Normal Approximation to Multivariate Sample Measures of Kurtosis

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In this paper, we consider the multivariate sample measures of kurtosis defined by Mardia (1970) and Srivastava (1984). Under normality, asymptotic expansions for the first, second and third moments of Mardia's and Srivastava's multivariate sample measures of kurtosis are given. Asymptotic expansions for the distributions of these sample measures of kurtosis are also given. By using these asymptotic expansions, normalizing transformation for these sample measures of kurtosis can be derived. Finally, we investigate the approximation accuracy of the normalizing transformations to the normal distribution by Monte Carlo simulation for some selected parameters.

Keywords: asymptotic distribution, asymptotic expansion, multivariate normal distribution, multivariate sample kurtosis, normalizing transformation.

1 Introduction

In multivariate statistical analysis, the test for multivariate normality is an important problem and has been studied by many authors. To assess multivariate normality, for example, the multivariate sample measures of skewness and kurtosis and their asymptotic distributions using expectations and variances of them are given in Mardia (1970, 1974). Srivastava (1984) has proposed another definition for the multivariate sample measures of skewness and kurtosis and derived their asymptotic distributions. Also, Song (2001) gave different definition for the multivariate sample measure of kurtosis. Multivariate extensions of Shapiro-Wilk test(Shapiro and Wilk (1965)) have been given by Malkovich and Afifi (1973), Royston (1983), Srivastava and Hui (1987) and so on. Small (1980) gave multivariate extensions of univariate skewness and kurtosis. For a comparison of these methods, see, Looney (1995).

Mardia and Kanazawa (1983) has proposed the normal approximation for Mardia's multivariate sample measure of kurtosis by using Wilson-Hilferty transformation and derived the asymptotic expansion of the third moment. For the asymptotic distributions of Mardia's and Srivastava's multivariate sample measures of kurtosis under elliptical populations, see, e.g., Berkane and Bentler (1990), Seo and Toyama (1996), Maruyama (2005a, 2005b). Seo and Ariga (2006) discussed the asymptotic distribution of Srivastava's multivariate sample measure of kurtosis under normal population for two cases when the population covariance matrix is known and unknown, but the part of this result has error in calculation. Seo and Ariga (2009) gave the correct result and the normalizing transformational statistic for Srivastava's multivariate sample measure of kurtosis when the population covariance matrix is unknown.

The limit distributions of Mardia's multivariate measures of kurtosis under Watson rotational symmetric distributions were discussed by Zhao and Konishi (1997). Henze (1994) discussed with the asymptotic distributions for Mardia's multivariate measures of kurtosis under non-normal populations. For a survey on multivariate measure of kurtosis, see, Schwager (1985).

Recently, Koizumi, Okamoto and Seo (2009) has given the multivariate Jaque-Bera statistics based on Mardia's and Srivastava's multivariate skewnesses and Kurtosises.

In this paper, we consider distribution of the multivariate sample measures of kurtosis defined by Mardia (1970) and Srivastava (1984) in two cases when the population covariance matrix Σ is known and unknown under normality. In each case asymptotic expansions for the first, second and third moments of the multivariate sample measures of kurtosis are obtained by perturbation method. Further, by using these expansions and the results in Mardia and Kanazawa (1983), standardized statistics and normalizing transformational statistics can be derived. Finally, we investigate the approximation accuracy of the normalizing transformational statistics and the standardized statistics to the normal distribution by Monte Carlo simulation for some selected parameters.

2 Multivariate kurtosis

Let $\boldsymbol{x}$ be a random p-vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma = \Gamma D_\lambda \Gamma'$, where $\Gamma = \left(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_p\right)$ is an orthogonal matrix and $D_\lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Note that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the characteristic roots of Σ. Then Mardia (1970) and Srivastava (1984) have defined the population measures of multivariate kurtosis as

$$\beta_M = \mathrm{E}\left[\left\{(\boldsymbol{x}-\boldsymbol{\mu})' \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right\}^2\right],$$

$$\beta_S = \frac{1}{p}\sum_{i=1}^{p} \frac{\mathrm{E}\left[(y_i-\theta_i)^4\right]}{\lambda_i^2},$$

respectively, where $y_i = \boldsymbol{\gamma}_i'\boldsymbol{x}$ and $\theta_i = \boldsymbol{\gamma}_i'\boldsymbol{\mu}$, $i = 1, 2, \ldots, p$. We note that $\beta_M = p(p+2)$ and $\beta_S = 3$ under a multivariate normal population.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be N random sample vectors from a multivariate population. Let $\overline{\boldsymbol{x}}$ and $S = N^{-1}\sum_{i=1}^{N}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})' = HD_\omega H'$ be the sample mean vector and the sample covariance matrix based on sample size N, where $H = (\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_p)$ is an orthogonal matrix and $D_\omega = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_p)$. We note that $\omega_1, \omega_2, \ldots, \omega_p$ are the characteristic roots of S. Then, by Mardia (1970) and Srivastava (1984), the sample measures of multivariate kurtosis are defined as

$$b_M = \frac{1}{N}\sum_{i=1}^{N}\left\{(\boldsymbol{x}_i-\overline{\boldsymbol{x}})' S^{-1} (\boldsymbol{x}_i-\overline{\boldsymbol{x}})\right\}^2, \tag{1}$$

$$b_S = \frac{1}{pN}\sum_{i=1}^{p}\frac{1}{\omega_i^2}\sum_{j=1}^{N}(y_{ij}-\overline{y}_i)^4, \tag{2}$$

respectively, where $y_{ij} = \boldsymbol{h}_i'\boldsymbol{x}_j$ and $\overline{y}_i = N^{-1}\sum_{j=1}^{N} y_{ij}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, N$.

Further Mardia (1970) and Srivastava (1984) have obtained asymptotic distributions of b_M and b_S and used them to test the multivariate normality and given the following Theorems 2.1 and 2.2.

Theorem 2.1 (Mardia (1970)) *Let b_M be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is unknown. Then*

$$z_M = \frac{b_M - p(p+2)}{\sqrt{\dfrac{8p(p+2)}{N}}}$$

is asymptotically distributed as $\mathrm{N}(0,1)$.

Theorem 2.2 (Srivastava (1984)) *Let b_S be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is unknown. Then*

$$z_S = \frac{b_S - 3}{\sqrt{\dfrac{24}{pN}}}$$

is asymptotically distributed as $\mathrm{N}(0,1)$.

In this paper, without loss of generality, we may assume that $\Sigma = I$ and $\boldsymbol{\mu} = \mathbf{0}$ when we consider the sample measures of multivariate kurtosis (1) and (2).

3 First, second and third moments of sample measures of multivariate kurtosis

In this section we consider asymptotic expansions of first, second and third moments of Mardia's multivariate sample kurtosis b_M in (1) and Srivastava's multivariate sample kurtosis b_S in (2) under normality in two cases when Σ is known and unknown.

3.1 Mardia's multivariate sample kurtosis

For the case of known Σ, Mardia's multivariate sample kurtosis can be written as

$$b_{M,k} = \frac{1}{N}\sum_{\alpha=1}^{N} \chi_\alpha^4,$$

where $\chi_\alpha^2 = (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}})'(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}})$. In order to avoid the dependence of $\boldsymbol{x}_\alpha$ and $\overline{\boldsymbol{x}}$, let $\overline{\boldsymbol{x}}^{(\alpha)}$ be a sample mean vector defined on the subset of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ by deleting some $\boldsymbol{x}_\alpha$, that is,

$$\overline{\boldsymbol{x}}^{(\alpha)} = \frac{1}{N-1}\sum_{i=1, i\neq\alpha}^{N} \boldsymbol{x}_i.$$

Then we can write

$$\chi_\alpha^2 = \left(1 - \frac{1}{N}\right)^2 \left(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)}\right)' \left(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(\alpha)}\right).$$

Note that $\boldsymbol{x}_\alpha$ is independent of $\overline{\boldsymbol{x}}^{(\alpha)}$. To obtain the expectation of $b_{M,k}$, we put

$$\overline{\boldsymbol{x}}^{(\alpha)} = \frac{1}{\sqrt{N-1}}\boldsymbol{y},$$

where $\boldsymbol{y}$ is independent of $\boldsymbol{x}_\alpha$, and both $\boldsymbol{y}$ and $\boldsymbol{x}_\alpha$ are distributed as a p-dimensional standard normal distribution. Therefore calculating the expectation with respect to $\boldsymbol{x}_\alpha$ and $\boldsymbol{y}$, we obtain

$$\begin{aligned}
\mathrm{E}\left[b_{M,\mathrm{k}}\right] &= \mathrm{E}\left[\chi_\alpha^4\right] \\
&= \mathrm{E}\left[\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\right] - 2\mathrm{E}\left[2\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2 - \left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{y}'\boldsymbol{y}\right) - 2\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)^2\right]\frac{1}{N} \\
&\quad + \mathrm{E}\left[6\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2 - \left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{y}'\boldsymbol{y}\right) + \left(\boldsymbol{y}'\boldsymbol{y}\right)^2 - 12\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)^2\right]\frac{1}{N^2} + O\left(N^{-3}\right) \\
&= p\left(p+2\right) - 2p\left(p+2\right)\frac{1}{N} + p\left(p+2\right)\frac{1}{N^2} + O\left(N^{-3}\right).
\end{aligned}$$

Next, we consider $\mathrm{Var}\left[b_{M,\mathrm{k}}\right] = \mathrm{E}\left[b_{M,\mathrm{k}}^2\right] - \left\{\mathrm{E}\left[b_{M,\mathrm{k}}\right]\right\}^2$, where

$$\mathrm{E}\left[b_{M,\mathrm{k}}^2\right] = \frac{1}{N}\mathrm{E}\left[\left(\chi_\alpha^4\right)^2\right] + \left(1 - \frac{1}{N}\right)\mathrm{E}\left[\chi_\alpha^4\chi_\beta^4\right].$$

First, we consider $\mathrm{E}\left[\left(\chi_\alpha^4\right)^2\right]$. Calculating the expectation with respect to $\boldsymbol{x}_\alpha$ and $\boldsymbol{y}$, we obtain

$$\begin{aligned}
\mathrm{E}\left[\left(\chi_\alpha^4\right)^2\right] &= \mathrm{E}\left[\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^4\right] \\
&\quad - 4\mathrm{E}\left[2\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^4 - \left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^3\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right) - 6\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{y}'\boldsymbol{y}\right)^2\right]\frac{1}{N} + O\left(N^{-2}\right) \\
&= -4p\left(p+2\right)\left(p+4\right)\left(p+6\right) + p\left(p+2\right)\left(p+4\right)\left(p+6\right)\frac{1}{N} + O\left(N^{-2}\right).
\end{aligned}$$

Secondly, we consider $\mathrm{E}\left[\chi_\alpha^4\chi_\beta^4\right]$. In order to avoid the dependence of $\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta$ and $\overline{\boldsymbol{x}}$, let $\overline{\boldsymbol{x}}^{(\alpha,\beta)}$ be a sample mean vector defined on the subset of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ by deleting $\boldsymbol{x}_\alpha$ and $\boldsymbol{x}_\beta$, that is,

$$\overline{\boldsymbol{x}}^{(\alpha,\beta)} = \frac{1}{N-2}\sum_{i=1, i\neq\alpha, i\neq\beta}^{N}\boldsymbol{x}_i.$$

Then we can write

$$\begin{aligned}
\chi_\alpha^2 &= \frac{1}{N^2}\left\{\left(N-1\right)\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta - \left(N-2\right)\overline{\boldsymbol{x}}^{(\alpha,\beta)}\right\}'\left\{\left(N-1\right)\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta - \left(N-2\right)\overline{\boldsymbol{x}}^{(\alpha,\beta)}\right\} \\
\chi_\beta^2 &= \frac{1}{N^2}\left\{\left(N-1\right)\boldsymbol{x}_\beta - \boldsymbol{x}_\alpha - \left(N-2\right)\overline{\boldsymbol{x}}^{(\alpha,\beta)}\right\}'\left\{\left(N-1\right)\boldsymbol{x}_\beta - \boldsymbol{x}_\alpha - \left(N-2\right)\overline{\boldsymbol{x}}^{(\alpha,\beta)}\right\}
\end{aligned}$$

Note that $\boldsymbol{x}_\alpha$ and $\boldsymbol{x}_\beta$ are independent of $\overline{\boldsymbol{x}}^{(\alpha,\beta)}$. To obtain the expectation of $\chi_\alpha^4\chi_\beta^4$, we put

$$\overline{\boldsymbol{x}}^{(\alpha,\beta)} = \frac{1}{\sqrt{N-2}}\boldsymbol{y},$$

where $\boldsymbol{y}$ is independent of $\boldsymbol{x}_\alpha$ and $\boldsymbol{x}_\beta$. $\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta$ and $\boldsymbol{y}$ are distributed as a p-dimensional standard normal distribution. Therefore calculating the expectation with respect to $\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta$ and $\boldsymbol{y}$, we obtain

$$
\begin{aligned}
\mathrm{E}\left[\chi_\alpha^4\chi_\beta^4\right] =&\mathrm{E}\left[\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2\right]\\
&-2\mathrm{E}\Big[4\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2-\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{y}'\boldsymbol{y}\right)\\
&\qquad-\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2\left(\boldsymbol{y}'\boldsymbol{y}\right)-2\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)^2-2\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{y}\right)\Big]\frac{1}{N}\\
&+\mathrm{E}\Big[2\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^3\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)+28\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2+2\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^3\\
&\qquad-16\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{y}'\boldsymbol{y}\right)-16\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2\left(\boldsymbol{y}'\boldsymbol{y}\right)\\
&\qquad+\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{y}'\boldsymbol{y}\right)^2+4\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{y}'\boldsymbol{y}\right)^2+\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2\left(\boldsymbol{y}'\boldsymbol{y}\right)^2\\
&\qquad+4\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\beta\right)^2+16\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\beta\right)^2\\
&\qquad+4\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\beta\right)^2-16\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)^2\\
&\qquad-32\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)^2\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)^2+8\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{y}'\boldsymbol{y}\right)\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)^2\\
&\qquad-32\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\beta\right)\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)\left(\boldsymbol{x}_\beta'\boldsymbol{y}\right)-32\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\beta\right)\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)\left(\boldsymbol{x}_\beta'\boldsymbol{y}\right)\\
&\qquad-32\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{y}\right)^2-16\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{x}_\beta'\boldsymbol{x}_\beta\right)\left(\boldsymbol{x}_\beta'\boldsymbol{y}\right)^2\\
&\qquad+8\left(\boldsymbol{x}_\alpha'\boldsymbol{x}_\alpha\right)\left(\boldsymbol{y}'\boldsymbol{y}\right)\left(\boldsymbol{x}_\beta'\boldsymbol{y}\right)^2+16\left(\boldsymbol{x}_\alpha'\boldsymbol{y}\right)^2\left(\boldsymbol{x}_\beta'\boldsymbol{y}\right)^2\Big]\frac{1}{N^2}+O\left(N^{-3}\right)\\
=&\,p^2\left(p+2\right)^2-4p^2\left(p+2\right)^2\frac{1}{N}+2p\left(p+2\right)^2\left(3p+4\right)\frac{1}{N^2}+O\left(N^{-3}\right).
\end{aligned}
$$

Therefore we may obtain

$$
\mathrm{Var}\left[b_{M,\mathrm{k}}\right]=8p\left(p+2\right)\left(p+3\right)\frac{1}{N}-8p\left(p+2\right)\left(3p+10\right)\frac{1}{N^2}+O\left(N^{-3}\right).
$$

Further, after a great deal of calculation, we can obtain $\mathrm{E}\left[b_{M,\mathrm{k}}^3\right]$ as

$$
\begin{aligned}
\mathrm{E}\left[b_{M,\mathrm{k}}^3\right]=&\,p^3\left(p+2\right)^3-6p^2\left(p+2\right)^2\left(p^2-2p-12\right)\frac{1}{N}\\
&+p\left(p+2\right)\left(15p^4-60p^3-404p^2+320p+1920\right)\frac{1}{N^2}+O\left(N^{-3}\right).
\end{aligned}
$$

Therefore we can obtain the following theorems;

Theorem 3.1 *Let b_M be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p\left(\boldsymbol{\mu},\Sigma\right)$ where Σ is known. Then*

$$
z_{M,k}=\frac{b_M-p\left(p+2\right)}{\sqrt{\dfrac{8p\left(p+2\right)\left(p+3\right)}{N}}}
$$

is asymptotically distributed as $\mathrm{N}\left(0,1\right)$.

Theorem 3.2 *Let b_M be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is known. Then*

$$z^*_{M,k} = \frac{b_M - \left(p(p+2) - 2p(p+2)\dfrac{1}{N} + p(p+2)\dfrac{1}{N^2} \right)}{\sqrt{\dfrac{8p(p+2)(p+3)}{N} - \dfrac{8p(p+2)(3p+10)}{N^2}}}$$

is asymptotically distributed as $\mathrm{N}(0,1)$.

For the case of unknown Σ, Mardia (1970, 1974) and Mardia and Kanazawa (1983) have given the following exact mean and variance and asymptotic expansion for third moment of b_M when the population is $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$;

$$\begin{aligned}
\mathrm{E}[b_M] &= p(p+2)\frac{N-1}{N+1}, \\
\mathrm{Var}[b_M] &= 8p(p+2)\frac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)}, \\
\mathrm{E}\left[b_M^3\right] &= p^3(p+2)^3 - 12p^2(p+2)^3(p-1)\frac{1}{N} \\
&\quad + 2p(p+2)\left(9p^4 + 12p^3 - 192p^2 - 328p + 256\right)\frac{1}{N^2} + O\left(N^{-3}\right).
\end{aligned}$$

Also, the following theorem is given by Mardia (1974) and Siotani, Hayakawa and Fujikoshi (1985).

Theorem 3.3 (Mardia (1974)) *Let b_M be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is unknown. Then*

$$z^*_M = \frac{b_M - p(p+2)\dfrac{N-1}{N+1}}{\sqrt{8p(p+2)\dfrac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)}}}$$

is asymptotically distributed as $\mathrm{N}(0,1)$.

3.2 Srivastava's multivariate sample kurtosis

The first, second and third moments for Srivastava's multivariate sample measure of kurtosis in two cases when Σ is known and unknown are discussed by Seo and Ariga (2006, 2009).

Assuming that Σ is known, we can write

$$b_{S,\mathrm{k}} = \frac{1}{p}\sum_{i=1}^{p} m_{4i},$$

where $m_{4i} = N^{-1}\sum_{j=1}^{N}(y_{ij} - \overline{y}_i)^4$. In order to avoid the dependence of $y_{i\alpha}$ and $\overline{y}_i$, let $\overline{y}_i^{(\alpha)}$ be a mean defined on the subset of $y_{i1}, y_{i2}, \ldots, y_{iN}$ by deleting $y_{i\alpha}$, that is,

$$\overline{y}_i^{(\alpha)} = \frac{1}{N-1}\sum_{j=1, j\neq\alpha}^{N} y_{ij}.$$

Then we can write

$$m_{4i} = \frac{1}{N}\left(1-\frac{1}{N}\right)^4 \sum_{\alpha=1}^{N}\left(y_{i\alpha} - \overline{y}_i^{(\alpha)}\right)^4.$$

Note that $y_{i\alpha}$ is independent of $\overline{y}_i^{(\alpha)}$. Putting $\overline{y}_i^{(\alpha)} = z/\sqrt{N-1}$, we have

$$\mathrm{E}\left[b_{S,\mathrm{k}}\right] = \mathrm{E}\left[m_{4i}\right] = \mathrm{E}\left[\frac{1}{N}\left(1-\frac{1}{N}\right)^4 \sum_{\alpha=1}^{N}\left(y_{i\alpha} - \frac{1}{\sqrt{N-1}}z\right)^4\right],$$

where z is independent of $y_{i\alpha}$, and both z and $y_{i\alpha}$ are distributed as a standard normal distribution. Therefore calculating the expectation $\mathrm{E}\left[b_{S,\mathrm{k}}\right]$ with respect to $y_{i\alpha}$ and z, we obtain

$$\mathrm{E}\left[b_{S,\mathrm{k}}\right] = \mathrm{E}\left[m_{4i}\right] = 3 - \frac{6}{N} + \frac{3}{N^2}. \tag{3}$$

Similarly, we may obtain the variance for the kurtosis $b_{S,k}$ as

$$\begin{aligned}\mathrm{Var}\left[b_{S,\mathrm{k}}\right] &= \mathrm{E}\left[b_{S,k}^2\right] - \left\{\mathrm{E}\left[b_{S,k}\right]\right\}^2 \\ &= \frac{1}{p}\mathrm{E}\left[m_{4i}^2\right] + \left(1-\frac{1}{p}\right)\mathrm{E}\left[m_{4i}\right]\mathrm{E}\left[m_{4j}\right] - \left\{\mathrm{E}\left[m_{4i}\right]\right\}^2 \\ &= \frac{96}{pN} - \frac{312}{pN^2} + \frac{360}{pN^3} + O\left(N^{-4}\right).\end{aligned}$$

Also, after a great deal of calculation, we obtain $\mathrm{E}\left[b_{S,\mathrm{k}}^3\right]$ as

$$\mathrm{E}\left[b_{S,\mathrm{k}}^3\right] = 27 - \frac{54}{pN}(3p-16) + \frac{27}{p^2N^2}\left(15p^2 - 168p + 352\right) + O\left(N^{-3}\right).$$

With the use of these results, the following theorems are given.

Theorem 3.4 (Seo and Ariga (2009)) *Let b_S be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is known. Then*

$$z_{S,k} = \frac{b_S - 3}{\sqrt{\dfrac{96}{pN}}}$$

is asymptotically distributed as $\mathrm{N}(0,1)$.

Theorem 3.5 (Seo and Ariga (2009)) *Let b_S be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is known. Then*

$$z_{S,k}^* = \frac{b_S - \left(3 - \dfrac{6}{N} + \dfrac{3}{N^2}\right)}{\sqrt{\dfrac{96}{pN} - \dfrac{312}{pN^2} + \dfrac{360}{pN^3}}}$$

is asymptotically distributed as $\mathrm{N}(0,1)$.

Next, we consider asymptotic expansions of first, second and third moments for b_S when Σ is unknown. First we note that

$$\omega_i = \boldsymbol{h}_i' S \boldsymbol{h}_i = \frac{1}{N}\sum_{j=1}^{N}\left(y_{ij}-\overline{y}_i\right)^2, \quad i=1,2,\ldots,p.$$

Then since S is defined as the maximum likelihood estimator, we can write

$$b_S = \frac{1}{p}\sum_{i=1}^{p}\frac{m_{4i}}{m_{2i}^2},$$

where

$$m_{4i} = \frac{1}{N}\sum_{j=1}^{N}\left(y_{ij}-\overline{y}_i\right)^4, \quad m_{2i}^2 = \left\{\frac{1}{N}\sum_{j=1}^{N}\left(y_{i\alpha}-\overline{y}_i\right)^2\right\}^2.$$

Under normality, $y_{i1}, y_{i2}, \ldots, y_{iN}$ are independently normally distributed. By Srivastava (1984), for large N, we note that

$$\mathrm{E}\left[\frac{m_{\nu i}^k}{m_{2i}^{\nu k/2}}\right] = \frac{\mathrm{E}\left[m_{\nu i}^k\right]}{\mathrm{E}\left[m_{2i}^{\nu k/2}\right]}.$$

Hence, for large N,

$$\mathrm{E}\left[b_S\right] = \frac{\mathrm{E}\left[m_{4i}\right]}{\mathrm{E}\left[m_{2i}^2\right]}.$$

Since $\mathrm{E}\left[m_{4i}\right]$ is given (3) and we can obtain $\mathrm{E}\left[m_{2i}^2\right]$ as

$$\mathrm{E}\left[m_{2i}^2\right] = 1 - \frac{1}{N^2} \tag{4}$$

in a similar way to derive $\mathrm{E}\left[m_{4i}\right]$, for large N, the expectation for the kurtosis b_S is given by

$$\begin{aligned}\mathrm{E}\left[b_S\right] &= \frac{3\left(N-1\right)}{N+1} \qquad (5)\\ &= 3 - \frac{6}{N} + \frac{6}{N^2} - \frac{6}{N^3} + O\left(N^{-4}\right).\end{aligned}$$

Similarly, we may obtain the variance $\mathrm{Var}\left[b_S\right] = \mathrm{E}\left[b_S^2\right] - \left\{\mathrm{E}\left[b_S\right]\right\}^2$. First, we note that

$$\mathrm{E}\left[b_S^2\right] = \frac{1}{p}\mathrm{E}\left[\frac{m_{4i}^2}{m_{2i}^4}\right] + \left(1-\frac{1}{p}\right)\mathrm{E}\left[\frac{m_{4i}}{m_{2i}^2}\right]\mathrm{E}\left[\frac{m_{4j}}{m_{2j}^2}\right].$$

For large N,

$$\mathrm{Var}\left[b_S\right] = \frac{1}{p}\left\{\frac{\mathrm{E}\left[m_{4i}^2\right]}{\mathrm{E}\left[m_{2i}^4\right]} - \left(\frac{\mathrm{E}\left[m_{4i}\right]}{\mathrm{E}\left[m_{2i}^2\right]}\right)^2\right\}.$$

Since $\mathrm{E}\left[m_{4i}\right]$, $\mathrm{E}\left[m_{2i}^2\right]$ and $\mathrm{E}\left[m_{4i}^2\right]$ are given by (3),(4) and (5), and we can obtain $\mathrm{E}\left[m_{2i}^4\right]$ as

$$\mathrm{E}\left[m_{2i}^4\right] = \frac{\left(N-1\right)^9}{N^{16}}\left(N^3+9N^2+23N+15\right)$$

in a similar way to derive $\mathrm{E}\left[m_{4i}\right]$, we have

$$\begin{aligned}\mathrm{Var}\left[b_S\right] &= \frac{24N\left(N^2-5N+6\right)}{p\left(N+1\right)^2\left(N^2+8N+15\right)}\\ &= \frac{24}{pN}-\frac{360}{pN^2}+\frac{2976}{pN^3}+O\left(N^{-4}\right).\end{aligned}$$

Similarly, after a great deal of calculation for the expectations, for large N, we obtain $\mathrm{E}\left[b_S^3\right]$ as

$$\begin{aligned}\mathrm{E}\left[b_S^3\right] = 27 &- \frac{54}{pN}\left(3p-4\right)+\frac{54}{p^2N^2}\left(9p^2-68p+32\right)\\ &+\frac{54}{p^2N^3}\left(19p^2-624p+734\right)+O\left(N^{-4}\right).\end{aligned}$$

With the use of these results, the following theorem is given.

Theorem 3.6 (Seo and Ariga (2009)) *Let b_S be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p\left(\boldsymbol{\mu},\Sigma\right)$ where Σ is unknown. Then*

$$z_S^* = \frac{b_S-\left(3-\dfrac{6}{N}+\dfrac{6}{N^2}-\dfrac{6}{N^3}\right)}{\sqrt{\dfrac{24}{pN}-\dfrac{360}{pN^2}+\dfrac{2976}{pN^3}}}$$

is asymptotically distributed as $\mathrm{N}\left(0,1\right)$.

4 Normalizing transformation

In this section, we drive the normalizing transformational statistics for b_M and b_S. The method of normalizing transformation for some statistics in multivariate analysis has been discussed by Konishi (1981), and Seo, Kanda and Fujikoshi (1994) and so on.

4.1 Normalizing transformation for b_M

Let $Y_M=\sqrt{N}\left(b_M-\beta_M\right)$. Then the distribution function for Y_M can be expanded as

$$\Pr\left[\frac{\sqrt{N}\left(b_M-\beta_M\right)}{\sigma}\le y_M\right]=\Phi\left(y_M\right)-\frac{1}{\sqrt{N}}\left\{\frac{a_1}{\sigma}\Phi^{(1)}\left(y\right)+\frac{a_3}{\sigma^3}\Phi^{(3)}\left(y_M\right)\right\}+O\left(N^{-1}\right),$$

where a_1,σ^2, and a_3 are coefficients of the first three cumulants of Y_M;

$$\begin{aligned}\kappa_1\left(Y_M\right) &= \frac{a_1}{\sqrt{N}}+O\left(N^{-\frac{3}{2}}\right),\\ \kappa_2\left(Y_M\right) &= \sigma^2+O\left(N^{-1}\right),\\ \kappa_3\left(Y_M\right) &= \frac{6}{\sqrt{N}}a_3+O\left(N^{-\frac{3}{2}}\right).\end{aligned}$$

Further, if we put $f\left(b_M\right)$ to fill following differential equation;

$$\frac{a_3}{\sigma^3}+\frac{\sigma f'\left(b_M\right)}{2f''\left(b_M\right)}=0,$$

we can obtain

$$\Pr\left[\frac{\sqrt{N}\left\{f\left(b_M\right)-f\left(\beta_M\right)-c/N\right\}}{f'\left(\beta_M\right)\sigma}\leq y_M\right]=\Phi\left(y\right)+O\left(N^{-1}\right),$$

where

$$c=\left(a_1-\frac{a_3}{\sigma^2}\right)f'\left(\beta_M\right).$$

Assuming Σ is known, $a_{1,\mathrm{k}}, \sigma^2_{\mathrm{k}}, a_{3,\mathrm{k}}, f_{\mathrm{k}}\left(b_M\right)$ and c_{k} are given by

$$a_{1,\mathrm{k}}=-2p\left(p+2\right),\ \ \sigma^2_{\mathrm{k}}=8p\left(p+2\right)\left(p+3\right),\ \ a_{3,\mathrm{k}}=\frac{16}{3}p\left(p+2\right)\left(5p^2+34p+60\right),$$

$$\begin{aligned}f_{\mathrm{k}}\left(b_M\right)&=-\frac{6p\left(p+2\right)\left(p+3\right)^2}{5p^2+34p+60}\exp\left[-\frac{5p^2+34p+60}{6p\left(p+2\right)\left(p+3\right)^2}b_M\right],\\ c_{\mathrm{k}}&=-\frac{2\left(3p^3+20p^2+52p+60\right)}{3p\left(p+3\right)}\exp\left[-\frac{5p^2+34p+60}{6\left(p+3\right)^2}\right].\end{aligned}$$

Therefore we can obtain the following theorem;

Theorem 4.1 *Let b_M be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p\left(\boldsymbol{\mu},\Sigma\right)$ where Σ is known. Then*

$$z_{MNT,k}=\frac{\sqrt{N}\left\{d_k\cdot\exp\left[\frac{b_M}{d_k}\right]-d_k\cdot\exp\left[\frac{p\left(p+2\right)}{d_k}\right]-\frac{c_k}{N}\right\}}{\sqrt{8p\left(p+2\right)\left(p+3\right)}\exp\left[\frac{p\left(p+2\right)}{d_k}\right]}$$

is asymptotically distributed as $N(0,1)$, where

$$c_k=-\frac{2\left(3p^3+20p^2+52p+60\right)}{3p\left(p+3\right)}\exp\left[-\frac{5p^2+34p+60}{6\left(p+3\right)^2}\right]$$

and

$$d_k=-\frac{6p\left(p+2\right)\left(p+3\right)^2}{5p^2+34p+60}.$$

Next, we consider the case when Σ is unknown. Then $a_1, \sigma^2, a_3, f\left(b_M\right)$ and c_1 are given by

$$a_1=-2p\left(p+2\right),\ \ \sigma^2=8p\left(p+2\right),\ \ a_3=64p\left(p+2\right)\left(p+8\right),$$

$$\begin{aligned}f\left(b_M\right)&=\frac{3p\left(p+2\right)}{p+8}\exp\left[-\frac{p+8}{3p\left(p+2\right)}\right],\\ c&=-\frac{2}{3}\left(3p^2+8p+16\right)\exp\left[-\frac{p+8}{3}\right].\end{aligned}$$

Therefore we can obtain the following theorem;

Theorem 4.2 *Let b_M be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is unknown. Then*

$$z_{MNT} = \frac{\sqrt{N}\left\{d \cdot \exp\left[\frac{b_M}{d}\right] - d \cdot \exp\left[\frac{p(p+2)}{d}\right] - \frac{c}{N}\right\}}{\sqrt{8p(p+2)}\exp\left[\frac{p(p+2)}{d}\right]}$$

is asymptotically distributed as $N(0,1)$, where

$$d = -\frac{3p(p+2)}{p+8}$$

and

$$c = -\frac{2}{3}\left(3p^2 + 8p + 16\right)\exp\left[-\frac{p+8}{3}\right].$$

4.2 Normalizing transformation for b_S

The normalizing transformation for b_M was discussed by Seo and Ariga (2006, 2009). Let $Y_S = \sqrt{N}(b_S - \beta_S)$. Then the distribution function for Y_S can be expanded as

$$\Pr\left[\frac{\sqrt{N}(b_S - \beta_S)}{\sigma} \leq y_S\right] = \Phi(y_S) - \frac{1}{\sqrt{N}}\left\{\frac{\tilde{a}_1}{\tilde{\sigma}}\Phi^{(1)}(y_S) + \frac{\tilde{a}_3}{\tilde{\sigma}^3}\Phi^{(3)}(y_S)\right\} + O\left(N^{-1}\right),$$

where $\tilde{a}_1, \tilde{\sigma}^2$, and $\tilde{a}_3$ are coefficients of the first three cumulants of Y_S. Further, if we put $g(b_S)$ to fill following differential equation;

$$\frac{\tilde{a}_3}{\tilde{\sigma}^3} + \frac{\tilde{\sigma}g'(b_S)}{2g''(b_S)} = 0,$$

we can obtain

$$\Pr\left[\frac{\sqrt{N}\{g(b_S) - g(\beta_S) - \tilde{c}/N\}}{g'(\beta_S)\tilde{\sigma}} \leq y_S\right] = \Phi(y_S) + O\left(N^{-1}\right),$$

where

$$\tilde{c} = \left(\tilde{a}_1 - \frac{\tilde{a}_3}{\tilde{\sigma}^2}\right)g'(\beta_S).$$

Assuming Σ is known, $\tilde{a}_{1,\mathrm{k}}, \tilde{\sigma}^2_{\mathrm{k}}, \tilde{a}_{3,\mathrm{k}}, g_{\mathrm{k}}(b_S)$ and $\tilde{c}_{\mathrm{k}}$ are given by

$$\tilde{a}_{1,\mathrm{k}} = -6,\ \tilde{\sigma}^2_{\mathrm{k}} = \frac{96}{p},\ \tilde{a}_{3,\mathrm{k}} = \frac{1584}{p^2},$$

$$g_{\mathrm{k}}(b_S) = -\frac{32}{11}\exp\left[-\frac{11}{32}b_S\right],$$

$$\tilde{c}_{\mathrm{k}} = -\left(6 + \frac{33}{2p}\right)\exp\left[-\frac{33}{32}\right].$$

Therefore the following theorem is given.

Theorem 4.3 (Seo and Ariga (2006)) *Let b_S be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is known. Then*

$$z_{SNT,\mathrm{k}} = \frac{\sqrt{pN}\left\{-\frac{32}{11}\exp\left[-\frac{11}{32}b_S\right] + \frac{32}{11}\exp\left[-\frac{33}{32}\right] - \frac{\tilde{c}_\mathrm{k}}{N}\right\}}{\sqrt{96}\exp\left[-\frac{33}{32}\right]}$$

is asymptotically distributed as $\mathrm{N}(0,1)$*, where*

$$\tilde{c}_k = -\left(6 + \frac{33}{2p}\right)\exp\left[-\frac{33}{32}\right].$$

Next, we consider the case when Σ is unknown. Then $\tilde{a}_1, \tilde{\sigma}^2, a_3, g(b_S)$ and $\tilde{c}$ are given by

$$\tilde{a}_1 = -6,\ \tilde{\sigma}^2 = \frac{24}{p},\ \tilde{a}_3 = \frac{288}{p^2},$$

$$g(b_S) = -\exp[-b_S],$$

$$\tilde{c} = -6e^{-3}\left(1 + \frac{2}{p}\right).$$

Therefore the following theorem is given by Seo and Ariga (2009).

Theorem 4.4 (Seo and Ariga (2009)) *Let b_S be the multivariate sample measure of kurtosis based on a random sample of size N drawn from $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$ where Σ is unknown. Then*

$$z_{SNT} = \frac{\sqrt{pN}\left\{-\exp[-b_S] + e^{-3} + \frac{\tilde{c}}{N}\right\}}{\sqrt{24}e^{-3}}$$

is asymptotically distributed as $\mathrm{N}(0,1)$*, where*

$$\tilde{c} = -6e^{-3}\left(1 + \frac{2}{p}\right).$$

5 Simulation study

In this section, we investigate the accuracy of the normalizing transformational statistics and the standardized statistics for the multivariate sample measures of kurtosis to the normal distribution by Monte Carlo simulation for some selected values of parameters.

Computations are made for $p = 3, 5, 7, 10;\ N = 20, 50, 100, 200, 400$ for each cases when the population covariance matrix Σ is known and unknown for multivariate normal populations. Without any loss of generality, we may assume that $\Sigma = I$. Simulation results based on $1,000,000$ simulations.

Tables 1 $\sim$ 12 give the values of expectation, variance, skewness and kurtosis, and the $\alpha\,(= 0.95, 0.99)$ percentiles for $z_{M,\mathrm{k}}$, $z^*_{M,\mathrm{k}}$, $z_{MNT,\mathrm{k}}$, z_M, z^*_M, z_{MNT}, $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$, $z_{SNT,\mathrm{k}}$, z_S, z^*_S and z_{SNT} in theorems 2.1$\sim$4.4. Finally, some histograms for these statistics by simulation are given in Figures 1 $\sim$ 16. We note that $z_{M,\mathrm{k}}$, z_M, $z_{S,\mathrm{k}}$ and z_S are the standardized statistics by using the values of the limiting terms for the expectations and the variance, and $z^*_{M,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and z^*_S are the standardized statistics by using the values of the asymptotic expansions for the expectations and the variance derived in this paper, and z^*_M is the standardized statistic by using the exact values of the expectation and the variance derived in Mardia and Kanazawa (1983), and $z_{MNT,\mathrm{k}}$, z_{MNT}, $z_{SNT,\mathrm{k}}$ and z_{SNT} are the normalizing transformational statistics derived in this paper. From Table 1, the expectation and variance of $z^*_{M,\mathrm{k}}$ are closer to values of the normal distribution than those of $z_{M,\mathrm{k}}$, and those of $z_{MNT,\mathrm{k}}$ become closer with increasing p and N. Also, from Table 2, the skewness and kurtosis of $z_{MNT,\mathrm{k}}$ are closer to values of the normal distribution than those of $z_{M,\mathrm{k}}$ and $z^*_{M,\mathrm{k}}$. These considerations apply to z_M, z^*_M, z_{MNT}, $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$, $z_{SNT,\mathrm{k}}$, z_S, z^*_S and z_{SNT} from Tables 4, 5, 7, 8, 10 and 11. Comparing $z_{MNT,\mathrm{k}}$, z_{MNT}, $z_{SNT,\mathrm{k}}$ and z_{SNT}, we can confirm that when N is not large to the p enough, the approximation accuracy of $z_{M,\mathrm{k}}$ and z_{MNT} are remarkably low when N is not large to p, and those of $z_{SNT,\mathrm{k}}$ and z_{SNT} is steady. Also, from Tables 3, 6, 9 and 12, it may be noted that the α percentiles of the normalizing transformational statistics become closer to values of the normal distribution with p and N, and the α percentiles of normalizing transformational statistics are closer than those of the standardized statistics. The above-mentioned considerations can be confirmed from Figures 1 $\sim$ 16.

In conclusion, it may be noted that the normalizing transformational statistics are considerably good normal approximation and are useful for multivariate normality test. However, when we treat the data with not large sample size to the number of dimension, it is necessary to note it.

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Table 1: Expectation and variance for $z_{M,\mathrm{k}}$, $z^*_{M,\mathrm{k}}$ and $z_{MNT,\mathrm{k}}$

p=3	Expectation			Variance		
N	$z_{M,\mathrm{k}}$	$z^*_{M,\mathrm{k}}$	$z_{MNT,\mathrm{k}}$	$z_{M,\mathrm{k}}$	$z^*_{M,\mathrm{k}}$	$z_{MNT,\mathrm{k}}$
20	−0.2437	0.0001	−0.2709	0.8507	1.0107	0.8525
50	−0.1558	0.0008	−0.1788	0.9361	0.9994	0.9319
100	−0.1126	−0.0014	−0.1307	0.9688	1.0005	0.9657
200	−0.0779	0.0010	−0.0913	0.9851	1.0010	0.9823
400	−0.0561	−0.0002	−0.0658	0.9923	1.0002	0.9908
p=5						
20	−0.3210	0.0016	−0.3657	0.8552	1.0136	0.9183
50	−0.2069	0.0002	−0.2351	0.9359	0.9982	0.9624
100	−0.1459	0.0013	−0.1661	0.9684	0.9996	0.9812
200	−0.1043	0.0000	−0.1187	0.9843	0.9999	0.9907
400	−0.0733	0.0005	−0.0835	0.9929	1.0007	0.9959
p=7						
20	−0.3866	0.0003	−0.4338	0.8543	1.0110	0.9458
50	−0.2486	−0.0001	−0.2761	0.9398	1.0019	0.9793
100	−0.1761	0.0005	−0.1948	0.9669	0.9978	0.9870
200	−0.1242	0.0010	−0.1374	0.9855	1.0010	0.9953
400	−0.0893	−0.0006	−0.0986	0.9942	1.0019	0.9993
p=10						
20	−0.4685	−0.0001	−0.5162	0.8554	1.0109	0.9681
50	−0.3012	−0.0005	−0.3267	0.9402	1.0019	0.9884
100	−0.2147	−0.0009	−0.2314	0.9664	0.9971	0.9912
200	−0.1508	0.0008	−0.1623	0.9870	1.0024	0.9994
400	−0.1066	0.0006	−0.1146	0.9921	0.9997	0.9983

Table 2: Skewness and kurtosis for $z_{M,\mathrm{k}}$, $z^*_{M,\mathrm{k}}$ and $z_{MNT,\mathrm{k}}$

p=3	Skewness		Kurtosis	
N	$z_{M,\mathrm{k}}, z^*_{M,\mathrm{k}}$	$z_{MNT,\mathrm{k}}$	$z_{M,\mathrm{k}}, z^*_{M,\mathrm{k}}$	$z_{MNT,\mathrm{k}}$
20	1.1664	0.0524	5.5489	2.6170
50	0.7267	0.0043	3.9808	2.8215
100	0.5131	0.0004	3.4797	2.9000
200	0.3673	0.0018	3.2604	2.9550
400	0.2602	0.0025	3.1288	2.9777
p=5				
20	0.8535	0.0566	4.3149	2.7537
50	0.5348	0.0124	3.5299	2.8995
100	0.3791	0.0064	3.2681	2.9448
200	0.2691	0.0056	3.1262	2.9666
400	0.1920	0.0048	3.0658	2.9845
p=7				
20	0.6966	0.0516	3.8707	2.8210
50	0.4355	0.0116	3.3448	2.9279
100	0.3059	0.0027	3.1750	2.9692
200	0.2206	0.0041	3.0992	2.9904
400	0.1547	0.0019	3.0448	2.9911
p=10				
20	0.5566	0.0419	3.5231	2.8680
50	0.3529	0.0132	3.2072	2.9392
100	0.2477	0.0040	3.1033	2.9692
200	0.1760	0.0018	3.0539	2.9842
400	0.1227	−0.0011	3.0321	2.9985

Table 3: Percentile for $z_{M,\mathrm{k}}, z^*_{M,\mathrm{k}}$ and $z_{MNT,\mathrm{k}}$ ($\alpha = 0.95, 0.99$)

p=3	$z\,(0.95) = 1.645$			$z\,(0.99) = 2.326$		
N	$z_{M,\mathrm{k}}$	$z^*_{M,\mathrm{k}}$	$z_{MNT,\mathrm{k}}$	$z_{M,\mathrm{k}}$	$z^*_{M,\mathrm{k}}$	$z_{MNT,\mathrm{k}}$
20	1.478	1.877	1.276	2.639	3.142	1.807
50	1.595	1.810	1.416	2.607	2.856	2.026
100	1.631	1.770	1.488	2.548	2.703	2.130
200	1.649	1.741	1.541	2.499	2.598	2.202
400	1.650	1.712	1.571	2.453	2.519	2.245
p=5						
20	1.374	1.847	1.235	2.390	2.953	1.837
50	1.511	1.775	1.386	2.424	2.718	2.031
100	1.568	1.743	1.466	2.418	2.607	2.129
200	1.601	1.718	1.522	2.400	2.524	2.192
400	1.616	1.697	1.559	2.390	2.473	2.241
p=7						
20	1.284	1.818	1.186	2.229	2.845	1.820
50	1.452	1.756	1.356	2.321	2.653	2.019
100	1.552	1.726	1.442	2.337	2.553	2.112
200	1.567	1.706	1.505	2.350	2.494	2.185
400	1.592	1.687	1.545	2.342	2.441	2.224
p=10						
20	1.182	1.794	1.119	2.059	2.748	1.770
50	1.383	1.738	1.313	2.213	2.595	1.987
100	1.471	1.711	1.411	2.247	2.500	2.077
200	1.529	1.693	1.480	2.290	2.460	2.161
400	1.563	1.677	1.527	2.307	2.423	2.213

Table 4: Expectation and variance for z_M, z_M^* and z_{MNT}

p=3	Expectation			Variance		
N	z_M	z_M^*	z_{MNT}	z_M	z_M^*	z_{MNT}
20	−0.5841	−0.0014	0.0588	0.3868	1.0018	0.7051
50	−0.3807	−0.0012	0.0256	0.6857	1.0019	0.8331
100	−0.2709	0.0003	0.0121	0.8282	1.0012	0.8952
200	−0.1920	0.0007	0.0056	0.9084	0.9988	0.9377
400	−0.1353	0.0013	0.0029	0.9565	1.0030	0.9688
p=5						
20	−0.8922	−0.0025	−0.0392	0.2998	0.9981	0.6456
50	−0.5796	0.0005	0.0011	0.6269	0.9994	0.8310
100	−0.4140	0.0002	0.0022	0.7924	0.9988	0.9031
200	−0.2937	0.0007	0.0016	0.8928	1.0019	0.9478
400	−0.2086	0.0001	0.0005	0.9448	1.0008	0.9726
p=7						
20	−1.1949	0.0008	−0.1468	0.2257	1.0022	0.5583
50	−0.7783	−0.0002	−0.0257	0.5715	0.9977	0.8100
100	−0.5560	−0.0003	−0.0076	0.7598	0.9993	0.8980
200	−0.3946	0.0003	−0.0021	0.8745	1.0019	0.9482
400	−0.2796	0.0003	−0.0003	0.9327	0.9981	0.9701
p=10						
20	−1.6499	−0.0008	−0.3604	0.1328	1.0004	0.4007
50	−1.0743	−0.0004	−0.0738	0.4957	1.0001	0.7686
100	−0.7675	−0.0007	−0.0242	0.7105	0.9978	0.8807
200	−0.5439	0.0012	−0.0069	0.8497	1.0044	0.9435
400	−0.3882	−0.0019	−0.0045	0.9167	0.9961	0.9660

Table 5: Skewness and kurtosis for z_M, z_M^* and z_{MNT}

p=3	Skewness		Kurtosis	
N	z_M, z_M^*	z_{MNT}	z_M, z_M^*	z_{MNT}
20	0.8549	−0.2081	4.2278	2.7464
50	0.8363	−0.1548	4.4588	2.8711
100	0.6918	−0.0963	4.0954	2.9446
200	0.5251	−0.0507	3.6193	2.9800
400	0.3862	−0.0227	3.3519	3.0035
p=5				
20	0.6171	−0.0990	3.5544	2.7500
50	0.6231	−0.0877	3.7597	2.8834
100	0.5176	−0.0566	3.5714	2.9497
200	0.4077	−0.0214	3.3672	2.9790
400	0.2971	−0.0119	3.2034	2.9970
p=7				
20	0.5292	−0.0098	3.3715	2.7747
50	0.5267	−0.0487	3.5089	2.8912
100	0.4435	−0.0331	3.4000	2.9493
200	0.3409	−0.0205	3.2514	2.9864
400	0.2543	−0.0072	3.1379	2.9938
p=10				
20	0.4973	0.1292	3.3269	2.8428
50	0.4451	−0.0151	3.3330	2.8947
100	0.3782	−0.0172	3.2704	2.9481
200	0.2976	−0.0087	3.1770	2.9764
400	0.2170	−0.0069	3.0943	2.9885

Table 6: Percentile for z_M, z_M^* and z_{MNT} $(\alpha = 0.95, 0.99)$

p=3	$z\,(0.95) = 1.645$			$z\,(0.99) = 2.326$		
N	z_M	z_M^*	z_{MNT}	z_M	z_M^*	z_{MNT}
20	0.562	1.843	1.389	1.251	2.952	1.792
50	1.121	1.814	1.490	2.053	2.941	2.004
100	1.355	1.788	1.544	2.307	2.834	2.129
200	1.490	1.764	1.586	2.397	2.716	2.215
400	1.563	1.746	1.614	2.425	2.623	2.275
p=5						
20	0.095	1.799	1.260	0.623	2.762	1.708
50	0.835	1.786	1.478	1.625	2.784	2.031
100	1.158	1.765	1.552	2.004	2.715	2.162
200	1.352	1.744	1.596	2.198	2.641	2.249
400	1.463	1.721	1.619	2.271	2.552	2.284
p=7						
20	−0.347	1.787	1.082	0.087	2.701	1.540
50	0.563	1.773	1.445	1.274	2.711	2.009
100	0.971	1.751	1.542	1.762	2.657	2.162
200	1.224	1.733	1.592	2.022	2.587	2.245
400	1.376	1.713	1.620	2.148	2.512	2.280
p=10						
20	−1.003	1.774	0.706	−0.673	2.681	1.148
50	0.164	1.757	1.365	0.793	2.623	1.933
100	0.700	1.738	1.515	1.429	2.603	2.134
200	1.043	1.726	1.588	1.805	2.555	2.240
400	1.242	1.697	1.620	1.998	2.486	2.277

Table 7: Expectation and variance for $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and $z_{SNT,\mathrm{k}}$

p=3	Expectation			Variance		
N	$z_{S,\mathrm{k}}$	$z^*_{S,\mathrm{k}}$	$z_{SNT,\mathrm{k}}$	$z_{S,\mathrm{k}}$	$z^*_{S,\mathrm{k}}$	$z_{SNT,\mathrm{k}}$
20	−0.2328	−0.0017	0.0293	0.8417	0.9939	0.8302
50	−0.1494	−0.0009	0.0078	0.9352	0.9986	0.9203
100	−0.1071	−0.0016	0.0015	0.9681	1.0002	0.9577
200	−0.0748	0.0000	0.0011	0.9850	1.0012	0.9783
400	−0.0534	−0.0004	0.0001	0.9902	0.9983	0.9868
p=5						
20	−0.2992	−0.0007	0.0130	0.8491	1.0026	0.8991
50	−0.1908	0.0009	0.0051	0.9344	0.9977	0.9520
100	−0.1369	−0.0006	0.0009	0.9666	0.9987	0.9748
200	−0.0987	−0.0022	−0.0017	0.9845	1.0007	0.9887
400	−0.0691	−0.0008	−0.0006	0.9918	0.9999	0.9940
p=7						
20	−0.3534	−0.0002	0.0069	0.8436	0.9962	0.9296
50	−0.2265	0.0004	0.0022	0.9369	1.0004	0.9715
100	−0.1623	−0.0011	−0.0006	0.9693	1.0015	0.9872
200	−0.1140	0.0003	0.0006	0.9828	0.9990	0.9913
400	−0.0803	0.0006	0.0007	0.9939	1.0020	0.9981
p=10						
20	−0.4209	0.0014	0.0021	0.8479	1.0012	0.9584
50	−0.2709	0.0002	0.0003	0.9370	1.0005	0.9838
100	−0.1918	0.0008	0.0009	0.9678	1.0000	0.9916
200	−0.1375	−0.0009	−0.0010	0.9850	1.0012	0.9973
400	−0.0966	0.0002	0.0002	0.9904	0.9985	0.9964

Table 8: Skewness and kurtosis for $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and $z_{SNT,\mathrm{k}}$

p=3	Skewness		Kurtosis	
N	$z_{S,\mathrm{k}}, z^*_{S,\mathrm{k}}$	$z_{SNT,\mathrm{k}}$	$z_{S,\mathrm{k}}, z^*_{S,\mathrm{k}}$	$z_{SNT,\mathrm{k}}$
20	1.3066	0.0183	6.3465	2.5801
50	0.8233	−0.0152	4.3473	2.8023
100	0.5826	−0.0097	3.6723	2.8971
200	0.4133	−0.0029	3.3315	2.9459
400	0.2909	−0.0021	3.1653	2.9748
p=5				
20	1.0254	0.0268	5.0835	2.7192
50	0.6413	−0.0003	3.8252	2.8749
100	0.4547	−0.0009	3.4170	2.9430
200	0.3213	−0.0002	3.2077	2.9751
400	0.2230	−0.0029	3.0934	2.9828
p=7				
20	0.8457	0.0262	4.3819	2.7892
50	0.5381	−0.0003	3.5680	2.9180
100	0.3784	−0.0040	3.2858	2.9607
200	0.2696	−0.0007	3.1435	2.9821
400	0.1899	−0.0017	3.0752	2.9951
p=10				
20	0.7224	0.0314	4.0216	2.8558
50	0.4548	0.0063	3.4122	2.9460
100	0.3197	0.0007	3.2050	2.9787
200	0.2252	−0.0002	3.0973	2.9840
400	0.1607	0.0005	3.0589	3.0014

Table 9: Percentile for $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and $z_{SNT,\mathrm{k}}$ $(\alpha = 0.95, 0.99)$

p=3	$z\,(0.95) = 1.645$			$z\,(0.99) = 2.326$		
N	$z_{S,\mathrm{k}}$	$z^*_{S,\mathrm{k}}$	$z_{SNT,\mathrm{k}}$	$z_{S,\mathrm{k}}$	$z^*_{S,\mathrm{k}}$	$z_{SNT,\mathrm{k}}$
20	1.480	1.860	1.546	2.697	3.182	2.042
50	1.606	1.813	1.586	2.674	2.917	2.181
100	1.643	1.777	1.609	2.609	2.759	2.249
200	1.660	1.749	1.628	2.540	2.636	2.288
400	1.661	1.721	1.635	2.474	2.538	2.309
p=5						
20	1.400	1.846	1.591	2.496	3.036	2.163
50	1.540	1.790	1.613	2.508	2.789	2.243
100	1.587	1.752	1.624	2.480	2.659	2.282
200	1.616	1.727	1.635	2.444	2.562	2.302
400	1.628	1.703	1.638	2.409	2.488	2.310
p=7						
20	1.324	1.822	1.608	2.342	2.926	2.214
50	1.488	1.772	1.625	2.412	2.727	2.276
100	1.553	1.743	1.635	2.406	2.610	2.298
200	1.587	1.715	1.638	2.397	2.532	2.316
400	1.610	1.698	1.643	2.383	2.474	2.324
p=10						
20	1.240	1.806	1.625	2.200	2.849	2.265
50	1.429	1.757	1.635	2.312	2.669	2.302
100	1.508	1.729	1.640	2.332	2.556	2.312
200	1.554	1.705	1.642	2.335	2.492	2.316
400	1.584	1.687	1.642	2.338	2.445	2.322

Table 10: Expectation and variance for z_S, z_S^* and z_{SNT}

p=3	Expectation			Variance		
N	z_S	z_S^*	z_{SNT}	z_S	z_S^*	z_{SNT}
20	−0.4524	−0.0008	0.0813	0.4815	0.8598	0.7567
50	−0.2941	0.0000	0.0318	0.7418	0.9896	0.8462
100	−0.2103	−0.0003	0.0132	0.8609	0.9983	0.9001
200	−0.1502	−0.0010	0.0044	0.9269	0.9987	0.9402
400	−0.1057	0.0002	0.0022	0.9626	0.9993	0.9664
p=5						
20	−0.5841	−0.0011	0.0321	0.4832	0.8628	0.7911
50	−0.3804	−0.0008	0.0140	0.7451	0.9940	0.8797
100	−0.2708	0.0004	0.0068	0.8627	1.0003	0.9260
200	−0.1941	−0.0015	0.0014	0.9234	0.9949	0.9533
400	−0.1368	−0.0002	0.0009	0.9621	0.9988	0.9753
p=7						
20	−0.6900	0.0003	0.0055	0.4818	0.8603	0.8055
50	−0.4481	0.0014	0.0075	0.7441	0.9926	0.8946
100	−0.3219	−0.0012	0.0019	0.8608	0.9981	0.9370
200	−0.2272	0.0008	0.0020	0.9285	1.0004	0.9648
400	−0.1618	−0.0003	0.0002	0.9625	0.9991	0.9806
p=10						
20	−0.8242	0.0010	−0.0228	0.4830	0.8625	0.8197
50	−0.5368	0.0002	−0.0019	0.7421	0.9900	0.9045
100	−0.3837	−0.0003	−0.0004	0.8615	0.9990	0.9459
200	−0.2717	0.0008	0.0008	0.9300	1.0020	0.9723
400	−0.1953	−0.0022	−0.0020	0.9615	0.9981	0.9829

Table 11: Skewness and kurtosis for z_S, z_S^* and z_{SNT}

p=3	Skewness		Kurtosis	
N	z_S, z_S^*	z_{SNT}	z_S, z_S^*	z_{SNT}
20	1.0066	−0.2929	4.8734	2.8340
50	0.9096	−0.2009	4.7890	2.9076
100	0.7391	−0.1183	4.2715	2.9441
200	0.5585	−0.0629	3.7597	2.9914
400	0.4083	−0.0272	3.3975	3.0049
p=5				
20	0.7802	−0.2317	4.1059	2.8779
50	0.7117	−0.1340	4.1050	2.9341
100	0.5717	−0.0790	3.7599	2.9771
200	0.4274	−0.0394	3.4326	2.9949
400	0.3198	−0.0135	3.2538	3.0117
p=7				
20	0.6530	−0.2024	3.7744	2.9048
50	0.5919	−0.1162	3.7560	2.9561
100	0.4752	−0.0609	3.4958	2.9782
200	0.3668	−0.0266	3.3253	3.0076
400	0.2649	−0.0152	3.1817	3.0176
p=10				
20	0.5518	−0.1699	3.5721	2.9275
50	0.4996	−0.0871	3.5403	2.9667
100	0.4063	−0.0440	3.3964	3.0015
200	0.3068	−0.0194	3.2299	3.0073
400	0.2250	−0.0066	3.1215	3.0063

Table 12: Percentile for z_S, z_S^* and z_{SNT} $(\alpha = 0.95, 0.99)$

p=3	$z\,(0.95) = 1.645$			$z\,(0.99) = 2.326$		
N	z_S	z_S^*	z_{SNT}	z_S	z_S^*	z_{SNT}
20	0.830	1.713	1.436	1.668	2.833	1.821
50	1.269	1.806	1.495	2.269	2.960	1.991
100	1.449	1.787	1.542	2.448	2.862	2.120
200	1.546	1.760	1.580	2.491	2.741	2.212
400	1.603	1.741	1.611	2.481	2.636	2.269
p=5						
20	0.683	1.692	1.438	1.438	2.701	1.890
50	1.167	1.786	1.521	2.082	2.844	2.077
100	1.372	1.770	1.569	2.284	2.751	2.180
200	1.481	1.737	1.594	2.347	2.636	2.239
400	1.553	1.721	1.620	2.380	2.563	2.286
p=7						
20	0.560	1.669	1.431	1.266	2.613	1.917
50	1.084	1.771	1.533	1.942	2.762	2.112
100	1.309	1.755	1.579	2.169	2.681	2.203
200	1.442	1.734	1.609	2.282	2.606	2.268
400	1.515	1.709	1.622	2.321	2.529	2.297
p=10						
20	0.412	1.653	1.424	1.079	2.544	1.940
50	0.981	1.754	1.540	1.795	2.694	2.141
100	1.232	1.739	1.586	2.064	2.635	2.231
200	1.388	1.723	1.616	2.191	2.557	2.277
400	1.476	1.701	1.627	2.250	2.490	2.297

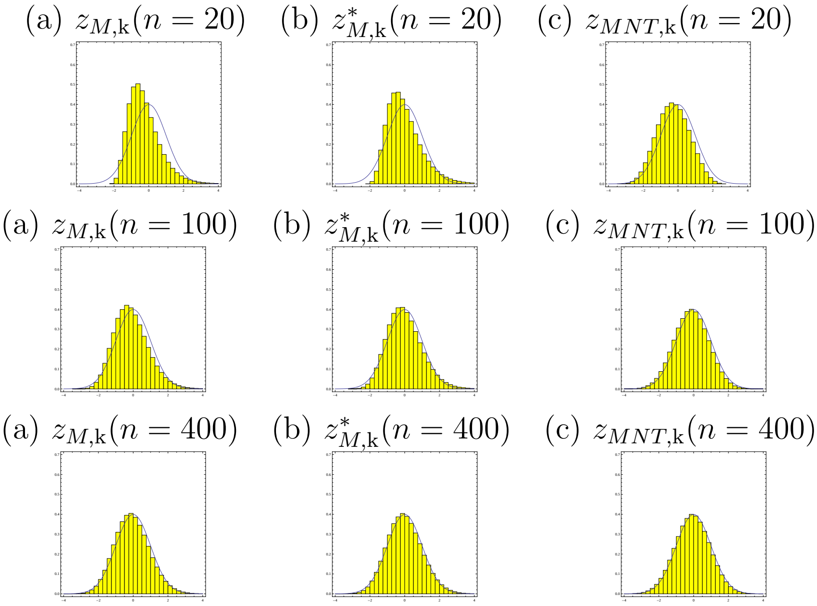

Figure 1: The sample distributions for $z_{M,\mathrm{k}}$, $z^*_{M,\mathrm{k}}$ and $z_{MNT,\mathrm{k}}$ by simulation, and the density plot for standard normal distribution($p=3$).

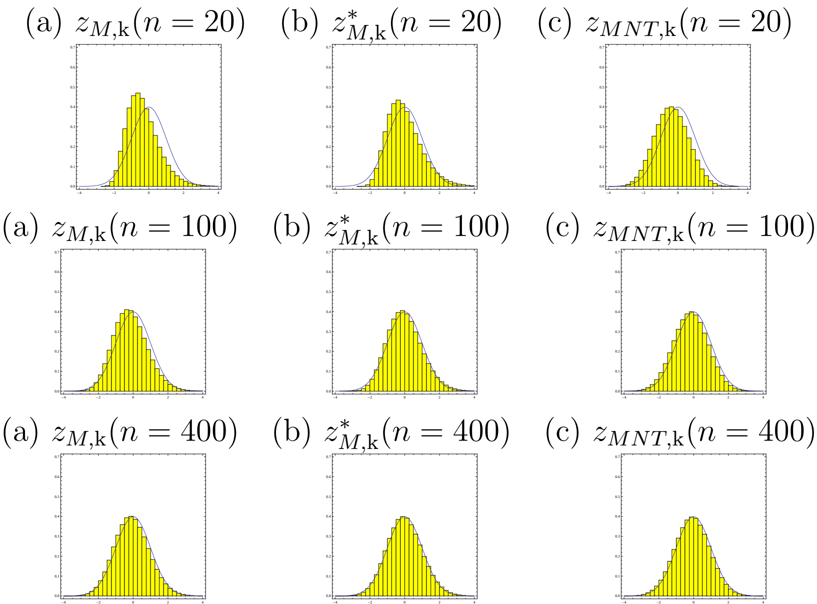

Figure 2: The sample distributions for $z_{M,\mathrm{k}}$, $z^*_{M,\mathrm{k}}$ and $z_{MNT,\mathrm{k}}$ by simulation, and the density plot for standard normal distribution($p=5$).

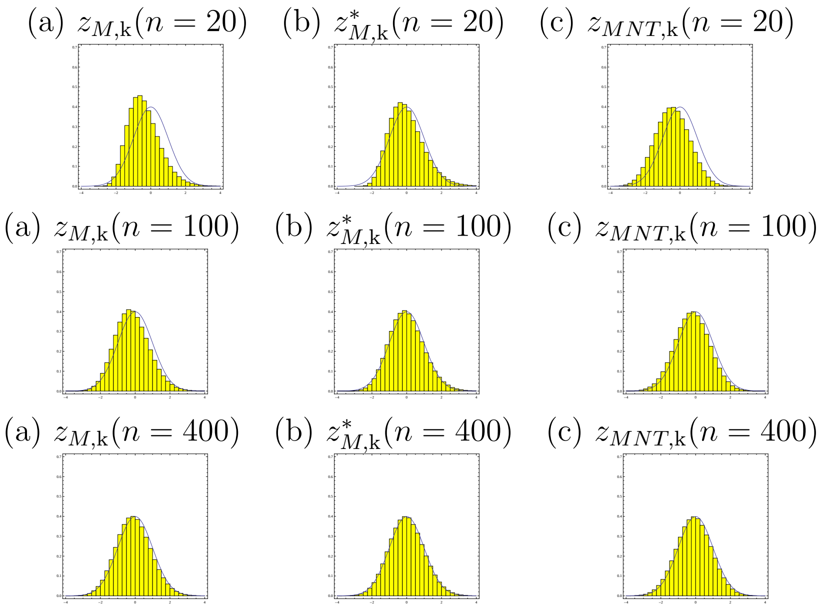

Figure 3: The sample distributions for $z_{M,\mathrm{k}}$, $z^*_{M,\mathrm{k}}$ and $z_{MNT,\mathrm{k}}$ by simulation, and the density plot for standard normal distribution($p = 7$).

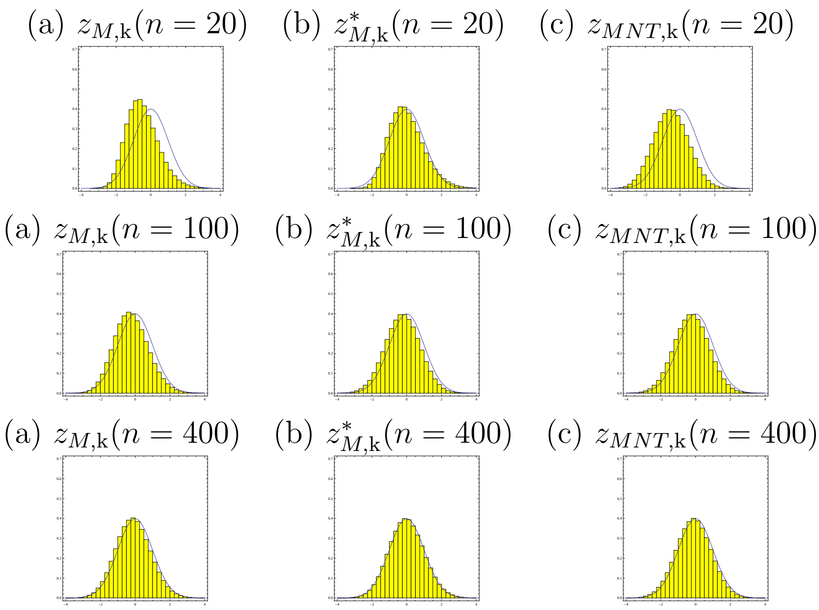

Figure 4: The sample distributions for $z_{M,\mathrm{k}}$, $z^*_{M,\mathrm{k}}$ and $z_{MNT,\mathrm{k}}$ by simulation, and the density plot for standard normal distribution($p = 10$).

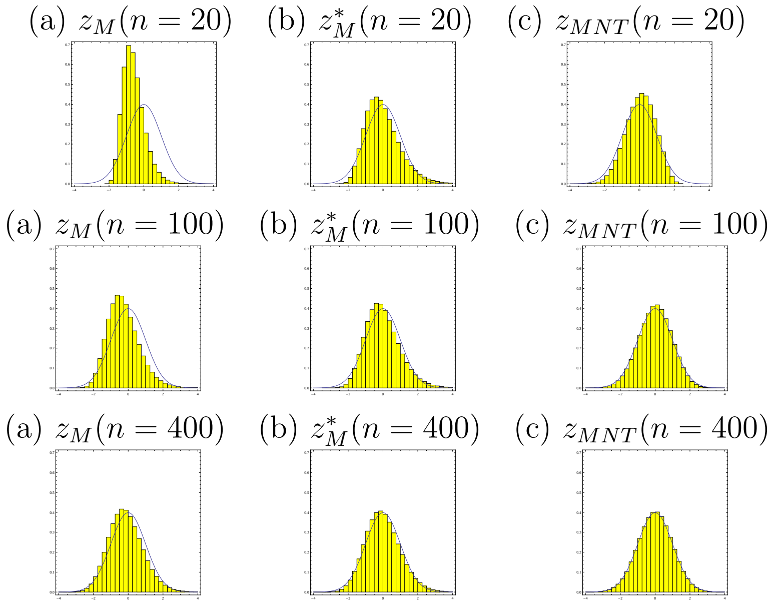

Figure 5: The sample distributions for z_M, z_M^* and z_{MNT} by simulation, and the density plot for standard normal distribution($p = 3$).

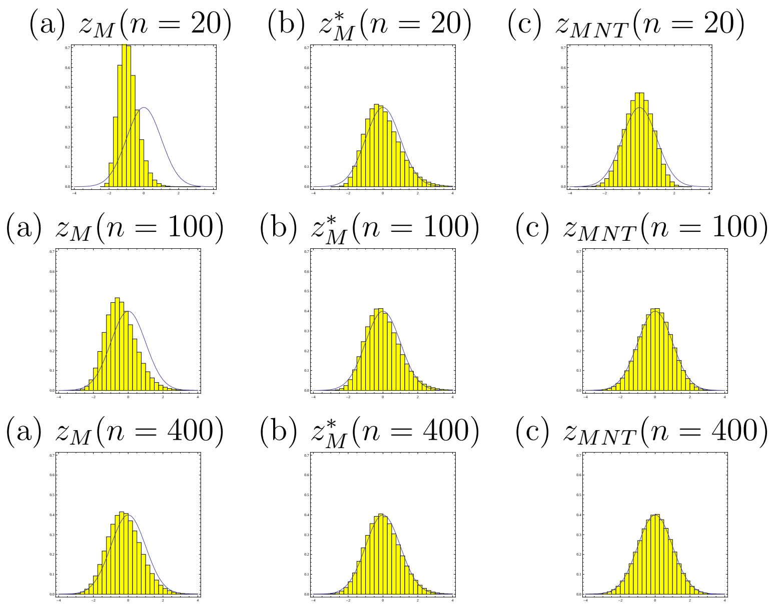

Figure 6: The sample distributions for z_M, z_M^* and z_{MNT} by simulation, and the density plot for standard normal distribution($p = 5$).

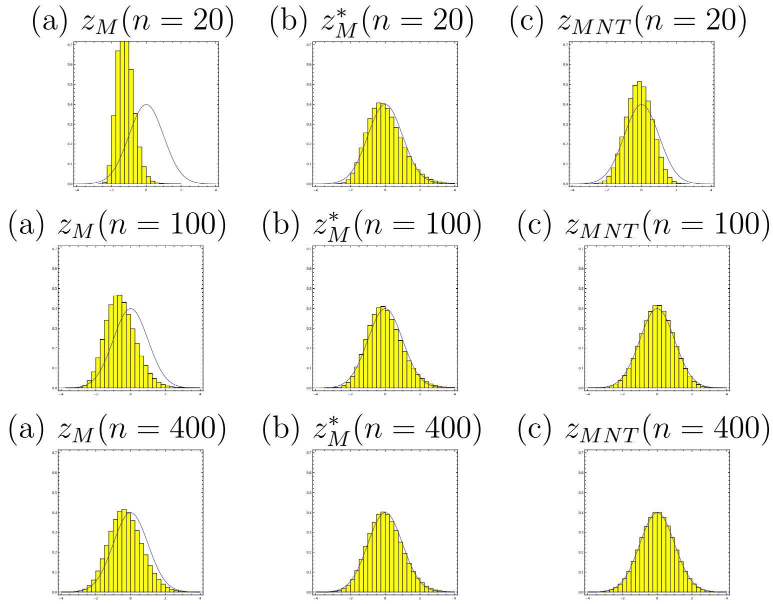

Figure 7: The sample distributions for z_M, z_M^* and z_{MNT} by simulation, and the density plot for standard normal distribution($p=7$).

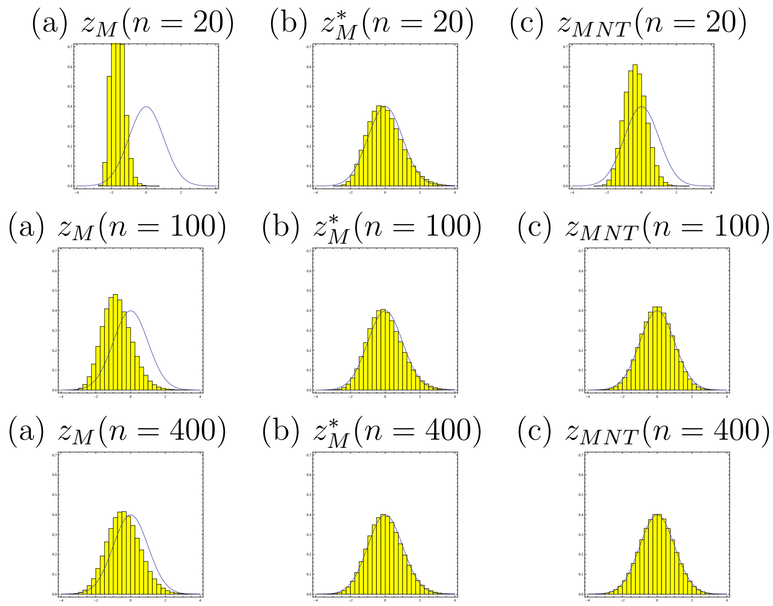

Figure 8: The sample distributions for z_M, z_M^* and z_{MNT} by simulation, and the density plot for standard normal distribution($p=10$).

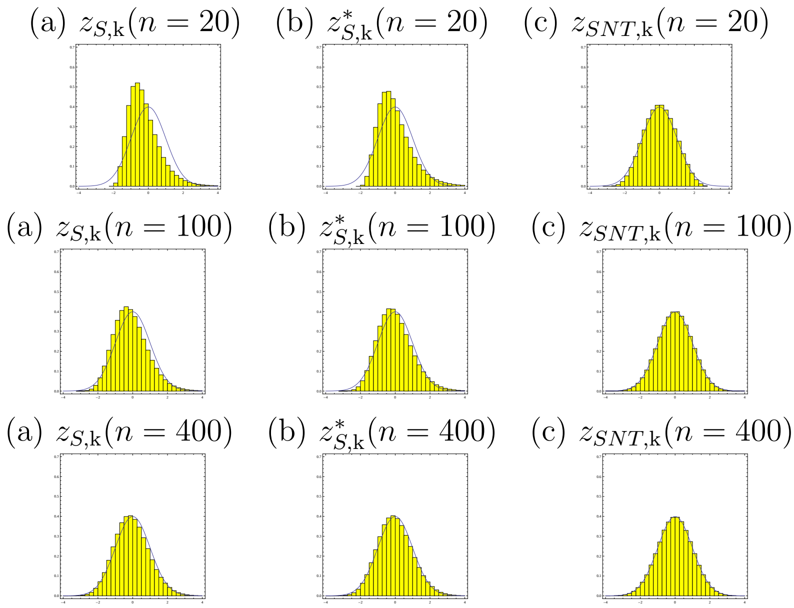

Figure 9: The sample distributions for $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and $z_{SNT,\mathrm{k}}$ by simulation, and the density plot for standard normal distribution($p = 3$).

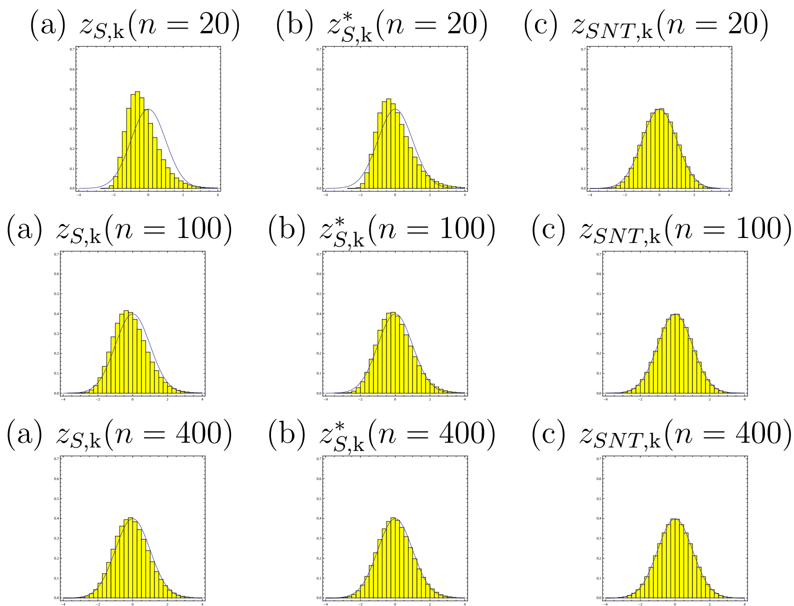

Figure 10: The sample distributions for $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and $z_{SNT,\mathrm{k}}$by simulation, and the density plot for standard normal distribution($p = 5$).

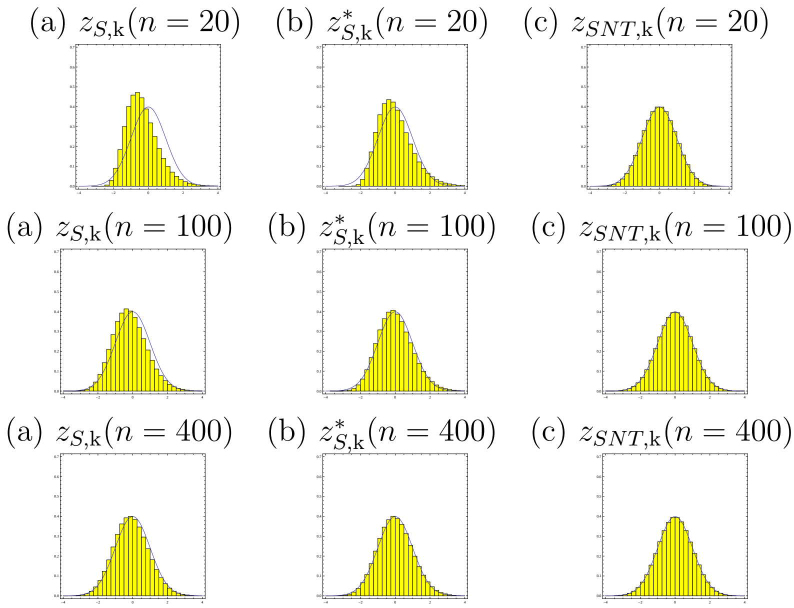

Figure 11: The sample distributions for $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and $z_{SNT,\mathrm{k}}$ by simulation, and the density plot for standard normal distribution($p = 7$).

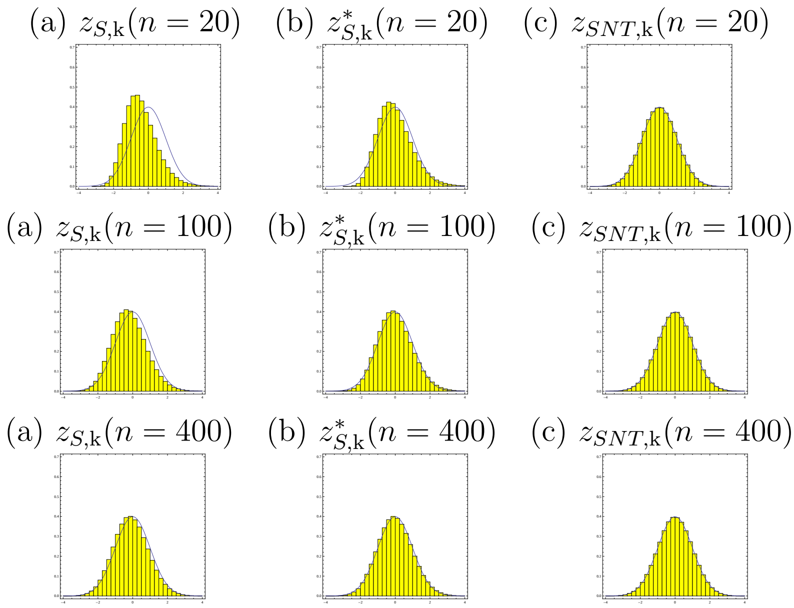

Figure 12: The sample distributions for $z_{S,\mathrm{k}}$, $z^*_{S,\mathrm{k}}$ and $z_{SNT,\mathrm{k}}$ by simulation, and the density plot for standard normal distribution($p = 10$).

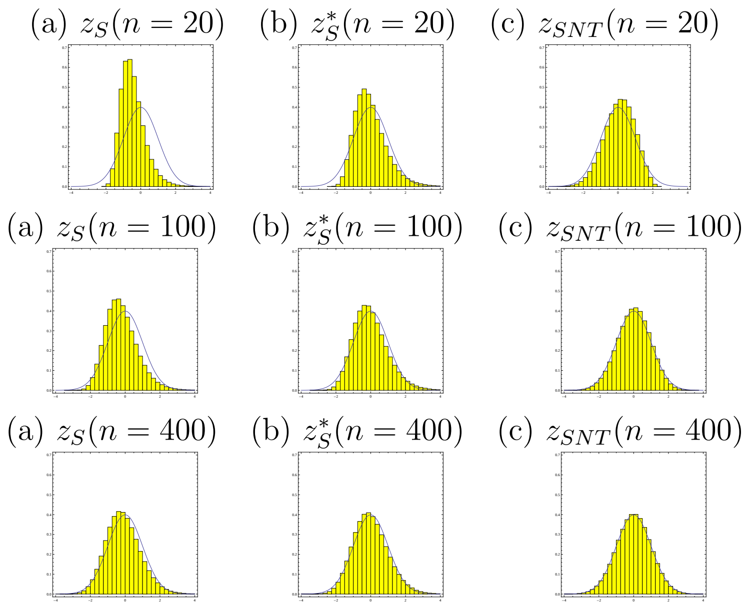

Figure 13: The sample distributions for z_S, z_S^* and z_{SNT} by simulation, and the density plot for standard normal distribution($p = 3$).

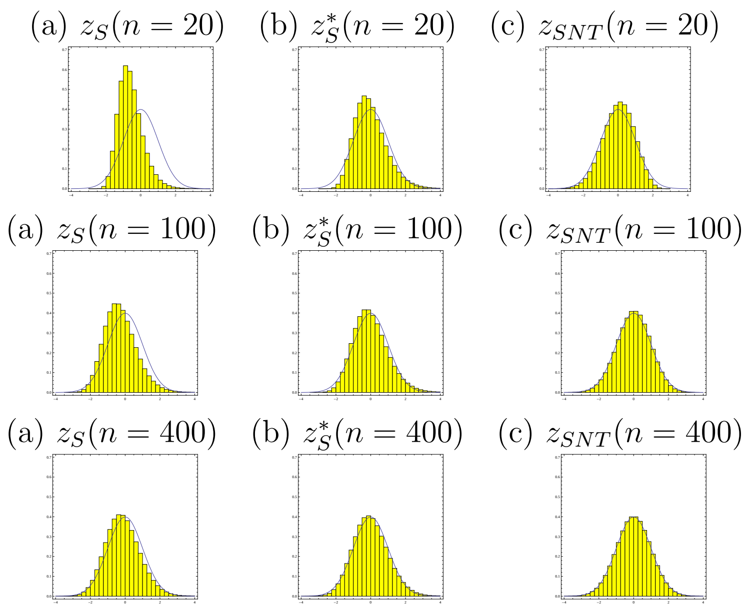

Figure 14: The sample distributions for z_S, z_S^* and z_{SNT} by simulation, and the density plot for standard normal distribution($p = 5$).

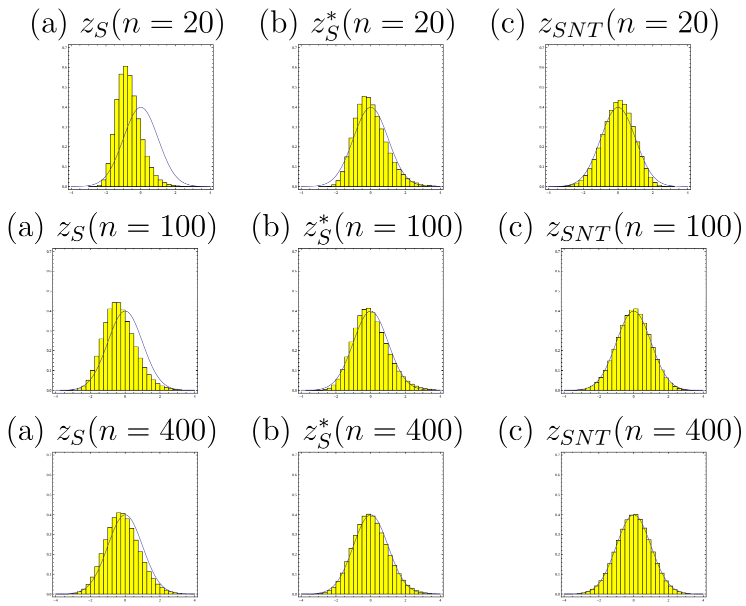

Figure 15: The sample distributions for z_S, z_S^* and z_{SNT} by simulation, and the density plot for standard normal distribution($p = 7$).

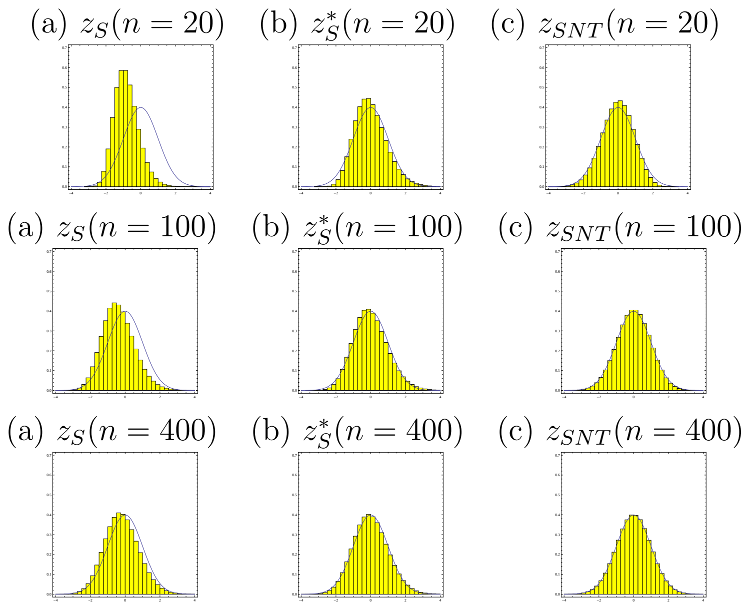

Figure 16: The sample distributions for z_S, z_S^* and z_{SNT} by simulation, and the density plot for standard normal distribution($p = 10$).