Asymptotic expansions for a class of tests for a general covariance structure under a local alternative

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Abstract

Let $\boldsymbol{S}$ be a $p \times p$ random matrix having a Wishart distribution $W_p(n, n^{-1}\boldsymbol{\Sigma})$. For testing a general covariance structure $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi})$, we consider a class of test statistics $T_h = n\rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}(\hat{\boldsymbol{\xi}}))$, where $\rho_h(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) = \sum_{i=1}^{p} h(\lambda_i)$ is a distance measure from $\boldsymbol{\Sigma}_1$ to $\boldsymbol{\Sigma}_2$, λ_i's are the eigenvalues of $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$, and h is a given function with certain properties. Wakaki, Eguchi, Fujikoshi (1990) suggested this class and gave an asymptotic expansion of the null distribution of T_h. This paper gives an asymptotic expansion of the non-null distribution of T_h under a sequence of alternatives. By using results, we derive the power, and compare the power asymptotically in the class. Especially we investigate the power of the sphericity tests.

1 Introduction

Let $\boldsymbol{S}$ be a $p \times p$ random matrix having a Wishart distribution $W_p(n, n^{-1}\boldsymbol{\Sigma})$. It is assumed that $n \geq p$. We consider the problem of testing

$$H_0 : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi}) \quad \text{against} \quad H_1 : \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}(\boldsymbol{\xi}),$$

where $\boldsymbol{\xi} \in \Xi$. Here, Ξ is an open subset of $\mathbb{R}^q$. We assume that

A1. All the elements of $\boldsymbol{\Sigma}(\boldsymbol{\xi})$ are known C^4-class functions on Ξ, and the Jacobian matrix of $\boldsymbol{\Sigma}(\boldsymbol{\xi})$ is of full rank.

$\boldsymbol{\Sigma}(\Xi)$ is a smooth subsurface in $\mathbb{R}^{p(p+1)/2}$ with coordinates $\boldsymbol{\xi} = (\xi^1, \cdots, \xi^q)'$. The hypothesis H_0 represents various covariance structures as special cases.

We consider a class of test statistics via minimization of the following divergence measure from $\boldsymbol{S}$ to $\boldsymbol{\Sigma}(\boldsymbol{\xi})$. Let h be a C^4-function on $(0, \infty)$ satisfying that

A2. $h(1) = 0$, $h_1 = 0$, and $h_2 = 1$,
A3. $h(\lambda) > 0$ for any $\lambda \neq 1$,

where h_r denotes the rth derivative of h at $\lambda = 1$. For arbitrary two matrices $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ we define a distance measure from $\boldsymbol{\Sigma}_1$ to $\boldsymbol{\Sigma}_2$ by

$$\rho_h(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) = \sum_{i=1}^{p} h(\lambda_i),$$

where λ_i's are the eigenvalues of $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$. Note that $\rho_h(\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2) \geq 0$ with equality if and only if $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ because of A3. However, in general, ρ_h is not symmetric and does not satisfy the triangle law.

Wakaki, Eguchi, Fujikoshi [10] suggested a class of test statistics

$$T_h = n \inf_{\boldsymbol{\xi} \in \Xi} \rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}(\boldsymbol{\xi})) = n\rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}(\hat{\boldsymbol{\xi}})), \tag{1.1}$$

where $\hat{\boldsymbol{\xi}}$ is the minimizing point. For example, using $h(\lambda_i) = -\log \lambda_i + \lambda_i - 1$, ρ_h is the Kullback divergence and the corresponding statistic T_h is just based on the log-likelihood ratio criterion. Another typical example is $h(\lambda_i) = (\lambda_i - 1)^2/2$.

It may be noted that the asymptotic expansions of the null distributions of T_h's in some special cases have been obtained by many authors (e.g., Anderson [1], Muirhead [4], Siotani, Hayakawa, Fujikoshi [7], etc.). An emphasis in Wakaki, Eguchi, Fujikoshi [10] is put on an asymptotic expansion of the null distribution of T_h in a general case. Many authors also gave the asymptotic expansions of the non-null distributions of T_h's in some special cases (e.g., Hayakawa [2], Nagao [5], Sugiura [8], etc.). This paper gives an asymptotic expansion of the non-null distribution of T_h in a general case under a sequence of alternatives converging to the null hypothesis with the rate of convergence $n^{-1/2}$. In Section 2 we give stochastic expansions of $\hat{\boldsymbol{\xi}}$ as well as T_h. In Section 3 we obtain an asymptotic expansion of the non-null distribution of T_h under the local alternatives up to the order $n^{-1/2}$. In Section 4 we derive the power, and compare the power asymptotically in the class. Especially we consider the power of the sphericity tests.

2 Stochastic Expansion of T_h

We consider a sequence of alternative hypotheses

$$H_n : \boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi}_0) + \frac{1}{\sqrt{n}}\boldsymbol{\Sigma}(\boldsymbol{\xi}_0)^{1/2}\boldsymbol{\Delta}\boldsymbol{\Sigma}(\boldsymbol{\xi}_0)^{1/2}$$

for $\boldsymbol{\Sigma} \notin \boldsymbol{\Sigma}(\Xi)$, where $\boldsymbol{\Delta}$ is a symmetric matrix and $\boldsymbol{\xi}_0 \in \Xi$. For simplicity, let us denote as $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}(\boldsymbol{\xi}_0)$ and $\hat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}(\hat{\boldsymbol{\xi}})$. We shall expand T_h in terms of

$$\boldsymbol{V} = \sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{S} - \boldsymbol{\Sigma})(\boldsymbol{\Sigma}^{-1/2})' \tag{2.1}$$

which is $O_p(1)$.

First we summarize the notations used in this paper. Let

$$\partial_a = \frac{\partial}{\partial \xi^a}, \quad \boldsymbol{J}_{ab\cdots} = \boldsymbol{\Sigma}_0^{1/2} \left[\partial_a\partial_b \cdots \boldsymbol{\Sigma}(\boldsymbol{\xi})^{-1}\right]_{\boldsymbol{\xi}=\boldsymbol{\xi}_0} \boldsymbol{\Sigma}_0^{1/2},$$

$$\widehat{\boldsymbol{J}}_{ab\cdots} = \boldsymbol{\Sigma}_0^{1/2} \left[\partial_a\partial_b \cdots \boldsymbol{\Sigma}(\boldsymbol{\xi})^{-1}\right]_{\boldsymbol{\xi}=\widehat{\boldsymbol{\xi}}} \boldsymbol{\Sigma}_0^{1/2},$$

$$\boldsymbol{V} = \sqrt{n}\boldsymbol{\Sigma}^{-1/2}(\boldsymbol{S} - \boldsymbol{\Sigma})(\boldsymbol{\Sigma}^{-1/2})',$$

$$s_a = -\frac{1}{2}\mathrm{tr}(\boldsymbol{J}_a\boldsymbol{V}), \quad (a = 1, \cdots, q),$$

and

$$\boldsymbol{G} = (g_{ab}), \quad g_{ab} = \mathrm{E}[s_a s_b] = \frac{1}{2}\mathrm{tr}(\boldsymbol{J}_a \boldsymbol{J}_b), \quad (a, b = 1, \cdots, q),$$

It follows from A1 that $\boldsymbol{G}$ is nonsingular. Let g^{ab} is the (a, b) element of $\boldsymbol{G}^{-1}$. As another version of $\boldsymbol{J}_{ab}$, let

$$\boldsymbol{J}_{[ab]} = \boldsymbol{J}_{ab} - \frac{1}{2}\boldsymbol{J}_c g^{cd}\mathrm{tr}(\boldsymbol{J}_d \boldsymbol{J}_{ab}),$$

with Einstein's summation convention. The summation convention is used throughout this paper. For example, $\boldsymbol{J}_c g^{cd}$ means $\sum_{c=1}^{q} \boldsymbol{J}_c g^{cd}$.

Considering the Taylor expansion of h around $\lambda_i = 1$, we have

$$\rho_h(\boldsymbol{S}, \boldsymbol{\Sigma}) = \mathrm{tr}\left[\frac{1}{2}(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^2 + \frac{1}{3!}h_3(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^3 + \frac{1}{4!}h_4(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^4\right] + O\left(\mathrm{tr}\left\{(\boldsymbol{S}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^4\right\}\right). \tag{2.2}$$

Let

$$\boldsymbol{\Lambda} = \sqrt{n}\boldsymbol{\Sigma}_0^{-1/2}(\boldsymbol{S}\widehat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{I}_p)\boldsymbol{\Sigma}_0^{1/2}. \tag{2.3}$$

Then we obtain an expansion of T_h,

$$T_h = \mathrm{tr}\left[\frac{1}{2}\boldsymbol{\Lambda}^2 + \frac{1}{3!\sqrt{n}}h_3\boldsymbol{\Lambda}^3 + \frac{1}{4!n}h_4\boldsymbol{\Lambda}^4\right] + O_p(n^{-3/2}). \tag{2.4}$$

In order to obtain an explicit expansion of T_h, it is necessary to obtain an expansion of $\boldsymbol{\Lambda}$. It is shown similarly as in Swain [9] that

$$\bar{\xi}^a = \sqrt{n}(\hat{\xi}^a - \xi_0^a)$$

is asymptotically normal and hence $O_p(1)$. The Taylor expansion of $\widehat{\boldsymbol{\Sigma}}^{-1}$ around $\boldsymbol{\xi}_0$ is given by

$$\sqrt{n}\boldsymbol{\Sigma}_0^{1/2}(\hat{\boldsymbol{\Sigma}}^{-1} - \boldsymbol{\Sigma}_0^{-1})\boldsymbol{\Sigma}_0^{1/2} = \boldsymbol{J}_b\bar{\xi}^b + \frac{1}{2\sqrt{n}}\boldsymbol{J}_{bc}\bar{\xi}^b\bar{\xi}^c + O_p(n^{-1}). \tag{2.5}$$

Using (2.1),

$$\boldsymbol{S} = \boldsymbol{\Sigma}_0^{1/2}\left\{\boldsymbol{I}_p + \frac{1}{\sqrt{n}}(\boldsymbol{V} + \boldsymbol{\Delta}) + \frac{1}{n}(\boldsymbol{V}\boldsymbol{\Delta} + \boldsymbol{\Delta}\boldsymbol{V})\right\}\boldsymbol{\Sigma}_0^{1/2} + O_p(n^{-3/2}). \tag{2.6}$$

Then using (2.5), and (2.6), (2.3) is expanded as

$$\begin{aligned}\mathbf{\Lambda} &= \boldsymbol{V} + \mathbf{\Delta} + \boldsymbol{J}_b \bar{\xi}^b \\ &\quad + \frac{1}{\sqrt{n}} \left\{ \frac{1}{2} \boldsymbol{J}_{bc} \bar{\xi}^b \bar{\xi}^c + (\boldsymbol{V} + \mathbf{\Delta}) \boldsymbol{J}_b \bar{\xi}^b + \frac{1}{2} (\boldsymbol{V} \mathbf{\Delta} + \mathbf{\Delta} \boldsymbol{V}) \right\} + O_p(n^{-1}).\end{aligned} \tag{2.7}$$

In order to obtain an explicit expansion of $\mathbf{\Lambda}$, it is necessary to obtain an expansion of $\bar{\xi}^a$. The estimates $\hat{\xi}^a, (a = 1, \cdots, q)$, satisfy the system of equations

$$[\partial_a \rho(\boldsymbol{S}, \mathbf{\Sigma})]_{\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}} = 0, \quad (a = 1, \cdots, q).$$

Using (2.2) it can be seen that $\hat{\xi}^a$'s satisfy

$$\mathrm{tr} \left[\boldsymbol{S} [\partial_a \mathbf{\Sigma}^{-1}]_{\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}} \left\{ \boldsymbol{S} \hat{\mathbf{\Sigma}}^{-1} - \boldsymbol{I}_p + \frac{1}{2} h_3 (\boldsymbol{S} \hat{\mathbf{\Sigma}}^{-1} - \boldsymbol{I}_p)^2 \right\} \right] = O_p(n^{-3/2}),$$

or equivalently

$$\mathrm{tr} \left[\left\{ \boldsymbol{I}_p + \frac{1}{\sqrt{n}} (\boldsymbol{V} + \mathbf{\Delta}) \right\} \hat{\boldsymbol{J}}_a (\mathbf{\Lambda} + \frac{1}{2\sqrt{n}} h_3 \mathbf{\Lambda}^2) \right] = O_p(n^{-1}). \tag{2.8}$$

Substituting (2.7) and

$$\hat{\boldsymbol{J}}_a = \boldsymbol{J}_a + \frac{1}{\sqrt{n}} \boldsymbol{J}_{ab} \bar{\xi}^b + O_p(n^{-1})$$

into (2.8), it is seen that $\bar{\xi}^a$'s satisfy

$$\begin{aligned}&\mathrm{tr} \left[\boldsymbol{J}_a (\boldsymbol{V} + \mathbf{\Delta} + \boldsymbol{J}_b \bar{\xi}^b) \right] + \frac{1}{\sqrt{n}} \mathrm{tr} \left[\widetilde{h}_3 \boldsymbol{J}_a (\boldsymbol{V} + \mathbf{\Delta} + \boldsymbol{J}_b \bar{\xi}^b)^2 + \boldsymbol{J}_a (\frac{1}{2} \boldsymbol{J}_{bc} - \boldsymbol{J}_b \boldsymbol{J}_c) \bar{\xi}^b \bar{\xi}^c \right. \\ &\quad \left. + \boldsymbol{J}_{ab} (\boldsymbol{V} + \mathbf{\Delta} + \boldsymbol{J}_c \bar{\xi}^c) \bar{\xi}^b + \boldsymbol{J}_a \boldsymbol{V} \mathbf{\Delta} \right] = O_p(n^{-1}), \quad (a = 1, \cdots, q),\end{aligned} \tag{2.9}$$

where $\widetilde{h}_3 = 1 + \frac{1}{2} h_3$. The solution of $\bar{\xi}^a$ in (2.9) can be found in an expanded form

$$\bar{\xi}^a = \kappa^a + \frac{1}{\sqrt{n}} \varepsilon^a + O_p(n^{-1}). \tag{2.10}$$

In fact, substituting (2.10) into (2.9), we obtain

$$\kappa^a = e^a + \delta^a, \quad \varepsilon^a = -\frac{1}{2} g^{ab} \mathrm{tr}\left[\boldsymbol{J}_b \boldsymbol{M} + \boldsymbol{J}_{bc} \widetilde{\boldsymbol{W}} (e^c + \delta^c) \right], \tag{2.11}$$

where

$$e^a = g^{ab} s_b, \quad \delta^a = -\frac{1}{2} g^{ab} \mathrm{tr}(\boldsymbol{J}_b \boldsymbol{\Delta}), \quad \widetilde{\boldsymbol{W}} = \boldsymbol{W} + \boldsymbol{W}_\delta, \quad \boldsymbol{W} = \boldsymbol{V} + \boldsymbol{J}_b e^b,$$

$$\boldsymbol{W}_\delta = \boldsymbol{\Delta} + \boldsymbol{J}_b \delta^b, \quad \boldsymbol{M} = \widetilde{h}_3 \widetilde{\boldsymbol{W}}^2 + (\frac{1}{2} \boldsymbol{J}_{bc} - \boldsymbol{J}_b \boldsymbol{J}_c)(e^b + \delta^b)(e^c + \delta^c) + \boldsymbol{V} \boldsymbol{\Delta}.$$

Hence, from (2.4), (2.7), and (2.11), we obtain an expansion of T_h given by

$$T_h = \frac{1}{2} \mathrm{tr}(\widetilde{\boldsymbol{W}}^2) + \frac{1}{\sqrt{n}} T_1(\boldsymbol{V}) + O_p(n^{-1}), \tag{2.12}$$

where

$$T_1(\boldsymbol{V}) = -\frac{1}{2} g^{ab} \mathrm{tr}\left[\boldsymbol{J}_a \boldsymbol{M} + \boldsymbol{J}_{ab} \widetilde{\boldsymbol{W}} (e^b + \delta^b) \right] \mathrm{tr}(\boldsymbol{J}_b \widetilde{\boldsymbol{W}}) + \mathrm{tr}\left\{ (\frac{1}{2} \boldsymbol{J}_{bc} - \boldsymbol{J}_b \boldsymbol{J}_c) \widetilde{\boldsymbol{W}} \right\}$$

$$\times (e^b + \delta^b)(e^c + \delta^c) + \mathrm{tr}(\boldsymbol{J}_b \widetilde{\boldsymbol{W}}^2)(e^b + \delta^b) + \mathrm{tr}(\boldsymbol{V} \boldsymbol{\Delta} \widetilde{\boldsymbol{W}}) + \frac{1}{6} h_3 \mathrm{tr}(\widetilde{\boldsymbol{W}}^3). \tag{2.13}$$

3 Asymptotic Expansion of the Non-null Distribution of T_h under the Local Alternative

We can write the characteristic function of T_h as

$$\phi(t) = \mathrm{E}[\exp(itT_h)] = \mathrm{E}\left[\left\{ \mathrm{etr}(\frac{1}{2} \theta \widetilde{\boldsymbol{W}}^2) \right\} T(\boldsymbol{V}) \right] + O(n^{-1}), \tag{3.1}$$

where

$$\theta = it, \quad T(\boldsymbol{V}) = 1 + \frac{1}{\sqrt{n}} \theta T_1(\boldsymbol{V}), \tag{3.2}$$

with the expression $T_1(\boldsymbol{V})$ in (2.13). The probability density function (pdf) of $\boldsymbol{V}$ is expressed as (see e.g., Siotani, Hayakawa, Fujikoshi [7, p.160])

$$f(\boldsymbol{V}) = f_0(\boldsymbol{V}) Q(\boldsymbol{V}) + O(n^{-3/2}), \tag{3.3}$$

where

$f_0(\boldsymbol{V}) = a_p \mathrm{etr}(-\frac{1}{4}\boldsymbol{V}^2), \quad a_p = \pi^{-p(p+1)/4} 2^{-p(p+1)/4},$

$Q(\boldsymbol{V}) = 1 + \frac{1}{\sqrt{n}} Q_1(\boldsymbol{V}) + \frac{1}{n} Q_2(\boldsymbol{V}),$

$Q_1(\boldsymbol{V}) = -\frac{1}{2}(p+1)\mathrm{tr}(\boldsymbol{V}) + \frac{1}{6}\mathrm{tr}(\boldsymbol{V}^3), \qquad (3.4)$

$Q_2(\boldsymbol{V}) = \frac{1}{2}\left\{Q_1(\boldsymbol{V})\right\}^2 - \frac{1}{24}p(2p^2+3p-1) + \frac{1}{4}(p+1)\mathrm{tr}(\boldsymbol{V}^2) - \frac{1}{8}\mathrm{tr}(\boldsymbol{V}^4).$

Therefore, we have

$$\phi(t) = \int a_p \left\{ \mathrm{etr}\left(-\frac{1}{4}\boldsymbol{V}^2 + \frac{1}{2}\theta \widetilde{\boldsymbol{W}}^2\right)\right\} Q(\boldsymbol{V}) T(\boldsymbol{V}) d\boldsymbol{V} + O(n^{-1}), \qquad (3.5)$$

where $d\boldsymbol{V} = dv_{11} dv_{12} \cdots dv_{p-1,p} d_{p,p}.$

We prepare some lemmas useful for reductions of (3.5). Note that $\boldsymbol{G}^{-1} = (g^{ab})$ exists. Let

$$e^a = -\frac{1}{2} g^{ab} \mathrm{tr}(\boldsymbol{J}_b \boldsymbol{V}), \quad \boldsymbol{U} = -\boldsymbol{J}_a e^a, \quad \text{and} \quad \boldsymbol{W} = \boldsymbol{V} - \boldsymbol{U}, \qquad (3.6)$$

and similarly

$$\delta^a = -\frac{1}{2} g^{ab} \mathrm{tr}(\boldsymbol{J}_b \boldsymbol{\Delta}), \quad \boldsymbol{U}_\delta = -\boldsymbol{J}_a \delta^a, \quad \text{and} \quad \boldsymbol{W}_\delta = \boldsymbol{\Delta} - \boldsymbol{U}_\delta. \qquad (3.7)$$

Further, let

$$\boldsymbol{M} = (\mathrm{vec}^*(\boldsymbol{J}_1), \cdots, \mathrm{vec}^*(\boldsymbol{J}_q)),$$

where for any $p \times p$ symmetric matrix $\boldsymbol{A} = (a_{ij})$,

$$\mathrm{vec}^*(\boldsymbol{A}) = \left(\frac{a_{11}}{\sqrt{2}}, \cdots, \frac{a_{pp}}{\sqrt{2}}, a_{12}, \cdots, a_{p-1,p}\right)'.$$

Note that $\{\mathrm{vec}^*(\boldsymbol{A})\}' \mathrm{vec}^*(\boldsymbol{B}) = \frac{1}{2}\mathrm{tr}(\boldsymbol{AB})$. We obtain the following lemmas.

Lemma 3.1 *Let* $\boldsymbol{P_M} = \boldsymbol{M}(\boldsymbol{M}'\boldsymbol{M})^{-1}\boldsymbol{M}'$. *Then,*

$$\boldsymbol{e} = (e^1, \cdots, e^q)' = -(\boldsymbol{M}'\boldsymbol{M})^{-1}\boldsymbol{M}\mathrm{vec}^*(\boldsymbol{V}),$$

$$\boldsymbol{\delta} = (\delta^1, \cdots, \delta^q)' = -(\boldsymbol{M}'\boldsymbol{M})^{-1}\boldsymbol{M}'\mathrm{vec}^*(\boldsymbol{\Delta}),$$

$$\mathrm{vec}^*(\boldsymbol{U}) = \boldsymbol{P_M}\mathrm{vec}^*(\boldsymbol{V}),$$

$$\mathrm{vec}^*(\boldsymbol{U}_\delta) = \boldsymbol{P_M}\mathrm{vec}^*(\boldsymbol{\Delta}),$$

$$\mathrm{vec}^*(\boldsymbol{W}) = (\boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M})\mathrm{vec}^*(\boldsymbol{V}),$$

$$\mathrm{vec}^*(\boldsymbol{W}_\delta) = (\boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P_M})\mathrm{vec}^*(\boldsymbol{\Delta}).$$

Lemma 3.2 *Let θ be any complex number whose real part is smaller than $-\frac{1}{2}$. Let $g(\boldsymbol{V},\boldsymbol{U},\boldsymbol{W})$ be a function of $\boldsymbol{V}$, $\boldsymbol{U}$, and $\boldsymbol{W}$. Then,*

$$\int \mathrm{etr}\left\{-\frac{1}{4}\boldsymbol{V}^2+\frac{1}{2}\theta(\boldsymbol{W}+\boldsymbol{W}_\delta)^2\right\}\times g(\boldsymbol{V},\boldsymbol{U},\boldsymbol{W})d\boldsymbol{V}=(1-2\theta)^{-r/2}$$
$$\times\exp\left[\theta(1-2\theta)^{-1}\left\{\frac{1}{2}\mathrm{tr}(\boldsymbol{W}_\delta^2)\right\}\right]\times\int\left(-\frac{1}{4}\boldsymbol{V}^2\right)g(\dot{\boldsymbol{V}},\boldsymbol{U},\dot{\boldsymbol{W}})d\boldsymbol{V},\quad (3.8)$$

where $r=p(p+1)/2-q$, $\dot{\boldsymbol{V}}=\boldsymbol{U}+(1-2\theta)^{-1/2}\boldsymbol{W}+2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta$,
$\dot{\boldsymbol{W}}=(1-2\theta)^{-1/2}\boldsymbol{W}+2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta$.

Proof. We shall show that (3.8) is obtained by considering the transformation $\boldsymbol{V}\to\breve{\boldsymbol{V}}$, where

$$\breve{\boldsymbol{V}}=\boldsymbol{U}+(1-2\theta)^{1/2}\boldsymbol{W}-2\theta(1-2\theta)^{-1/2}\boldsymbol{W}_\delta.\quad (3.9)$$

Using Lemma 3.1, we have

$$\mathrm{vec}^*(\breve{\boldsymbol{V}})=\left\{\boldsymbol{P}_{\boldsymbol{M}}+(1-2\theta)^{1/2}(\boldsymbol{I}_{p(p+1)/2}-\boldsymbol{P}_{\boldsymbol{M}})\right\}$$
$$\times\left\{\mathrm{vec}^*(\boldsymbol{V})-2\theta(1-2\theta)^{-1}\mathrm{vec}^*(\boldsymbol{W}_\delta)\right\}.$$

This implies that the inverse transformation is

$$\mathrm{vec}^*(\boldsymbol{V})=\left\{\boldsymbol{P}_{\boldsymbol{M}}+(1-2\theta)^{-1/2}(\boldsymbol{I}_{p(p+1)/2}-\boldsymbol{P}_{\boldsymbol{M}})\right\}\mathrm{vec}^*(\breve{\boldsymbol{V}})$$
$$+2\theta(1-2\theta)^{-1}\mathrm{vec}^*(\boldsymbol{W}_\delta).$$

It is equivalent to

$$\boldsymbol{V}=\breve{\boldsymbol{U}}+(1-2\theta)^{-1/2}\breve{\boldsymbol{W}}+2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta,$$

where $\breve{\boldsymbol{U}}=\frac{1}{2}\boldsymbol{J}_a g^{ab}\mathrm{tr}(\boldsymbol{J}_b\breve{\boldsymbol{V}})$, and $\breve{\boldsymbol{W}}=\breve{\boldsymbol{V}}-\breve{\boldsymbol{U}}$. Therefore, the Jacobian of the transformation (3.9) is

$$J(\boldsymbol{V}\to\breve{\boldsymbol{V}})=|\boldsymbol{P}_{\boldsymbol{M}}+(1-2\theta)^{-1/2}(\boldsymbol{I}_{p(p+1)/2}-\boldsymbol{P}_{\boldsymbol{M}})|=(1-2\theta)^{-r/2},$$

since the characteristic roots of $\boldsymbol{P}_{\boldsymbol{M}}$ are one or zero and $\mathrm{rank}(\boldsymbol{P}_{\boldsymbol{M}})=q$. Further, it holds that $\boldsymbol{U}=\breve{\boldsymbol{U}}$, and $\boldsymbol{W}=(1-2\theta)^{-1/2}\breve{\boldsymbol{W}}+2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta$, since $\mathrm{vec}^*(\breve{\boldsymbol{U}})=\boldsymbol{P}_{\boldsymbol{M}}\mathrm{vec}^*(\breve{\boldsymbol{V}})=\mathrm{vec}^*(\boldsymbol{U})$, and $\breve{\boldsymbol{W}}=(1-2\theta)^{1/2}\boldsymbol{W}-2\theta(1-2\theta)^{-1/2}\boldsymbol{W}_\delta$. These complete the proof.

Lemma 3.3 *Let $\boldsymbol{V}$ be a $p \times p$ symmetric random matrix with pdf $f_0(\boldsymbol{V})$ in* (3.3)*. Let e^a, $\boldsymbol{U}$, and $\boldsymbol{W}$ be the random variables defined by* (3.6)*. Then*
(1) *$\boldsymbol{e} = (e^1, \cdots, e^q)'$ and $\boldsymbol{W}$ are independent,*
(2) *$\boldsymbol{e}$ is distributed as $N_q(\boldsymbol{0}, \boldsymbol{G}^{-1})$,*
(3) *$\mathrm{vec}^*(\boldsymbol{U})$ and $\mathrm{vec}^*(\boldsymbol{W})$ are independently distributed as $N_{p(p+1)/2}(\boldsymbol{0}, \boldsymbol{P}_{\boldsymbol{M}})$ and $N_{p(p+1)/2}(\boldsymbol{0}, \boldsymbol{I}_{p(p+1)/2} - \boldsymbol{P}_{\boldsymbol{M}})$, respectively.*

Proof. The results are easily obtained by using Lemma 3.1 and the fact that $\mathrm{vec}^*(\boldsymbol{V})$ is distributed as $N_{p(p+1)/2}(\boldsymbol{0}, \boldsymbol{I}_{p(p+1)/2})$.

Using Lemmas 3.2 and 3.3, we can write the characteristic function (3.5) as

$$\phi(t) = (1-2\theta)^{-r/2} \exp\left[\theta(1-2\theta)^{-1}\left\{\frac{1}{2}\mathrm{tr}(\boldsymbol{W}_\delta^2)\right\}\right] \times \mathrm{E}\left[Q(\dot{\boldsymbol{V}})T(\dot{\boldsymbol{V}})\right] + O(n^{-1}), \tag{3.10}$$

where $\dot{\boldsymbol{V}}$ is given by Lemma 3.2.
Here the expectation in (3.10) is taken with respect to the distribution of $\boldsymbol{U}$ (or $\boldsymbol{e}$) and $\boldsymbol{W}$ given in Lemma 3.3. After calculation of these expectations, we obtain

$$\begin{aligned}\phi(t) &= (1-2\theta)^{-r/2} \exp\left[\theta(1-2\theta)^{-1}\left\{\frac{1}{2}\mathrm{tr}(\boldsymbol{W}_\delta^2)\right\}\right] \\ &\quad \times \left\{1 + \frac{1}{\sqrt{n}}\sum_{j=0}^{3} c_j (1-2\theta)^{-j}\right\} + O(n^{-1}),\end{aligned} \tag{3.11}$$

where the coefficients c_j's are given by

$$\begin{aligned} c_0 &= \tfrac{1}{2}(g^{ab} + \delta^a\delta^b)\boldsymbol{K}_{ab\delta} - \tfrac{1}{4}(g^{ab} + \delta^a\delta^b)\boldsymbol{K}_{(ab)\delta} - \tfrac{1}{2}\boldsymbol{K}_{a\delta^2}\delta^a + \tfrac{1}{3}\boldsymbol{K}_{\delta^3}, \\ c_1 &= \tfrac{1}{4}(g^{ab} + \delta^a\delta^b)\boldsymbol{K}_{(ab)\delta} + (\tfrac{1}{4}h_3 g^{ab} - \tfrac{1}{2}\delta^a\delta^b)\boldsymbol{K}_{ab\delta} - \tfrac{1}{2}\widetilde{h}_3(p+1)\boldsymbol{K}_\delta \\ &\quad + \tfrac{1}{2}\boldsymbol{K}_{a\delta^2}\delta^a - \tfrac{1}{2}\boldsymbol{K}_{\delta^3}, \\ c_2 &= \tfrac{1}{2}\widetilde{h}_3\left\{(p+1)\boldsymbol{K}_\delta - g^{ab}\boldsymbol{K}_{ab\delta}\right\} - \tfrac{1}{12}h_3\boldsymbol{K}_{\delta^3}, \quad c_3 = \tfrac{1}{6}\widetilde{h}_3\boldsymbol{K}_{\delta^3}. \end{aligned} \tag{3.12}$$

Here we use the following notations:

$$\boldsymbol{K}_{ab\delta} = \mathrm{tr}(\boldsymbol{J}_a\boldsymbol{J}_b\boldsymbol{W}_\delta),\ \boldsymbol{K}_{(ab)\delta} = \mathrm{tr}(\boldsymbol{J}_{(ab)}\boldsymbol{W}_\delta),\ \boldsymbol{K}_{\delta^k} = \mathrm{tr}(\boldsymbol{W}_\delta^k),$$

and so on. The formulae needed for calculating expectations are given in Appendix A. By inverting the characteristic function term by term, we obtain an expansion of the non-null distribution of T_h under the local alternative as in the following theorem.

Theorem 3.1 *Let T_h be the test statistic given by* (1.1) *with a function h satisfying A2 and A3. Suppose that a given covariance structure $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\xi})$ satisfies A1. Then under the local alternative hypothesis H_n, the distribution of T_h can be expanded for large n as*

$$\mathrm{P}(T_h \leq x) = G_r(x;\tau) + \frac{1}{\sqrt{n}}\sum_{j=0}^{3} c_j G_{r+2j}(x;\tau) + O(n^{-1}), \tag{3.13}$$

where $r = p(p+1)/2 - q$, $\tau = \mathrm{tr}(\boldsymbol{W}_\delta^2)/2$, $G_k(\cdot;\tau)$ is the noncentral χ^2 distribution function with k degrees of freedom and the noncentrality parameter τ, and the coefficients c_j's are given by (3.12).

4 Applications

4.1 Power Comparisions

Wakaki, Eguchi, Fujikoshi [10] gave an asymptotic expansion of the null distribution of T_h in a general case as

$$\mathrm{P}(T_h \leq x|H_0) = G_r(x) + \frac{1}{n}\sum_{j=0}^{3} a_j G_{r+2j}(x) + O(n^{-3/2}), \tag{4.1}$$

where $G_k(\cdot)$ is the χ^2 distribution function with k degrees of freedom, the coefficients a_j's are given by

$$\begin{aligned}
a_0 &= \tfrac{1}{72}\left\{-3p(p^2+3p-1) - 9g^{abcd}\boldsymbol{K}_{abcd} + g^{abcdef}\boldsymbol{K}_{abc,def}\right\} \\
&\quad + \tfrac{1}{16}g^{ab}g^{cd}\left\{4\boldsymbol{K}_{[ab]cd} - \boldsymbol{K}_{[ab][cd]} + 2\boldsymbol{K}_{[ab][cd]}\right\}, \\
a_1 &= -a_0 + \widetilde{h}_3^{\,2} C - (h_4 - 6)B + \widetilde{h}_3 D, \\
a_2 &= -\widetilde{h}_3^{\,2}(A+C) + (h_4-6)B - \widetilde{h}_3 D, \qquad a_3 = \widetilde{h}_3^{\,2} A,
\end{aligned} \tag{4.2}$$

and the coefficients $A,\cdots,D$ are given by

$$\begin{aligned}
A &= \tfrac{1}{72}\left\{6p(4p^2+9p+7) - 36q(3p+4) - 9(p^2+2p+3)g^{ab}\boldsymbol{K}_{a,b}\right. \\
&\quad \left. + 6(p+1)g^{abcd}\boldsymbol{K}_{abc,d} + 18g^{abcd}\boldsymbol{K}_{abcd} - g^{abcdef}\boldsymbol{K}_{abc,def}\right\}, \\
B &= \tfrac{1}{48}\left\{p(p^2+5p+5) - 4q(2p+3) - 2g^{ab}\boldsymbol{K}_{a,b} + g^{abcd}\boldsymbol{K}_{abcd}\right\}, \\
C &= \tfrac{1}{12}\left\{p(4p^2+9p+7) - 12q(p+1) - 3g^{ab}g^{cd}\boldsymbol{K}_{abcd}\right. \\
&\quad \left. - 2g^{ab}g^{cd}g^{ef}\boldsymbol{K}_{ace,bdf}\right\}, \\
D &= -\tfrac{1}{6}p(p^2+3p+4) + q(2p+3) + \tfrac{1}{2}g^{ab}\boldsymbol{K}_{a,b} \\
&\quad - \tfrac{1}{4}(p+1)g^{ab}g^{cd}\boldsymbol{K}_{abc,d} - \tfrac{1}{2}g^{abcd}\boldsymbol{K}_{abcd} + \tfrac{1}{36}g^{abcdef}\boldsymbol{K}_{abc,def} \\
&\quad - \tfrac{1}{4}(p+1)g^{ab}\boldsymbol{K}_{[ab]} + \tfrac{1}{4}g^{ab}g^{cd}\boldsymbol{K}_{[ab]cd}.
\end{aligned} \tag{4.3}$$

Here we use the following notations:

$$g^{abcd} = \sum_{[3]} g^{ab} g^{cd},\ g^{abcdef} = \sum_{[5]} g^{ab} g^{cdef},\ \boldsymbol{K}_{abc\cdots} = \mathrm{tr}(\boldsymbol{J}_a \boldsymbol{J}_b \boldsymbol{J}_c \cdots),$$

$$\boldsymbol{K}_{[ab]cd} = \mathrm{tr}(\boldsymbol{J}_{[ab]} \boldsymbol{J}_c \boldsymbol{J}_d),\ \boldsymbol{K}_{abc,def} = \boldsymbol{K}_{abc} \boldsymbol{K}_{def},$$

and so on.

Let t_α be the upper 100α percent point of the null distribution of T_h and χ^2_α be the upper 100α percent point of the χ^2 distribution with r degrees of freedom. By a Cornish-Fisher expansion, we obtain

$$\begin{aligned} t_\alpha &= \chi^2_\alpha - \frac{1}{n}\left\{ \frac{1}{g_r(\chi^2_\alpha)} \sum_{j=0}^{3} a_j G_{r+2j}(\chi^2_\alpha) \right\} + O(n^{-3/2}) \\ &= \chi^2_\alpha + O(n^{-1}). \end{aligned} \tag{4.4}$$

Using (3.13), (4.1), and (4.4), we can calculate the power β_h,

$$\beta_h = \mathrm{P}(T_h > t_\alpha | H_1) = 1 - G_r(\chi^2_\alpha; \tau) - \frac{1}{\sqrt{n}} \sum_{j=0}^{3} c_j G_{r+2j}(\chi^2_\alpha; \tau) + O(n^{-1}). \tag{4.5}$$

We use useful formulas for reductions of (4.5). Noncentral χ^2 distribution function and χ^2 distribution can be expanded as (see e.g., Muirhead [4])

$$G_r(x;\tau) = \sum_{k=1}^{\infty} P_k G_{r+2k}(x), \quad \text{where} \quad P_k = \frac{e^{-\tau/2}(\frac{1}{2}\tau)^k}{k!}, \tag{4.6}$$

$$G_{r+2}(x) = -2 g_{r+2}(x) + G_r(x), \tag{4.7}$$

respectively, where $g_k(\cdot)$ is the pdf of the χ^2 distribution with k degrees of freedom. Using (4.6) and (4.7), we can obtain

$$\begin{aligned} \sum_{j=0}^{3} c_j G_{r+2j}(\chi^2_\alpha; \tau) &= (c_1 + c_2 + c_3 + c_4) \sum_{k=0}^{\infty} P_k G_{r+2k}(\chi^2_\alpha) - 2(c_1 + c_2 + c_3) \\ &\times \sum_{k=0}^{\infty} P_k g_{r+2k+2}(\chi^2_\alpha) - 2(c_2 + c_3) \sum_{k=0}^{\infty} P_k g_{r+2k+4}(\chi^2_\alpha) - 2c_3 \sum_{k=0}^{\infty} P_k g_{r+2k+6}(\chi^2_\alpha), \end{aligned} \tag{4.8}$$

where coefficients c_j's are given by (3.12). After calculating (4.8), we can rewrite (4.5) as

$$\beta_h = \frac{1}{\sqrt{n}}\widetilde{h_3}\left[\left\{(p+1)\boldsymbol{K}_\delta - g^{ab}\boldsymbol{K}_{ab\delta}\right\} g_{r+4}(\chi^2_\alpha;\tau) + \frac{1}{3}\boldsymbol{K}_{\delta^3} g_{r+6}(\chi^2_\alpha;\tau)\right] + \beta_{LR} + O(n^{-1}), \tag{4.9}$$

where $g_k(\cdot;\tau)$ is the pdf of the noncentral χ^2 distribution with k degrees of freedom and the noncentrality parameter τ and β_{LR} is the power of the likelihood ratio statistic.

4.2 Linear Structures

We consider the structure : $\boldsymbol{\Sigma}$ is a linear combination of matrices,

$$\boldsymbol{\Sigma}(\boldsymbol{\xi}) = \xi^1\boldsymbol{G}_1 + \xi^2\boldsymbol{G}_2 + \cdots + \xi^q\boldsymbol{G}_q,$$

where $\boldsymbol{G}_a$'s are given $p\times p$ symmetric matrices which are linearly independent, satisfying that

$$\boldsymbol{G}_j^2 = \boldsymbol{G}_j, \qquad \boldsymbol{G}_i\boldsymbol{G}_j = O\ (i \neq j),$$

and ξ^a's are unknown such that $\boldsymbol{\Sigma}(\boldsymbol{\xi})$ is positive definite. We note that this structure includes sphericity structure, intraclass correlation structure, and so on.

We can easily calculate $\boldsymbol{J}_a$ and $\boldsymbol{K}_\delta$, $\boldsymbol{K}_{ab\delta}$ in this case as

$$\boldsymbol{J}_a = -\boldsymbol{G}_a, \qquad \boldsymbol{K}_\delta = 0, \quad \boldsymbol{K}_{ab\delta} = 0. \tag{4.10}$$

Hence we can write the power (4.9) as

$$\beta_h = \frac{1}{3\sqrt{n}}\widetilde{h_3}\boldsymbol{K}_{\delta^3} g_{r+6}(\chi^2_\alpha;\tau) + \beta_{LR} + O(n^{-1}).$$

It is equivalent to

$$\sqrt{n}(\beta_h - \beta_{LR}) \to \frac{1}{3}\widetilde{h_3}\boldsymbol{K}_{\delta^3} g_{r+6}(\chi^2_\alpha;\tau) \quad (n \to \infty). \tag{4.11}$$

This shows that LR statistic has greater power than statistics with negative values of $\widetilde{h_3}$ if $\boldsymbol{K}_{\delta^3} > 0$.

In a special case that $q=1$, this structure is the sphericity structure. Since we can choose an arbitrary parametrization, we use $\mathbf{\Sigma}(\boldsymbol{\xi})=\{\exp(\xi^1)\}\,\boldsymbol{I}_p$, then,

$$\boldsymbol{J}_1=-\boldsymbol{I}_p,\quad \boldsymbol{K}_{\delta^3}=\sum_{i=1}^{p}(\nu_i-\overline{\nu})^3, \tag{4.12}$$

where ν_i's are the eigenvalues of $\mathbf{\Delta}$ and $\overline{\nu}=\frac{1}{p}\sum_{i=1}^{p}\nu_i$. From (4.11) and (4.12), when $\boldsymbol{K}_{\delta^3}\neq 0$, power comparisions of sphericity test in the class depend on a kind of skewness of $\mathbf{\Delta}$'s eigenvalues. When $\boldsymbol{K}_{\delta^3}=0$, we can not compare the power asymptotically in the class on order $n^{-1/2}$. So we consider an asymptotic expansion of the non-null distribution of T_h under the local alternatives up to the order n^{-1} taking focus on h_3 and h_4, we have

$$\mathrm{P}(T_h\leq x)=G_r(x;\tau)+\frac{1}{\sqrt{n}}\sum_{j=0}^{1}\tilde{c}_jG_{r+2j}(x;\tau)+\frac{1}{n}\left\{\sum_{j=1}^{5}d_jG_{r+2j}(x;\tau)\right.$$
$$\left.+\sum_{j=1}^{4}f_jG_{r+2j}(x;\tau)+bG_r(x;\tau)\right\}+O(n^{-3/2}), \tag{4.13}$$

where b does not depend on h_3 and h_4, the coefficients $\tilde{c}_j$'s, d_j's, f_j's and the formulae needed for calculating expectations are given in Appendix A.

Using (3.13), (4.13), (4.6), and (4.7) with noting (4.4), we can also calculate the power β_h as

$$\beta_h=\frac{1}{n}\left[\sum_{j=2}^{5}e_jg_{r+2j}(\chi^2_\alpha;\tau)+\left\{(h_3^2+4h_3)E+h_3D\right\}\frac{(\chi^2_\alpha)^2}{r(r+2)}g_r(\chi^2_\alpha;\tau)\right]$$
$$+\frac{1}{n}\left[\sum_{j=2}^{4}g_jg_{r+2j}(\chi^2_\alpha;\tau)-2h_4B\frac{(\chi^2_\alpha)^2}{r(r+2)}g_r(\chi^2_\alpha;\tau)\right]+c+O(n^{-3/2}), \tag{4.14}$$

where c does not depend on h_3 and h_4, and E are given by

$$c=1-G_r(\chi^2_\alpha;\tau)-\frac{1}{n}\left(bG_r(\chi^2_\alpha;\tau)\right),\quad E=\frac{1}{2}\left(C-\frac{\chi^2_\alpha}{r+4}A\right),$$

and the coefficients e_j's, g_j's are given in Appendix B, the coefficients $A,\cdots,D$ are given by (4.3). The difference of local powers among the class is complex. We can examine the difference numerically for specified values of p, α and $\mathbf{\Delta}$.

Hayakawa [3], Pillai and Jayachandran [6] also gave the numerical examples about the power of T_h in some special cases.

We have shown that the difference of the asymptotic power in our class of the test for linear structures which depend on only $\boldsymbol{K}_{\delta^3}$. Other important covariance structures arise when we treat covariance structure analysis (system of equation model) (see e.g., [11]). In this case, we have to consider non-linear covariance structures. Sometimes the domain $\boldsymbol{\Xi}$ of $\boldsymbol{\Sigma}(\boldsymbol{\Xi})$ is not an open set. If the minimizing point lies on the boundary, the asymptotic expansion formulas derived in this paper are not applied. The problems of deriving asymptotic expansion formulas in such case are left for the future.

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Appendix A

Let $\boldsymbol{V}$ be a $p \times p$ symmetric random matrix normal with pdf $f_0(\boldsymbol{V})$ in (3.3). Let $\boldsymbol{e} = (e^1, \cdots, e^q)'$ and $\boldsymbol{W}$ be the random vector and matrix defined by (3.6). Then, it holds that any $p \times p$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$\mathrm{E}[e^a e^b] = g^{ab}, \quad \mathrm{E}[e^a e^b e^c e^d] = g^{abcd}, \quad \mathrm{E}[e^a e^b e^c e^d e^e e^f] = g^{abcdef},$$

$$\mathrm{E}[\mathrm{tr}(\boldsymbol{AW})\mathrm{tr}(\boldsymbol{BW})] = 2\mathrm{tr}(\boldsymbol{AB}) - g^{ab} tr(\boldsymbol{AJ}_a)\mathrm{tr}(\boldsymbol{BJ}_b),$$
$$\mathrm{E}[\mathrm{tr}(\boldsymbol{AWBW})] = \mathrm{tr}\boldsymbol{A}\mathrm{tr}\boldsymbol{B}' + \mathrm{tr}(\boldsymbol{AB}') - g^{ab}\mathrm{tr}(\boldsymbol{AJ}_a\boldsymbol{BJ}_b),$$

$$
\begin{aligned}
\mathrm{E}[\mathrm{tr}(\boldsymbol{A}\boldsymbol{W}^2)\mathrm{tr}(\boldsymbol{B}\boldsymbol{W}^2)] &= 4\mathrm{tr}(\boldsymbol{A}\bar{\boldsymbol{B}}) + (p^2+2p+1)\mathrm{tr}\boldsymbol{A}\mathrm{tr}\boldsymbol{B} \\
&\quad -(p+1)g^{ab}\left\{\mathrm{tr}\boldsymbol{A}\mathrm{tr}(\boldsymbol{B}\boldsymbol{J}_a\boldsymbol{J}_b) + \mathrm{tr}\boldsymbol{B}\mathrm{tr}(\boldsymbol{A}\boldsymbol{J}_a\boldsymbol{J}_b)\right\} \\
&\quad -8g^{ab}\mathrm{tr}(\boldsymbol{A}\boldsymbol{J}_a\bar{\boldsymbol{B}}\boldsymbol{J}_b) + g^{abcd}\mathrm{tr}(\boldsymbol{A}\boldsymbol{J}_a\boldsymbol{J}_b)\mathrm{tr}(\boldsymbol{B}\boldsymbol{J}_c\boldsymbol{J}_d), \\
\mathrm{E}[\mathrm{tr}(\boldsymbol{A}\boldsymbol{W})\mathrm{tr}\boldsymbol{W}^3] &= 6(p+1)\mathrm{tr}\boldsymbol{A} - 6g^{ab}\mathrm{tr}(\bar{\boldsymbol{A}}\boldsymbol{J}_a\boldsymbol{J}_b) \\
&\quad -3(p+1)g^{ab}\mathrm{tr}(\boldsymbol{A}\boldsymbol{J}_a)\boldsymbol{K}_b + g^{abcd}\mathrm{tr}(\boldsymbol{A}\boldsymbol{J}_a\boldsymbol{K}_{bcd}), \\
\mathrm{E}[\boldsymbol{W}^4] &= p(2p^2+5p+5) - 4q(2p+3) \\
&\quad -2g^{ab}\boldsymbol{K}_{a,b} + g^{abcd}\boldsymbol{K}_{abcd}, \\
\mathrm{E}[(\boldsymbol{W}^3)^2] &= 6p(4p^2+9p+7) \\
&\quad -24q(2p+3) - 12g^{ab}(2p+3)\boldsymbol{K}_{ab} \\
&\quad -3g^{ab}(3p^2+6p+7)\boldsymbol{K}_{a,b} + 6(p+1)g^{abcd}\boldsymbol{K}_{a,bcd} \\
&\quad +18g^{abcd}\boldsymbol{K}_{abcd} - g^{abcdef}\boldsymbol{K}_{abc,def},
\end{aligned}
$$

where $\bar{\boldsymbol{A}} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}')$. The expectations are obtained by using Lemma 3.3 and the fact that $\mathrm{vec}^*(\boldsymbol{V})$ is distributed as $N_{p(p+1)/2}(\boldsymbol{0}, \boldsymbol{I}_{p(p+1)/2})$. The calculations can be simplified by using the properties such as

$$
\mathrm{E}[\mathrm{tr}\boldsymbol{W}^2\mathrm{tr}\boldsymbol{W}^2] = \mathrm{E}[\mathrm{tr}(\boldsymbol{W}^2\ddot{\boldsymbol{W}}^2) + 2\mathrm{tr}(\boldsymbol{W}\ddot{\boldsymbol{W}})\mathrm{tr}(\boldsymbol{W}\ddot{\boldsymbol{W}})],
$$

where $\ddot{\boldsymbol{W}}$ is a symmetric random matrix having the same distribution $\boldsymbol{W}$ and being independent of $\boldsymbol{W}$.

Appendix B

The coefficients b and $\tilde{c}_j$'s, d_j's, f_j's are given by

$\tilde{c}_0 = \frac{1}{2}\overline{\nu}\boldsymbol{K}_{\delta^2}, \tilde{c}_1 = -\tilde{c}_0,$

$f_1 = -\frac{1}{2}g_2, f_2 = \frac{1}{2}(g_2 - g_3), f_3 = \frac{1}{2}(g_3 - g_4), f_4 = \frac{1}{2}g_4,$

$d_1 = -\frac{1}{2}e_2, d_2 = \frac{1}{2}(e_2 - e_3), d_3 = \frac{1}{2}(e_3 - e_4), d_4 = \frac{1}{2}(e_4 - e_5), d_5 = \frac{1}{2}e_5,$

$b = \mathrm{E}\left[Q_2(\boldsymbol{V}_1) + \theta Q_1(\boldsymbol{V}_1)\mathrm{tr}(\boldsymbol{W}_1\boldsymbol{K}) + \frac{1}{2}\theta^2\left\{\mathrm{tr}(\boldsymbol{W}_1\boldsymbol{M}_1)\right\}^2 + \frac{1}{2}\mathrm{tr}(\boldsymbol{M}_1^2)\right],$

where $Q_1(\cdot)$ and $Q_2(\cdot)$ are given by (3.4), and the coefficients e_j's, g_j's, $\boldsymbol{V}_1$, $\boldsymbol{W}_1$, $\boldsymbol{M}_1$ are given by

$e_2 = -\frac{1}{24}h_3^2(4p^3 + 9p^2 - 13p - 12 + 4p^{-1}) + h_3 p\boldsymbol{K}_{\delta^2},$

$e_3 = -\frac{1}{4}h_3^2(p + 2 - 2p^{-1})\boldsymbol{K}_{\delta^2} + \frac{1}{2}h_3 p^{-1}\boldsymbol{K}_{\delta^2}\boldsymbol{K}_{\delta^2} - \frac{1}{2}h_3\boldsymbol{K}_{\delta^4}$
$\quad + \frac{1}{144}(h_3^2 + 4h_3)(6p^3 + 18p^2 - 24p - 72 + 96p^{-1}),$

$e_4 = \frac{1}{4}(h_3^2 + 4h_3)\boldsymbol{K}_{\delta^2} - \frac{1}{8}h_3^2\boldsymbol{K}_{\delta^4},$

$e_5 = \frac{1}{8}(h_3^2 + 4h_3)(\boldsymbol{K}_{\delta^4} - p^{-1}\boldsymbol{K}_{\delta^2}\boldsymbol{K}_{\delta^2}),$

$$
\begin{aligned}
&g_2 = \tfrac{1}{24}h_4(2p^3 + 5p^2 - 7p - 12 + 12p^{-1}),\\
&g_3 = \tfrac{1}{12}h_4(2p + 3 - 6p^{-1})\boldsymbol{K}_{\delta^2}, \quad g_4 = \tfrac{1}{24}h_4\boldsymbol{K}_{\delta^4},\\
&\boldsymbol{V}_1 = \boldsymbol{U} + (1-2\theta)^{-\frac{1}{2}}\boldsymbol{W} + 2\theta(1-2\theta)^{-1}\boldsymbol{W}_\delta,\\
&\boldsymbol{W}_1 = (1-2\theta)^{-\frac{1}{2}}\boldsymbol{W} + (1-2\theta)^{-1}\boldsymbol{W}_\delta,\\
&\boldsymbol{M}_1 = -p^{-1}\left\{\mathrm{tr}(\boldsymbol{W}_1^2)\right\}\boldsymbol{I}_p - p^{-1}\left\{\mathrm{tr}(\boldsymbol{V}_1\boldsymbol{\Delta})\right\}(e^1+\delta^1)\boldsymbol{I}_p\\
&\qquad -p^{-1}\left\{\mathrm{tr}(\boldsymbol{W}_1)\right\}(e^1+\delta^1)\boldsymbol{I}_p + (e^1+\delta^1)^2\boldsymbol{I}_p\\
&\qquad -(e^1+\delta^1)(\boldsymbol{V}_1+\boldsymbol{\Delta}) + \tfrac{1}{2}(\boldsymbol{V}_1\boldsymbol{\Delta}+\boldsymbol{\Delta}\boldsymbol{V}_1).
\end{aligned}
$$

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