Hence, we obtain the following expectations when the null hypothesis (1.6) and the distributional assumption $\boldsymbol{\varepsilon} \sim \mathrm{N}_p(\boldsymbol{0}_p, \boldsymbol{I}_p)$ are true:

$$\begin{aligned}
&\mathrm{E}[\boldsymbol{v}'\boldsymbol{\Xi v}] = p^* - q, && \mathrm{E}[\boldsymbol{v}'\boldsymbol{\Xi Q}_1\boldsymbol{\Xi v}] = \frac{1}{\sqrt{n}}(p^* - q)(p^* - q + 1),\\
&\mathrm{E}[\boldsymbol{v}'\boldsymbol{\Xi h}] = \frac{1}{\sqrt{n}}(p^* - q), && \mathrm{E}[\boldsymbol{v}'\boldsymbol{\Xi Q}_1\boldsymbol{\Xi Q}_1\boldsymbol{\Xi v}] = (p^* - q)(p^* - q + 1),\\
&\mathrm{E}[\boldsymbol{v}'\boldsymbol{\Xi Q}_2\boldsymbol{\Xi v}] = -(p^* - q)^2, && \mathrm{E}[\boldsymbol{v}'\boldsymbol{\Xi Q}_1\boldsymbol{\Xi h}] = o(1),\\
&\mathrm{E}[\boldsymbol{h}'\boldsymbol{\Xi h}] = p^* - q.
\end{aligned}$$

By using the above expectations, we derive expectations of $T_0$, $T_1$ and $T_2$ in (2.9) as

$$\mathrm{E}[T_0] = p^* - q, \quad \mathrm{E}[T_1] = \frac{1}{\sqrt{n}}(p^* - q)(p^* - q - 1), \quad \mathrm{E}[T_2] = 2(p^* - q)(p^* - q + 2) + o(1).$$

From the expectations above and the stochastic expansion (2.8), $\mathrm{E}[T]$ may be expanded as

$$\mathrm{E}[T] = (p^* - q)\left\{1 + \frac{3(p^* - q + 1)}{n}\right\} + o(n^{-1}). \tag{2.12}$$

The equation (2.12) indicates the difference between the means of the null distribution of the ADF statistic and the chi-square distribution with $p^* - q$ degrees of freedom. The difference will become large as $p$ is increased, and will become small as $q$ is increased. Moreover, we can see that the mean of $T$ tends to be larger than the mean of the chi-square distribution with $p^* - q$ degrees of freedom, because $(p^* - q)\{1 + 3(p^* - q + 1)/n\} > p^* - q$ holds. This implies that the actual test size tends to be larger than the nominal test size when we use the chi-square distribution with $p^* - q$ degrees of freedom as the null distribution.

The Bartlett correction is a linear transformation applied to move the mean of a test statistic closer to the mean of its asymptotic distribution. Hence, the asymptotic expansion of $\mathrm{E}[T]$ leads us to the following improved ADF test statistic with the Bartlett correction:

$$T_{\mathrm{B}} = \frac{nT}{n + 3(p^* - q + 1)}. \tag{2.13}$$

It is easy to see that $\mathrm{E}[T_{\mathrm{B}}] = p^* - q + o(n^{-1})$ holds when the null hypothesis and the assumption of normality are true.

## 3. Numerical Studies

In this section, we examine the performance of $T_{\mathrm{B}}$ by conducting numerical experiments. Covariance matrix structures considered in this examination are

$$\text{Model A: } \boldsymbol{\Sigma}(\boldsymbol{\theta}) = \theta \boldsymbol{I}_p, \quad \text{Model B: } \boldsymbol{\Sigma}(\boldsymbol{\theta}) = \theta_1 \boldsymbol{I}_p + \theta_2(\mathbf{1}_p \mathbf{1}_p' - \boldsymbol{I}_p),$$

where $\mathbf{1}_p$ is a $p$-dimensional vector all of whose elements are 1. Simulation data $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ were generated from $\boldsymbol{y} = \mathbf{1}_p + \boldsymbol{\Sigma}_*^{1/2}\boldsymbol{\varepsilon}$, where $\boldsymbol{\Sigma}_*$ is the true covariance matrix. Matrices $\boldsymbol{I}_p$ and $\boldsymbol{I}_p + (0.5)(\mathbf{1}_p\mathbf{1}_p' - \boldsymbol{I}_p)$ were used as $\boldsymbol{\Sigma}_*$ in the cases of models A and B, respectively. Although our Bartlett correction has been obtained under normality, we also examine the situation when the assumption of normality is not satisfied. For generating multivariate nonnormal data, we consider a data model introduced by Yuan and Bentler (1997).

**Data Model**: *Let* $x_1, \ldots, x_m$ $(m \geq p)$ *be independent random variables with* $\mathrm{E}[x_j] = 0$, $\mathrm{E}[x_j^2] = 1$ *and* $\boldsymbol{x} = (x_1, \ldots, x_m)'$. *Further, let* $r$ *be a random variable which is independent of* $\boldsymbol{x}$, $\mathrm{E}[r^2] = 1$. *Then, we generate an error vector by*

$$\boldsymbol{\varepsilon} = r\boldsymbol{B}'\boldsymbol{x}, \tag{3.1}$$

*where* $\boldsymbol{B}$ *is a* $m \times p$ *matrix with full rank* $p$ *and* $\boldsymbol{B}'\boldsymbol{B} = \boldsymbol{I}_p$.

Let $w$ be a random variable from the chi-square distribution with 8 degrees of freedom, and let $\boldsymbol{B}_0$ be the $(p+1) \times p$ matrix defined by $\boldsymbol{B}_0 = (\boldsymbol{I}_p, \mathbf{1}_p)'(\boldsymbol{I}_p + \mathbf{1}_p\mathbf{1}_p')^{-1/2}$. By using the data model in (3.1), we generated error vectors for the following five models (this setting is the same as in Yanagihara, 2005; 2007b):

Model 1 (*Normal Distribution*): $x_j \sim \mathrm{N}(0,1)$, $r = 1$ and $\boldsymbol{B} = \boldsymbol{I}_p$. ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} = 0$ and $\kappa_4^{(1)} = 0$).

Model 2 (*t-Distribution*): $x_j \sim \mathrm{N}(0,1)$ , $r = \sqrt{6/w}$ and $\boldsymbol{B} = \boldsymbol{I}_p$ ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} = 0$ and $\kappa_4^{(1)} = p(p+2)/2$).

Model 3 (*Uniform Distribution*): $x_j$ is generated from the uniform (-5,5) distribution divided by the standard deviation $5/\sqrt{3}$, $r = 1$ and $\boldsymbol{B} = \boldsymbol{B}_0$ ($\kappa_{3,3}^{(1)} = \kappa_{3,3}^{(2)} = 0$ and $\kappa_4^{(1)} = -1.2 \times p^2(p+1)^{-1}$).

Model 4 (*Chi-Square Distribution*): $x_j$ is generated from a chi-squared distribution with 4 degrees of freedom standardized with a mean of 4 and standard deviation $2\sqrt{2}$, $r = \sqrt{6/w}$ and $\boldsymbol{B} = \boldsymbol{B}_0$ ($\kappa_{3,3}^{(1)} \approx 0.15 \times p(p^3 + p^2 - p + 3)(p+1)^{-3}$, $\kappa_{3,3}^{(2)} \approx 0.60 \times p^3(p+1)^{-3}$ and $\kappa_4^{(1)} = 4.5 \times p^2(p+1)^{-1} + p(p+2)/2$).

Model 5 (*Log-Normal Distribution*): $x_j$ is generated from a lognormal distribution such that $\log x_j \sim \mathrm{N}(0, 1/4)$ standardized with a mean of $e^{1/8}$ and standard deviation $e^{1/8}\sqrt{e^{1/4} - 1}$, $r = \sqrt{6/w}$

and $\boldsymbol{B} = \boldsymbol{B}_0$ ($\kappa_{3,3}^{(1)} \approx 0.45 \times p(p^3 + p^2 - p + 3)(p + 1)^{-3}$, $\kappa_{3,3}^{(2)} \approx 1.80 \times p^3(p + 1)^{-3}$ and $\kappa_4^{(1)} \approx 8.8 \times p^2(p + 1)^{-1} + p(p + 2)/2$).

We can measure the influence of nonnormality by the multivariate skewness $\kappa_{3,3}^{(1)}$ and $\kappa_{3,3}^{(2)}$, and multivariate kurtosis $\kappa_4^{(1)}$, which were proposed by Mardia (1970). It is easy to see that data models 1, 2 and 3 are symmetric distributions, and data models 4 and 5 are skewed distributions. Moreover, the sizes of the kurtosis in each model satisfy kurtosis(model 3) $<$ kurtosis(model 1) $<$ kurtosis(model 2) $<$ kurtosis(model 4) $<$ kurtosis(model 5), and the sizes of the skewness in each model satisfy skewness(model 1) $=$ skewness(model 2) $=$ skewness(model 3) $<$ skewness(model 4) $<$ skewness(model 5).

There were other adjusted ADF test statistics proposed by Yuan and Bentler (1998). These are defined by

$$T_{\mathrm{YB}} = \frac{T}{1 + nT/(n-1)^2}, \quad T_{\mathrm{C}} = \left(1 - \frac{p^* - q + 1}{n - 1}\right) T.$$

Moreover, Yuan and Belter (1998, 1999b) proposed an $F$-test based on the ADF test statistic. The test statistic used is defined by

$$T_{\mathrm{F}} = \frac{(n - p^* + q)T}{(n-1)(p^* - q)}.$$

They approximated the null distribution of $T_{\mathrm{F}}$ by the $F$-distribution with $p^* - q$ and $n - p^* + q$ degrees of freedoms. We compared the actual sizes (or the type I errors of tests) for $T$, $T_{\mathrm{B}}$, $T_{\mathrm{YB}}$, $T_{\mathrm{C}}$ and $T_{\mathrm{F}}$, which are defined by

$$\alpha_1 = \mathrm{P}(T > \chi^2_\alpha), \qquad \alpha_2 = \mathrm{P}(T_{\mathrm{B}} > \chi^2_\alpha), \qquad \alpha_3 = \mathrm{P}(T_{\mathrm{YB}} > \chi^2_\alpha),$$
$$\alpha_4 = \mathrm{P}(T_{\mathrm{C}} > \chi^2_\alpha), \qquad \alpha_5 = \mathrm{P}(T_{\mathrm{F}} > f_\alpha),$$

where $\chi^2_\alpha$ and $f_\alpha$ are the upper $\alpha$ percentage points of the chi-square distribution with $p^* - q$ degrees of freedom, and the $F$-distribution with $p^* - q$ and $n - p^* + q$ degrees of freedoms, respectively. We chose $\alpha = 10$, 5 and 1 and used 30,000 replications to estimate $\alpha_1$ to $\alpha_5$ for each condition. Many conditions on $p$ and $n$ were simulated. Except for the simple chi-square approximation, all the procedures were very stable across different conditions. To save space, we only report the results corresponding to: $n = 100$, 200 and 500 when $p = 5$.

Tables 1 and 2 contain empirical sizes of the five tests corresponding to nominal sizes 10%, 5% and 1%, in the cases of the models A and B, respectively. The closest to the nominal size in each test is in bold. The last row provides the average absolute discrepancy (AAD) between the nominal sizes and the empirical sizes over the 15 conditions as given by AAD $= \sum |\hat{\alpha} - \alpha|/15$. From the tables, we can see that $T_{\mathrm{B}}$ given in (2.13), with empirical size $\hat{\alpha}_2$, performs well even under nonnormality, although $T_{\mathrm{B}}$ was proposed under the assumption of normality. In particular, $T_{\mathrm{B}}$ performs best according to AAD. The simple chi-square approximation, with empirical size $\hat{\alpha}_1$, was the poorest. The $T_{\mathrm{C}}$ and $T_{\mathrm{F}}$, with empirical size $\hat{\alpha}_4$ and $\hat{\alpha}_5$, improved the simple chi-square approximation greatly but still not sufficiently. The $T_{\mathrm{YB}}$, with empirical size $\hat{\alpha}_3$, outperformed $T_{\mathrm{C}}$ and $T_{\mathrm{F}}$, sometimes $T_{\mathrm{B}}$. Actually $T_{\mathrm{YB}}$ performed the second best according to AAD. When $n$ is large, $q$ is small or $\alpha$ is small, $T_{\mathrm{B}}$ tends to outperforms $T_{\mathrm{YB}}$.

Please insert Tables 1 and 2 around here

Although our Bartlett correction was obtained for the case of a covariance matrix with a linear structure, we study the performance of $T_{\mathrm{B}}$ when we deal with a covariance matrix without such a linear structure. The covariance structure considered is

$$\text{Model C: } \boldsymbol{\Sigma}(\boldsymbol{\theta}) = \boldsymbol{\theta}_p\boldsymbol{\theta}_p' + \theta_{p+1}\boldsymbol{I}_p,$$

where $\boldsymbol{\theta}_p$ is a $p$-dimensional unknown parameter vector. The matrix $\boldsymbol{c}\boldsymbol{c}' + (0.5)\boldsymbol{I}_5$ was used as the true covariance matrix $\boldsymbol{\Sigma}_*$, where $\boldsymbol{c} = (0.8, 0.8, 0.4, 0.2, 0.2)'$. Table 3 describes the empirical sizes of the five tests corresponding to nominal sizes 10%, 5% and 1%. From the table, we can see that our Bartlett correction performs well even when the structure of $\boldsymbol{\Sigma}$ is not linear. Unfortunately, $T_{\mathrm{B}}$ did not outperform $T_{\mathrm{YB}}$ when $\alpha = 10$ and 5. However, its differences were small, comparing the cases of models A and B. Thus, our results suggest that this Bartlett correction is available for testing in the case of a general covariance structure.

Please insert Table 3 around here

## 4. Conclusion and Discussion

In this paper, we calculated the asymptotic expansion of the expectation of the ADF test statistic for a covariance matrix with a linear structure under the assumption that the true distribution is the multivariate normal distribution. From this expansion, we have proposed the use of $T_{\mathrm{B}}$ in (2.13), which is a version of the ADF test statistic adjusted with the Bartlett correction in the case of a normal distribution. Although $T_{\mathrm{B}}$ was obtained under the assumption of normality, we verified, by conducting numerical examinations, that $T_{\mathrm{B}}$ performed well, even when the true distribution is not normal and the covariance structure is not linear. Since the correction term depends on only $p$, $q$ and $n$, it is easy to calculate $T_{\mathrm{B}}$ when the value of the ADF test statistic has already been derived. Furthermore, the actual size of $T_{\mathrm{B}}$ tends to be conservative. It is considered that conservativeness is an important property for a hypothesis test. From the discussion above, we appears that $T_{\mathrm{B}}$ will be helpful in actual data analysis.

On the other hand, we can derive the Bartlett correction without an assumption of normality. Then, the correction term will depend on the higher-order moments of the true distribution. Thus, estimating the higher-order moments in the nonnormal case is required to use such a Bartlett correction in practice. However, it is well known that we cannot obtain good estimates of higher-order moments without a huge sample size (as for an estimation of kurtosis, see e.g., Yanagihara, 2007). When the sample size is huge, the Bartlett correction becomes meaningless because the correction term unboundedly approaches 1. Hence, the Bartlett correction under nonnormality will become useful only when good estimators of the higher-order moments are found.

By comparing the simulation results for each distribution, we can see that the influence of nonnormality is not so great on the actual test size of $T$. This fact may support the idea that the effect of large $p$ is more important than the effect of nonnormality. Even if the effect of nonnormality cannot be removed completely, at least by using $T_{\rm B}$, we can remove the bad effect caused by an increase in $p$.

## Appendix

In order to obtain the stochastic expansion of the test statistic $T$ in (2.3), we expand $\boldsymbol{S}_Y$ in (1.4) at the beginning. Note that sample covariance matrix $\boldsymbol{S}$ in (1.2) can be expressed as

$$\boldsymbol{S} = \frac{n}{n-1}\left(\boldsymbol{I}_p + \frac{1}{\sqrt{n}}\boldsymbol{V} - \frac{1}{n}\boldsymbol{z}\boldsymbol{z}'\right),$$

where $\boldsymbol{z}$ and $\boldsymbol{V}$ are given by (2.4). This expression implies that

$$\boldsymbol{s} = \boldsymbol{a} + \frac{1}{\sqrt{n}}\boldsymbol{v} - \frac{1}{n}\boldsymbol{h} + O_p(n^{-3/2}),$$

where $\boldsymbol{a}$ is given by (2.6), and $\boldsymbol{v}$ and $\boldsymbol{h}$ are given by and (2.7). On the other hand, $\boldsymbol{S}_Y$ can be expressed as

$$\begin{aligned}
\boldsymbol{S}_Y &= \frac{1}{n}\sum_{i=1}^{n}\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i') - \frac{1}{n\sqrt{n}}\sum_{i=1}^{n}\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')' \\
&\quad -\frac{1}{n\sqrt{n}}\sum_{i=1}^{n}\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')' + \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')' \\
&\quad +\frac{n-1}{n^2}\left(\boldsymbol{s}\boldsymbol{h}' + \boldsymbol{h}\boldsymbol{s}'\right) - \frac{3}{n^2}\boldsymbol{h}\boldsymbol{h}' - \left(1 - \frac{2}{n}\right)\boldsymbol{s}\boldsymbol{s}'. 
\end{aligned} \tag{A.1}$$

From the property of the vech operator (see e.g., Harville, 1997, Chapter 16), we obtain

$$\begin{aligned}
&\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')' = \mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')\boldsymbol{\varepsilon}_i'\left\{(\boldsymbol{z}\otimes\boldsymbol{I}_p) + (\boldsymbol{I}_p\otimes\boldsymbol{z})\right\}'\boldsymbol{D}_p^{+'}, \\
&\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')' = \boldsymbol{D}_p^{+}\left\{(\boldsymbol{z}\otimes\boldsymbol{I}_p) + (\boldsymbol{I}_p\otimes\boldsymbol{z})\right\}\boldsymbol{\varepsilon}_i\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')', \\
&\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')\mathrm{vech}(\boldsymbol{\varepsilon}_i\boldsymbol{z}' + \boldsymbol{z}\boldsymbol{\varepsilon}_i')' = \boldsymbol{D}_p^{+}\left\{(\boldsymbol{z}\otimes\boldsymbol{I}_p) + (\boldsymbol{I}_p\otimes\boldsymbol{z})\right\}\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\left\{(\boldsymbol{z}\otimes\boldsymbol{I}_p) + (\boldsymbol{I}_p\otimes\boldsymbol{z})\right\}'\boldsymbol{D}_p^{+'},
\end{aligned}$$

where $\boldsymbol{D}_p$ is the duplication matrix and $\boldsymbol{D}_p^{+}$ is the Moore-Penrose inverse of $\boldsymbol{D}_p$, which are defined by (2.11). Using the above equations, the equation (A.1) can be rewritten as

$$\boldsymbol{S}_Y = \boldsymbol{\Psi} + \frac{1}{\sqrt{n}}\boldsymbol{Q}_1 + \frac{1}{n}\boldsymbol{Q}_2 + O_p(n^{-3/2}), \tag{A.2}$$

where $\boldsymbol{\Psi}$ is given by (2.6), and $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are given by (2.10). Using the expansion (A.2), the stochastic expansion of $\boldsymbol{S}_Y^{-1}$ can be derived as

$$\boldsymbol{S}_Y^{-1} = \boldsymbol{\Psi}^{-1} - \frac{1}{\sqrt{n}}\boldsymbol{\Psi}^{-1}\boldsymbol{Q}_1\boldsymbol{\Psi}^{-1} + \frac{1}{n}\left(\boldsymbol{\Psi}^{-1}\boldsymbol{Q}_1\boldsymbol{\Psi}^{-1}\boldsymbol{Q}_1\boldsymbol{\Psi}^{-1} - \boldsymbol{\Psi}^{-1}\boldsymbol{Q}_2\boldsymbol{\Psi}^{-1}\right) + O_p(n^{-3/2}). \tag{A.3}$$

Substituting (A.3) into $\boldsymbol{S}_Y^{-1} - \boldsymbol{S}_Y^{-1}\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{S}_Y^{-1}$ yields

$$\boldsymbol{S}_Y^{-1} - \boldsymbol{S}_Y^{-1}\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{S}_Y^{-1} = \boldsymbol{\Xi} - \frac{1}{\sqrt{n}}\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi} + \frac{1}{n}\left(\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi} - \boldsymbol{\Xi}\boldsymbol{Q}_2\boldsymbol{\Xi}\right) + O_p(n^{-3/2}),$$

where $\boldsymbol{\Xi}$ is given by (2.6). Let $\boldsymbol{U}_1 = \boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi}$ and $\boldsymbol{U}_2 = \boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi}\boldsymbol{Q}_1\boldsymbol{\Xi} - \boldsymbol{\Xi}\boldsymbol{Q}_2\boldsymbol{\Xi}$. The test statistic $T$ in (2.3) can be rewritten as

$$\begin{aligned} T &= n\boldsymbol{s}'\left\{\boldsymbol{S}_Y^{-1} - \boldsymbol{S}_Y^{-1}\boldsymbol{C}(\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}'\boldsymbol{S}_Y^{-1}\right\}\boldsymbol{s} \\ &= \left(1 - \frac{1}{n}\right)^{-2}\left\{\boldsymbol{v} - \frac{1}{\sqrt{n}}\mathrm{vech}(\boldsymbol{z}\boldsymbol{z}')\right\}'\left(\boldsymbol{\Xi} + \frac{1}{\sqrt{n}}\boldsymbol{U}_1 + \frac{1}{n}\boldsymbol{U}_2\right)\left\{\boldsymbol{v} - \frac{1}{\sqrt{n}}\mathrm{vech}(\boldsymbol{z}\boldsymbol{z}')\right\} + O_p(n^{-3/2}). \end{aligned}$$

After coordinating the $O_p(1)$, $O_p(n^{-1/2})$ and $O_p(n^{-1})$ terms in the above equation, we obtain the stochastic expansion of $T$ as (2.8).

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**Table 1**. Actual test sizes in the case of model A.

| Dist. | Nominal 10% test | | | | | Nominal 5% test | | | | | Nominal 1% test | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ |
| | $n = 100$ | | | | | | | | | | | | | | |
| 1 | 31.07 | **7.93** | 14.06 | 18.50 | 16.73 | 21.98 | **4.31** | 7.13 | 12.16 | 9.85 | 9.86 | **1.33** | 1.50 | 4.31 | 2.84 |
| 2 | 31.75 | 6.86 | **12.93** | 18.13 | 16.02 | 21.47 | 3.59 | **6.13** | 10.89 | 8.61 | 8.61 | 0.74 | **0.90** | 3.59 | 2.17 |
| 3 | 30.33 | **8.21** | 13.92 | 18.58 | 16.73 | 21.65 | **4.89** | 7.43 | 11.97 | 9.93 | 9.94 | **1.65** | 1.87 | 4.89 | 3.36 |
| 4 | 33.04 | **6.74** | 13.63 | 18.98 | 17.00 | 22.65 | 3.36 | **5.77** | 11.23 | 8.86 | 8.88 | 0.87 | **0.97** | 3.36 | 2.04 |
| 5 | 33.22 | **7.26** | 13.49 | 18.56 | 16.52 | 22.14 | 3.33 | **6.31** | 11.11 | 9.01 | 9.01 | 0.44 | **0.59** | 3.33 | 1.96 |
| | $n = 200$ | | | | | | | | | | | | | | |
| 1 | 19.82 | **8.04** | 12.55 | 14.55 | 13.84 | 12.27 | **4.20** | 6.77 | 8.45 | 7.64 | 4.11 | **0.99** | 1.39 | 2.49 | 1.93 |
| 2 | 20.46 | 7.64 | **12.28** | 14.45 | 13.57 | 11.99 | 3.80 | **5.97** | 7.91 | 6.99 | 3.70 | **0.91** | 1.31 | 2.28 | 1.89 |
| 3 | 19.70 | **8.55** | 12.60 | 14.78 | 13.83 | 12.37 | **4.34** | 6.92 | 8.86 | 8.05 | 4.22 | **1.18** | 1.44 | 2.57 | 1.99 |
| 4 | 21.05 | **8.58** | 13.34 | 15.51 | 14.51 | 13.14 | **4.23** | 6.75 | 8.84 | 7.92 | 4.10 | **0.98** | 1.31 | 2.50 | 1.98 |
| 5 | 21.61 | **8.10** | 13.35 | 15.90 | 14.97 | 13.05 | **4.20** | 6.38 | 8.33 | 7.59 | 4.11 | **0.87** | 1.21 | 2.32 | 1.72 |
| | $n = 500$ | | | | | | | | | | | | | | |
| 1 | 13.66 | 8.71 | **11.02** | 11.75 | 11.47 | 7.50 | 4.36 | **5.61** | 6.33 | 6.10 | 2.03 | **1.12** | 1.34 | 1.66 | 1.50 |
| 2 | 13.74 | 8.79 | **11.00** | 11.81 | 11.53 | 7.29 | 4.47 | **5.52** | 6.17 | 5.85 | 1.95 | **0.97** | 1.21 | 1.60 | 1.38 |
| 3 | 13.42 | **9.01** | 11.18 | 11.73 | 11.53 | 7.51 | **4.66** | 5.73 | 6.31 | 6.00 | 2.01 | **0.97** | 1.26 | 1.60 | 1.45 |
| 4 | 14.99 | **9.53** | 12.00 | 12.77 | 12.44 | 7.85 | **4.50** | 5.93 | 6.59 | 6.28 | 1.89 | **0.94** | 1.24 | 1.50 | 1.36 |
| 5 | 15.06 | **9.60** | 12.26 | 13.06 | 12.75 | 7.99 | **4.66** | 5.79 | 6.47 | 6.18 | 2.04 | **1.11** | 1.38 | 1.75 | 1.52 |
| | AAD | | | | | | | | | | | | | | |
| | 12.19 | **1.76** | 2.64 | 5.27 | 4.23 | 9.06 | **0.81** | 1.28 | 3.77 | 2.66 | 4.10 | **0.18** | 0.33 | 1.65 | 0.94 |

**Table 2**. Actual test sizes in the case of model B.

| Dist. | Nominal 10% test | | | | | Nominal 5% test | | | | | Nominal 1% test | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ |
| | $n = 100$ | | | | | | | | | | | | | | |
| 1 | 29.97 | **8.01** | 13.84 | 18.23 | 16.40 | 20.32 | **4.50** | 7.08 | 11.45 | 9.54 | 8.57 | **1.27** | 1.53 | 4.20 | 2.75 |
| 2 | 29.26 | **7.13** | 13.06 | 17.88 | 16.12 | 19.77 | 3.63 | **6.02** | 10.53 | 8.46 | 7.63 | 0.68 | **0.81** | 3.27 | 2.12 |
| 3 | 29.21 | **8.64** | 14.12 | 18.43 | 16.82 | 20.33 | **4.96** | 7.56 | 11.81 | 10.08 | 9.29 | **1.62** | 1.84 | 4.70 | 3.36 |
| 4 | 29.14 | 6.56 | **12.64** | 17.03 | 15.46 | 19.17 | 3.01 | **5.51** | 9.89 | 8.17 | 7.18 | 0.64 | **0.71** | 2.79 | 1.69 |
| 5 | 29.77 | 6.55 | **12.06** | 17.04 | 15.43 | 19.33 | 3.04 | **5.56** | 9.84 | 7.93 | 7.10 | 0.52 | **0.61** | 2.72 | 1.63 |
| | $n = 200$ | | | | | | | | | | | | | | |
| 1 | 18.94 | **8.19** | 12.39 | 14.19 | 13.49 | 11.54 | **4.34** | 6.31 | 8.07 | 7.31 | 3.74 | **1.01** | 1.34 | 2.32 | 1.91 |
| 2 | 19.00 | 7.99 | **11.78** | 13.86 | 13.01 | 11.17 | **3.93** | 6.14 | 7.87 | 7.18 | 3.36 | **0.86** | 1.16 | 2.08 | 1.69 |
| 3 | 18.53 | **8.45** | 12.22 | 14.04 | 13.34 | 11.54 | **4.78** | 6.85 | 8.29 | 7.78 | 4.10 | **1.21** | 1.68 | 2.77 | 2.26 |
| 4 | 19.38 | 7.93 | **12.04** | 14.09 | 13.22 | 11.36 | 3.83 | **6.04** | 7.79 | 7.11 | 3.24 | 0.67 | **1.09** | 2.02 | 1.49 |
| 5 | 18.69 | 7.44 | **11.77** | 13.76 | 13.03 | 10.84 | 3.56 | **5.58** | 7.35 | 6.59 | 3.07 | 0.80 | **1.07** | 1.85 | 1.45 |
| | $n = 500$ | | | | | | | | | | | | | | |
| 1 | 14.05 | **9.65** | 11.75 | 12.47 | 12.19 | 7.67 | **4.99** | 6.00 | 6.56 | 6.33 | 2.09 | **1.11** | 1.36 | 1.62 | 1.53 |
| 2 | 13.92 | **9.41** | 11.48 | 12.21 | 11.95 | 7.67 | **4.74** | 5.84 | 6.61 | 6.22 | 1.62 | 0.82 | **0.96** | 1.29 | 1.16 |
| 3 | 12.75 | 8.67 | **10.71** | 11.22 | 11.00 | 7.05 | 4.34 | **5.41** | 5.97 | 5.77 | 1.87 | **1.01** | 1.26 | 1.51 | 1.39 |
| 4 | 13.78 | **9.43** | 11.49 | 12.03 | 11.81 | 7.61 | **4.85** | 5.94 | 6.55 | 6.27 | 1.84 | 0.84 | **1.06** | 1.39 | 1.23 |
| 5 | 13.92 | **9.46** | 11.36 | 12.09 | 11.75 | 7.33 | 4.24 | **5.42** | 6.00 | 5.77 | 1.58 | 0.86 | **1.08** | 1.28 | 1.15 |
| | AAD | | | | | | | | | | | | | | |
| | 10.69 | **1.77** | 2.18 | 4.57 | 3.67 | 7.85 | **0.82** | 1.08 | 3.31 | 2.37 | 3.42 | **0.24** | 0.29 | 1.39 | 0.79 |

**Table 3**. Actual test sizes in the case of model C.

| Dist. | Nominal 10% test $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ | Nominal 5% test $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ | Nominal 1% test $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ | $\hat{\alpha}_4$ | $\hat{\alpha}_5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n = 100$ | | | | | | | | | | | | | | |
| 1 | 20.63 | **8.28** | 12.17 | 14.77 | 13.94 | 13.30 | **4.36** | 6.17 | 8.70 | 7.54 | 4.70 | **1.13** | 1.23 | 2.73 | 1.95 |
| 2 | 19.35 | 6.80 | **10.61** | 13.23 | 12.29 | 11.57 | 3.06 | **4.81** | 7.28 | 6.17 | 3.38 | 0.57 | **0.63** | 1.83 | 1.19 |
| 3 | 21.41 | **8.48** | 12.32 | 15.19 | 14.24 | 13.46 | **4.79** | 6.47 | 8.97 | 7.85 | 5.09 | **1.45** | 1.55 | 3.20 | 2.38 |
| 4 | 20.17 | 6.51 | **10.52** | 13.48 | 12.46 | 11.54 | 2.89 | **4.37** | 6.87 | 5.91 | 3.23 | 0.43 | **0.49** | 1.54 | 0.94 |
| 5 | 19.80 | 6.51 | **10.24** | 13.18 | 12.14 | 11.36 | 3.00 | **4.52** | 7.16 | 5.87 | 3.23 | 0.40 | **0.42** | 1.56 | 0.92 |
| | $n = 200$ | | | | | | | | | | | | | | |
| 1 | 15.47 | **9.04** | 11.48 | 12.67 | 12.25 | 8.98 | **4.61** | 5.99 | 7.14 | 6.50 | 2.50 | **0.95** | 1.16 | 1.88 | 1.54 |
| 2 | 15.55 | **8.82** | 11.61 | 12.70 | 12.31 | 8.74 | 4.14 | **5.48** | 6.74 | 6.34 | 2.32 | 0.83 | **1.01** | 1.73 | 1.45 |
| 3 | 15.35 | **9.13** | 11.63 | 12.95 | 12.43 | 9.12 | **4.67** | 6.02 | 7.11 | 6.61 | 2.73 | **1.25** | 1.40 | 2.08 | 1.72 |
| 4 | 15.83 | **8.84** | 11.78 | 13.04 | 12.57 | 8.78 | **4.72** | 5.74 | 6.79 | 6.40 | 2.67 | **1.03** | 1.25 | 1.86 | 1.60 |
| 5 | 15.61 | **8.56** | 11.63 | 13.00 | 12.46 | 8.52 | 4.28 | **5.51** | 6.72 | 6.16 | 2.29 | 0.80 | **0.98** | 1.55 | 1.25 |
| | $n = 500$ | | | | | | | | | | | | | | |
| 1 | 12.02 | **9.45** | 10.67 | 11.12 | 10.90 | 6.47 | **4.81** | 5.50 | 5.85 | 5.68 | 1.60 | **1.06** | 1.17 | 1.36 | 1.28 |
| 2 | 12.07 | 9.24 | **10.58** | 11.13 | 10.88 | 6.15 | 4.60 | **5.15** | 5.45 | 5.29 | 1.45 | 0.94 | **1.03** | 1.19 | 1.14 |
| 3 | 12.04 | **9.54** | 10.71 | 11.18 | 11.00 | 6.49 | **4.89** | 5.59 | 5.98 | 5.83 | 1.48 | **1.01** | 1.16 | 1.40 | 1.30 |
| 4 | 12.69 | **9.94** | 11.34 | 11.78 | 11.66 | 6.65 | **5.00** | 5.60 | 6.00 | 5.78 | 1.55 | **1.05** | 1.18 | 1.44 | 1.35 |
| 5 | 12.52 | **9.73** | 11.04 | 11.55 | 11.30 | 6.62 | **4.83** | 5.37 | 5.85 | 5.65 | 1.56 | **0.95** | 1.10 | 1.38 | 1.21 |
| | AAD | | | | | | | | | | | | | | |
| | 6.03 | 1.41 | **1.22** | 2.73 | 2.19 | 4.18 | 0.69 | **0.66** | 1.84 | 1.24 | 1.65 | **0.21** | 0.25 | 0.78 | 0.43 |