Approximation to the upper percentiles of the statistic for pairwise comparison among components of mean vector in elliptical distributions

Sho Takahashi[a], Takahiro Nishiyama[b], Takashi Seo[b] *

[a] Graduate School of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 Japan
[b] Department of Mathematical Information Science , Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 Japan

Abstract

In this paper, we consider approximation to the upper percentiles of the statistic for pairwise comparison among components of mean vector in elliptical distributions. The first order and modified second order approximations based on the Bonferroni inequalities are given by asymptotic expansion procedure. We investigate the effects of nonnormality on upper percentiles of this statistic in elliptical distribution. Finally, numerical results by Monte Carlo simulations are given.

Key words and phrases: Asymptotic expansion, Bonferroni inequality, Pairwise comparison, Elliptical distribution, Monte Carlo simulation.

1. Introduction

Let us consider the simultaneous confidence intervals for pairwise comparisons among components of the mean vector. Such a situation arises, for example, in multiple comparisons of the components of repeated measurements of the same quantity in different conditions. Under the multivariate normal population, these simultaneous confidence intervals are discussed by many authors. Lin, Seppänen

*Corresponding author. Tel.: +81-3-5228-8199; fax: +81-3-3260-4293.
E-mail address: seo@rs.kagu.tus.ac.jp.

and Uusipaikka (1990) and Nishiyama (2009) considered the simultaneous confidence intervals by Tukey-Kramer type procedure. Also, Seo (1995) considered the simultaneous confidence intervals by asymptotic expansion procedure. Here, we discuss the simultaneous confidence intervals under the elliptical populations.

Let Π be the population distributed as a p-dimensional elliptical distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, i.e., $\mathrm{E}_p(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ (see, e.g., Muirhead (1982), Fang, Kotz and Ng (1989)). A probability density function and characteristic function of the elliptical distribution are of the form

$$f(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\Lambda}) = c_p|\boldsymbol{\Lambda}|^{-1/2} g\left\{(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Lambda}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\},$$

for some nonnegative function g, where c_p is the normalizing constant and $\boldsymbol{\Lambda}$ is positive definite, and

$$\phi(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\boldsymbol{\Lambda}\boldsymbol{t}),$$

for some function ψ, where $i = \sqrt{-1}$, respectively. Note that $\mathrm{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{\Sigma} = -2\psi'(0)\boldsymbol{\Lambda}$. We also define the kurtosis parameter by $\kappa = \{\psi''(0)/(\psi'(0))^2\} - 1$. Elliptical distributions include the multivariate normal, the multivariate t, the contaminated normal distributions and so on.

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ be N independent sample vectors from $\mathrm{E}_p(\boldsymbol{\mu}, \boldsymbol{\Lambda})$. Then the sample mean vector and the sample covariance matrix are

$$\begin{aligned}\overline{\boldsymbol{x}} &= \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i, \\ \boldsymbol{S} &= \frac{1}{N-1}\sum_{i=1}^{N}(\boldsymbol{x}_i-\overline{\boldsymbol{x}})(\boldsymbol{x}_i-\overline{\boldsymbol{x}})',\end{aligned}$$

respectively. In general, simultaneous confidence intervals for pairwise comparisons among components of mean vector are of the form

$$\boldsymbol{b}_{\ell m}'\boldsymbol{\mu} \in \left[\boldsymbol{b}_{\ell m}'\overline{\boldsymbol{x}} \pm w\sqrt{\boldsymbol{b}_{\ell m}'\boldsymbol{S}\boldsymbol{b}_{\ell m}/N}\right],\ 1 \le \ell < m \le p,$$

where $\boldsymbol{b}_{\ell m} = \boldsymbol{e}_\ell - \boldsymbol{e}_m$ and $\boldsymbol{e}_\ell$ is a unit vector of the p-dimensional space having 1 at ℓ-th component and 0 at others. We note that the value w (> 0) is the upper α percentile of the $F^2_{\max\cdot \mathrm{p}}$ statistics,

$$F^2_{\max\cdot \mathrm{p}} = \max_{1\le\ell<m\le p} \left\{ \frac{\boldsymbol{b}'_{\ell m}\boldsymbol{z}\boldsymbol{z}'\boldsymbol{b}_{\ell m}}{\boldsymbol{b}'_{\ell m}\boldsymbol{S}\boldsymbol{b}_{\ell m}} \right\},$$

where $\boldsymbol{z} = \sqrt{N}(\overline{\boldsymbol{x}} - \boldsymbol{\mu})$. In order to construct actually simultaneous confidence intervals with the confidence level α, it is necessary to find the value w. However, it is difficult to find the exact value w even under the multivariate normality.

In this paper, we consider approximation to the upper percentiles of $F^2_{\max\cdot \mathrm{p}}$ statistics based on the Bonferroni inequalities to construct approximate simultaneous confidence intervals in elliptical distributions. Note that, under the elliptical populations, approximate simultaneous confidence intervals for pairwise comparisons among mean vectors based on the Bonferroni inequalities are discussed by Seo (2002), Okamoto (2005) and so on.

The organization of this paper is as follows. In Section 2, the First order and modified second order approximations to the upper percentiles of $F^2_{\max\cdot \mathrm{p}}$ statistic based on the Bonferroni inequalities are described. In Section 3, the First order and modified second order approximate upper percentiles of $F^2_{\max\cdot \mathrm{p}}$ statistic by asymptotic expansion procedure are given. Finally, the accuracy of the approximations is investigated by Monte Carlo simulations for selected parameters in Section4.

2. Approximation procedures based on Bonferroni inequalities

In this section, to construct approximate simultaneous confidence intervals, we describe first order and modified second order approximations based on the Bonferroni inequalities (see, e.g., Siotani (1959), Seo (2002)). By the first order Bonferroni inequality for $\Pr(F^2_{\max\cdot \mathrm{p}} > w^2)$;

$$\Pr(F^2_{\max\cdot \mathrm{p}} > w^2) < \sum\sum_{\ell<m} \Pr(F^2_{\ell m} > w^2),$$

where

$$F_{\ell m}^2 = \frac{\boldsymbol{b}_{\ell m}'\boldsymbol{z}\boldsymbol{z}'\boldsymbol{b}_{\ell m}}{\boldsymbol{b}_{\ell m}'\boldsymbol{S}\boldsymbol{b}_{\ell m}},$$

the first order approximation w_1^2 is given as a critical value that satisfies the equality

$$\sum\sum_{\ell<m} \Pr(F_{\ell m}^2 > w_1^2) = \alpha.$$

We note taht w_1^2 is overestimated, and the statistic $F_{\ell m}^2$ is essentially distributed as F distribution under normality. However, under the class of the elliptical distributions, $F_{\ell m}^2$ is not distributed as F distribution. Hence the first order approximation cannot be exactly expressed as the upper percentiles of F distribution. Therefore, we discuss an asymptotic expansion for the first order approximation in Section 3.

Next, a modified second order Bonferroni procedure is described to improve the first order approximation.

Let $\boldsymbol{a}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2$, $\boldsymbol{a}_2 = \boldsymbol{e}_1 - \boldsymbol{e}_3$, $\ldots$, $\boldsymbol{a}_M = \boldsymbol{e}_{p-1} - \boldsymbol{e}_p$ and $M = p(p-1)/2$. By the Bonferroni inequalities for $\Pr(F_{\max\cdot\mathrm{p}}^2 > w^2)$, i.e.,

$$\sum\sum_{\ell<m} \Pr(F_{\ell m}^2 > w^2) - \beta(w^2) < \Pr(F_{\max\cdot\mathrm{p}}^2 > w^2) < \sum\sum_{\ell<m} \Pr(F_{\ell m}^2 > w^2),$$

where

$$\beta(w^2) = \sum\sum_{j<k} \Pr\left\{ \frac{\boldsymbol{a}_j'\boldsymbol{z}\boldsymbol{z}'\boldsymbol{a}_j}{\boldsymbol{a}_j'\boldsymbol{S}\boldsymbol{a}_j} > w^2, \frac{\boldsymbol{a}_k'\boldsymbol{z}\boldsymbol{z}'\boldsymbol{a}_k}{\boldsymbol{a}_k'\boldsymbol{S}\boldsymbol{a}_k} > w^2 \right\},$$

the modified second order approximation w_M^2 by the modified second Bonferroni procedure is defined as a critical value that satisfies the equality

$$\sum\sum_{\ell<m} \Pr(F_{\ell m}^2 > w_M^2) - \beta(w_1^2) = \alpha.$$

In order to obtain the modified second order approximation w_M^2, we discuss the evaluation of $\beta(w_1^2)$. To evaluate $\beta(w_1^2)$, consider two cases of joint probabilities, that is,

$$\beta_1(w_1^2) = \Pr\left\{ F_{ij}^2 > w_1^2, F_{k\ell}^2 > w_1^2 \right\}, \ (i \neq j \neq k \neq \ell),$$

and

$$\beta_2(w_1^2) = \Pr\left\{F_{ij}^2 > w_1^2, F_{ik}^2 > w_1^2\right\}, \ (i \neq j \neq k),$$

under the elliptical distribution set up. We note that

$$\beta(w_1^2) = \frac{1}{8}p(p-1)(p-2)(p-3)\beta_1(w_1^2) + \frac{1}{2}p(p-1)(p-2)\beta_1(w_2^2).$$

For large sample approximations, in Section 3, asymptotic expansions of these joint probabilities obtained by using the perturbation method are presented.

3. First order and modified second order approximations

3.1 First order approximation

In this subsection, we give an asymptotic expansion for the first order Bonferroni approximation by using the perturbation method. Note that

$$(N-1)\boldsymbol{S} = N\boldsymbol{W} - \boldsymbol{z}\boldsymbol{z}',$$

where

$$\boldsymbol{W} = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j - \boldsymbol{\mu})(\boldsymbol{x}_j - \boldsymbol{\mu})'.$$

Without loss of generality, we can assume $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. Let

$$\overline{\boldsymbol{x}} = \boldsymbol{\mu} + \frac{1}{\sqrt{N}}\boldsymbol{z}, \quad \boldsymbol{W} = \boldsymbol{I}_p + \frac{1}{\sqrt{N}}\boldsymbol{Z},$$

then we can write

$$\boldsymbol{b}_{\ell m}'\boldsymbol{S}\boldsymbol{b}_{\ell m} = \frac{N}{N-1}\left(1 + \frac{1}{2\sqrt{N}}\boldsymbol{b}_{\ell m}'\boldsymbol{Z}\boldsymbol{b}_{\ell m} - \frac{1}{2N}\boldsymbol{b}_{\ell m}'\boldsymbol{z}\boldsymbol{z}'\boldsymbol{b}_{\ell m}\right).$$

Therefore,

$$(\boldsymbol{b}_{\ell m}'\boldsymbol{S}\boldsymbol{b}_{\ell m})^{-1} = \frac{1}{2}\left\{1 - \frac{1}{\sqrt{N}}a_{\ell m}^{(0)} + \frac{1}{N}\left(a_{\ell m}^{(0)^2} + a_{\ell m}^{(1)^2} - 1\right) + o_p(N^{-1})\right\},$$

where

$$a_{\ell m}^{(0)} = \frac{1}{2}\boldsymbol{b}_{\ell m}'\boldsymbol{Z}\boldsymbol{b}_{\ell m}, \quad a_{\ell m}^{(1)} = \frac{1}{\sqrt{2}}\boldsymbol{b}_{\ell m}'\boldsymbol{z}.$$

Hence, calculating the characteristic function of $F_{\ell m}^2$ with $\boldsymbol{z}$ and $\boldsymbol{Z}$ by using the joint density of $\boldsymbol{z}$ and $\boldsymbol{Z}$ given by Iwashita (1997), we obtain

$$\mathrm{E}[\exp(itF_{\ell m}^2)] = (1-2it)^{-1/2}\left\{1+\frac{1}{4N}(c_0+c_1(1-2it)^{-1}+c_2(1-2it)^{-2})+o(N^{-1})\right\},$$

where

$$c_0 = -1-\frac{15}{4}\kappa, \quad c_1 = -2+\frac{15}{2}\kappa, \quad c_2 = 3-\frac{15}{4}\kappa.$$

Therefore, inverting the characteristic function, we have the following theorem.

Theorem 1. *The distribution of $F_{\ell m}^2$ can be expanded as*

$$\Pr\left\{F_{\ell m}^2 > w^2\right\} = \Pr\left\{\chi_1^2 > w^2\right\} + \frac{1}{4N}\sum_{j=0}^{2} c_j \Pr\left\{\chi_{1+2j}^2 > w^2\right\} + o(N^{-1}),$$

and also its upper α percentiles can be expanded as

$$w_{\ell m}^2(\alpha) = \chi_1^2(\alpha) - \frac{1}{2N}\chi_1^2(\alpha)\left\{c_0 - \frac{1}{3}c_2\chi_1^2(\alpha)\right\} + o(N^{-1}),$$

where $\chi_1^2(\alpha)$ is the upper α percentiles of χ^2 distribution with 1 degrees of freedom.

Since $F_{\ell m}^2$ is essentially distributed as F distribution under normality, we have

$$w_{\ell m}^2(\alpha) = F_{1,\nu}(\alpha) - \frac{1}{2N}\chi_1^2(\alpha)\left\{(c_0+1) - \left(\frac{1}{3}c_2 - 1\right)\chi_1^2(\alpha)\right\} + o(N^{-1}),$$

where $\nu = N-1$ and $F_{1,\nu}(\alpha)$ is the upper α percentiles of F distribution with 1 and ν degrees of freedom.

For large N, an asymptotic expansion for w_1^2, we can write

$$w_1^2 = \chi_1^2(\alpha^*) - \frac{1}{2N}\chi_1^2(\alpha^*)\left\{c_0 - \frac{1}{3}c_2\chi_1^2(\alpha^*)\right\} + o(N^{-1}),$$

where $\alpha^* = \alpha/M$ and $M = p(p-1)/2$, and an another expression, we can write

$$w_1^2 = F_{1,\nu}(\alpha^*) - \frac{1}{2N}\chi_1^2(\alpha^*)\left\{(c_0+1) - \left(\frac{1}{3}c_2 - 1\right)\chi_1^2(\alpha^*)\right\} + o(N^{-1}).$$

Thus, the approximate simultaneous confidence intervals for pairwise comparisons among components of mean vector are given by

$$\boldsymbol{b}_{\ell m}'\boldsymbol{\mu} \in \left[\boldsymbol{b}_{\ell m}'\overline{\boldsymbol{x}} \pm w_1\sqrt{\boldsymbol{b}_{\ell m}'\boldsymbol{S}\boldsymbol{b}_{\ell m}/N}\right],\ 1 \leq \ell < m \leq p.$$

3.2 Modified second order approximation

In this subsection, we give an asymptotic expansion for the modified second order Bonferroni approximation. In order to obtain the modified second order approximation w_M^2, we discuss the evaluation of

$$\beta(w_1^2) = \sum\sum_{j<k} \Pr\left\{\frac{\boldsymbol{a}_j'\boldsymbol{z}\boldsymbol{z}'\boldsymbol{a}_j}{\boldsymbol{a}_j'\boldsymbol{S}\boldsymbol{a}_j} > w_1^2, \frac{\boldsymbol{a}_k'\boldsymbol{z}\boldsymbol{z}'\boldsymbol{a}_k}{\boldsymbol{a}_k'\boldsymbol{S}\boldsymbol{a}_k} > w_1^2\right\},$$

that is, we consider the following two cases of joint probability;

$$\beta_1(w_1^2) = \Pr\left\{F_{ij}^2 > w_1^2, F_{k\ell}^2 > w_1^2\right\},\ (i \neq j \neq k \neq \ell),$$

and

$$\beta_2(w_1^2) = \Pr\left\{F_{ij}^2 > w_1^2, F_{ik}^2 > w_1^2\right\},\ (i \neq j \neq k).$$

At first, we discuss an asymptotic expansion for $\beta_1(w_1^2)$. On the same line in Section 3.1, let $\overline{\boldsymbol{x}} = \boldsymbol{\mu} + \frac{1}{\sqrt{N}}\boldsymbol{z}$, $\boldsymbol{W} = \boldsymbol{I}_p + \frac{1}{\sqrt{N}}\boldsymbol{Z}$. For convenience, we consider the joint characteristic function of F_{12}^2 and F_{34}^2. Then the joint characteristic function can be written as

$$C_1(it_1, it_2) = \mathrm{E}[\exp(it_1F_{12}^{(1)} + it_2F_{34}^{(1)})]\left[1 + \frac{1}{\sqrt{N}}B_1 + \frac{1}{N}B_2\right] + o(N^{-1}),$$

where

$$B_1 = (-it_1)F_{12}^{(2)} + (-it_2)F_{34}^{(2)},$$
$$B_2 = it_1F_{12}^{(3)} + \frac{(it_1)^2}{2}(F_{12}^{(2)})^2 + it_2F_{34}^{(3)} + \frac{(it_2)^2}{2}(F_{34}^{(2)})^2 + (it_1)(it_2)F_{12}^{(2)}F_{34}^{(2)},$$

and

$$F_{12}^{(1)} = (a_{12}^{(1)})^2, \quad F_{12}^{(2)} = a_{12}^{(0)}(a_{12}^{(1)})^2, \quad F_{12}^{(3)} = (a_{12}^{(1)})^2\left((a_{12}^{(0)})^2 + (a_{12}^{(1)})^2 - 1\right),$$
$$F_{34}^{(1)} = (a_{34}^{(1)})^2, \quad F_{34}^{(2)} = a_{34}^{(0)}(a_{34}^{(1)})^2, \quad F_{34}^{(3)} = (a_{34}^{(1)})^2\left((a_{34}^{(0)})^2 + (a_{34}^{(1)})^2 - 1\right).$$

Calculating the expectation $\mathrm{E}[\exp(it_1F_{12}^2+it_2F_{34})]$ with respect to $\boldsymbol{z}$ and $\boldsymbol{Z}$, we have

$$C_1(it_1, it_2) = (u_1u_2)^{-1/2} + \frac{1}{N}(u_1u_2)^{-1/2}$$
$$\times\left\{A_1 + (A_{21}u_1^{-1} + A_{22}u_2^{-1}) + (A_{31}u_1^{-2} + A_{32}u_2^{-2}) + A_4(u_1u_2)^{-1}\right\} + o(N^{-1}),$$

where

$$A_1 = -\frac{1}{2} - \frac{17}{8}\kappa, \quad A_{21} = A_{22} = -\frac{1}{2} + \frac{17}{8}\kappa,$$
$$A_{31} = A_{32} = \frac{3}{4} - \frac{15}{4}\kappa, \quad A_4 = -\frac{1}{4}\kappa,$$

and $u_1 = 1 - 2it_1$, $u_2 = 1 - 2it_2$.

Inverting this joint characteristic function $C_1(it_1, it_2)$, we have the following theorem.

Theorem 2. *For large N, an asymptotic expansion for the joint probability $\beta_1(w_1^2)$ is given by*

$$\Pr\left\{F_{ij}^2 > w_1^2, F_{k\ell}^2 > w_1^2\right\} = G_{1/2}^2(\eta_1) + \frac{1}{N}\left\{d_1g_{1/2}(\eta_1)G_{1/2}(\eta_1) + d_2g_{1/2}^2(\eta_1)\right\} + o(N^{-1}),$$

where

$$\eta_1 = \frac{1}{2}w_1^2, \quad G_{1/2}(\eta_1) = \int_{\eta_1}^{\infty} g_{1/2}(t)dt, \quad g_{1/2}(\eta_1) = \frac{1}{\Gamma(\frac{1}{2})}t^{-1/2}e^{-t},$$

and

$$d_1 = \frac{\eta_1}{4}\left\{8\eta_1 + 4 + (-10\eta_1 + 15)\kappa\right\}, \quad d_2 = -\eta_1^2\kappa.$$

Secondly, we consider $\beta_2(w_1^2)$. In this case, the joint characteristic function $C_2(it_1, it_2) = \mathrm{E}[\exp(it_1 F_{12}^2 + it_2 F_{13}^2)]$ can be written as

$$C_2(it_1, it_2) = \mathrm{E}[\exp(it_1 F_{12}^{(1)} + it_2 F_{13}^{(1)})] \left[1 + \frac{1}{\sqrt{N}} D_1 + \frac{1}{N} D_2\right] + o(N^{-1}),$$

where

$$\begin{aligned}
D_1 &= (-it_1) F_{12}^{(2)} + (-it_2) F_{13}^{(2)}, \\
D_2 &= it_1 F_{12}^{(3)} + \frac{(it_1)^2}{2} (F_{12}^{(2)})^2 + it_2 F_{13}^{(3)} + \frac{(it_2)^2}{2} (F_{13}^{(2)})^2 + (it_1)(it_2) F_{12}^{(2)} F_{13}^{(2)},
\end{aligned}$$

and

$$\begin{aligned}
F_{12}^{(1)} &= (a_{12}^{(1)})^2, \quad F_{12}^{(2)} = a_{12}^{(0)} (a_{12}^{(1)})^2, \quad F_{12}^{(3)} = (a_{12}^{(1)})^2 \left((a_{12}^{(0)})^2 + (a_{12}^{(1)})^2 - 1 \right), \\
F_{13}^{(1)} &= (a_{13}^{(1)})^2, \quad F_{13}^{(2)} = a_{13}^{(0)} (a_{13}^{(1)})^2, \quad F_{13}^{(3)} = (a_{13}^{(1)})^2 \left((a_{13}^{(0)})^2 + (a_{13}^{(1)})^2 - 1 \right).
\end{aligned}$$

Calculating the expectation $\mathrm{E}[\exp(it_1 F_{12}^2 + it_2 F_{13})]$ with respect to $\boldsymbol{z}$ and $\boldsymbol{Z}$, we have

$$C_2(it_1, it_2) = \phi^{-1/2} \left[1 + \frac{1}{N} (u_1 \phi^{-1} + u_2 \phi^{-2})\right] + o(N^{-1}),$$

where

$$\begin{aligned}
\phi &= \frac{1}{3}(4\lambda_1\lambda_2 - 1), \\
u_1 &= \frac{1}{6} \left[1 + 7(\lambda_1 + \lambda_2) - 15\lambda_1\lambda_2 + \{3 + 17(\lambda_1 + \lambda_2) - 37\lambda_1\lambda_2\} \kappa\right], \\
u_2 &= \frac{1}{18} \{3b_{21} + 3b_{22} + (b_{23} + b_{24})\kappa\},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_1 = 1 - \frac{3}{2} it_1, \quad \lambda_2 = 1 - \frac{3}{2} it_2, \\
&b_{21} = 8(\lambda_1^2 + \lambda_2^2) - \lambda_1 - \lambda_2 + 1, \quad b_{22} = \lambda_1\lambda_2 \{16\lambda_1\lambda_2 - 16(\lambda_1 + \lambda_2) + 1\}, \\
&b_{23} = -18(\lambda_1^2 + \lambda_2^2) + 7(\lambda_1 + \lambda_2) - 1, \quad b_{24} = \lambda_1\lambda_2 \{88\lambda_1\lambda_2 - 28(\lambda_1 + \lambda_2) - 9\}.
\end{aligned}$$

Therefore, we have the following theorem.

Theorem 3. *For large N, an asymptotic expansion for the joint probability $\beta_2(w_1^2)$ is given by*

$$\Pr\left\{F_{ij}^2 > w_1^2, F_{ik}^2 > w_1^2\right\} = \left(\frac{3}{4}\right)^{1/2} \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m}{m!}\left(\frac{1}{4}\right)^m \times \left[G_{m+1/2}^2(\eta_2) + \frac{1}{N}\left\{d_1 g_{m+1/2}(\eta_2) G_{m+1/2}(\eta_2) + d_2 g_{m+1/2}^2(\eta_2)\right\}\right] + o(N^{-1}),$$

where

$$\eta_2 = \frac{2}{3}w_1^2, \quad G_{m+1/2}(\eta_2) = \int_{\eta_2}^{\infty} g_{m+1/2}(t)dt, \quad g_{m+1/2}(\eta_2) = \frac{1}{\Gamma(m+\frac{1}{2})} t^{m-1/2} e^{-t},$$

and

$$d_1 = \eta_2(2\eta_2 - 2m + 1) + \frac{\eta_2}{4}(-4\eta_2 - 14m + 19)\kappa,$$

$$d_2 = \frac{\eta_2^2}{6(2m+1)(2m+3)}\left[3(\eta_2^2+2m+3)+\left\{-\eta_2^2+18\eta_2+4m^2+6(5+2\eta_2)m\right\}\kappa\right].$$

For large N, an asymptotic expansion for w_M^2 is given by

$$w_M^2 = \chi_1^2(\alpha^{**}) - \frac{1}{2N}\chi_1^2(\alpha^{**})\left\{c_0 - \frac{1}{3}c_2\chi_1^2(\alpha^{**})\right\} + o(N^{-1}),$$

where $\alpha^{**} = (\alpha + \beta(w_1^2))/M$ and $M = p(p-1)/2$, and an another expression, we can write

$$w_M^2 = F_{1,\nu}(\alpha^{**}) - \frac{1}{2N}\chi_1^2(\alpha^{**})\left\{(c_0+1) - \left(\frac{1}{3}c_2 - 1\right)\chi_1^2(\alpha^{**})\right\} + o(N^{-1}).$$

Thus, the approximate simultaneous confidence intervals for pairwise comparisons among components of mean vector are given by

$$\boldsymbol{b}_{\ell m}'\boldsymbol{\mu} \in \left[\boldsymbol{b}_{\ell m}'\overline{\boldsymbol{x}} \pm w_M\sqrt{\boldsymbol{b}_{\ell m}'\boldsymbol{S}\boldsymbol{b}_{\ell m}/N}\right], \ 1 \le \ell < m \le p.$$

4. Numerical examinations

In this section, we examine accuracy of the obtained approximation. We give some numerical results of the upper percentiles of $F_{\max\cdot\mathrm{p}}$ $(=\sqrt{F^2_{\max\cdot\mathrm{p}}})$ statistic by Monte Carlo simulation. The Monte Carlo simulations are made from 10^6 trials for selected values of parameters; p, κ, α and N.

Tables 1, 2 and 3 give four kinds of approximation as follows;

(i) $w_{1\cdot\chi}$: the first order approximation based on χ^2 distribution,

(ii) $w_{1\cdot F}$: the first order approximation based on F distribution,

(iii) $w_{M\cdot\chi}$: the modified second order approximation based on χ^2 distribution,

(iv) $w_{M\cdot F}$: the modified second order approximation based on F distribution.

Each value is calculated for the following combinations of parameter values: $p = 3, 5, 10$, $N = 20, 40, 80$ and $\alpha = 0.1, 0.05, 0.01$. For the distributions of population, the multivariate normal ($\kappa = 0$), the ε-contaminated normal ($\varepsilon = 0.1, \sigma = 3 : \kappa = 1.78$) and the ε-contaminated normal ($\varepsilon = 0.1, \sigma = 4 : \kappa = 3.24$) are treated.

For the multivariate normal case, it can be seen from Tables 1 that the values of $w_{1\cdot F}$ are always become conservative and modified second order approximation is close to the simulated value. We note that there is a trend that when p is small, the approximate values become be better. And it can be seen from Tables 1 that the approximate values converge to the simulated values when the sample size is large. Also, when α is small, $w_{M\cdot F}$ is better than $w_{M\cdot\chi}$. But, when α and sample size are small, it can be seen that first order approximation is better than the modified second order approximation.

For the contaminated normal case, it can be seen from Tables 2 and 3 that the first order and modified second order approximations in this paper become better when κ is small. But, we note that there is a case that the first order approximation is smaller that the simulated value. So, in these cases, there is a trend that the modified second order approximation is become worth.

Acknowledgements

The research of the second author was supported in part by Grant-in-Aid for Young Scientists (B) under Contract Number 21700311.

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Table 1: Approximate and simulated values.

Multivariate normal distribution ($\kappa = 0$)							
p	N	α	$w_{1\cdot\chi}$	$w_{1\cdot F}$	$w_{M\cdot\chi}$	$w_{M\cdot F}$	w
3	20	0.01	3.269	3.354	3.247	3.393	3.334
		0.05	2.588	2.679	2.545	2.634	2.571
		0.1	2.270	2.343	2.210	2.279	2.213
	40	0.01	3.107	3.127	3.085	3.138	3.104
		0.05	2.493	2.530	2.450	2.487	2.450
		0.1	2.200	2.232	2.140	2.170	2.128
	80	0.01	3.022	3.027	3.001	3.023	3.007
		0.05	2.444	2.461	2.402	2.418	2.395
		0.1	2.164	2.179	2.105	2.118	2.088
5	20	0.01	3.746	3.883	3.708	3.910	3.848
		0.05	3.103	3.235	3.029	3.154	3.090
		0.1	2.811	2.918	2.714	2.814	2.736
	40	0.01	3.525	3.558	3.483	3.551	3.523
		0.05	2.959	3.009	2.886	2.933	2.894
		0.1	2.696	2.738	2.600	2.640	2.587
	80	0.01	3.410	3.418	3.369	3.396	3.383
		0.05	2.884	2.905	2.813	2.833	2.807
		0.1	2.637	2.655	2.541	2.559	2.521
10	20	0.01	4.315	4.543	4.278	4.574	4.502
		0.05	3.705	3.906	3.607	3.796	3.736
		0.1	3.432	3.599	3.301	3.453	3.386
	40	0.01	4.016	4.070	3.954	4.046	4.025
		0.05	3.490	3.559	3.382	3.446	3.424
		0.1	3.251	3.310	3.117	3.172	3.133
	80	0.01	3.857	3.871	3.792	3.826	3.827
		0.05	3.377	3.405	3.274	3.300	3.288
		0.1	3.156	3.181	3.027	3.050	3.022

Table 2: Continued.

Contaminated normal distribution ($\kappa = 1.78$)							
p	N	α	$w_{1\cdot\chi}$	$w_{1\cdot F}$	$w_{M\cdot\chi}$	$w_{M\cdot F}$	w
3	20	0.01	2.828	2.926	2.782	2.941	3.172
		0.05	2.414	2.454	2.364	2.456	2.481
		0.1	2.184	2.209	2.124	2.195	2.157
	40	0.01	2.882	2.904	2.847	2.904	2.991
		0.05	2.404	2.413	2.357	2.394	2.395
		0.1	2.156	2.162	2.096	2.126	2.099
	80	0.01	2.909	2.914	2.881	2.904	2.935
		0.05	2.399	2.401	2.354	2.371	2.366
		0.1	2.142	2.144	2.082	2.096	2.076
5	20	0.01	3.053	3.221	2.917	3.114	3.614
		0.05	2.737	2.817	2.615	2.741	2.911
		0.1	2.562	2.616	2.433	2.533	2.587
	40	0.01	3.174	3.210	3.081	3.152	3.340
		0.05	2.772	2.790	2.671	2.720	2.763
		0.1	2.569	2.581	2.453	2.493	2.479
	80	0.01	3.233	3.241	3.168	3.196	3.250
		0.05	2.790	2.794	2.703	2.724	2.713
		0.1	2.572	2.575	2.466	2.484	2.441
10	20	0.01	3.251	3.548	2.947	3.154	4.261
		0.05	3.036	3.196	2.765	2.920	3.533
		0.1	2.911	3.030	2.646	2.777	3.232
	40	0.01	3.479	3.541	3.266	3.352	3.834
		0.05	3.150	3.185	2.947	3.009	3.289
		0.1	2.986	3.012	2.776	2.829	3.027
	80	0.01	3.587	3.601	3.453	3.487	3.690
		0.05	3.206	3.214	3.054	3.080	3.202
		0.1	3.023	3.029	2.853	2.875	2.959

Table 3: Continued.

Contaminated normal distribution ($\kappa = 3.24$)							
p	N	α	$w_{1\cdot\chi}$	$w_{1\cdot F}$	$w_{M\cdot\chi}$	$w_{M\cdot F}$	w
3	20	0.01	2.406	2.520	2.381	2.535	3.077
		0.05	2.261	2.304	2.213	2.307	2.416
		0.1	2.110	2.136	2.055	2.127	2.113
	40	0.01	2.684	2.707	2.639	2.698	2.906
		0.05	2.328	2.228	2.279	2.317	2.349
		0.1	2.119	2.125	2.060	2.091	2.073
	80	0.01	2.812	2.818	2.779	2.802	2.875
		0.05	2.361	2.364	2.315	2.332	2.341
		0.1	2.124	2.125	2.064	2.078	2.063
5	20	0.01	2.334	2.549	2.300	2.414	3.498
		0.05	2.395	2.486	2.280	2.389	2.813
		0.1	2.337	2.396	2.207	2.300	2.503
	40	0.01	2.853	2.893	2.725	2.793	3.228
		0.05	2.609	2.628	2.489	2.537	2.683
		0.1	2.459	2.472	2.331	2.371	2.417
	80	0.01	3.079	3.088	2.991	3.020	3.165
		0.05	2.710	2.714	2.611	2.632	2.658
		0.1	2.518	2.521	2.403	2.421	2.400
10	20	0.01	1.986	2.442	1.588	1.629	4.145
		0.05	2.347	2.551	2.074	2.148	3.450
		0.1	2.399	2.542	2.115	2.194	3.238
	40	0.01	2.964	3.037	2.691	2.755	3.708
		0.05	2.841	2.879	2.579	2.633	3.191
		0.1	2.749	2.777	2.488	2.536	2.947
	80	0.01	3.348	3.363	3.147	3.179	3.585
		0.05	3.058	3.067	2.863	2.888	3.132
		0.1	2.908	2.914	2.703	2.726	2.906