Pairwise comparisons among components of mean vector in elliptical distributions

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Abstract

In this paper, we consider approximation to the upper percentiles of the statistic for pairwise comparisons among components of mean vector in elliptical distributions. The first order approximation based on Bonferroni's inequality is given by asymptotic expansion procedure. Also, we investigate the effects of nonnormality on the upper percentiles of this statistic in elliptical distributions. Finally, numerical results by Monte Carlo simulations are given.

Key words and phrases: Asymptotic expansion, Bonferroni's inequality, Elliptical distribution, Monte Carlo simulation, Pairwise comparison.

1. Introduction

Let us consider the simultaneous confidence intervals for pairwise comparisons among components of mean vector. Such a situation arises, for example, in multiple comparisons of the components of repeated measurements of the same quantity in different conditions. Under the multivariate normal population, these simultaneous confidence intervals are discussed by many authors. Lin, Seppänen and Uusipaikka

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(1990) and Nishiyama (2009) considered the approximate simultaneous confidence intervals by Tukey-Kramer type procedure. Also, Seo (1995) considered the simultaneous confidence intervals by asymptotic expansion procedure. In this paper, we discuss these simultaneous confidence intervals under the elliptical population.

This paper gives an extension of Seo (1995) to the case of elliptical distributions. We consider approximation to the upper percentiles of $F^2_{\max\cdot \mathrm{p}}$ statistics based on Bonferroni's inequality to construct approximate simultaneous confidence intervals in elliptical distributions and investigate the effect of nonnormality. It should be noted that, under the elliptical populations, the approximate simultaneous confidence intervals for pairwise comparisons among mean vectors based on Bonferroni's inequality are discussed by Seo (2002), Okamoto (2005) and so on.

The organization of this paper is as follows. In Section 2, the approximations to the upper percentiles of $F^2_{\max\cdot \mathrm{p}}$ statistic based on Bonferroni's inequality are described. In Section 3, the first order approximate upper percentiles of $F^2_{\max\cdot \mathrm{p}}$ statistic by asymptotic expansion procedure are given. Finally, the accuracy of the approximations is investigated by Monte Carlo simulations for selected parameters in Section 4.

2. Approximate procedure based on Bonferroni's inequality

Let Π be the population distributed as a p-dimensional elliptical distribution with parameters $\boldsymbol{\mu}$ and Λ, i.e., $E_p(\boldsymbol{\mu}, \Lambda)$ (see, e.g., Muirhead (1982), Fang, Kotz and Ng (1990)). A probability density function of a $p \times 1$ random vector $\boldsymbol{x}$ from $E_p(\boldsymbol{\mu}, \Lambda)$ is of the form

$$f(\boldsymbol{x}; \boldsymbol{\mu}, \Lambda) = c_p |\Lambda|^{-1/2} g\left\{(\boldsymbol{x} - \boldsymbol{\mu})' \Lambda^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right\},$$

for some nonnegative function g, where c_p is the normalizing constant and Λ is a positive definite. The characteristic function of vector $\boldsymbol{x}$ is

$$\phi(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{\mu})\psi(\boldsymbol{t}'\Lambda\boldsymbol{t}),$$

for some function ψ, where $i = \sqrt{-1}$. It should be noted that $\mathrm{E}(\boldsymbol{x}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\boldsymbol{x}) = \Sigma = -2\psi'(0)\Lambda$. Throughout this paper, we set down the following assumption:

(A1) $\boldsymbol{X} = (\boldsymbol{x}', \{\mathrm{vech}(\boldsymbol{x}\boldsymbol{x}' - \Sigma)\}')'$ satisfies Cramér condition

$$\limsup_{||\xi||\to\infty} |\mathrm{E}[\exp(i\boldsymbol{\xi}'\boldsymbol{X})]| < 1, \quad \boldsymbol{\xi} \in \mathbb{R}^{p+\frac{p(p+1)}{2}}$$

(see, e.g., Bhattacharya and Rao (1976)).

Further, in addition to (A1), we set down the following assumptions if it is required:

(A2) a 8-th absolute moment is finite, that is, $\mathrm{E}[||\boldsymbol{x}||^8] < \infty$,

(A3) a 12-th absolute moment is finite, that is, $\mathrm{E}[||\boldsymbol{x}||^{12}] < \infty$.

We also define the kurtosis parameter by $\kappa = \{\psi''(0)/\ (\psi'(0))^2\} - 1$. Elliptical distributions include the multivariate normal, the multivariate t, the ε-contaminated normal distributions and so on.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be N independent sample vectors from $E_p(\boldsymbol{\mu}, \Lambda)$. Then the sample mean vector and the sample covariance matrix are

$$\begin{aligned} \overline{\boldsymbol{x}} &= \frac{1}{N}\sum_{j=1}^{N} \boldsymbol{x}_j, \\ S &= \frac{1}{N-1}\sum_{j=1}^{N} (\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})', \end{aligned}$$

respectively. In general, the simultaneous confidence intervals for pairwise multiple comparisons among components of mean vector are given by

$$\boldsymbol{b}_{\ell m}'\boldsymbol{\mu} \in \left[\boldsymbol{b}_{\ell m}'\overline{\boldsymbol{x}} \pm w\sqrt{\boldsymbol{b}_{\ell m}'S\boldsymbol{b}_{\ell m}/N}\right],\ 1 \le \ell < m \le p,$$

where $\boldsymbol{b}_{\ell m} = \boldsymbol{e}_\ell - \boldsymbol{e}_m$, $\boldsymbol{e}_\ell$ is a unit vector of the p-dimensional space having 1 at ℓ-th component and 0 at others, and the value w (> 0) satisfies as follows:

$$\Pr\{F^2_{\max\cdot \mathrm{p}} > w^2\} = \alpha,$$

where

$$F^2_{\max\cdot\mathrm{p}} = \max_{1\le\ell<m\le p}\left\{\frac{N\boldsymbol{b}'_{\ell m}(\overline{\boldsymbol{x}}-\boldsymbol{\mu})(\overline{\boldsymbol{x}}-\boldsymbol{\mu})'\boldsymbol{b}_{\ell m}}{\boldsymbol{b}'_{\ell m}S\boldsymbol{b}_{\ell m}}\right\}.$$

In order to construct these simultaneous confidence intervals with the confidence level $1-\alpha$, it is required to find the value w. However, it is difficult to find the exact value w even under the multivariate normality. Therefore, we construct approximate simultaneous confidence intervals. Here, we describe the first order approximation based on Bonferroni's inequality (see, e.g., Siotani (1959), Seo (2002)). By Bonferroni's inequality for $\Pr(F^2_{\max\cdot\mathrm{p}} > w^2)$,

$$\Pr(F^2_{\max\cdot\mathrm{p}} > w^2) < \sum_{\ell=1}^{p-1}\sum_{m=\ell+1}^{p}\Pr(F^2_{\ell m} > w^2),$$

where

$$F^2_{\ell m} = \frac{N\boldsymbol{b}'_{\ell m}(\overline{\boldsymbol{x}}-\boldsymbol{\mu})(\overline{\boldsymbol{x}}-\boldsymbol{\mu})'\boldsymbol{b}_{\ell m}}{\boldsymbol{b}'_{\ell m}S\boldsymbol{b}_{\ell m}},$$

and the first order approximation w_1^2 is given as a critical value that satisfies the equality

$$\sum_{\ell=1}^{p-1}\sum_{m=\ell+1}^{p}\Pr(F^2_{\ell m} > w_1^2) = \alpha.$$

It should be noted that w_1^2 is overestimated, and the statistic $F^2_{\ell m}$ is essentially distributed as F-distribution under the multivariate normality. However, under the class of the elliptical distributions, $F^2_{\ell m}$ is not distributed as F-distribution. Hence, the first order approximation cannot be exactly expressed as the upper percentiles of F-distribution. Therefore, we discuss an asymptotic expansion for the first order approximation in Section 3.

3. The first order Bonferroni approximation for the upper percentile of the statistic

3.1. Asymptotic expansion using Iwashita (1997)

In this subsection, we discuss under the assumption (A2). Takahashi, Nishiyama and Seo (2010) derived the first order Bonferroni approximation for the upper percentiles of $F^2_{\max\cdot\mathrm{p}}$ statistic. Unfortunately, this result included some miscalculations. So, we correct the asymptotic expansion for $F^2_{\ell m}$. Here, we assume $\Sigma = I_p$. Let

$$(N-1)S = NW - N(\overline{\boldsymbol{x}}-\boldsymbol{\mu})(\overline{\boldsymbol{x}}-\boldsymbol{\mu})',$$

where

$$W = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j-\boldsymbol{\mu})(\boldsymbol{x}_j-\boldsymbol{\mu})',$$

and

$$\overline{\boldsymbol{x}} = \boldsymbol{\mu} + \frac{1}{\sqrt{N}}\boldsymbol{z}, \quad W = I_p + \frac{1}{\sqrt{N}}Z.$$

Then we can write

$$\boldsymbol{b}'_{\ell m}S\boldsymbol{b}_{\ell m} = \frac{N}{N-1}\left(1 + \frac{1}{2\sqrt{N}}\boldsymbol{b}'_{\ell m}Z\boldsymbol{b}_{\ell m} - \frac{1}{2N}\boldsymbol{b}'_{\ell m}\boldsymbol{z}\boldsymbol{z}'\boldsymbol{b}_{\ell m}\right).$$

Therefore,

$$\left(\boldsymbol{b}'_{\ell m}S\boldsymbol{b}_{\ell m}\right)^{-1} = \frac{1}{2}\Big\{1 - \frac{1}{\sqrt{N}}Y_{\ell m} + \frac{1}{N}\Big(Y^2_{\ell m} + y^2_{\ell m} - 1\Big) + o_p(N^{-1})\Big\},$$

where

$$Y_{\ell m} = \frac{1}{2}\boldsymbol{b}'_{\ell m}Z\boldsymbol{b}_{\ell m}, \quad y_{\ell m} = \frac{1}{\sqrt{2}}\boldsymbol{b}'_{\ell m}\boldsymbol{z}.$$

Hence, calculating the characteristic function of $F^2_{\ell m}$ with $\boldsymbol{z}$ and Z by using the joint density function of $\boldsymbol{z}$ and Z given in Iwashita (1997), we obtain

$$\mathrm{E}[\exp(itF^2_{\ell m})] = u^{-\frac{1}{2}}\left\{1 + \frac{1}{4N}(c_0 + c_1u^{-1} + c_2u^{-2}) + o(N^{-1})\right\},$$

where $u = 1 - 2it$, and

$$c_0 = -1 - 3\kappa, \quad c_1 = -2 + 6\kappa, \quad c_2 = 3 - 3\kappa.$$

Therefore, inverting this characteristic function, we have the following theorem.

Theorem 1. *The distribution of $F_{\ell m}^2$ can be expanded as*

$$\Pr\left\{F_{\ell m}^2 > w^2\right\} = \Pr\left\{\chi_1^2 > w^2\right\} + \frac{1}{4N}\sum_{j=0}^{2} c_j \Pr\left\{\chi_{1+2j}^2 > w^2\right\} + o(N^{-1}),$$

and also its upper 100α percentile can be expanded as

$$w_{\ell m}^2(\alpha) = \chi_1^2(\alpha) - \frac{1}{2N}\chi_1^2(\alpha)\left\{c_0 - \frac{1}{3}c_2\chi_1^2(\alpha)\right\} + o(N^{-1}),$$

where $\chi_1^2(\alpha)$ is the upper 100α percentiles of χ^2 distribution with 1 degree of freedom.

Since $F_{\ell m}^2$ is essentially distributed as F-distribution under the multivariate normality, we also have the following theorem.

Theorem 2. *The upper 100α percentile of $F_{\ell m}^2$ can be also expanded as*

$$w_{\ell m}^2(\alpha) = F_{1,N-1}(\alpha) - \frac{1}{2N}\chi_1^2(\alpha)\left\{(c_0 + 1) - \left(\frac{1}{3}c_2 - 1\right)\chi_1^2(\alpha)\right\} + o(N^{-1}),$$

where $F_{1,N-1}(\alpha)$ is the upper 100α percentile of F-distribution with 1 and $N-1$ degrees of freedom.

Therefore, for large N, the first order Bonferroni approximate upper 100α percentiles of $F_{\max\cdot\mathrm{p}}^2$, that is, $w_{1\cdot\chi^2}^2 \equiv w_{1\cdot\chi^2}^2(\alpha)$ and $w_{1\cdot F}^2 \equiv w_{1\cdot F}^2(\alpha)$ are obtained as follows:

$$\begin{aligned} w_{1\cdot\chi^2}^2 &= \chi_1^2(\alpha^*) - \frac{1}{2N}\chi_1^2(\alpha^*)\left\{c_0 - \frac{1}{3}c_2\chi_1^2(\alpha^*)\right\} + o(N^{-1}), \\ w_{1\cdot F}^2 &= F_{1,N-1}(\alpha^*) - \frac{1}{2N}\chi_1^2(\alpha^*)\left\{(c_0 + 1) - \left(\frac{1}{3}c_2 - 1\right)\chi_1^2(\alpha^*)\right\} + o(N^{-1}), \end{aligned}$$

where $\alpha^* = \alpha/M$ and $M = p(p-1)/2$.

3.2. Asymptotic expansion using Iwashita and Seo (2002)

In this subsection, we discuss under the assumption (A3). Here, we give the first order Bonferroni approximation up to the terms of order N^{-2} for the upper percentiles of $F^2_{\max\cdot\mathrm{p}}$ statistic. Since

$$\begin{aligned}(\boldsymbol{b}'_{\ell m}S\boldsymbol{b}_{\ell m})^{-1} &= \frac{1}{2}\Big\{1-\frac{1}{\sqrt{N}}Y_{\ell m}+\frac{1}{N}(Y^2_{\ell m}+y^2_{\ell m}-1)\\ &\quad -\frac{1}{N\sqrt{N}}(Y^3_{\ell m}+2Y_{\ell m}y^2_{\ell m}-Y_{\ell m})\\ &\quad +\frac{1}{N^2}(Y^4_{\ell m}+3Y^2_{\ell m}y^2_{\ell m}+y^4_{\ell m}-Y^2_{\ell m}-y^2_{\ell m})+o_p(N^{-2})\Big\},\end{aligned}$$

$F^2_{\ell m}$ can be expanded as

$$F^2_{\ell m}=y^2_{\ell m}-\frac{1}{\sqrt{N}}A_1+\frac{1}{N}A_2-\frac{1}{N\sqrt{N}}A_3+\frac{1}{N^2}A_4+o_p(N^{-2}),$$

where

$$\begin{aligned}A_1 &= Y_{\ell m}y^2_{\ell m},\\ A_2 &= Y^2_{\ell m}y^2_{\ell m}+y^4_{\ell m}-y^2_{\ell m},\\ A_3 &= Y^3_{\ell m}y^2_{\ell m}+2Y_{\ell m}y^4_{\ell m}-Y_{\ell m}y^2_{\ell m},\\ A_4 &= Y^4_{\ell m}y^2_{\ell m}+3Y^2_{\ell m}y^4_{\ell m}-Y^2_{\ell m}y^2_{\ell m}+y^6_{\ell m}-y^4_{\ell m}.\end{aligned}$$

Therefore,

$$\begin{aligned}\exp(itF^2_{\ell m}) &= \exp(ity^2_{\ell m})\\ &\quad\times\Big[1-\frac{1}{\sqrt{N}}itA_1+\frac{1}{N}\Big\{itA_2+\frac{(it)^2}{2}A_1^2\Big\}\\ &\quad -\frac{1}{N\sqrt{N}}\Big\{itA_3+(it)^2A_1A_2+\frac{(it)^3}{6}A_1^3\Big\}\\ &\quad +\frac{1}{N^2}\Big\{itA_4+(it)^2\Big(A_1A_3+\frac{1}{2}A_2^2+\frac{(it)^3}{2}A_1A_2+\frac{(it)^4}{24}A_1^4\Big)\Big\}\\ &\quad +o_p(N^{-2})\Big].\end{aligned}$$

In order to calculate the characteristic function of $F^2_{\ell m}$, we use the joint characteristic function of $\boldsymbol{z}$ and Z, and the marginal characteristic function of $\boldsymbol{z}$ given in Iwashita and Seo (2002). Then we obtain

$$\mathrm{E}[\exp(itF_{\ell m}^2)] = u^{-\frac{1}{2}} + \frac{1}{4N}\sum_{j=0}^{2} d_{1j}u^{-\frac{1}{2}-j} + \frac{1}{32N^2}\sum_{j=0}^{4} d_{2j}u^{-\frac{1}{2}-j} + o(N^{-2}).$$

where

$$\begin{aligned}
&d_{10} = -1-3\kappa, \quad d_{11} = 2(-1+3\kappa), \quad d_{12} = 3(1-\kappa),\\
&d_{20} = -7+80\beta-210\kappa-111\kappa^2, \quad d_{21} = 12(-1+8\kappa+\kappa^2),\\
&d_{22} = 6(9-40\beta+54\kappa+69\kappa^2), \quad d_{23} = 20(-7+8\beta-21\kappa^2),\\
&d_{24} = 105(1-\kappa)^2,
\end{aligned}$$

and $\beta = \psi'''(0)/\{\psi'(0)\}^3 - 1$. Therefore, inverting the characteristic function, we have the following theorem.

Theorem 3. *The distribution of $F_{\ell m}^2$ can be expanded as*

$$\begin{aligned}
\Pr\{F_{\ell m}^2 > w^2\} = \Pr\{\chi_1^2 > w^2\} &+ \frac{1}{4N}\sum_{j=0}^{2} d_{1j}\Pr\{\chi_{1+2j}^2 > w^2\}\\
&+\frac{1}{32N^2}\sum_{j=0}^{4} d_{2j}\Pr\{\chi_{1+2j}^2 > w^2\} + o(N^{-2}),
\end{aligned}$$

and also its 100α percentile can be expanded as

$$\begin{aligned}
\widetilde{w}_{\ell m}^2(\alpha) &= \chi_1^2(\alpha) - \frac{1}{2N}\chi_1^2(\alpha)q_1(\alpha)\\
&\quad -\frac{1}{16N^2}\chi_1^2(\alpha)\Big[\{1+\chi_1^2(\alpha)\}\{q_1(\alpha)\}^2 - 4q_1(\alpha)q_2(\alpha) + q_3(\alpha)\Big] + o(N^{-2}),\\
\widetilde{w}_{\ell m}^2(\alpha) &= F_{1,\nu}(\alpha) - \frac{1}{2N}\chi_1^2(\alpha)r_1(\alpha)\\
&\quad -\frac{1}{16N^2}\chi_1^2(\alpha)\Big[\{1+\chi_1^2(\alpha)\}r_2(\alpha) - 4r_3(\alpha) + r_4(\alpha)\Big] + o(N^{-2}),
\end{aligned}$$

where

$$\begin{aligned}
q_1(\alpha) &= d_{10} - \frac{1}{3}d_{12}\chi_1^2(\alpha),\\
q_2(\alpha) &= d_{10} - \frac{2}{3}d_{12}\chi_1^2(\alpha),\\
q_3(\alpha) &= d_{20} - \frac{1}{3}(d_{22}+d_{23}+d_{24})\chi_1^2(\alpha)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{15}(d_{23}+d_{24})\{\chi_1^2(\alpha)\}^2-\frac{1}{105}d_{24}\{\chi_1^2(\alpha)\}^3, \\
r_1(\alpha) &= q_1(\alpha)+1+\chi_1^2(\alpha), \\
r_2(\alpha) &= \{q_1(\alpha)\}^2-\{1+\chi_1^2(\alpha)\}^2, \\
r_3(\alpha) &= q_1(\alpha)q_2(\alpha)-\{1+\chi_1^2(\alpha)\}\{1+2\chi_1^2(\alpha)\}, \\
r_4(\alpha) &= q_3(\alpha)+7+\frac{19}{3}\chi_1^2(\alpha)-\frac{7}{3}\{\chi_1^2(\alpha)\}^2+\{\chi_1^2(\alpha)\}^3,
\end{aligned}
$$

and $\chi_1^2(\alpha)$ and $F_{1,N-1}(\alpha)$ are the upper 100α percentile of χ^2 distribution with 1 degree of freedom and that of F-distribution with 1 and $N-1$ degrees of freedom, respectively.

Therefore, for large N, the first order Bonferroni approximate upper 100α percentiles of $F_{\max}^2$ up to the terms of order N^{-2}, that is, $\widetilde{w}_{1\cdot\chi^2}^2 \equiv \widetilde{w}_{1\cdot\chi^2}^2(\alpha)$ and $\widetilde{w}_{1\cdot F}^2 \equiv \widetilde{w}_{1\cdot F}^2(\alpha)$ are obtained as follows:

$$
\begin{aligned}
\widetilde{w}_{1\cdot\chi^2}^2 &= \chi_1^2(\alpha^*)-\frac{1}{2N}\chi_1^2(\alpha^*)q_1(\alpha^*) \\
&\quad -\frac{1}{16N^2}\chi_1^2(\alpha^*)\Big[\{1+\chi_1^2(\alpha^*)\}\{q_1(\alpha^*)\}^2-4q_1(\alpha^*)q_2(\alpha^*)+q_3(\alpha^*)\Big] \\
&\quad +o(N^{-2}), \\
\widetilde{w}_{1\cdot F}^2 &= F_{1,\nu}(\alpha^*)-\frac{1}{2N}\chi_1^2(\alpha^*)r_1(\alpha^*) \\
&\quad -\frac{1}{16N^2}\chi_1^2(\alpha^*)\Big[\{1+\chi_1^2(\alpha^*)\}r_2(\alpha^*)-4r_3(\alpha^*)+r_4(\alpha^*)\Big]+o(N^{-2}).
\end{aligned}
$$

It should be noted that $w_{1\cdot F}^2=\widetilde{w}_{1\cdot F}^2=F_{1,N-1}(\alpha^*)$ under the multivariate normality.

4. Numerical examinations

We evaluate the accuracy of the obtained approximations by Monte Carlo simulation. Monte Carlo simulation of the upper percentiles of $F_{\max\cdot\mathrm{p}}$ statistic is implemented from 10^6 trials for selected values of parameters p, N, α and κ

Tables 1-6 list the simulated and approximate values of the upper percentiles of $F_{\max\cdot\mathrm{p}}$ $(=\sqrt{F_{\max\cdot\mathrm{p}}^2})$ statistic for the combinations of following parameter values: $p=3,5,10,\ N=10,20,40,80,200\ (p<N)$ and $\alpha=0.05$. For the distributions of

population, we adopt the following three distributions; the multivariate normal ($\kappa = 0$), the ε-contaminated normal ($\varepsilon = 0.1, \sigma = 3 : \kappa = 1.78$) and the ε-contaminated normal ($\varepsilon = 0.1, \sigma = 4 : \kappa = 3.24$).

In Tables 1-6, $w_{1\cdot\chi^2}$, $w_{1\cdot F}$, $\widetilde{w}_{1\cdot\chi^2}$ and $\widetilde{w}_{1\cdot F}$ stand for $\sqrt{w^2_{1\cdot\chi^2}}$, $\sqrt{w^2_{1\cdot F}}$, $\sqrt{\widetilde{w}^2_{1\cdot\chi^2}}$ and $\sqrt{\widetilde{w}_{1\cdot F}}$ respectively. Also, $P(w^2_{1\cdot\chi^2})$, $P(w^2_{1\cdot F})$, $P(\widetilde{w}^2_{1\cdot\chi^2})$ and $P(\widetilde{w}^2_{1\cdot F})$ denote $\Pr\{F^2_{\max\cdot\mathrm{p}} < w^2_{1\cdot\chi^2}\}$, $\Pr\{F^2_{\max\cdot\mathrm{p}} < w^2_{1\cdot F}\}$, $\Pr\{F^2_{\max\cdot\mathrm{p}} < \widetilde{w}^2_{1\cdot\chi^2}\}$ and $\Pr\{F^2_{\max\cdot\mathrm{p}} < \widetilde{w}^2_{1\cdot F}\}$, respectively. It should be noted that w^* is a simulated value of the upper percentiles of $F^2_{\max\cdot\mathrm{p}}$ statistic, that is, $\Pr\{F^2_{\max\cdot\mathrm{p}} < w^{*2}\} = 1 - \alpha$.

In Tables 1 and 2, numerical results for the multivariate normal case ($\kappa = 0$) are given. It can be observed from these Tables that the values of $w_{1\cdot F} = \widetilde{w}_{1\cdot F}$ are always larger than the values of w^*. So, it should be noted that always $P(w^2_{1\cdot F}) = P(\widetilde{w}^2_{1\cdot F}) \geq 1 - \alpha$. Also, when N becomes large, the values of $w^2_{1\cdot\chi^2}$ and $\widetilde{w}^2_{1\cdot\chi^2}$ are larger than that of w^*. Besides, it should be noted that the values of $\widetilde{w}^2_{1\cdot\chi^2}$ are always larger than the values of $w^2_{1\cdot\chi^2}$, that is, $P(\widetilde{w}^2_{1\cdot\chi^2}) \geq P(w^2_{1\cdot\chi^2})$.

Tables 3 and 4 and Tables 5 and 6 give numerical results for the case that $\kappa = 1.78$ and $\kappa = 3.24$, respectively. From these Tables, when $p = 3$, it should be noted that the values of $w_{1\cdot F}$ are greater than or equal to that of $\widetilde{w}_{1\cdot F}$. However, when $p = 5$ and 10, $\widetilde{w}_{1\cdot F}$ are always greater than $w_{1\cdot F}$. Besides, it should be noted that $P(\widetilde{w}_{1\cdot F}) \geq 1 - \alpha$ for almost all case. Also, it can be observed from these Tables, when $\kappa = 1.78$, $\widetilde{w}_{1\cdot\chi^2}$ are always greater than or equal to $w_{1\cdot\chi^2}$. However, when $\kappa = 3.24$ and $p = 3$, $\widetilde{w}_{1\cdot\chi^2}$ are always smaller than or equal to $w_{1\cdot\chi^2}$.

From Tables 1-6, it should be noted that when κ becomes large, $w_{1\cdot\chi^2}$, $w_{1\cdot F}$, $\widetilde{w}_{1\cdot\chi^2}$ and $\widetilde{w}_{1\cdot F}$, that is, $P(w^2_{1\cdot\chi^2})$, $P(w^2_{1\cdot F})$, $P(\widetilde{w}^2_{1\cdot\chi^2})$ and $P(\widetilde{w}^2_{1\cdot F})$ become small. Also, it can be observed that always $P(w^2_{1\cdot\chi^2}) \leq P(w^2_{1\cdot F})$ and $P(\widetilde{w}^2_{1\cdot\chi^2}) \leq P(\widetilde{w}^2_{1\cdot F})$.

Acknowledgements

The research of the second author was supported in part by Grant-in-Aid for Young Scientists (B) under Contract Number 21700311.

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Table 1. The simulated and approximate values up to the terms of order N^{-1} for the multivariate normal distribution ($\kappa = 0$).

$\kappa = 0,\ \alpha = 0.05$						
p	N	$w_{1\cdot\chi^2}$	$w_{1\cdot F}$	w^*	$P(w_{1\cdot\chi^2}^2)$	$P(w_{1\cdot F}^2)$
3	10	2.768	2.933	2.871	0.941	0.955
	20	2.588	2.625	2.572	0.952	0.955
	40	2.493	2.502	2.453	0.954	0.955
	80	2.444	2.446	2.395	0.955	0.956
	200	2.414	2.414	2.362	0.956	0.956
5	10	3.373	3.690	3.600	0.931	0.956
	20	3.103	3.174	3.092	0.951	0.958
	40	2.959	2.976	2.893	0.957	0.959
	80	2.884	2.888	2.809	0.959	0.959
	200	2.838	2.839	2.758	0.960	0.960
10	10	–	–	–	–	–
	20	3.705	3.837	3.736	0.947	0.959
	40	3.490	3.521	3.425	0.957	0.961
	80	3.377	3.385	3.289	0.961	0.962
	200	3.308	3.309	3.210	0.963	0.963

Table 2. The simulated and approximate values up to the terms of order N^{-2} for the multivariate normal distribution ($\kappa = 0$).

$\kappa = 0,\ \alpha = 0.05$						
p	N	$\widetilde{w}_{1\cdot\chi^2}$	$\widetilde{w}_{1\cdot F}$	w^*	$P(\widetilde{w}_{1\cdot\chi^2}^2)$	$P(\widetilde{w}_{1\cdot F}^2)$
3	10	2.885	2.933	2.871	0.951	0.955
	20	2.619	2.625	2.572	0.955	0.955
	40	2.501	2.502	2.453	0.955	0.955
	80	2.446	2.446	2.395	0.956	0.956
	200	2.414	2.414	2.362	0.956	0.956
5	10	3.583	3.690	3.600	0.949	0.956
	20	3.161	3.174	3.092	0.957	0.958
	40	2.974	2.976	2.893	0.959	0.959
	80	2.888	2.888	2.809	0.959	0.959
	200	2.839	2.839	2.758	0.960	0.960
10	10	–	–	–	–	–
	20	3.810	3.837	3.736	0.957	0.959
	40	3.518	3.521	3.425	0.960	0.961
	80	3.385	3.385	3.289	0.962	0.962
	200	3.309	3.309	3.210	0.963	0.963

Table 3. The simulated and approximate values up to the terms of order N^{-1} for the ε-contaminated normal distribution ($\kappa = 1.78$).

$\kappa = 1.78$, $\alpha = 0.05$						
p	N	$w_{1\cdot\chi^2}$	$w_{1\cdot F}$	w^*	$P(w_{1\cdot\chi^2}^2)$	$P(w_{1\cdot F}^2)$
3	10	2.504	2.686	2.740	0.926	0.945
	20	2.449	2.489	2.480	0.947	0.951
	40	2.422	2.431	2.396	0.953	0.954
	80	2.408	2.410	2.368	0.955	0.955
	200	2.400	2.400	2.351	0.956	0.956
5	10	2.821	3.193	3.408	0.876	0.930
	20	2.814	2.892	2.954	0.932	0.943
	40	2.811	2.828	2.804	0.951	0.953
	80	2.809	2.813	2.754	0.957	0.958
	200	2.808	2.808	2.734	0.959	0.959
10	10	–	–	–	–	–
	20	3.181	3.334	3.551	0.889	0.920
	40	3.221	3.255	3.288	0.940	0.945
	80	3.241	3.249	3.203	0.955	0.956
	200	3.253	3.254	3.175	0.961	0.961

Table 4. The simulated and approximate values up to the terms of order N^{-2} for the ε-contaminated normal distribution ($\kappa = 1.78$).

$\kappa = 1.78$, $\alpha = 0.05$						
p	N	$\widetilde{w}_{1\cdot\chi^2}$	$\widetilde{w}_{1\cdot F}$	w^*	$P(\widetilde{w}_{1\cdot\chi^2}^2)$	$P(\widetilde{w}_{1\cdot F}^2)$
3	10	2.587	2.641	2.740	0.935	0.941
	20	2.471	2.477	2.480	0.949	0.950
	40	2.427	2.428	2.396	0.954	0.954
	80	2.409	2.409	2.368	0.955	0.955
	200	2.400	2.400	2.351	0.956	0.956
5	10	3.556	3.664	3.408	0.960	0.966
	20	3.015	3.028	2.954	0.956	0.958
	40	2.862	2.864	2.804	0.957	0.957
	80	2.822	2.822	2.754	0.959	0.959
	200	2.810	2.810	2.734	0.960	0.960
10	10	–	–	–	–	–
	20	3.805	3.833	3.551	0.972	0.974
	40	3.386	3.390	3.288	0.962	0.962
	80	3.283	3.283	3.203	0.961	0.961
	200	3.260	3.260	3.175	0.962	0.962

Table 5. The simulated and approximate values up to the terms of order N^{-1} for the ε-contaminated normal distribution ($\kappa = 3.24$).

$\kappa = 3.24,\ \alpha = 0.05$						
p	N	$w_{1\cdot\chi^2}$	$w_{1\cdot F}$	w^*	$P(w^2_{1\cdot\chi^2})$	$P(w^2_{1\cdot F})$
3	10	2.264	2.463	2.670	0.899	0.929
	20	2.330	2.371	2.417	0.939	0.944
	40	2.362	2.372	2.350	0.952	0.953
	80	2.378	2.380	2.341	0.955	0.955
	200	2.388	2.388	2.340	0.956	0.956
5	10	2.269	2.718	3.323	0.743	0.870
	20	2.552	2.638	2.871	0.897	0.915
	40	2.683	2.701	2.735	0.942	0.945
	80	2.746	2.750	2.710	0.955	0.956
	200	2.783	2.783	2.714	0.959	0.959
10	10	–	–	–	–	–
	20	2.674	2.855	3.448	0.743	0.819
	40	2.982	3.019	3.191	0.909	0.918
	80	3.124	3.133	3.132	0.949	0.950
	200	3.207	3.208	3.141	0.960	0.960

Table 6. The simulated and approximate values up to the terms of order N^{-2} for the ε-contaminated normal distribution ($\kappa = 3.24$).

$\kappa = 3.24,\ \alpha = 0.05$						
p	N	$\widetilde{w}_{1\cdot\chi^2}$	$\widetilde{w}_{1\cdot F}$	w^*	$P(\widetilde{w}^2_{1\cdot\chi^2})$	$P(\widetilde{w}^2_{1\cdot F})$
3	10	2.086	2.152	2.670	0.863	0.878
	20	2.288	2.294	2.417	0.933	0.934
	40	2.352	2.353	2.350	0.950	0.950
	80	2.376	2.376	2.341	0.954	0.954
	200	2.387	2.387	2.340	0.956	0.956
5	10	3.227	3.345	3.323	0.942	0.952
	20	2.798	2.812	2.871	0.941	0.943
	40	2.743	2.745	2.735	0.951	0.951
	80	2.761	2.761	2.710	0.957	0.957
	200	2.785	2.785	2.714	0.959	0.959
10	10	–	–	–	–	–
	20	3.682	3.711	3.448	0.971	0.973
	40	3.239	3.243	3.191	0.957	0.957
	80	3.188	3.188	3.132	0.958	0.958
	200	3.217	3.217	3.141	0.961	0.961