On the conservative multivariate multiple comparison procedure of correlated mean vectors with a control

Takahiro Nishiyama[a] *

[a] Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 Japan

Abstract

In this paper, we consider the simultaneous confidence intervals for multiple comparisons with a control among mean vectors from the multivariate normal distributions. We discuss the approximate simultaneous confidence procedure proposed by Seo (1995). Seo (1995) conjectured that this procedure always construct the conservative approximate simultaneous confidence intervals. In this paper, we give the affirmative proof of this conjecture and give the upper bound for the conservativeness of this procedure in the case of five correlated mean vectors. Finally, numerical results by Monte Carlo simulation are given.

Key Words: Comparisons with a control; Conservativeness; Coverage probability; Monte Carlo simulation.

1. Introduction

Consider the simultaneous confidence intervals for multiple comparisons among mean vectors from the multivariate normal populations. Let $\boldsymbol{M} = [\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k]$ be the unknown $p \times k$ matrix of k mean vectors corresponding to the k treatments, where $\boldsymbol{\mu}_i$ is the mean vector from i-th popultation. And let $\widehat{\boldsymbol{M}} = [\widehat{\boldsymbol{\mu}}_1, \ldots, \widehat{\boldsymbol{\mu}}_k]$ be an estimator of $\boldsymbol{M}$ such that $\mathrm{vec}(\boldsymbol{X})$ has $\mathrm{N}_{kp}(\boldsymbol{0}, \boldsymbol{V} \otimes \boldsymbol{\Sigma})$, where $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k] =$

* *E-mail address*: nishiyam@rs.kagu.tus.ac.jp

$\widehat{\boldsymbol{M}} - \boldsymbol{M}$, $\boldsymbol{V} = [v_{ij}]$ is a known $k \times k$ positive definite matrix and $\boldsymbol{\Sigma}$ is an unknown $p \times p$ positive definite matrix, and vec$(\cdot)$ denotes the column vector formed by stacking the columns of the matrix under each other. Further, we assume that $\boldsymbol{S}$ is an unbiased estimator of $\boldsymbol{\Sigma}$ such that $\nu\boldsymbol{S}$ is independent of $\widehat{\boldsymbol{M}}$ and is distributed as a Wishart distribution $\mathrm{W}_p(\boldsymbol{\Sigma}, \nu)$. Then, in general, the simultaneous confidence intervals for multiple comparisons among mean vectors are given by the following form:

$$\boldsymbol{a}'\boldsymbol{M}\boldsymbol{b} \in \left[\ \boldsymbol{a}'\widehat{\boldsymbol{M}}\boldsymbol{b} \pm t(\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b})^{1/2}(\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a})^{1/2}\right], \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\},\ \boldsymbol{b} \in \mathbb{B}, \tag{1}$$

where $\mathbb{R}^p - \{\boldsymbol{0}\}$ is a set of any nonzero real p-dimensional vectors, $\mathbb{B}$ is a subset in k-dimensional space. Also, the square of value $t(> 0)$ in (1) is the upper 100α percentiles of $T^2_{\max}$-type statistic defined by

$$T^2_{\max} = \max_{\boldsymbol{b} \in \mathbb{B}} \left\{ \frac{(\boldsymbol{X}\boldsymbol{b})'\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{b}}{\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b}} \right\}.$$

Then we note that the coverage probability for (1) is exactly $1 - \alpha$.

In order to construct the actually simultaneous confidence intervals (1) with the confidence level α, it is necessary to find the value of t. However, it is difficult to find the exact value. So some approximations for the upper 100α percentiles of $T^2_{\max}$-type statistic have been discussed by Siotani (1959a,b, 1960), Krishnaiah (1979), Siotani, Hayakawa and Fujikoshi (1985), Seo and Siotani (1992, 1993), Seo, Mano and Fujikoshi (1994), Seo (1995) and so on. Also, under elliptical distributions and general distributions, some approximations based on the asymptotic expansion have been discussed by Seo (2002), Okamoto and Seo (2004), Okamoto (2005) and Kakizawa (2006).

In this paper, we discuss the comparisons with a control among mean vectors. Here, we assume that k-th treatment is a control treatment. In the case of comparisons with a control, a subset $\mathbb{B}$ is given by

$$\mathbb{B} = \mathbb{C} \equiv \{\boldsymbol{c} \in \mathbb{R}^k : \boldsymbol{c} = \boldsymbol{e}_i - \boldsymbol{e}_k,\ i = 1, \ldots, k-1\}, \tag{2}$$

where $\boldsymbol{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)'$ is a k-dimensional unit vector which having 1 at i-th component.

Therefore we have the simultaneous confidence intervals for comparisons with a control among mean vectors given by

$$\boldsymbol{a}'\boldsymbol{M}\boldsymbol{c} \in \left[\ \boldsymbol{a}'\widehat{\boldsymbol{M}}\boldsymbol{c} \pm t_{\mathrm{c}\cdot V}(\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c})^{1/2}(\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a})^{1/2}\right], \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\},\ \boldsymbol{c} \in \mathbb{C}, \qquad (3)$$

where $t^2_{\mathrm{c}\cdot V}$ is the upper 100α percentiles of $T^2_{\max\cdot\mathrm{c}}$ statistic defined by

$$\begin{aligned} T^2_{\max\cdot\mathrm{c}} &= \max_{\boldsymbol{c}\in\mathbb{C}} \left\{ \frac{(\boldsymbol{X}\boldsymbol{c})'\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{c}}{\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c}} \right\} \\ &= \max_{i=1,\ldots,k-1} \{(\boldsymbol{x}_i - \boldsymbol{x}_k)'(d_{ik}\boldsymbol{S})^{-1}(\boldsymbol{x}_i - \boldsymbol{x}_k)\}, \end{aligned}$$

and $d_{ik} = v_{ii} - 2v_{ik} + v_{kk}$. We note that the simultaneous confidence intervals (3) can be expressed as

$$\begin{aligned} \boldsymbol{a}'(\boldsymbol{\mu}_i - \boldsymbol{\mu}_k) \in [\ \boldsymbol{a}'(\widehat{\boldsymbol{\mu}}_i - \widehat{\boldsymbol{\mu}}_k) \pm t_{\mathrm{c}\cdot V}\ (d_{ik}\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a})^{1/2}]\,, \\ \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\},\ i = 1, \ldots, k-1. \qquad (4) \end{aligned}$$

In the case of comparisons with a control, some approximations for the upper 100α percentiles of $T^2_{\max\cdot\mathrm{c}}$ statistic have been discussed by Seo and Siotani (1993), Seo (1995) and so on. In particular, Seo (1995) proposed a conservative approximate simultaneous confidence procedure which concerning to the multivariate Tukey-Kramer procedure (see, Seo, Mano and Fujikoshi (1994)). In the case of three and four correlated mean vectors (that is, $k = 3$ and 4), its conservativeness has been affirmatively proved by Seo (1995) and Nishiyama (2007), respectively. Also, Seo and Nishiyama (2008) and Nishiyama (2007) gave the upper bound for the conservativeness of this procedure for the case $k = 3$ and 4, respectively.

In this paper, we discuss the conservativeness of the approximate simultaneous confidence procedure for comparisons with a control proposed by Seo (1995) in the case of five correlated mean vectors. Further, we give the upper bound for the conservativeness of the procedure. In the case of pairwise comparisons, conservativeness of multivariate Tukey-Kramer procedure has been affirmatively proved by Seo, Mano and Fujikoshi (1994) and Nishiyama and Seo (2008) when $k = 3$ and 4, respectively. However, it is left as a future problem for the case $k \geq 5$.

The organization of this paper is as follows. In Section 2, we describe the approximate simultaneous confidence procedure for comparisons with a control. In Section 3, we give the affirmatively proof of the conservativeness and upper bound for the conservativeness. Finally, in Section 4, some numerical results by Monte Carlo simulation are given.

2. A conservative procedure for comparisons with a control

In this section, we describe the approximate simultaneous confidence procedure for comparisons with a control. For $k \geq 3$, Seo (1995) proposed the conservative approximate simultaneous confidence intervals given by

$$\boldsymbol{a}'(\boldsymbol{\mu}_i - \boldsymbol{\mu}_k) \in \left[\, \boldsymbol{a}'(\widehat{\boldsymbol{\mu}}_i - \widehat{\boldsymbol{\mu}}_k) \pm t_{\mathrm{c}\cdot V_1}\ (d_{ik}\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a})^{1/2} \right], \quad \forall \boldsymbol{a} \in \mathbb{R}^p - \{\boldsymbol{0}\},\ i = 1, \ldots, k-1, \tag{5}$$

where $t^2_{\mathrm{c}\cdot V_1}$ is the upper 100α percentiles of $T^2_{\max\cdot \mathrm{c}}$ statistic with $\boldsymbol{V} = \boldsymbol{V}_1$, and $\boldsymbol{V}_1$ satisfies with $d_{ij} = d_{ik} + d_{jk}$, $1 \leq i < j \leq k-1$. Further, Seo (1995) gave the conjecture that these simultaneous confidence intervals are always conservative. For this conjecture, its proof for the case $k = 3$ and 4 is given by Seo (1995) and Nishiyama (2007), respectively. Also, Seo and Nishiyama (2008) and Nishiyama (2007) gave the upper bound for the conservativeness for the case $k = 3$ and 4, respectively.

We consider the probability

$$Q(q, \boldsymbol{V}, \mathbb{B}) = \Pr\{(\boldsymbol{X}\boldsymbol{b})'(\nu\boldsymbol{S})^{-1}(\boldsymbol{X}\boldsymbol{b}) \leq q(\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b}),\ \forall \boldsymbol{b} \in \mathbb{B}\}, \tag{6}$$

where $q(> 0)$ is any fixed constant. Without loss of generality, we assume $\boldsymbol{\Sigma} = \boldsymbol{I}_p$.

We note that when $\mathbb{B} = \mathbb{C}$ and $q = t_{\mathrm{c}}^{*}(\equiv t_{\mathrm{c}\cdot V_1}^{2})$, the coverage probability (6) is the same as one of (5). Then, concerning to the coverage probability, Nishiyama (2007) gave the following conjecture for the case $k \geq 3$.

Conjecture 1. (Nishiyama (2007)) *Let $Q(q, \boldsymbol{V}, \mathbb{C})$ be the coverage probability (6) with a known matrix $\boldsymbol{V}$. Then*

$$1 - \alpha = Q(t_{\mathrm{c}}^{*}, \boldsymbol{V}_1, \mathbb{C}) \leq Q(t_{\mathrm{c}}^{*}, \boldsymbol{V}, \mathbb{C}) < Q(t_{\mathrm{c}}^{*}, \boldsymbol{V}_2, \mathbb{C}),$$

holds for any positive definite matrix $\boldsymbol{V}$, where $t_{\mathrm{c}}^{*} = t_{\mathrm{c}\cdot V_1}^{2}/\nu$, $\mathbb{C} = \{\boldsymbol{c} \in \mathbb{R}^k : \boldsymbol{c} = \boldsymbol{e}_i - \boldsymbol{e}_k,\ i = 1, \ldots, k-1\}$ and $\boldsymbol{V}_1$ satisfies with $d_{ij} = d_{ik} + d_{jk}$ for all i, j $(1 \leq i < j \leq k-1)$ and $\boldsymbol{V}_2$ satisfies with $\sqrt{d_{ij}} = |\sqrt{d_{ik}} - \sqrt{d_{jk}}|$ for all i, j $(1 \leq i < j \leq k-1)$.

In Conjecture 1, we note that there does not exist a positive definite matrix such that $\sqrt{d_{ij}} = |\sqrt{d_{ik}} - \sqrt{d_{jk}}|$. However, we can find $\boldsymbol{V}_2$ as a positive semi-definite matrix. For example, in the case $k = 5$, the one of such matrix $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are given by

$$\boldsymbol{V}_1 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{V}_2 = \begin{bmatrix} 4 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 & 4 \end{bmatrix}.$$

3. Proof of conjecture for the case $k = 5$

In this section, we give the affirmative proof of Conjecture 1 for the case $k = 5$ by using same idea of Seo (1995) and Nishiyama (2007).

Let $\boldsymbol{A}$ be $k \times k$ nonsingular matrix such that $\boldsymbol{V} = \boldsymbol{A}'\boldsymbol{A}$. Then, by the transformation $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{A}^{-1}$, $\mathrm{vec}(\boldsymbol{Y}) \sim \mathrm{N}_{kp}(\boldsymbol{0}, \boldsymbol{I}_k \otimes \boldsymbol{I}_p)$. Further, let

$$\Gamma = \{\, \boldsymbol{\gamma}\ ;\ \boldsymbol{\gamma} = (\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b})^{-1/2}\boldsymbol{A}\boldsymbol{b},\ \forall \boldsymbol{b} \in \mathbb{B}\}, \tag{7}$$

which is a subset of unit vectors in $\mathbb{R}^k$. Then we can write the coverage probability

$Q(q, \boldsymbol{V}, \mathbb{B})$ as

$$\begin{aligned} Q(q, \boldsymbol{V}, \mathbb{B}) &= \Pr\{(\boldsymbol{YAb})'(\nu\boldsymbol{S})^{-1}(\boldsymbol{YAb}) \le q(\boldsymbol{b}'\boldsymbol{Vb}),\ \forall\boldsymbol{b} \in \mathbb{B}\} \\ &= \Pr\{(\boldsymbol{Y\gamma})'(\nu\boldsymbol{S})^{-1}(\boldsymbol{Y\gamma}) \le q,\ \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}. \end{aligned}$$

Further, we consider the transformation $\boldsymbol{S}$ to $\boldsymbol{L} = \mathrm{diag}(\ell_1, \ldots, \ell_{\mathrm{p}})$, $\ell_1 \ge \ldots \ge \ell_p$ and $p \times p$ orthogonal matrix $\boldsymbol{H}_1$ such that $\nu\boldsymbol{S} = \boldsymbol{H}_1\boldsymbol{L}\boldsymbol{H}_1'$. Then $\boldsymbol{H}_1$ is a $p \times p$ orthogonal matrix and $\boldsymbol{L}$ and $\boldsymbol{H}_1$ are independent. Then we note that $\boldsymbol{L}$ and $\boldsymbol{H}_1$ are independent and the p.d.f. of $\boldsymbol{L}$ is given by

$$\frac{\pi^{p^2/2}}{2^{p\nu/2}\Gamma_p(\frac{1}{2}\nu)\Gamma_p(\frac{1}{2}p)} \exp\left(-\frac{1}{2}\sum_{i=1}^{p}\ell_i\right) \prod_{i=1}^{p}\ell_i^{(\nu-p-1)/2} \prod_{i<j}^{p}(\ell_i - \ell_j), \tag{8}$$

where $\Gamma_p(n) = \pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(n-(i-1)/2)$ (see, e.g., Siotani, Hayakawa and Fujikoshi (1985)). Summarizing these results, we have the following theorem given by Seo, Mano and Fujikoshi (1994).

Theorem 2. (Seo, Mano and Fujikoshi (1994)) *Let $\boldsymbol{\Gamma}$ be a subset of unit vector in $\mathbb{R}^k$ defined by (7). Then the coverage probability (6) can be expressed as*

$$Q(q, \boldsymbol{V}, \mathbb{B}) = E_{\boldsymbol{L}}[\Pr\{(\boldsymbol{Y\gamma})'\boldsymbol{L}^{-1}(\boldsymbol{Y\gamma}) \le q,\ \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}],$$

where the probability density function of $\boldsymbol{L} = \mathrm{diag}(\ell_1, \ldots, \ell_p)$ is given by (8), $\boldsymbol{L}$ is independent of $\boldsymbol{Y} = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k]$ and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k$ are independent identically distributed as $N_p(\boldsymbol{0}, \boldsymbol{I}_p)$.

Next we consider a special case when $\mathbb{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m\}$ and the dimension of the space spanned by $\mathbb{B}$ equals 4. Let $\boldsymbol{\gamma}_i = (\boldsymbol{b}_i'\boldsymbol{V}\boldsymbol{b}_i)^{-1/2}\boldsymbol{A}\boldsymbol{b}_i$, $i = 1, \ldots, m$ and $\boldsymbol{\Gamma} = \{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_m\}$. Since the dimension of the space spanned by $\mathbb{B}$ equals 4, there exists a $k \times k$ orthogonal matrix $\boldsymbol{H}_2$ such that

$$\boldsymbol{\gamma}_i'\boldsymbol{H}_2 = [\boldsymbol{\delta}_i', 0, \ldots, 0],\ i = 1, \ldots, m,$$

where $\boldsymbol{\delta}_i (= (\delta_{i1}, \ldots, \delta_{i4})')$ is a 4-dimensional vector and satisfy $\boldsymbol{\delta}_i' \boldsymbol{\delta}_i = 1$. Therefore we can write

$$\boldsymbol{\delta}_i = \begin{pmatrix} \sin\beta_{i1} \sin\beta_{i2} \sin\beta_{i3} \\ \sin\beta_{i1} \sin\beta_{i2} \cos\beta_{i3} \\ \sin\beta_{i1} \cos\beta_{i2} \\ \cos\beta_{i1} \end{pmatrix}, \ i = 1, \ldots, m,$$

where $0 \leq \beta_{i1} < \pi$, $0 \leq \beta_{i2} < \pi$ and $0 \leq \beta_{i3} < 2\pi$.

Further, we consider the transformation from $\boldsymbol{Y}$ to $\boldsymbol{Y}\boldsymbol{H}_2 = [\boldsymbol{U}, \widetilde{\boldsymbol{U}}]$, where $\boldsymbol{U}$ is $p \times 4$. Letting $\boldsymbol{U} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p]'$, $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p$ are independent identically distributed as $\mathrm{N}_4(\mathbf{0}, \boldsymbol{I}_4)$. Then we can write

$$\boldsymbol{u}_s = r_s \begin{pmatrix} \sin\theta_{s1} \sin\theta_{s2} \sin\theta_{s3} \\ \sin\theta_{s1} \sin\theta_{s2} \cos\theta_{s3} \\ \sin\theta_{s1} \cos\theta_{s2} \\ \cos\theta_{s1} \end{pmatrix}, \ s = 1, \ldots, p,$$

where $r_s^2 = \boldsymbol{u}_s' \boldsymbol{u}_s$, θ_{s1}, θ_{s2} and θ_{s3} are independently distributed as χ^2 distribution with four degrees of freedom, uniform distribution on $\mathrm{U}[0, \pi)$, on $\mathrm{U}[0, \pi)$ and on $\mathrm{U}[0, 2\pi)$, respectively. Summarizing these results, from Theorem 2, we have the following theorem.

Theorem 3. *Suppose that dimension of the space spanned by $\mathbb{B}$ equals 4. Then the coverage probability (6) can be expressed as*

$$\begin{aligned} Q(q, \boldsymbol{V}, \mathbb{B}) = E_{L,R} \Bigg[\Pr \Bigg\{ & \sum_{s=1}^{p} \frac{r_s^2}{\ell_s} (\sin\theta_{s1} \sin\theta_{s2} \sin\theta_{s3} \sin\beta_{i1} \sin\beta_{i2} \sin\beta_{i3} \\ & + \sin\theta_{s1} \sin\theta_{s2} \cos\theta_{s3} \sin\beta_{i1} \sin\beta_{i2} \cos\beta_{i3} + \sin\theta_{s1} \cos\theta_{s2} \sin\beta_{i1} \cos\beta_{i2} \\ & + \cos\theta_{s1} \cos\beta_{i1})^2 \leq q \quad \textit{for } i = 1, \ldots, m \Bigg\} \Bigg], \end{aligned}$$

where $\boldsymbol{R} = diag(r_1, \ldots, r_p)$ and r_s^2 $(s = 1, \ldots, p)$ are independently identically distributed as χ_4^2. Also, θ_{s1}, θ_{s2} and θ_{s3} are independent of $\boldsymbol{R}$ and $\boldsymbol{L}$ and independently identically distributed as uniform distribution on $U[0, \pi)$, on $U[0, \pi)$ and on $U[0, 2\pi)$, respectively. Further, $\boldsymbol{R}$ is independent of $\boldsymbol{L}$ whose probability density function is given by (8).

Next we consider a case when $\mathbb{B} = \mathbb{C}$ defined by (2) and $k = 5$, that is, $m = k - 1 = 4$. In this case, without loss of generality we can assume

$$\begin{aligned} \mathbb{C} &= \{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4\} \\ &= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}, \end{aligned} \tag{9}$$

and let $\boldsymbol{\gamma}_i = (\boldsymbol{c}_i'\boldsymbol{V}\boldsymbol{c}_i)^{-1/2}\boldsymbol{A}\boldsymbol{c}_i$, $i = 1, \ldots, 4$.

Relating the coverage probability $Q(q, \boldsymbol{V}, \mathbb{C})$ in Theorem 3, we consider the following probability:

$$\begin{aligned} G(\boldsymbol{\beta}) = \Pr \Bigg\{ \sum_{s=1}^{p} \frac{r_s^2}{\ell_s} (&\sin\theta_{s1}\sin\theta_{s2}\sin\theta_{s3}\sin\beta_{i1}\sin\beta_{i2}\sin\beta_{i3} \\ &+ \sin\theta_{s1}\sin\theta_{s2}\cos\theta_{s3}\sin\beta_{i1}\sin\beta_{i2}\cos\beta_{i3} + \sin\theta_{s1}\cos\theta_{s2}\sin\beta_{i1}\cos\beta_{i2} \\ &+ \cos\theta_{s1}\cos\beta_{i1})^2 \le q \quad \text{for } i = 1, \ldots, 4 \Bigg\}, \end{aligned} \tag{10}$$

where $\boldsymbol{\beta} = (\beta_{11}, \ldots, \beta_{41}, \beta_{12}, \ldots, \beta_{42}, \beta_{13}, \ldots, \beta_{43})'$. We consider the case that probability (10) achieves maximum and minimum, respectively. So we define the volume Ω and D_i as follows:

$$\begin{aligned} \Omega &= \{(\theta_{s1}, \theta_{s2}, \theta_{s3})^p : 0 < \theta_{s1} < \pi, 0 < \theta_{s2} < \pi, 0 < \theta_{s3} < 2\pi\}, \\ D_i &= \Bigg\{ (\theta_{s1}, \theta_{s2}, \theta_{s3})^p \in \Omega : \sum_{s=1}^{p} \frac{r_s^2}{\ell_s} (\sin\theta_{s1}\sin\theta_{s2}\sin\theta_{s3}\sin\beta_{i1}\sin\beta_{i2}\sin\beta_{i3} \\ &\quad + \sin\theta_{s1}\sin\theta_{s2}\cos\theta_{s3}\sin\beta_{i1}\sin\beta_{i2}\cos\beta_{i3} + \sin\theta_{s1}\cos\theta_{s2}\sin\beta_{i1}\cos\beta_{i2} \\ &\quad + \cos\theta_{s1}\cos\beta_{i1})^2 > q \Bigg\} \quad \text{for } i = 1, \ldots, 4. \end{aligned}$$

Then we note that the probability (10) is equivalent to $1 - \text{volume}\left[\cup_{i=1}^4 D_i\right]/(2\pi^3)^p$. Therefore to minimize $G(\boldsymbol{\beta})$ is equivalent to maximizing the value for volume $[\cup_{i=1}^4 D_i]$. Similarly to maximize $G(\boldsymbol{\beta})$ is equivalent to minimizing the value for volume $[\cup_{i=1}^4 D_i]$.

At first we consider the case that volume $[\cup_{i=1}^4 D_i]$ achieves maximum. Assuming that $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ and $\boldsymbol{\delta}_4$ are orthogonal each other, we can put

$$\boldsymbol{\delta}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \boldsymbol{\delta}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{\delta}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\delta}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then we can get

$$\begin{cases} \beta_{11} = 0, & \beta_{12} = 0, & \beta_{13} = 0, \\ \beta_{21} = \pi/2, & \beta_{22} = 0, & \beta_{23} = 0, \\ \beta_{31} = \pi/2, & \beta_{32} = \pi/2, & \beta_{23} = 0, \\ \beta_{41} = \pi/2, & \beta_{42} = \pi/2, & \beta_{43} = \pi/2. \end{cases}$$

For example, putting $p = 1$, $r_1^2/\ell_1 = 1$ and $q = 0.5$, we have

$$\begin{aligned} G(\boldsymbol{\beta}) = \Pr \Big\{ & (\sin\theta_{11} \sin\theta_{12} \sin\theta_{13} \sin\beta_{i1} \sin\beta_{i2} \sin\beta_{i3} \\ & + \sin\theta_{11} \sin\theta_{12} \cos\theta_{13} \sin\beta_{i1} \sin\beta_{i2} \cos\beta_{i3} + \sin\theta_{11} \cos\theta_{12} \sin\beta_{i1} \cos\beta_{i2} \\ & + \cos\theta_{11} \cos\beta_{i1})^2 \le 0.5 \ \text{ for } i = 1, \ldots, 4 \Big\}, \end{aligned}$$

and

$$\begin{aligned} D_i = \Big\{ & (\theta_{11}, \theta_{12}, \theta_{13})^p \in \Omega : (\sin\theta_{11} \sin\theta_{12} \sin\theta_{13} \sin\beta_{i1} \sin\beta_{i2} \sin\beta_{i3} \\ & + \sin\theta_{11} \sin\theta_{12} \cos\theta_{13} \sin\beta_{i1} \sin\beta_{i2} \cos\beta_{i3} + \sin\theta_{11} \cos\theta_{12} \sin\beta_{i1} \cos\beta_{i2} \\ & + \cos\theta_{11} \cos\beta_{i1})^2 > 0.5 \Big\} \quad \text{for } i = 1, \ldots, 4. \end{aligned}$$

Then we evaluate volume $[\cup_{i=1}^4 D_i]$. Figure 1 $\sim$ Figure 6 shows the area of union of D_i's when $\theta_{13} = 0$, $\pi/6$, $\pi/5$, $\pi/4$, $\pi/3$, $\pi/2$, respectively. From Figure 1 $\sim$ Figure 6, we note that D_i's don't overlap when $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ and $\boldsymbol{\delta}_4$ are orthogonal each other.

On the other hands, we choose

$$\boldsymbol{\delta}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \ \boldsymbol{\delta}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{\delta}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\delta}_4' = \begin{pmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix}.$$

In this case we can get $\beta_{41} = \pi/2$, $\beta_{42} = \pi/6$, $\beta_{43} = 0$. Then we compare volume $[D_1 \cup D_2 \cup D_3 \cup D_4]$ with volume $[D_1 \cup D_2 \cup D_3 \cup D_4']$ where D_4' is concerned with $\boldsymbol{\delta}_4'$. Figure 7 $\sim$ Figure 12 shows area $[D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_4']$ when $\theta_{13} = 0$,

$\pi/6$, $\pi/5$, $\pi/4$, $\pi/3$, $\pi/2$, respectively. From Figure 7 $\sim$ Figure 12, we note that D_4' becomes small. Also, D_2 and D_4' always overlap each other and finally D_2 contain D_4'. On the other hands, D_4 doesn't overlap other areas and becomes large. So, we note that the volume of complement of $D_1 \cup D_2 \cup D_3 \cup D_4 \leq$ the volume of complement of $D_1 \cup D_2 \cup D_3 \cup D_4'$. Therefore volume $[\cup_{i=1}^{4} D_i]$ achieves maximum, that is, $Q(q, \boldsymbol{V}, \mathbb{C})$ achieves minimum when $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ and $\boldsymbol{\delta}_4$ are orthogonal each other.

When $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ and $\boldsymbol{\delta}_4$ are orthogonal each other, $\boldsymbol{\delta}_\ell' \boldsymbol{\delta}_m = 0$ $(\ell \neq m)$, that is, $\boldsymbol{\gamma}_\ell' \boldsymbol{\gamma}_m = 0$ $(\ell \neq m)$. We can show that $\boldsymbol{\gamma}_i' \boldsymbol{\gamma}_j = 0$ if and only if $v_{ij} - v_{i5} - v_{j5} + v_{55} = 0$ for $1 \leq i < j \leq 4$. Therefore we can get the condition $d_{ij} = d_{i5} + d_{j5}$ for $1 \leq i < j \leq 4$. Summarizing these results, we have following Lemma.

Lemma 4. *Let $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, $\boldsymbol{c}_3$ and $\boldsymbol{c}_4$ be the vectors defined by (9) and let $\boldsymbol{\gamma}_i = (\boldsymbol{c}_i' \boldsymbol{V} \boldsymbol{c}_i)^{-1/2} \boldsymbol{A} \boldsymbol{c}_i$, $i = 1, \ldots, 4$, where $\boldsymbol{V} = [v_{ij}]$ is a 5×5 positive definite matrix and $\boldsymbol{A}$ is a nonsingular matrix such that $\boldsymbol{V} = \boldsymbol{A}'\boldsymbol{A}$. Then $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$ and $\boldsymbol{\gamma}_4$ are orthogonal each other if and only if $d_{ij} = v_{ii} - 2v_{ij} + v_{jj} = d_{i5} + d_{j5}$ $(1 \leq i < j \leq 4)$.*

Secondly, we consider the case that volume $[\cup_{i=1}^{4} D_i]$ achieves minimum, that is, $Q(q, \boldsymbol{V}, \mathbb{C})$ achieves maximum. By using same procedure, in this case, we note that $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_3$ and $\boldsymbol{\delta}_4$ are all same. So $\boldsymbol{\delta}_\ell' \boldsymbol{\delta}_m = \boldsymbol{\delta}_\ell' \boldsymbol{\delta}_\ell = 1$ $(\ell \neq m)$, that is, $\boldsymbol{\gamma}_\ell' \boldsymbol{\gamma}_m = \boldsymbol{\gamma}_\ell' \boldsymbol{\gamma}_\ell = 1$ $(\ell \neq m)$. We can show that $\boldsymbol{\gamma}_i' \boldsymbol{\gamma}_j = 1$ if and only if $v_{ij} - v_{i5} - v_{j5} + v_{55} = \sqrt{d_{i5}}\sqrt{d_{j5}}$ for $1 \leq i < j \leq 4$. Therefore we can get the condition $\sqrt{d_{ij}} = |\sqrt{d_{i5}} - \sqrt{d_{j5}}|$ for $1 \leq i < j \leq 4$. Summarizing these results, we have following Lemma.

Lemma 5. *Let $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, $\boldsymbol{c}_3$ and $\boldsymbol{c}_4$ be the vectors defined by (9) and let $\boldsymbol{\gamma}_i = (\boldsymbol{c}_i' \boldsymbol{V} \boldsymbol{c}_i)^{-1/2} \boldsymbol{A} \boldsymbol{c}_i$, $i = 1, \ldots, 4$, where $\boldsymbol{V} = [v_{ij}]$ is a 5×5 positive definite matrix and $\boldsymbol{A}$ is a nonsingular matrix such that $\boldsymbol{V} = \boldsymbol{A}'\boldsymbol{A}$. Then $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$ and $\boldsymbol{\gamma}_4$ are* *all same if and only if* $\sqrt{d_{ij}} = \sqrt{v_{ii} - 2v_{ij} + v_{jj}} = |\sqrt{d_{i5}} - \sqrt{d_{j5}}|$ $(1 \leq i < j \leq 4)$.

From Lemma 4 and Lemma 5, we we have following Theorem.

Theorem 6. *Let* $Q(q, \boldsymbol{V}, \mathbb{C})$ *be the coverage probability (6) with a known matrix* $\boldsymbol{V}$ *for the case* $k = 5$*. Then*

$$1 - \alpha = Q(t_{\mathrm{c}}^*, \boldsymbol{V}_1, \mathbb{C}) \leq Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{C}) < Q(t_{\mathrm{c}}^*, \boldsymbol{V}_2, \mathbb{C}),$$

holds for any positive definite matrix $\boldsymbol{V}$*, where* $t_{\mathrm{c}}^* = t_{\mathrm{c}\cdot V_1}^2/\nu$*,* $\mathbb{C} = \{\boldsymbol{c} \in \mathbb{R}^k : \boldsymbol{c} = \boldsymbol{e}_i - \boldsymbol{e}_k,\ i = 1, \ldots, k-1\}$ *and* $\boldsymbol{V}_1$ *satisfies with* $d_{ij} = d_{i5} + d_{j5}$ *for all* i*,* j $(1 \leq i < j \leq 4)$ *and* $\boldsymbol{V}_2$ *satisfies with* $\sqrt{d_{ij}} = |\sqrt{d_{i5}} - \sqrt{d_{j5}}|$ *for all* i*,* j $(1 \leq i < j \leq 4)$.

4. Numerical examinations

This section gives some numerical results of the coverage probability $Q(t_{\mathrm{c}}^*, V, \mathbb{C})$ and the upper 100α percentiles of $T_{\max\cdot\mathrm{c}}^2$ statistics by Monte Carlo simulation. The Monte Carlo simulations are made from 10^6 trials for each of parameters based on normal random vectors from $N_{kp}(\boldsymbol{0}, \boldsymbol{V} \otimes \boldsymbol{I}_p)$. The sample covariance matrix $\boldsymbol{S}$ is computed on the basis of random vectors from $N_p(\boldsymbol{0}, \boldsymbol{I}_p)$. Also we note that $\boldsymbol{S}$ is formed independently in each time with ν degrees of freedom.

Table 1 gives the simulated upper 100α percentiles $t_{\mathrm{c}\cdot V}$ of $T_{\max\cdot\mathrm{c}}(= \sqrt{T_{\max\cdot\mathrm{c}}^2})$ and simulated coverage probability $Q(t_{\mathrm{c}}^*, V, \mathbb{C})(\equiv \mathrm{CP}(\boldsymbol{V}))$ when the simulated values of $t_{\mathrm{c}\cdot V}$ are substituted for $t_{\mathrm{c}\cdot V_1}$ for the following parameters: $\alpha = 0.1, 0.05, 0.01$, $p = 1, 2, 5$, $k = 5$, $\nu = 20, 40, 60$, and $\boldsymbol{V} = \boldsymbol{I}$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ and $\boldsymbol{V}_3$, that is,

$$\boldsymbol{V}_1 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \boldsymbol{V}_2 = \begin{bmatrix} 4 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 & 4 \end{bmatrix}, \boldsymbol{V}_3 = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Here we note that $\boldsymbol{V}_1$ and $\boldsymbol{V}_3$ are positive definite matrices and $\boldsymbol{V}_2$, whose eigenvalues are $(10, 4, 0, 0, 0)$, is a positive semi-definite matrix. Also we note that $\boldsymbol{V}_1$ is a matrix such that $d_{ij} = d_{i5} + d_{j5}$ and $\boldsymbol{V}_2$ is a matrix such that $\sqrt{d_{ij}} = |\sqrt{d_{i5}} - \sqrt{d_{j5}}|$ for all $1 \leq i < j \leq 4$.

Since $\boldsymbol{V}$ is a positive semi-definite matrix, a distribution of vec$(\boldsymbol{X})$ is said to be singular or degenerate when the rank of $\boldsymbol{V}$ is less than k, say $r(< k)$. In this case, the total probability mass concentrates on a linear set of exactly $r \times p$ dimensions with probability one (see, Cramér (1946)). Therefore, in this case, the $k \times p$ dimensional random numbers are produced from the ones generated by $r \times p$ dimensional normal distribution (see, e.g., Seo and Nishiyama (2008)).

From Table 1, it can be seen that always $t_{\mathrm{c}\cdot V_2} < t_{\mathrm{c}\cdot V_3} \leq t_{\mathrm{c}\cdot I} \leq t_{\mathrm{c}\cdot V_1}$. So we note that $t_{\mathrm{c}\cdot V_2} < t_{\mathrm{c}\cdot V} \leq t_{\mathrm{c}\cdot V_1}$ for any positive definite matrix $\boldsymbol{V}$. Therefore, this result shows that the upper bound for the conservativeness of multiple comparisons with a control can be obtained. For example, from Theorem 6, when $p = 2$, $\nu = 20$ and $\alpha = 0.1$ we note that $0.900 \leq Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{C}) < 0.973$ for any positive definite matrix $\boldsymbol{V}$. Further it may be noted that from simulation results, coverage probabilities $Q(t_{\mathrm{c}}^*, \boldsymbol{V}, \mathbb{C})$ do not depend on the value p and ν.

In conclusion, it may be noted that the approximate simultaneous procedure which is discussed by this paper gives the conservative and good approximate simultaneous confidence intervals, and useful for the simultaneous confidence intervals estimation in the case of comparisons with a control among mean vectors.

Acknowledgements

The author would like to express his sincere graditude to Professor Takashi Seo for his useful suggestions. The research of the author was supported in part by Grant-in-Aid for Young Scientists (B) under Contract Number 21700311.

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Table 1: Upper 100α percentiles of $T_{\max\cdot\mathrm{c}}$ and coverage probabilities

p	ν	α	$t_{\mathrm{c}\cdot V_1}$	$t_{\mathrm{c}\cdot I}$	$t_{\mathrm{c}\cdot V_3}$	$t_{\mathrm{c}\cdot V_2}$	CP($\boldsymbol{I}$)	CP($\boldsymbol{V}_3$)	CP($\boldsymbol{V}_2$)
1	20	0.01	3.445	3.396	3.304	2.846	0.991	0.993	0.997
		0.05	2.722	2.652	2.567	2.086	0.957	0.964	0.987
		0.1	2.387	2.304	2.226	1.724	0.915	0.928	0.973
	40	0.01	3.223	3.188	3.104	2.702	0.991	0.993	0.997
		0.05	2.602	2.543	2.464	2.020	0.956	0.964	0.987
		0.1	2.304	2.230	2.155	1.683	0.915	0.928	0.974
	60	0.01	3.154	3.120	3.039	2.663	0.991	0.993	0.997
		0.05	2.565	2.509	2.431	2.001	0.956	0.964	0.987
		0.1	2.278	2.207	2.132	1.671	0.914	0.928	0.974
2	20	0.01	4.184	4.131	4.023	3.532	0.991	0.993	0.997
		0.05	3.399	3.329	3.231	2.722	0.956	0.964	0.987
		0.1	3.041	2.959	2.866	2.342	0.914	0.928	0.973
	40	0.01	3.790	3.756	3.667	3.265	0.991	0.993	0.997
		0.05	3.163	3.108	3.020	2.578	0.956	0.964	0.987
		0.1	2.863	2.794	2.708	2.240	0.914	0.929	0.974
	60	0.01	3.675	3.646	3.563	3.180	0.991	0.993	0.998
		0.05	3.090	3.040	2.956	2.532	0.956	0.964	0.987
		0.1	2.807	2.743	2.660	2.207	0.913	0.929	0.974
5	20	0.01	6.135	6.063	5.919	5.268	0.991	0.993	0.997
		0.05	5.091	5.001	4.868	4.223	0.956	0.964	0.987
		0.1	4.627	4.524	4.397	3.745	0.914	0.929	0.973
	40	0.01	5.036	5.004	4.900	4.449	0.991	0.993	0.997
		0.05	4.345	4.291	4.187	3.708	0.955	0.964	0.987
		0.1	4.020	3.951	3.848	3.343	0.913	0.929	0.974
	60	0.01	4.756	4.731	4.636	4.242	0.991	0.993	0.998
		0.05	4.149	4.101	4.006	3.570	0.955	0.964	0.987
		0.1	3.855	3.795	3.698	3.233	0.912	0.929	0.974

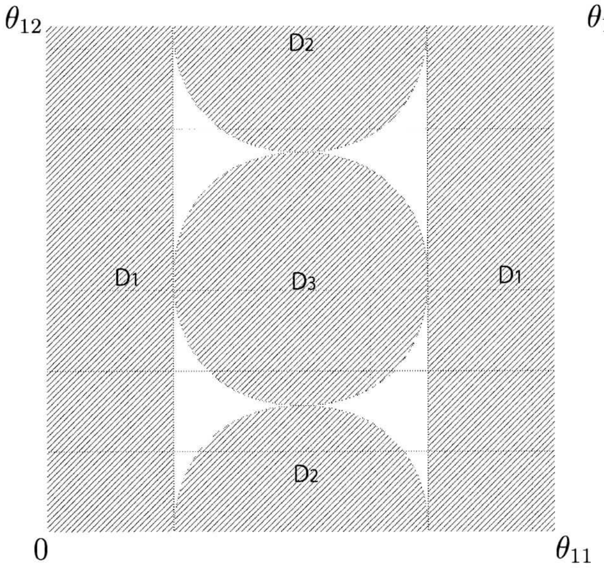

Figure 1. area$[D_1 \cup D_2 \cup D_3 \cup D_4]$
when $\theta_{13} = 0$.

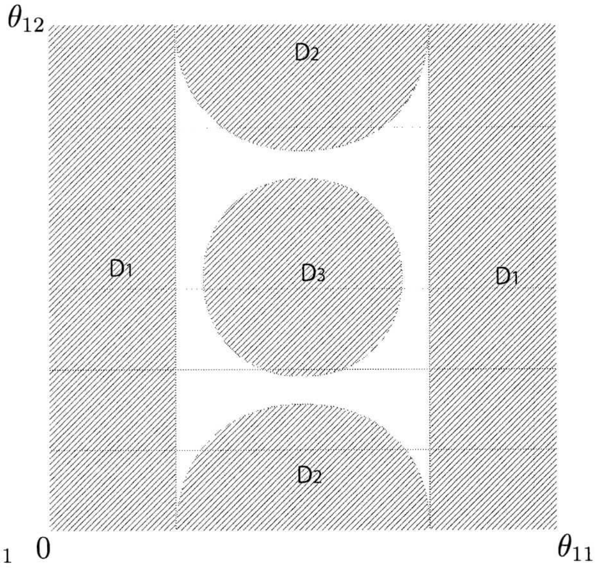

Figure 2. area$[D_1 \cup D_2 \cup D_3 \cup D_4]$
when $\theta_{13} = \pi/6$.

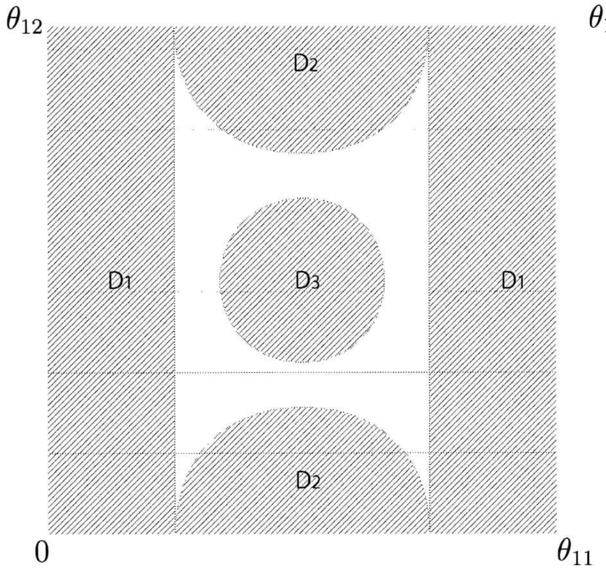

Figure 3. area$[D_1 \cup D_2 \cup D_3 \cup D_4]$
when $\theta_{13} = \pi/5$.

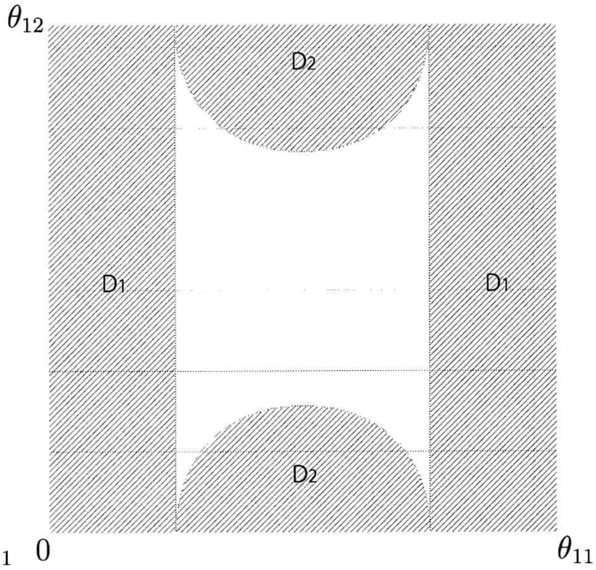

Figure 4. area$[D_1 \cup D_2 \cup D_3 \cup D_4]$
when $\theta_{13} = \pi/4$.

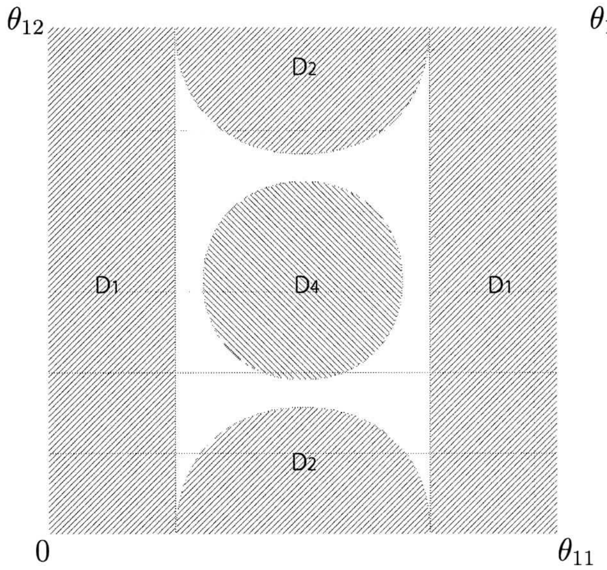

Figure 5. area$[D_1 \cup D_2 \cup D_3 \cup D_4]$ when $\theta_{13} = \pi/3$.

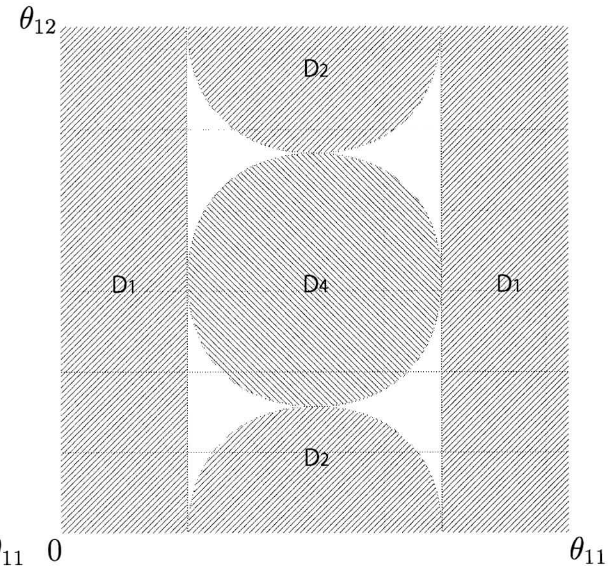

Figure 6. area$[D_1 \cup D_2 \cup D_3 \cup D_4]$ when $\theta_{13} = \pi/2$.

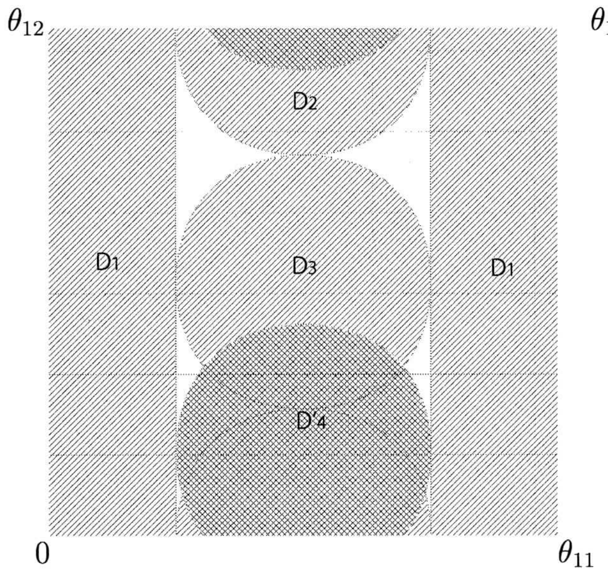

Figure 7. area$[D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_4']$ when $\theta_{13} = 0$.

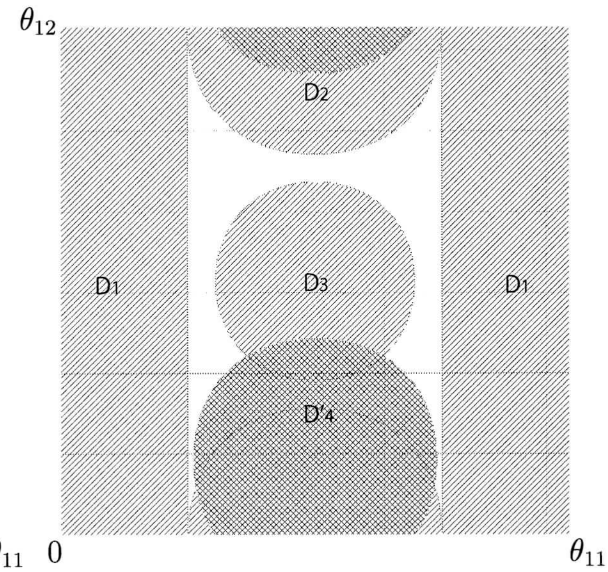

Figure 8. area$[D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_4']$ when $\theta_{13} = \pi/6$.

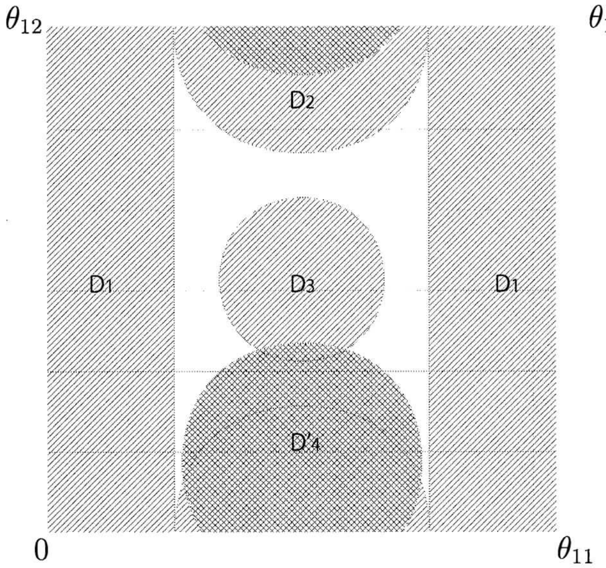

Figure 9. area$[D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_4']$ when $\theta_{13} = \pi/5$.

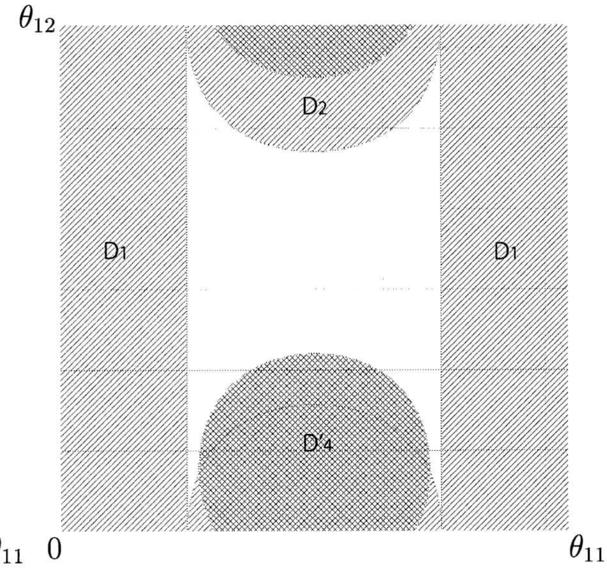

Figure 10. area$[D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_4']$ when $\theta_{13} = \pi/4$.

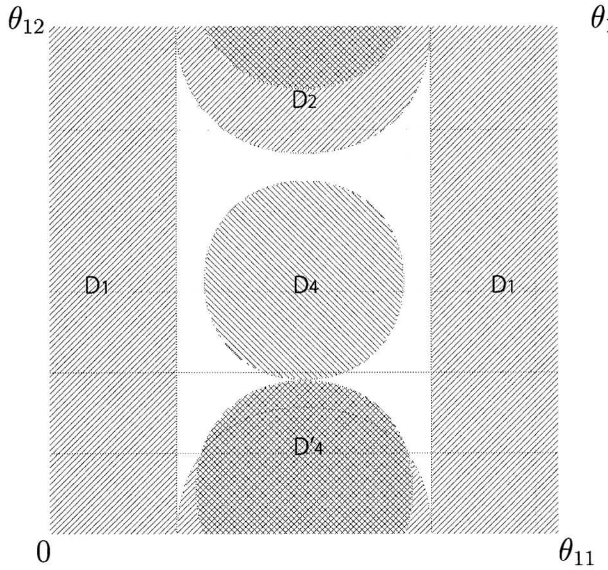

Figure 11. area$[D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_4']$ when $\theta_{13} = \pi/3$.

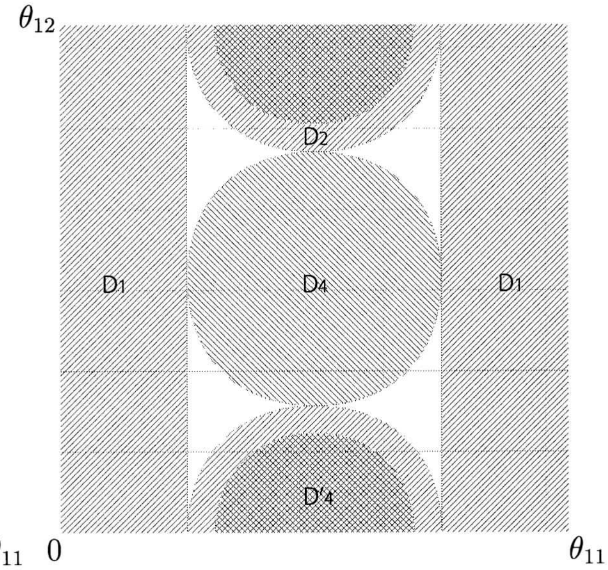

Figure 12. area$[D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_4']$ when $\theta_{13} = \pi/2$.