Asymptotic expansion for the distribution of Wald's classification statistic with two-step monotone missing data

Kazumasa Kurihara[1], Rie Enomoto[1], Nobumichi Shutoh[1,2,*]

[1] *Department of Mathematical Information Science, Graduate School of Science, Tokyo University of Science*
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, JAPAN
[2] *Research Fellow of the Japan Society for the Promotion of Science*

Abstract

In this paper, we discuss a high order asymptotic expansion for the distribution of the linear discriminant function with two-step monotone missing data. In discriminant analysis, asymptotic expansion plays important role in considering the probabilities of misclassification. We derive a high order asymptotic expansion based on two-step monotone missing data. Finally, we perform the Monte Carlo simulation in order to evaluate our result.

Key Words and Phrases: asymptotic expansion; Wishart distribution; probabilities of misclassification; two-step monotone missing data.

1 Introduction

Linear discriminant analysis is well known as one of multivariate statistical procedures to assign p–dimensional observation vector $\boldsymbol{x}$ which arises from one of some groups into one of them (see, e.g., Fisher [1], Wald [13]). In this paper, we discuss that $\boldsymbol{x}$ comes from one of two groups, i.e., $\Pi^{(1)} : N_p(\boldsymbol{\mu}^{(1)}, \Sigma)$ and $\Pi^{(2)} : N_p(\boldsymbol{\mu}^{(2)}, \Sigma)$. Then, the linear discriminant function W_1 based on the p–dimensional sample vectors $\boldsymbol{x}_j^{(g)} (j = 1, \ldots, N_1^{(g)},\ g = 1, 2)$ from $\Pi^{(1)}$ and $\Pi^{(2)}$ can be constructed as

$$W_1 = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-1} \Big[\boldsymbol{x} - \frac{1}{2}(\overline{\boldsymbol{x}}^{(1)} + \overline{\boldsymbol{x}}^{(2)})\Big],$$

where $\overline{\boldsymbol{x}}^{(g)}$ denotes the sample mean vector from $\Pi^{(g)}$ and S denotes the pooled sample covariance matrix. If $W_1 > c$, the sample vector $\boldsymbol{x}$ may be assigned to $\Pi^{(1)}$ and if $W_1 \leq c$, otherwise it may

*Corresponding author, *E-mail address* : nobumichi.shutoh@gmail.com

be assigned to $\Pi^{(2)}$, where c is a cut-off point. Consequently, the probabilities of misclassification can be represented as

$$e_1(2|1) \equiv \Pr(W_1 \le c|\boldsymbol{x} \in \Pi^{(1)}),$$
$$e_1(1|2) \equiv \Pr(W_1 > c|\boldsymbol{x} \in \Pi^{(2)}).$$

However, it is so difficult to obtain the exact misclassification probability since the discriminant function W_1 has complicated distributional expression. Alternatively, the asymptotic distribution of W_1 is well known under a couple of asymptotic frameworks.

First, we focus on a large-sample framework: a fixed p, $N_1^{(1)} \to \infty$, $N_1^{(2)} \to \infty$ and $N_1^{(1)}/N_1^{(2)} \to$ positive constant. Then, the limiting distribution of W_1 under $\boldsymbol{x} \in \Pi^{(g)}$ is $N((-1)^{g-1}(1/2)\Delta^2, \Delta^2)$ for $g = 1, 2$, where $\Delta^2 \equiv (\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})'\Sigma^{-1}(\boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)})$ is Mahalanobis squared distance between $\Pi^{(1)}$ and $\Pi^{(2)}$.

Under the framework, Okamoto [8] derived an asymptotic expansion for the distribution of W_1 under $\boldsymbol{x} \in \Pi^{(1)}$ up to the terms of the second order with respect to $(\{N_1^{(1)}\}^{-1}, \{N_1^{(2)}\}^{-1}, n_1^{-1})$:

$$\begin{aligned}
&\Pr\left[\frac{W_1 - (1/2)\Delta^2}{\Delta} \le u \middle| \boldsymbol{x} \in \Pi^{(1)}\right] \\
&= \Bigg[1 + (2N_1^{(1)}\Delta^2)^{-1}\{d^4 + p(d^2 + \Delta d)\} \\
&\quad + (2N_1^{(2)}\Delta^2)^{-1}\{(d^2 - \Delta d)^2 + p(d^2 - \Delta d)\} \\
&\quad + (4n_1)^{-1}\{(2d^2 - \Delta d)^2 + 2(p+1)(3d^2 - \Delta d)\} \\
&\quad + (4(N_1^{(1)})^2\Delta^4)^{-1}\Big\{2d^4(d^2 + \Delta d) + p(d^2 + \Delta d)^2 + \frac{1}{2}\{d^4 + p(d^2 + \Delta d)\}^2\Big\} \\
&\quad + (4(N_1^{(2)})^2\Delta^4)^{-1}\Big\{2(d^2 - \Delta d)^3 + p(d^2 - \Delta d)^2 + \frac{1}{2}\{(d^2 - \Delta d)^2 + p(d^2 - \Delta d)\}^2\Big\} \\
&\quad + (2N_1^{(1)}N_1^{(2)}\Delta^4)^{-1} \\
&\qquad \times \Big\{2d^4(d^2 - \Delta d) + pd^4 + \frac{1}{2}\{d^4 + p(d^2 + \Delta d)\}\{(d^2 - \Delta d)^2 + p(d^2 - \Delta d)\Big\} \\
&\quad + (2N_1^{(1)}n_1\Delta^2)^{-1}\Big\{4d^4(2d^2 - \Delta d) + 2(5p + 7)d^4 - \Delta^2 d^2 + (p^2 + p)(3d^2 + \Delta d) \\
&\qquad + \frac{1}{4}\{d^4 + p(d^2 + \Delta d)\}\{(2d^4 - \Delta d)^2 + 2(p+1)(3d^2 - \Delta d)\}\Big\} \\
&\quad + (2N_1^{(2)}n_1\Delta^2)^{-1}\Big\{2(d^2 - \Delta d)(2d^2 - \Delta d)^2 + 2(5p + 7)d^4 - 4(3p + 4)\Delta d^3 + (3p + 4)\Delta^2 d^2 \\
&\qquad + (p^2 + p)(3d^2 - \Delta d) + \frac{1}{4}\{(d^2 - \Delta d)^2 + p(d^2 - \Delta d)\} \\
&\qquad \times \{(2d^4 - \Delta d)^2 + 2(p+1)(3d^2 - \Delta d)\}\Big\} \\
&\quad + (12n_1^2)^{-1}\Big\{2(2d^2 - \Delta d)^2(7d^2 - 2\Delta d) + 9(15p + 13)d^4 - 24(4p + 3)\Delta d^3 + 3(5p + 3)\Delta^2 d^2
\end{aligned}$$

$$+ 6(6p^2 + 13p + 9)d^2 - 6(p+1)^2 \Delta d$$
$$\left. + \frac{3}{8}\{(2d^4 - \Delta d)^2 + 2(p+1)(3d^2 - \Delta d)\}^2 \right\} \Bigg] \Phi(u) + \mathrm{O}_3,$$

where $n_1 = N_1^{(1)} + N_1^{(2)} - 2$, $\Phi(\cdot)$ is the cumulative distribution function of $N(0,1)$, d denotes the differential operator d/du, u is a finite constant, and O_3 denotes the remainder terms of the third order. The same under $\boldsymbol{x} \in \Pi^{(2)}$ can be obtained by interchanging $N_1^{(1)}$ and $N_1^{(2)}$. Okamoto's [8] result can be consider as one of the asymptotic approximations for $e_1(2|1)$ and $e_1(1|2)$ under the large-sample framework.

In contrast, Fujikoshi and Seo [2] dealt with a high-dimensional asymptotic framework: $p \to \infty$, $N_1^{(1)} \to \infty$, $N_1^{(2)} \to \infty$, $n_1 - p \to \infty$, $N_1^{(1)}/N_1^{(2)} \to$ positive constant and $\Delta^2 = \mathrm{O}(1)$. Under the framework, Fujikoshi and Seo [2] proposed an asymptotic approximation for the expected probabilities of misclassification. In addition, Fujikoshi and Seo [2] also justified Lachenbruch's [7] asymptotic approximation under the framework. Thus, Lachenbruch [7] derived the expected probabilities of misclassification (EPMC) in linear discriminant function W_1 under the both of frameworks.

Furthermore, some authors discussed the asymptotic theory for linear discriminant analysis based on monotone missing data. For Lachenbruch's [7] type asymptotic approximation, Shutoh et al. [12], Kurihara et al. [6] and Shutoh [10] derived an asymptotic approximation for EPMC and the unbiased estimators for Mahalanobis squared distance based on two-step monotone missing data, three-step monotone missing data and k-step monotone missing data, respectively. For Okamoto's [8] asymptotic expansion, Shutoh [11] derived an asymptotic expansion for the distribution of linear discriminant function up to the terms of the first order based on k-step monotone missing data.

In this paper, our propose is to derive an asymptotic expansion for the distribution of linear discriminant function up to the terms of the second order based on two-step monotone missing data:

$$\begin{aligned}
\boldsymbol{x}_j^{(g)} &= (\boldsymbol{x}_{1j}^{(g)\prime}, \boldsymbol{x}_{2j}^{(g)\prime})' \sim N_p(\boldsymbol{\mu}^{(g)}, \Sigma) \ \ (g = 1, 2, \ j = 1, \ldots, N_1^{(g)}), \\
\boldsymbol{x}_{1j}^{(g)} &\sim N_{p_1}(\boldsymbol{\mu}_1^{(g)}, \Sigma_{11}) \ (g = 1, 2, \ j = N_1^{(g)} + 1, \ldots, N_1^{(g)} + N_2^{(g)}),
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{x}_{1j}^{(g)} &= (x_{j1}^{(g)}, \ldots, x_{j,p_1}^{(g)})' \ \ (g = 1, 2, \ j = 1, \ldots, N^{(g)}), \\
\boldsymbol{x}_{2j}^{(g)} &= (x_{j,p_1+1}^{(g)}, \ldots, x_{j,p}^{(g)})' \ \ (g = 1, 2, \ j = 1, \ldots, N_1^{(g)}), \\
N^{(g)} &= N_1^{(g)} + N_2^{(g)} \ \ (g = 1, 2), \\
\boldsymbol{\mu}^{(g)} &= \begin{pmatrix} \boldsymbol{\mu}_1^{(g)} \\ \boldsymbol{\mu}_2^{(g)} \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.
\end{aligned}$$

Then, $\boldsymbol{\mu}_\ell^{(g)}$ is p_ℓ–dimensional partitioned vector and $p_2 \equiv p - p_1$, $\Sigma_{\ell m}$ is $p_\ell \times p_m$ partitioned matrix.

The rest of this paper is organized as follows. Section 2 states the estimators of two-step monotone missing data. Section 3 derives an asymptotic approximation of the linear discriminant function. In Section 4, we perform the simulation studies to evaluate the main result. Section 5 concludes this paper and states the future problem. In Appendix, we present the main proofs.

2 The estimators based on two-step monotone missing data

In Section 2, we discuss the estimators based on two-step monotone missing data. Shutoh et al. [12] derived the MLEs under the similar setting for a data set. However, it is complicated to calculate an asymptotic expansion of W_2 using MLEs. Alternatively, in this paper, we consider the estimators similar to Shutoh [11] based on MLEs.

Then, we construct the respective estimators of $\boldsymbol{\mu}_i^{(g)}$ $(i = 1, 2)$ and

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{11}^{-1}\Sigma_{12} \\ \Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22\cdot 1} \end{pmatrix},$$

where $\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$. Similarly to the results obtained in Shutoh [11], the corrected MLEs can be also obtained as

$$\begin{aligned} \widehat{\boldsymbol{\mu}}_1^{(g)} &= \overline{\boldsymbol{x}}_1^{[g,2]}, \quad \widehat{\boldsymbol{\mu}}_2^{(g)} = \overline{\boldsymbol{x}}_2^{[g,1]} - \widehat{\Psi}_{21}(\overline{\boldsymbol{x}}_1^{[g,1]} - \widehat{\boldsymbol{\mu}}_1^{(g)}), \\ \widehat{\Psi}_{11} &= \frac{1}{n}(\Gamma_{11}^{(1)} + \Gamma^{(2)}), \quad \widehat{\Psi}_{12} = (\Gamma_{11}^{(1)})^{-1}\Gamma_{12}^{(1)}, \quad \widehat{\Psi}_{22} = \frac{1}{n_1}\Gamma_{22\cdot 1}^{(1)}, \\ \Gamma_{22\cdot 1}^{(1)} &= \Gamma_{22}^{(1)} - \Gamma_{21}^{(1)}(\Gamma_{11}^{(1)})^{-1}\Gamma_{12}^{(1)}, \end{aligned}$$

where

$$\begin{aligned} \overline{\boldsymbol{x}}_1^{[g,1]} &= \frac{1}{N_1^{(g)}}\sum_{j=1}^{N_1^{(g)}} \boldsymbol{x}_{1j}^{(g)}, \quad \overline{\boldsymbol{x}}_2^{[g,1]} = \frac{1}{N_2^{(g)}}\sum_{j=N_1^{(g)}+1}^{N^{(g)}}, \quad \overline{\boldsymbol{x}}_1^{[g,2]} = \frac{1}{N^{(g)}}\sum_{j=1}^{N^{(g)}} \boldsymbol{x}_{1j}^{(g)}, \\ S_{11} &= \frac{1}{n_1}\sum_{g=1}^{2}\sum_{j=1}^{N_1^{(g)}} (\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_1^{[g,1]})(\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_1^{[g,1]})', \\ S_{11}^{(2)} &= \frac{1}{n_2}\sum_{g=1}^{2}\sum_{j=N_1^{(g)}+1}^{N^{(g)}} (\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_1^{(g,2)})(\boldsymbol{x}_{1j}^{(g)} - \overline{\boldsymbol{x}}_1^{(g,2)})', \\ n_2 &= N_2^{(1)} + N_2^{(2)} - 2, \quad n = n_1 + n_2, \\ \Gamma^{(1)} &= \begin{pmatrix} \Gamma_{11}^{(1)} & \Gamma_{12}^{(1)} \\ \Gamma_{21}^{(1)} & \Gamma_{22}^{(1)} \end{pmatrix} = n_1 S, \quad \Gamma^{(2)} = n_2 S_{11}^{(2)}. \end{aligned}$$

3 An asymptotic expansion of the linear discriminant function

In Section 3, we construct the linear discriminant function and derive the asymptotic expansion of W_2 similar to Shutoh [11] under a large-sample framework, i,e.,

$$\text{fixed } p,\ N_i^{(g)} \to \infty\ (g = 1, 2,\ i = 1, 2),$$
$$N_1^{(2)}/N_1^{(1)} \to \text{positive const.},\ N^{(2)}/N^{(1)} \to \text{positive const.}$$

Using Shutoh's [11] technique, W_2 has another expression using W_1:

$$\begin{aligned} W_2 = W_1 &+ (\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]})'\widehat{\Psi}_{11}^{-1}(\boldsymbol{x}_1 - \overline{\boldsymbol{x}}_1^{[1,2]}) + \frac{1}{2}(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]})'\widehat{\Psi}_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]}) \\ &- (\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]})'S_{11}^{-1}(\boldsymbol{x}_1 - \overline{\boldsymbol{x}}_1^{[1,1]}) - \frac{1}{2}(\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]})'S_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]}), \end{aligned}$$

where

$$W_1 = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'S^{-1}(\boldsymbol{x} - \overline{\boldsymbol{x}}^{(2)}) + \frac{1}{2}(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'S^{-1}(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}).$$

In order to derive the asymptotic expansion of W_2, we consider the characteristic function $C(t)$ of $(W_2 - (1/2)\Delta^2)\Delta^{-1}$ under $\boldsymbol{x} \in \Pi^{(1)}$ as

$$C(t) = \mathrm{E}\Big[\exp\Big\{it\Delta^{-1}\Big(W_2 - \frac{1}{2}\Delta^2\Big)\Big|\boldsymbol{x} \in \Pi^{(1)}\Big\}\Big].$$

Then, the expectation with respect to $\boldsymbol{x}$ can be represented as

$$\begin{aligned} &\mathrm{E}_{\boldsymbol{x}}\left[\exp\Big(it\Delta^{-1}(W_2 - \frac{1}{2}\Delta^2)\Big)\Big|\boldsymbol{x} \in \Pi^{(1)}\right] \\ &= \exp\left(-\frac{1}{2}it\Delta - it\Delta^{-1}(-\boldsymbol{a}_1'\boldsymbol{\mu}^{(1)} + a_2) - \frac{1}{2}t^2\Delta^{-2}\boldsymbol{a}_1'\boldsymbol{a}_1 \right), \end{aligned}$$

where

$$\boldsymbol{a}_1' = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'S^{-1} + \Big[(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_2^{[2,2]})'\widehat{\Psi}_{11}^{-1} - (\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_2^{[2,1]})'S_{11}^{-1}\Big]B',$$

$$
\begin{aligned}
B &= \begin{pmatrix} I_{p_1} & O_{p_1\times p_2} \end{pmatrix}, \\
a_2 &= \frac{1}{2}(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-1} (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}) \\
&\quad + \frac{1}{2}\Big[(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_2^{[2,2]})'\widehat{\Psi}_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_2^{[2,2]}) - (\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_2^{[2,1]})' S_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_2^{[2,1]})\Big],
\end{aligned}
$$

$O_{p_1\times p_2}$ denotes $p_1 \times p_2$ matrix with 0's. Therefore, the expectation with respect to $\boldsymbol{x}$ can be represented as follows:

$$
\exp\left(-\frac{1}{2} it\Delta - it\Delta^{-1} b_1 + \frac{1}{2} it\Delta^{-2} b_2 - \frac{1}{2} t^2 \Delta^{-2} b_3 \right),
$$

where

$$
\begin{aligned}
b_1 &= (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-1} (\overline{\boldsymbol{x}}^{(1)} - \boldsymbol{\mu}^{(1)}) + (\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]})'\widehat{\Psi}_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,2]} - \boldsymbol{\mu}_1^{(1)}) \\
&\quad - (\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]})' S_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,1]} - \boldsymbol{\mu}_1^{(1)}), \\
b_2 &= (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-1} (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}) + (\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]})'\widehat{\Psi}_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]}) \\
&\quad - (\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]})' S_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]}), \\
b_3 &= (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-2} (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}) + 2\Big\{(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-1} B' \widehat{\Psi}_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]}) \\
&\quad - (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-1} B' S_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]})\Big\} + (\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]})'\widehat{\Psi}_{11}^{-2}(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]}) \\
&\quad - 2(\overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]})'\widehat{\Psi}_{11}^{-1} S_{11}^{-1}(\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]}) + (\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]}) S^{-2}(\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]}).
\end{aligned}
$$

We use the following random vectors and matrices:

$$
\begin{aligned}
&\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)} = \boldsymbol{\delta} + \frac{1}{\sqrt{n\rho_1}}\boldsymbol{z}^{[1]}, \quad \overline{\boldsymbol{x}}^{(1)} - \boldsymbol{\mu}^{(1)} = \frac{1}{\sqrt{n\rho_1}}\boldsymbol{y}^{[1]}, \\
&\overline{\boldsymbol{x}}_1^{[1,1]} - \overline{\boldsymbol{x}}_1^{[2,1]} = \boldsymbol{\delta}_1 + \frac{1}{\sqrt{n\rho_1}}\boldsymbol{z}_1^{[1]}, \quad \overline{\boldsymbol{x}}_1^{[1,2]} - \overline{\boldsymbol{x}}_1^{[2,2]} = \boldsymbol{\delta}_1 + \frac{1}{\sqrt{n}}\boldsymbol{z}_1^{[2]}, \\
&\overline{\boldsymbol{x}}_1^{[1,1]} - \boldsymbol{\mu}_1^{(1)} = \frac{1}{\sqrt{n\rho_1}}\boldsymbol{y}_1^{[1]}, \quad \overline{\boldsymbol{x}}_1^{[1,2]} - \boldsymbol{\mu}_1^{(1)} = \frac{1}{\sqrt{n}}\boldsymbol{y}_1^{[2]}, \\
&S = I_p + \frac{1}{\sqrt{n\rho_1}} T^{(1)}, \\
&S_{11} = I_{p_1} + \frac{1}{\sqrt{n\rho_1}} T_{11}^{(1)}, \quad S_{11}^{(2)} = I_{p_1} + \frac{1}{\sqrt{n\rho_2}} T_{11}^{(2)},
\end{aligned}
$$

where

$$\boldsymbol{\delta} = \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}, \boldsymbol{\delta}_1 = \boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}, \boldsymbol{\delta}_2 = \boldsymbol{\mu}_2^{(1)} - \boldsymbol{\mu}_2^{(2)}, \rho_1 = \frac{n_1}{n}, \rho_2 = \frac{n_2}{n}.$$

By using the above notations and

$$\begin{aligned}
\left(I + \frac{1}{\sqrt{m}}A\right)^{-1} &= I + \sum_{j=1}^{\infty}(-1)^j m^{-j/2} A^j, \\
\left(I + \frac{1}{\sqrt{m}}A\right)^{-2} &= I + \sum_{j=1}^{\infty}(-1)^j (j+1) m^{-j/2} A^j,
\end{aligned}$$

we can obtain the following terms of b_{ij} $(i = 1, 2, 3,\ j = 1, 2, 3, 4)$:

$$\begin{aligned}
b_1 &= \frac{1}{\sqrt{n}}b_{11} + \frac{1}{n}b_{12} + \frac{1}{n\sqrt{n}}b_{13} + \frac{1}{n^2}b_{14} + \mathrm{o}(n^{-2}), \\
b_2 &= \Delta^2 + \frac{1}{\sqrt{n}}b_{21} + \frac{1}{n}b_{22} + \frac{1}{n\sqrt{n}}b_{23} + \frac{1}{n^2}b_{24} + \mathrm{o}(n^{-2}), \\
b_3 &= \Delta^2 + \frac{1}{\sqrt{n}}b_{31} + \frac{1}{n}b_{32} + \frac{1}{n\sqrt{n}}b_{23} + \frac{1}{n^2}b_{34} + \mathrm{o}(n^{-2}).
\end{aligned}$$

Thus, by using Taylor expansion, we can also obtain

$$\begin{aligned}
C(t) = e^{\frac{1}{2}\theta^2}\mathrm{E}\Bigg[1 &+ \frac{1}{\sqrt{n}}\left\{\sum_{j=1}^{2}\theta^j\Delta^{-j}\beta_{1j}\right\} + \frac{1}{n}\left\{\sum_{j=1}^{4}\theta^j\Delta^{-j}\beta_{2j}\right\} \\
&+ \frac{1}{n\sqrt{n}}\left\{\sum_{j=1}^{6}\theta^j\Delta^{-j}\beta_{3j}\right\} + \frac{1}{n^2}\left\{\sum_{j=1}^{8}\theta^j\Delta^{-j}\beta_{4j}\right\}\Bigg] + \mathrm{o}(n^{-2}),
\end{aligned}$$

where

$$\begin{aligned}
&\theta = -it, \\
&\beta_{11} = b_{11} - \frac{1}{2}b_{21}, \quad \beta_{12} = \frac{1}{2}b_{31}, \\
&\beta_{21} = b_{12} - \frac{1}{2}b_{22}, \quad \beta_{22} = \frac{1}{8}(4b_{32} + 4b_{11}^2 - 4b_{11}b_{21} + b_{21}^2), \\
&\beta_{23} = \frac{1}{4}(2b_{11}b_{31} - b_{21}b_{31}), \quad \beta_{24} = \frac{1}{8}b_{31}^2, \\
&\beta_{31} = b_{13} - \frac{1}{2}b_{23}, \quad \beta_{32} = \frac{1}{4}(2b_{33} + 4b_{11}b_{12} + b_{21}b_{22} - 2b_{11}b_{22} - 2b_{21}b_{12}),
\end{aligned}$$

$$\beta_{33} = \frac{1}{48}(24b_{11}b_{32} - 12b_{21}b_{32} + 24b_{31}b_{12} - 12b_{31}b_{22} + 8b_{11}^3 - b_{21}^3 - 12b_{11}^2 b_{21} + 6b_{11}b_{21}^2),$$
$$\beta_{34} = \frac{1}{16}(4b_{31}b_{32} + 4b_{11}^2 b_{31} + b_{31}b_{21}^2 - 4b_{11}b_{21}b_{31}),$$
$$\beta_{35} = \frac{1}{16}(2b_{11}b_{31}^2 - b_{21}b_{31}^2), \quad \beta_{36} = \frac{1}{48}b_{31}^3,$$
$$\beta_{41} = b_{14} - \frac{1}{2}b_{24},$$
$$\beta_{42} = \frac{1}{8}(4b_{34} + 4b_{12}^2 + b_{22}^2 - 4b_{12}b_{22} + 8b_{11}b_{13} + 2b_{21}b_{23} - 4b_{11}b_{23} - 4b_{21}b_{13}),$$
$$\beta_{43} = \frac{1}{16}(8b_{12}b_{32} - 4b_{22}b_{32} + 8b_{11}b_{33} - 4b_{21}b_{33} + 8b_{31}b_{13} - 4b_{31}b_{23} + 8b_{11}^2 b_{12} - 4b_{11}^2 b_{22} + 2b_{21}^2 b_{12} - b_{21}^2 b_{22} - 8b_{11}b_{21}b_{12} + 4b_{11}b_{21}b_{22}),$$
$$\beta_{44} = \frac{1}{384}(48b_{32}^2 + 96b_{31}b_{33} + 96b_{11}^2 b_{32} + 24b_{21}^2 b_{32} - 96b_{11}b_{21}b_{32} + 192b_{11}b_{31}b_{12} - 96b_{11}b_{31}b_{22} - 96b_{21}b_{31}b_{12} + 48b_{21}b_{31}b_{22} + 16b_{11}^4 + b_{21}^4 - 32b_{11}^3 b_{21} + 24b_{11}^2 b_{21}^2 - 8b_{11}b_{21}^3),$$
$$\beta_{45} = \frac{1}{96}(12b_{31}^2 b_{12} - 6b_{31}^2 b_{22} + 24b_{11}b_{31}b_{32} - 12b_{21}b_{31}b_{32} + 8b_{11}^3 b_{31} - 12b_{11}^2 b_{21}b_{31} + 6b_{11}b_{21}^2 b_{31} - b_{21}^3 b_{31}),$$
$$\beta_{46} = \frac{1}{64}(4b_{31}^2 b_{32} + 4b_{11}^2 b_{31}^2 - 4b_{11}b_{21}b_{31}^2 + b_{21}^2 b_{31}^2),$$
$$\beta_{47} = \frac{1}{96}(2b_{11}b_{31}^3 - b_{21}b_{31}^3), \quad \beta_{48} = \frac{1}{384}b_{31}^4.$$

Then, we use the following Lemmas in order to derive an asymptotic expansion of W_2.

Lemma 3.1. *Suppose that $\boldsymbol{x}$ has $N_p(\boldsymbol{\mu}, \Sigma)$. Let both A and B be $p \times p$ constant matrices, respectively. Then the following expectations can be obtained:*

$$\mathrm{E}[\boldsymbol{x}'A\boldsymbol{x}] = \mathrm{tr}(\Sigma)A + \boldsymbol{\mu}'A\boldsymbol{\mu},$$
$$\mathrm{E}\Big[\boldsymbol{x}'A\boldsymbol{x}\boldsymbol{x}'B\boldsymbol{x}\Big] = \mathrm{tr}(\Sigma B'\Sigma A') + \mathrm{tr}(A\Sigma)\mathrm{tr}(B\Sigma) + \mathrm{tr}(A\Sigma B'\Sigma) + \mathrm{tr}(\Sigma B)\boldsymbol{\mu}'A\boldsymbol{\mu} + 2\boldsymbol{\mu}'A\Sigma B\boldsymbol{\mu} + 2\boldsymbol{\mu}'A\Sigma B'\boldsymbol{\mu} + \mathrm{tr}(A\Sigma)\boldsymbol{\mu}'B\boldsymbol{\mu} + \boldsymbol{\mu}'A\boldsymbol{\mu}\boldsymbol{\mu}'B\boldsymbol{\mu}.$$

Lemma 3.2. *Let A, B and C be $p \times p$ constant matrices, respectively. Then the following expectations of the random matrix $T^{(1)}$ can be obtained as follows*:

$$\mathrm{E}\Big[T^{(1)}AT^{(1)}BT^{(1)}\Big] = \frac{1}{\sqrt{n_1}}\Big(A'B' + BA + B'A + BA' + \mathrm{tr}(A)B' + \mathrm{tr}(B)A' + \mathrm{tr}(AB')I_p + \mathrm{tr}(A)\mathrm{tr}(B)I_p\Big),$$
$$\mathrm{E}\Big[T^{(1)}AT^{(1)}BT^{(1)}CT^{(1)}\Big]$$

$$= \Big(A'BC' + B'C'A + CA'B' + CBA + C'BA' + \mathrm{tr}(A)BC' + \mathrm{tr}(B)C'A' + \mathrm{tr}(C)A'B + \mathrm{tr}(CA')B' + \mathrm{tr}(C)\mathrm{tr}(A)B + \mathrm{tr}(AB'C)I_p + \mathrm{tr}(B)\mathrm{tr}(CA)I_p\Big) + \mathrm{o}(n_1^{-1}).$$

For a proof, refer to Appendix.

The expectations of β_{ij} $(i = 1,2,3,4,\ j = 1,2,3,4,5,6,7,8)$ can be derived by using both Lemma 3.1 and Lemma 3.2, and we can obtain the following Theorem.

Theorem 3.3. *The cumulative distribution function of* $(W_k - (1/2)\Delta^2)\Delta^{-1}$ *under* $\boldsymbol{x} \in \Pi^{(1)}$ *is expressed as*

$$\begin{aligned}
&\Pr\Big((W_2 - 1/2\Delta^2)\Delta^{-1} \le u \mid \boldsymbol{x} \in \Pi^{(1)}\Big) \\
&= \Bigg[1 + \frac{f_1}{2N_1^{(1)}\Delta^2} + \frac{f_2}{2N_1^{(2)}\Delta^2} + \frac{f_3}{4n_1} \\
&\quad + \frac{f_{11}}{4(N_1^{(1)})^2\Delta^4} + \frac{f_{22}}{4(N_1^{(2)})^2\Delta^4} + \frac{f_{12}}{2N_1^{(1)}N_1^{(2)}\Delta^4} \\
&\quad + \frac{f_{13}}{2n_1N_1^{(1)}\Delta^2} + \frac{f_{23}}{2n_1N_1^{(2)}\Delta^2} + \frac{f_{33}}{12n_1^2} \\
&\quad + \frac{1}{2\Delta^2}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)g_{4,1} + \frac{1}{2\Delta^2}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)g_{5,2} + \frac{1}{4}\left(\frac{1}{n} - \frac{1}{n_1}\right)g_{6,3} \\
&\quad + \frac{1}{4N_1^{(1)}\Delta^4}\left(\frac{1}{N_1^{(1)}} - \frac{1}{N^{(1)}}\right)h_{1(1,4)} + \frac{1}{4N_1^{(2)}\Delta^4}\left(\frac{1}{N_1^{(2)}} - \frac{1}{N^{(2)}}\right)h_{2(2,5)} \\
&\quad + \frac{1}{4N^{(1)}\Delta^4}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{4(4,1)} + \frac{1}{4N^{(2)}\Delta^4}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{5(5,2)} \\
&\quad + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N^{(1)}}\right)^2 - \left(\frac{1}{N_1^{(1)}}\right)^2\right\}h_{(44,11)} + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N^{(2)}}\right)^2 - \left(\frac{1}{N_1^{(2)}}\right)^2\right\}h_{(55,22)} \\
&\quad + \frac{1}{4N_1^{(1)}\Delta^4}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{1(5,2)} + \frac{1}{4N_1^{(2)}\Delta^4}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{2(4,1)} \\
&\quad + \frac{1}{4N^{(1)}\Delta^4}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{4(5,2)} + \frac{1}{4N^{(2)}\Delta^4}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{5(4,1)} \\
&\quad + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N^{(1)}}\right)\left(\frac{1}{N_1^{(2)}}\right) - \left(\frac{1}{N_1^{(1)}}\right)\left(\frac{1}{N^{(2)}}\right)\right\}h_{(42,15)} \\
&\quad + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N_1^{(1)}}\right)\left(\frac{1}{N_1^{(2)}}\right) - \left(\frac{1}{N^{(1)}}\right)\left(\frac{1}{N^{(2)}}\right)\right\}h_{(12,45)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4\Delta^2}\left(\frac{1}{n}\cdot\frac{1}{N^{(1)}}-\frac{1}{n_1}\cdot\frac{1}{N_1^{(1)}}\right)h_{(64,31)}+\frac{1}{4\Delta^2}\left(\frac{1}{n}\cdot\frac{1}{N^{(2)}}-\frac{1}{n_1}\cdot\frac{1}{N_1^{(2)}}\right)h_{(65,32)}\\
&+\frac{1}{4N^{(1)}\Delta^2}\Big(\frac{1}{n}-\frac{1}{n_1}\Big)h_{4(6,3)}+\frac{1}{4N^{(2)}\Delta^2}\Big(\frac{1}{n}-\frac{1}{n_1}\Big)h_{5(6,3)}\\
&+\frac{1}{4N_1^{(1)}\Delta^2}\Big(\frac{1}{n}-\frac{1}{n_1}\Big)h_{1(6,3)}+\frac{1}{4N_1^{(2)}\Delta^2}\Big(\frac{1}{n}-\frac{1}{n_1}\Big)h_{2(6,3)}\\
&+\frac{1}{4n_1\Delta^2}\left(\frac{1}{N^{(1)}}-\frac{1}{N_1^{(1)}}\right)h_{3(4,1)}+\frac{1}{4n_1\Delta^2}\left(\frac{1}{N^{(2)}}-\frac{1}{N_1^{(2)}}\right)h_{3(5,2)}\\
&+\frac{1}{4n\Delta^2}\left(\frac{1}{N^{(1)}}-\frac{1}{N_1^{(1)}}\right)h_{6(4,1)}+\frac{1}{4n\Delta^2}\left(\frac{1}{N^{(2)}}-\frac{1}{N_1^{(2)}}\right)h_{6(5,2)}\\
&+\Big(\frac{1}{n_1^2}-\frac{1}{n}\cdot\frac{1}{n_1}\Big)h_{(33,63)}+\Big(\frac{1}{n}\cdot\frac{1}{n_1}-\frac{1}{n^2}\Big)h_{(63,66)}\\
&+\Big(\frac{1}{n_1^2}-\frac{1}{n^2}\Big)h_{(33,66)}\Bigg]\Phi(u)+\mathrm{O}_3,
\end{aligned}
$$

where

$$
\begin{aligned}
f_1 &= d^4+p(d^2+\Delta d),\\
f_2 &= (d^2-\Delta d)^2+p(d^2-\Delta d),\\
f_3 &= (2d^2-\Delta d)^2+2(p+1)(3d^2-\Delta d),\\
f_{11} &= 2d^4(d^2+\Delta d)+p(d^2+\Delta d)^2+\frac{1}{2}\Big\{d^4+p(d^2+\Delta d)\Big\}^2,\\
f_{22} &= 2(d^2-\Delta d)^3+p(d^2-\Delta d)^2+\frac{1}{2}\Big\{(d^2-\Delta d)^2+p(d^2-\Delta d)\Big\}^2,\\
f_{12} &= 2d^4(d^2-\Delta d)+pd^4+\frac{1}{2}\Big\{d^4+p(d^2+\Delta d)\Big\}\Big\{(d^2-\Delta d)^2+p(d^2-\Delta d)\Big\},\\
f_{13} &= 4d^4(2d^2-\Delta d)+2(5p+7)d^4-\Delta^2d^2+(p^2+p)(3d^2+\Delta d)\\
&\quad+\frac{1}{4}\Big\{d^4+p(d^2+\Delta d)\Big\}\Big\{(2d^4-\Delta d)^2+2(p+1)(3d^2-\Delta d)\Big\},\\
f_{23} &= 2(d^2-\Delta d)(2d^2-\Delta d)^2+2(5p+7)d^4-4(3p+4)\Delta d^3+(3p+4)\Delta^2d^2\\
&\quad+(p^2+p)(3d^2-\Delta d)+\frac{1}{4}\Big\{(d^2-\Delta d)^2+p(d^2-\Delta d)\Big\}\\
&\qquad\times\Big\{(2d^4-\Delta d)^2+2(p+1)(3d^2-\Delta d)\Big\},\\
f_{33} &= 2(2d^2-\Delta d)^2(7d^2-2\Delta d)+9(15p+13)d^4-24(4p+3)\Delta d^3+3(5p+3)\Delta^2d^2\\
&\quad+6(6p^2+13p+9)d^2-6(p+1)^2\Delta d+\frac{3}{8}\Big\{(2d^4-\Delta d)^2+2(p+1)(3d^2-\Delta d)\Big\}^2,\\
g_{4,1} &= \frac{\delta^2}{\Delta^2}d^4+p_1(d^2+\Delta d),
\end{aligned}
$$

$$
\begin{aligned}
g_{5,2} &= \frac{\delta^2}{\Delta^2}(d^2-\Delta d)^2 + p_1(d^2-\Delta d),\\
g_{6,3} &= \frac{\delta^4}{\Delta^4}(2d^2-\Delta d)^2 + 2\cdot\frac{\delta^2}{\Delta^2}(p_1+1)(3d^2-\Delta d),\\
h_{1(1,4)} &= \frac{1}{2}p_1(p_1-2p)(\Delta^2 d^2+2\Delta d^3+2d^4) - \Big(p_1+p\cdot\frac{\delta^2}{\Delta^2}\Big)(\Delta d^5+d^6)\\
&\quad + \frac{\delta^2}{\Delta^2}(2\Delta^3 d^5 + p_1\Delta d^5 - 2\Delta^2 d^6 - d^8) + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}\cdot d^8,\\
h_{2(2,5)} &= \frac{1}{2}p_1(p_1-2p)(\Delta^2 d^2 - 2\Delta d^3) + p_1(p_1-2)d^4 + \Big(p_1+p\cdot\frac{\delta^2}{\Delta^2}\Big)(\Delta d - d^2)^3\\
&\quad + 3p_1\cdot\frac{\delta^2}{\Delta^2}(\Delta^2 d^4 - \Delta d^5) - \Big(\frac{\delta^2}{\Delta^2}+\frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}\Big)(\Delta d - d^2)^4,\\
h_{4(4,1)} &= \frac{1}{2}p_1^2(\Delta^2 d^2 + 2\Delta d^3 + 2d^4) + p_1\delta^2\Delta^{-1}d^5 + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}\cdot d^8,\\
h_{5(5,2)} &= \frac{1}{2}p_1^2(\Delta^2 d^2 + 2d^4) + 2p_1\Delta d^3 + p_1\delta^2(2\Delta^2 d^2 - 3\Delta d^3) + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}\Big(\Delta d - d^2\Big)^4,\\
h_{(44,11)} &= p_1(\Delta^2 d^2 + 2\Delta d^3 + 2d^4) + 2\frac{\delta^2}{\Delta^2}(\Delta d^5 + d^6) + 2p_1\frac{\delta^2}{\Delta^2}\cdot d^6,\\
h_{(55,22)} &= p_1(\Delta^2 d^2 + 2\Delta d^3) + \frac{\delta^2}{\Delta^2}(-2\Delta^3 d^3 - 6\Delta d^5 + (p_1+1)d^6)\\
&\quad + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}(12\Delta^4 d^4 - 4\Delta d^7 + d^8),\\
h_{1(5,2)} &= \Big\{p_1(p_1-p)\Delta^2 d^2 - p_1(p_1+2p)d^4\Big\} + p_1(2\Delta^3 d^3 + 2\Delta^2 d^4 + d^6)\\
&\quad + \frac{\delta^2}{\Delta^2}\Big\{-p_1\Delta^3 d^3 - 2(p+p_1)\Delta d^5 + pd^6 - 2\Delta d^7 + d^8\Big\},\\
h_{2(4,1)} &= \Big\{-p_1(p-1)\Delta^2 d^2 + p_1(2p-p_1)d^4 + p_1 d^6\Big\} + \delta^2\Big\{(3p+2p_1)\Delta^3 d^3 - (p-p_1)\Delta^2 d^4\\
&\quad - 2\Delta d^5 + 2d^6\Big\} + \frac{\delta^2}{\Delta^2}\Big\{pd^6 - 2\Delta d^7 + d^8\Big\} - \frac{\delta^4}{2\Delta^4}d^8,\\
h_{4(5,2)} &= \Big\{-p_1^2\Delta^2 d^2 + p_1(p_1+4)d^4\Big\} + \frac{\delta^2}{\Delta^2}\Big\{p_1\Delta^3 d^3 - 2p_1\Delta d^5 - \Delta^2 d^6\Big\}\\
&\quad + \frac{\delta^4}{2\Delta^4}\Big\{4\Delta^2 d^6 - 4\Delta d^7 + d^8\Big\} - \frac{1}{2}\delta^4 d^4,\\
h_{5(4,1)} &= p_1^2 d^4 + \delta^2(2p_1\Delta d^3 - p_1 d^4 - d^6) + \frac{1}{2}\delta^4 d^4 + \frac{\delta^4}{\Delta^4}(2\Delta^3 d^5 + d^8),\\
h_{(42,15)} &= \Big\{p_1\Delta^3 d^3 - 2p_1\Delta^2 d^4 + p_1\Delta d^5\Big\} + \delta^2\Big\{(2p_1-p)\Delta d^3 + 3p\Delta^2 d^4 - p\Delta d^5\Big\} + \frac{\delta^4}{\Delta^2}\cdot 2\Delta d^5,\\
h_{(12,45)} &= 2p_1\Delta d^5 + \frac{\delta^2}{\Delta^2}\Big\{\Delta d^5 + 2(p_1-1)d^6\Big\} + \delta^2\cdot\Delta d^3,\\
h_{(64,31)} &= \Big\{4p_1(p_1+1)\Delta d + 2p_1(p_1+1)d^2\Big\} + \delta^2\Big\{-p(p_1+1) + 2p_1(p_1+1) - 2\Big\}d^2\\
&\quad + \frac{\delta^2}{\Delta^2}\Big\{-6p_1^2 - 2p_1 + 38\Big\}d^4,\\
h_{(65,32)} &= \Big\{2p_1(p_1+1)(\Delta d + d^2)\Big\} + \frac{\delta^2}{\Delta^2}\Big\{-4(p_1-2)\Delta d^3 - 2(3p_1^2+3p_1-2)d^4\Big\}
\end{aligned}
$$

$$+ 2\delta^2 p_1 d^2 + \frac{\delta^4}{\Delta^2} 38 d^4,$$

$$h_{4(6,3)} = \Big\{4p_1(p_1+1)d^2\Big\} + \frac{\delta^2}{\Delta^2}\Big\{\frac{1}{2}(3p_1+2)\Delta d^3 + 6p_1(p_1+2)d^4\Big\}$$
$$+ \frac{\delta^4}{\Delta^4}\Big\{-(p_1+9)\Delta d^5 + 2(p_1+3)d^6\Big\} + \frac{\delta^6}{\Delta^6}\Big\{-2\Delta d^7 + 2d^8\Big\}$$
$$+ \Big\{\frac{\delta^4}{\Delta^2}\frac{1}{2}(p_1+4)d^4\Big\} + \frac{\delta^6}{2\Delta^4}\cdot d^6,$$

$$h_{5(6,3)} = \Big\{4p_1(p_1+1)d^2\Big\} + \frac{\delta^2}{\Delta^2}\Big\{-4(3p_1+4)\Delta d^3 + 6(p_1^2+3p_1+4)d^4\Big\}$$
$$+ \frac{\delta^4}{\Delta^4}\Big\{-(9p_1+109)\Delta d^5 + (2p_1+27)d^6\Big\} + \frac{\delta^6}{\Delta^6}\Big\{-6\Delta d^7 + 2d^8\Big\}$$
$$+ \delta^2\Big\{(p_1+4)(p_1+1)d^2\Big\} + \frac{\delta^4}{\Delta^2}\Big\{\frac{1}{2}(p_1+8)\Delta d^3 + \frac{1}{2}d^4 + 22d^4\Big\}$$
$$+ \frac{\delta^6}{\Delta^4}\Big\{\frac{1}{2}\Delta^2 d^4 - 3\Delta d^5 + \frac{13}{2}d^6\Big\},$$

$$h_{1(6,3)} = \frac{\delta^2}{\Delta^2}(p_1+1)(2p\Delta d^3 + 6pd^4 - \Delta d^5 + 3d^6) + \frac{\delta^4}{\Delta^4}\Big\{(2p-2p_1-3)d^6 - 2\Delta d^7 + 2d^8\Big\}$$
$$+ \frac{\delta^6}{\Delta^6}\Big\{2\Delta d^7 - 2d^8\Big\} + \frac{\delta^4}{\Delta^2}\Bigg\{(2p-3p_1-3)\Delta d^3 - \frac{1}{4}(29p_1+24)d^4$$
$$+ \frac{1}{2}(p-2p_1-2)\Delta d^5 + \frac{1}{2}d^6\Bigg\} - \frac{1}{2}\cdot\frac{\delta^6}{\Delta^4}d^6,$$

$$h_{2(6,3)} = \frac{\delta^2}{\Delta^2}\Big\{(p_1+1)(-4p\Delta d^3 + 6pd^4 - 7\Delta d^5 + 3d^6)\Big\}$$
$$+ \frac{\delta^4}{\Delta^4}\Big\{(-4p+3p_1-1)\Delta d^5 + (2p-2p_1-3)d^6 - 6\Delta d^7 + 2d^8\Big\} - \frac{\delta^6}{\Delta^6}2d^8$$
$$+ \delta^2(p_1+1)\Big\{(p_1-p)d^2 - \Delta d^3 + 5d^4\Big\} + \frac{\delta^4}{\Delta^2}\Big\{\frac{1}{2}(p-1)\Delta d^3 + \frac{1}{2}(5p-p_1)d^4$$
$$+ \frac{1}{2}\Delta^2 d^4 - 3\Delta d^5 + \frac{13}{2}d^6\Big\} + \frac{\delta^6}{\Delta^4}\Big\{\frac{1}{2}\Delta^2 d^4 + 3\Delta d^5 - \frac{13}{2}d^6\Big\},$$

$$h_{3(4,1)} = \Big\{-p_1(p+1)\Delta^2 d^2 + 2p_1(p_1+1)\Delta d^3 + \frac{1}{2}p_1\Delta^3 d^3 + 6p_1(p+1)d^4$$
$$- \frac{3}{2}p_1\Delta^2 d^4 + 2p_1 d^6\Big\} + \frac{\delta^2}{\Delta^2}\Big\{-(2p_1^2+23p_1+21)\Delta d^3 + 2(6p+7)d^4$$
$$-(p+9)\Delta d^5 + 3(p+3)d^6 - 2\Delta d^7 + 2d^8\Big\} + \delta^2\Big\{(p+1)\Delta d^3 + 2pd^4 + \frac{1}{2}d^6\Big\}$$
$$+ \frac{\delta^4}{\Delta^2}\Big\{-(p_1+1)\Delta d^3 + 2p_1 d^4\Big\},$$

$$h_{3(5,2)} = p_1\Big\{(p+1)(\Delta^2 d^2 - 4\Delta d^3 + 6d^4) - 4\Delta d^3 + \frac{1}{2}(-\Delta^3 d^3 + 5\Delta^2 d^4 - 8\Delta d^5 + 4d^6)\Big\}$$
$$+ \frac{\delta^2}{\Delta^2}\Big\{4(p+1)\Delta^2 d^2 + (4p_1^2+4p_1-9p-13)\Delta d^3 + 12(p+1)d^4 + 5(p+5)\Delta^2 d^4$$
$$-(10p+41)\Delta d^5 + 3(p+9)d^6 + \frac{13}{2}\Delta^2 d^6 - 6\Delta d^7 + 2d^8\Big\}$$

$$
\begin{aligned}
&\qquad + \frac{\delta^4}{\Delta^4}\Big\{\Big(\frac{1}{2}p - p_1 - 1\Big)\Delta d^3 - 2p_1\Delta^2 d^4 + 6(p_1+1)\Delta d^5\Big\} + \frac{1}{2}\Delta^2\delta^2 d^4,\\
h_{6(4,1)} &= \frac{\delta^2}{\Delta^2}\Big\{2p_1(p_1+1)\Delta d^3 + 12(p_1+1)d^4\Big\} - \frac{\delta^4}{\Delta^2}(p_1+1)\Delta d^3,\\
h_{6(5,2)} &= \frac{\delta^2}{\Delta^2}\Big\{-4p_1(p_1+1)\Delta d^3\Big\} + \frac{\delta^4}{\Delta^4}\Big\{-(p_1+1)\Delta^3 d^3 + 4p_1 d^4 - 6(p_1+1)\Delta d^5\Big\}\\
&\quad + \frac{\delta^6}{\Delta^6}6\Delta d^7,\\
h_{(33,63)} &= \frac{\delta^2}{\Delta^2}\Big\{-(6+3p+4p_1+3pp_1)d^2 + \frac{1}{2}(9p+7p_1+3p+3pp_1)(\Delta d^3 - d^4)\\
&\qquad - (p_1+1)\Big(\frac{7}{8}\Delta^2 d^4 - \Delta d^5 + \frac{3}{2}d^6\Big) - \frac{1}{4}(1+p_1+p+pp_1)\Delta^2 d^2 + \frac{1}{16}\Delta^3 d^3\Big\}\\
&\quad + \frac{\delta^4}{\Delta^4}\Big\{\frac{1}{8}(-3-4p+2p_1+p_1^2)\Delta^2 d^2 + \frac{1}{2}(7-7p_1+9p-p_1^2)\Delta d^3\\
&\qquad + \frac{1}{8}(p+9)\Delta^3 d^3 + \frac{1}{2}(-31-18p+4p_1+5p_1^2)d^4 - 7\Delta^2 d^4 - \frac{1}{16}\Delta^4 d^4\\
&\qquad\quad + \frac{1}{2}(5p-7)\Delta d^5 + \frac{1}{2}(3p-6p_1-19)d^6 - \frac{5}{2}\Delta^2 d^6 + 2\Delta d^7 - d^8\Big\}\\
&\quad + \frac{\delta^6}{\Delta^6}\Big\{-\frac{1}{8}(p_1+1)\Delta^3 d^3 + \frac{1}{8}(7p_1-61)\Delta^2 d^4 - (p_1+1)\Delta d^5 + \frac{1}{3}(42p_1+190)d^6\Big\}\\
&\quad + \frac{\delta^8}{\Delta^8}\Big\{\frac{1}{32}\Delta^4 d^4 - \frac{1}{4}\Delta^3 d^5 + \frac{3}{2}\Delta^2 d^6 - \Delta d^7 + \frac{1}{2}d^8\Big\},\\
h_{(63,66)} &= \frac{\delta^2}{\Delta^2}\Big\{-(3p_1^2+7p_1+6)d^2 + (p_1+1)\Delta d^3 + (8+8p_1+3p+3pp_1)\Delta^2 d^4\Big\}\\
&\quad + \frac{\delta^4}{4\Delta^4}\Big\{\frac{1}{2}(p_1^2+14p_1+29)\Delta^2 d^2 + (3p_1^2+14p_1+19)\Delta d^3\\
&\qquad - (5p_1^2+42p_1+25)d^4 + 16\Delta d^5\Big\}\\
&\quad + \frac{\delta^6}{24\Delta^6}\Big\{(3p_1+11)\Delta^3 d^3 - 3(7p_1+35)\Delta^2 d^4 - 24(2p_1+6)d^6\Big\}\\
&\quad + \frac{\delta^8}{32\Delta^8}\Big\{-\Delta^4 d^4 + 8\Delta^3 d^5 - 24\Delta^2 d^6 + 32\Delta d^7 - 16 d^8\Big\},\\
h_{(33,66)} &= \frac{\delta^2}{2\Delta^2}(5+5p_1+2p_1^2)\Delta d + \frac{\delta^2}{2\Delta^2}(p_1+3)d^2 + \frac{\delta^4}{2\Delta^4}(-37p_1-79)d^4 + \frac{\delta^6}{2\Delta^6}(p_1-3)d^6,
\end{aligned}
$$

where d denotes the differential operator d/du, $\delta^2 = (\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})'\Sigma_{11}^{-1}(\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)})$, u is a constant, and O_3 denotes the terms of the third order.

In addition, the cumulative distribution function of $(W_2 - (1/2)\Delta^2)\Delta^{-1}$ under $\boldsymbol{x} \in \Pi^{(2)}$ is also obtained by substituting $u = -u$ and interchanging $N_i^{(1)}$ and $N_i^{(2)}$ for $i = 1, 2$.

Corollary 3.4. *The probabilities of misclassification in linear discriminant function W_2 for $c = 0$ can be obtained as follows:*

$$
\begin{aligned}
&\Pr\Big(W_2 \le 0 | \boldsymbol{x} \in \Pi^{(1)}\Big) \\
&= \Phi\Big(-\frac{1}{2}\Delta\Big) + \frac{f_1^*}{2N_1^{(1)}\Delta^2} + \frac{f_2^*}{2N_1^{(2)}\Delta^2} + \frac{f_3^*}{2n_1} + \frac{f_{11}^*}{8(N_1^{(1)})^2\Delta^4} + \frac{f_{22}^*}{8(N_1^{(2)})^2\Delta^4} \\
&\quad + \frac{f_{12}^*}{4N_1^{(1)}N_1^{(2)}\Delta^4} + \frac{f_{13}^*}{4N_1^{(1)}n_1\Delta^2} + \frac{f_{23}^*}{4N_1^{(2)}n_1\Delta^2} + \frac{f_{33}^*}{8n_1^2} \\
&\quad + \Big(\frac{1}{2N^{(1)}\Delta^2} - \frac{1}{2N_1^{(1)}\Delta^2}\Big)g_{4,1}^* + \Big(\frac{1}{2N^{(2)}\Delta^2} - \frac{1}{2N_1^{(2)}\Delta^2}\Big)g_{5,2}^* + \Big(\frac{1}{2n} - \frac{1}{2n_1}\Big)g_{6,3}^* \\
&\quad + \frac{1}{4N_1^{(1)}\Delta^4}\left(\frac{1}{N_1^{(1)}} - \frac{1}{N^{(1)}}\right)h_{1(1,4)}^* + \frac{1}{4N_1^{(2)}\Delta^4}\left(\frac{1}{N_1^{(2)}} - \frac{1}{N^{(2)}}\right)h_{2(2,5)}^* \\
&\quad + \frac{1}{4N^{(1)}\Delta^4}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{4(4,1)}^* + \frac{1}{4N^{(2)}\Delta^4}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{5(5,2)}^* \\
&\quad + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N^{(1)}}\right)^2 - \left(\frac{1}{N_1^{(1)}}\right)^2\right\}h_{(44,11)}^* + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N^{(2)}}\right)^2 - \left(\frac{1}{N_1^{(2)}}\right)^2\right\}h_{(55,22)}^* \\
&\quad + \frac{1}{4N_1^{(1)}\Delta^4}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{1(5,2)}^* + \frac{1}{4N_1^{(2)}\Delta^4}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{2(4,1)}^* \\
&\quad + \frac{1}{4N^{(1)}\Delta^4}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{4(5,2)}^* + \frac{1}{4N^{(2)}\Delta^4}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{5(4,1)}^* \\
&\quad + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N^{(1)}}\right)\left(\frac{1}{N_1^{(2)}}\right) - \left(\frac{1}{N_1^{(1)}}\right)\left(\frac{1}{N^{(2)}}\right)\right\}h_{(42,15)}^* \\
&\quad + \frac{1}{4\Delta^4}\left\{\left(\frac{1}{N_1^{(1)}}\right)\left(\frac{1}{N_1^{(2)}}\right) - \left(\frac{1}{N^{(1)}}\right)\left(\frac{1}{N^{(2)}}\right)\right\}h_{(12,45)}^* \\
&\quad + \frac{1}{4\Delta^2}\left(\frac{1}{n}\cdot\frac{1}{N^{(1)}} - \frac{1}{n_1}\cdot\frac{1}{N_1^{(1)}}\right)h_{(64,31)}^* + \frac{1}{4\Delta^2}\left(\frac{1}{n}\cdot\frac{1}{N^{(2)}} - \frac{1}{n_1}\cdot\frac{1}{N_1^{(2)}}\right)h_{(65,32)}^* \\
&\quad + \frac{1}{4N^{(1)}\Delta^2}\Big(\frac{1}{n} - \frac{1}{n_1}\Big)h_{4(6,3)}^* + \frac{1}{4N^{(2)}\Delta^2}\Big(\frac{1}{n} - \frac{1}{n_1}\Big)h_{5(6,3)}^* \\
&\quad + \frac{1}{4N_1^{(1)}\Delta^2}\Big(\frac{1}{n} - \frac{1}{n_1}\Big)h_{1(6,3)}^* + \frac{1}{4N_1^{(2)}\Delta^2}\Big(\frac{1}{n} - \frac{1}{n_1}\Big)h_{2(6,3)}^* \\
&\quad + \frac{1}{4n_1\Delta^2}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{3(4,1)}^* + \frac{1}{4n_1\Delta^2}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{3(5,2)}^* \\
&\quad + \frac{1}{4n\Delta^2}\left(\frac{1}{N^{(1)}} - \frac{1}{N_1^{(1)}}\right)h_{6(4,1)}^* + \frac{1}{4n\Delta^2}\left(\frac{1}{N^{(2)}} - \frac{1}{N_1^{(2)}}\right)h_{6(5,2)}^* \\
&\quad + \Big(\frac{1}{n_1^2} - \frac{1}{n}\cdot\frac{1}{n_1}\Big)h_{(33,63)}^* + \Big(\frac{1}{n}\cdot\frac{1}{n_1} - \frac{1}{n^2}\Big)h_{(63,66)}^* + \Big(\frac{1}{n_1^2} - \frac{1}{n^2}\Big)h_{(33,66)}^* + \mathrm{O}_3,
\end{aligned}
$$

where $d_0^j = (d^j/du^j)\Phi(u)\Big|_{u=-\frac{1}{2}\Delta}$ $(j = 2, 4, 6, 8)$,

$$
\begin{aligned}
f_1^* &= d_0^4 + 3pd_0^2, \quad f_2^* = d_0^4 - (p-4)d_0^2, \quad f_3^* = (p-1)d_0^2,\\
f_{11}^* &= d_0^8 + 6(p+2)d_0^6 + (p+2)(9p+16)d_0^4 + 20p(p+2)d_0^2,\\
f_{22}^* &= d_0^8 - 2(p-10)d_0^6 + (p-6)(p-16)d_0^4 + 4(p-4)(p-6)d_0^2,\\
f_{12}^* &= d_0^8 + 2(p+8)d_0^6 - 3(p^2 - 10p - 16)d_0^4 - 12p(p-6)d_0^2,\\
f_{13}^* &= (p-1)\Big\{d_0^6 + 3(p+4)d_0^4 + 6(p+4)d_0^2\Big\},\\
f_{23}^* &= (p-1)\Big\{d_0^6 - 8(p-4)d_0^4 - 2(p-4)d_0^2\Big\},\\
f_{33}^* &= (p-1)\Big\{(p+1)d_0^4 + 4(p-12)d_0^2\Big\},\\
g_{4,1}^* &= \frac{\delta^2}{\Delta^2}\cdot d_0^4 + 3p_1d_0^2, \quad g_{5,2}^* = \frac{\delta^2}{\Delta^2}(d_0^4 + 4d_0^2) - p_1d_0^2, \quad g_{6,3}^* = \frac{\delta^2}{\Delta^2}\Big(p_1 + 1 - 2\cdot\frac{\delta^2}{\Delta^2}\Big)d_0^2,\\
h_{1(1,4)}^* &= \{-3p_1d_0^6 + p_1(5p_1 - 10p - 8)d_0^4 + 10p_1(p_1 - 2p)d_0^2\}\\
&\qquad + 8\cdot\delta^2d_0^4 + \frac{\delta^2}{\Delta^2}\{-d_0^8 + (2p_1 - 3p)d_0^6 + 8(p_1 - p)d_0^4\} + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}d_0^8,\\
h_{2(2,5)}^* &= \{p_1d_0^6 + p_1(p_1 + 10)d_0^4 + 2p_1(12 + p_1 - 2p)d_0^2\}\\
&\qquad + \frac{\delta^2}{\Delta^2}\{-d_0^8 + (7p_1 - 24)d_0^6 + 72(p_1 - 2)d_0^4 + 96(p_1 - 2)d_0^2\}\\
&\qquad\quad - \frac{\delta^4}{\Delta^4}(d_0^8 + 24d_0^6 + 144d_0^4 + 192d_0^2),\\
h_{4(4,1)}^* &= (5p_1^2d_0^4 + 10p_1^2d_0^2) + \frac{\delta^2}{\Delta^2}(2p_1d_0^6 + 8p_1d_0^4) + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}d_0^8,\\
h_{5(5,2)}^* &= \{p_1(3p_1 + 4)d_0^4 + 2p_1(3p_1 + 4)d_0^2\} + 2p_1\cdot\delta^2d_0^4 + \frac{1}{2}\cdot\frac{\delta^2}{\Delta^2}(d_0^8 + 24d_0^6 + 144d_0^4 + 192d_0^2),\\
h_{(44,11)}^* &= (10p_1d_0^4 + 20p_1d_0^2) + \frac{\delta^2}{\Delta^2}\{2(p_1 + 3)d_0^6 + 16d_0^4\},\\
h_{(55,22)}^* &= (6p_1d_0^4 + 16p_1d_0^2) + 6\cdot\delta^2d_0^4 + \frac{\delta^2}{\Delta^2}\{(p_1 - 27)d_0^6 - 190d_0^4 - 192d_0^2\}\\
&\qquad - \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}(7d_0^8 + 48d_0^6),\\
h_{1(5,2)}^* &= \{25p_1d_0^6 + p_1(3p_1 - 6p + 200)d_0^4 + 12p_1(p_1 - p + 20)d_0^2\}\\
&\qquad - \frac{\delta^2}{\Delta^2}\{3d_0^8 + 3(4p_1 + p + 8)d_0^6 + 8(11p_1 - 2p)d_0^4 - 92p_1d_0^2\}\\
&\qquad\quad + \frac{\delta^4}{\Delta^4}\Big\{\frac{15}{2}d_0^8 + 128d_0^6 + 456d_0^4 + 288d_0^2\Big\},\\
h_{2(4,1)}^* &= \{-p_1d_0^6 + p_1(2p + p_1 - 4)d_0^4 + 12p_1(p - p_1)d_0^2\}\\
&\qquad + \delta^2\{2d_0^6 - (5p + 5p_1 - 16)d_0^4 - (12p + 8p_1)d_0^2\}\\
&\qquad\quad + \frac{\delta^2}{\Delta^2}\{3d_0^8 + (24 - p)d_0^6\} + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}d_0^8,
\end{aligned}
$$

$$
\begin{aligned}
h^*_{4(5,2)} &= \{-p_1(3p_1-4)d_0^4 - 12p_1^2 d_0^2\} + \frac{\delta^2}{\Delta^2}\{-4d_0^8 + 4(p_1-11)d_0^6 + 8(7p_1-10)d_0^4 + 96p_1 d_0^2\} \\
&\quad - \frac{1}{2}\cdot\delta^4 d_0^4 + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}(9d_0^8 + 128d_0^6 + 320d_0^4), \\
h^*_{5(4,1)} &= p_1 d_0^4 + \delta^2(-d_0^6 + 3p_1 d_0^4 + 8p_1 d_0^2) + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}d_0^4 + \frac{\delta^4}{\Delta^4}(17d_0^8 + 240d_0^6 + 768d_0^4 + 384d_0^2), \\
h^*_{(42,15)} &= (2p_1 d_0^6 + 24p_1 d_0^4 + 48d_0^2) + \delta^2\{10pd_0^6 + 2(p_1+37)d_0^4 + 8(p_1+8)d_0^2\} \\
&\quad + \frac{\delta^4}{\Delta^2}(4d_0^6 + 16d_0^4), \\
h^*_{(12,45)} &= (4p_1 d_0^6 + 16p_1 d_0^4) + \delta^2(2d_0^4 + 4d_0^2) + \frac{\delta^2}{\Delta^2}(2p_1 d_0^6 + 8d_0^4), \\
h^*_{(64,31)} &= 10p_1(p_1+1)d_0^2 + \delta^2\{(2p_1-p)(p_1+1)-2\}d_0^2 - \frac{\delta^2}{\Delta^2}(6p_1^2 + 2p_1 - 38)d_0^4, \\
h^*_{(65,32)} &= 6p_1(p_1+1)d_0^2 - \frac{\delta^2}{\Delta^2}\{2(3p_1^2 + 7p_1 - 10)d_0^4 + 16(p_1-2)d_0^2\} \\
&\quad + 2p_1\cdot\delta^2 d_0^2 + 39\cdot\frac{\delta^4}{\Delta^2}d_0^4, \\
h^*_{4(6,3)} &= 4p_1(p_1+1)d_0^2 + \frac{\delta^2}{\Delta^2}\{(6p_1^2 + 15p_1 + 2)d_0^4 + (6p_1+4)d_0^2\} + \frac{1}{2}\cdot\frac{\delta^4}{\Delta^2}(p_1+4)d_0^4 \\
&\quad - \frac{\delta^4}{\Delta^4}\{12d_0^6 + 8(p_1+9)d_0^4\} + \frac{1}{2}\cdot\frac{\delta^6}{\Delta^4}d_0^6 - \frac{\delta^6}{\Delta^6}(2d_0^8 + 24d_0^6), \\
h^*_{5(6,3)} &= 4p_1(p_1+1)d_0^2 + (p_1+1)(p_1+4)\delta^2 d_0^2 + \frac{\delta^2}{\Delta^2}\{2(3p_1^2 + 3p_1 - 10)d_0^4 - 16(3p_1+4)d_0^2\} \\
&\quad + \frac{\delta^4}{\Delta^2}\left\{\frac{3}{2}(p_1+20)d_0^4 + 2(p_1+8)d_0^2\right\} - \frac{\delta^4}{\Delta^4}\{(20p_1+191)d_0^6 + 8(11p_1+109)d_0^4\} \\
&\qquad + \frac{\delta^6}{\Delta^4}\left\{\frac{5}{2}d_0^6 - 10d_0^4 + 12d_0^2\right\} - \frac{\delta^6}{\Delta^6}\{10d_0^8 + 72d_0^6\}, \\
h^*_{1(6,3)} &= \frac{\delta^2}{\Delta^2}\{(p_1+1)d_0^6 + 2(p_1+1)(5p-4)d_0^4 + 8p(p_1+1)d_0^2\} \\
&\quad + \frac{\delta^4}{\Delta^2}\left\{\frac{1}{2}(2p-4p_1-3)d_0^6 + \frac{1}{4}(32p-85p_1-64)d_0^4 + 4(2p-3p_1-1)d_0^2\right\} \\
&\qquad + \frac{\delta^4}{\Delta^4}\{-2d_0^8 + (2p-2p_1-27)d_0^6\} - \frac{1}{2}\cdot\frac{\delta^6}{\Delta^4}d_0^6 + \frac{\delta^6}{\Delta^6}(2d_0^8 + 24d_0^6), \\
h^*_{2(6,3)} &= \delta^2\{3(p_1+1)d_0^4 + (p_1+1)(p_1-p-4)d_0^2\} \\
&\quad - \frac{\delta^2}{\Delta^2}\{11(p_1+1)d_0^6 + 2(p_1+1)(p+42)d_0^4 + 16p(p_1+1)d_0^2\} \\
&\quad + \frac{\delta^4}{\Delta^2}\left\{\frac{5}{2}d_0^6 + \left(\frac{7}{2}p - \frac{1}{2}p_1 - 11\right)d_0^4 + 2(p+2)d_0^2\right\} \\
&\quad + \frac{\delta^4}{\Delta^4}\{-10d_0^8 + (-6p_1 + 4p_1 - 77)d_0^6 + 16(-4p + 3p_1 - 1)d_0^4\} \\
&\quad + \frac{\delta^6}{\Delta^4}\left\{\frac{3}{2}d_0^6 + 38d_0^4 + 12d_0^2\right\} - \frac{\delta^6}{\Delta^6}\cdot 2d_0^8, \\
h^*_{3(4,1)} &= \{2(pp_1 + 2p_1^2 + 3p + 3p_1)d_0^4 + 4(-3pp_1 + 2p_1^2 + 2p_1)d_0^2\}
\end{aligned}
$$

$$+\delta^2\Big\{\frac{1}{2}d_0^6+2(2p+1)d_0^4+4(p+1)d_0^2\Big\}-\frac{\delta^4}{\Delta^2}\{2d_0^4+4(p_1+1)d_0^2\},$$
$$-\frac{\delta^2}{\Delta^2}\{2d_0^8+(33-p)d_0^6+2(2p_1^2+23p_1-2p+50)d_0^4+4(p_1+1)(2p_1+21)d_0^2\},$$
$$h^*_{3(5,2)}=\Big\{\frac{1}{2}p_1(p-2)d_0^4-p_1(2p+3)d_0^2\Big\}+\delta^2\Big(\frac{1}{2}d_0^6+\frac{7}{2}d_0^4+3d_0^2\Big)$$
$$+\frac{\delta^2}{\Delta^2}\Big\{4d_0^8+\frac{1}{4}(3p+215)d_0^6+\frac{1}{2}(4p_1^2+4p_1+7p+435)d_0^4$$
$$+(4p_1^2+33p+4p_1+101)d_0^2\Big\}+\frac{\delta^4}{\Delta^4}\Big\{(p_1+3)d_0^6+\Big(2p_1-\frac{1}{4}p+\frac{23}{2}\Big)d_0^4$$
$$-\Big(\frac{1}{2}p+12p_1+1\Big)d_0^2\Big\},$$
$$h^*_{6(4,1)}=\frac{\delta^2}{\Delta^2}\{4(p_1+3)(p_1+1)d_0^4+8p_1(p_1+1)d_0^2\}-\frac{\delta^4}{\Delta^2}\{2(p_1+1)d_0^4+4(p_1+1)d_0^2\},$$
$$h^*_{6(5,2)}=-\frac{\delta^2}{\Delta^2}\{8p_1(p_1+1)d_0^4+16p_1(p_1+1)d_0^2\}-\frac{\delta^4}{\Delta^4}\{20(p_1+1)d_0^6+4(29p_1+30)d_0^4$$
$$+96(p_1+1)d_0^2\}+\frac{\delta^6}{\Delta^6}(12d_0^8+72d_0^6),$$
$$h^*_{(33,63)}=\frac{\delta^2}{\Delta^2}\Big\{-\frac{1}{2}(6p_1+5)d_0^6+\frac{1}{2}(pp_1+p-28p_1-17)d_0^4-2(7p_1+3)d_0^2\Big\},$$
$$-\frac{\delta^4}{\Delta^4}\Big\{8d_0^8-\frac{3}{2}(5p-2p_1-93)d_0^6+(6p_1^2+5p_1-37p+437)d_0^4$$
$$+\frac{1}{2}(33p_1^2+22p_1-48p+221)d_0^2\Big\}+\frac{\delta^6}{\Delta^6}\Big\{\frac{1}{6}(21p+66p_1+179)d_0^6$$
$$+\frac{1}{2}(49p-34p_1+15)d_0^4+(21p-12p_1-195)d_0^2\Big\}$$
$$+\frac{\delta^8}{\Delta^8}\Big\{3d_0^8+30d_0^6+\frac{123}{2}d_0^4-18d_0^2\Big\},$$
$$h^*_{(63,66)}=\frac{\delta^2}{\Delta^2}\{(10p_1+10+3p+3pp_1)d_0^4-(3p_1^2+3p_1+2)d_0^2\}$$
$$+\frac{\delta^4}{\Delta^4}\Big\{8d_0^6-\frac{1}{4}(3p_1^2+14p_1-196)d_0^4+\frac{1}{2}(9p_1^2+70p_1+125)d_0^2\Big\}$$
$$-\frac{\delta^6}{\Delta^6}\Big\{\frac{1}{6}(27p_1+119)d_0^6+\frac{1}{2}(31p_1+179)d_0^4+(9p_1+61)d_0^2\Big\}+\frac{\delta^8}{\Delta^8}\Big(\frac{3}{2}d_0^4+18d_0^2\Big),$$
$$h^*_{(33,66)}=\frac{\delta^2}{\Delta^2}(8+6p_1+2p_1^2)d_0^2-\frac{1}{2}\cdot\frac{\delta^4}{\Delta^4}(37p_1+79)d_0^4+\frac{1}{2}\cdot\frac{\delta^6}{\Delta^6}(p_1-3)d_0^2.$$

In addition, another probabilities of misclassification in linear discriminant function W_2 can be obtained by interchanging $N_i^{(1)}$ and $N_i^{(2)}$ for $i=1,2$.

4 Simulation studies

In this section, we perform Monte Carlo simulation in order to evaluatethe result stated in Theorem 3.3. In particular, we select some δ because δ depends on the result in the case of monotone missing data. We compare the accuracy of the result which is derived in Theorem 3.3 denoted by $\widehat{e}_K(2|1)$ with other asymptotic expansions, i.e., the result of Okamoto [8] denoted by $\widehat{e}_O(2|1)$ in the case of complete data, and Shutoh [11] denoted by $\widehat{e}_S(2|1)$ in the case of $k = 2$. As Monte Carlo simulation for $\widehat{e}_K(2|1)$, $\widehat{e}_O(2|1)$ and $\widehat{e}_S(2|1)$, we carry out 1,000,000 replications. Then, for the result of Okamoto [8], we use the estimators of Δ^2, i.e., $(n_1 - p - 1)D^2/n_1$, where $D^2 = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' S^{-1} (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})$. For the result of both Theorem 3.3 and Shutoh [11], we also use the estimators of $\widehat{\Delta}^2$ and $\widehat{\delta}^2$, i.e., $(n_1 - p - 1)d_{12}^2/n_1$ and $(n - p_1 - 1)d_{11}^2/n$, where $d_{12}^2 = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})' \widehat{\Sigma}^{-1} (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})$ and $d_{11}^2 = (\widehat{\boldsymbol{\mu}}_1^{(1)} - \widehat{\boldsymbol{\mu}}_1^{(2)})' \widehat{\Sigma}_{11}^{-1} (\widehat{\boldsymbol{\mu}}_1^{(1)} - \widehat{\boldsymbol{\mu}}_1^{(2)})$, respectively. Besides, the Mahalanobis distance is fixed as $\Delta = 1.05$ in all the tables and we select δ, dimensions, and sample sizes as follows:

$$\begin{aligned} \delta &= 0.42,\ 0.63,\ 1.00, \\ (p_1, p_2) &= (2,1),\ (3,2),\ (4,2),\ (4,3), \\ (M_1, M_2) &= (10,10),\ (20,20),\ (30,30),\ (40,40),\ (50,50), \end{aligned}$$

where $M_i = N_i^{(i)} = N_i^{(2)}$ $(i = 1, 2)$. Then, the results are presented in Table 1 – Table 6. For $\widehat{e}_O(2|1)$ and $e_1(2|1)$, we put $M_2 = 0$.

5 Conclusion and future problem

In this paper, we derived third moment and forth moment of Wishart matrix. By using these results, we also derived an asymptotic expansion for linear discriminant function W_2. Moreover, we compared our result with Okamoto's [8] expansion and Shutoh's [10] expansion by Monte Calro simulation. Then the expansion derived in this paper could be useful since we could observe our result provided more accurate approximation except for some cases. In the case of small sample sizes, it could be obtained that the both Okamoto's [8] and Shutoh's [10] expansions are more accurate than the derived result. Thereby, we can also consider the confidence interval for the misclassification probability.

Appendix. A

In Appendix, we prove Lemma 3.2. First, $T^{(1)}$ can be transformed as

$$T^{(1)} = \frac{1}{\sqrt{n_1}}(n_1 S) - \sqrt{n_1} I_p \ ,$$

where $n_1 S \sim W_p(n_1, \Sigma)$. Then, $T^{(1)} A T^{(1)} B T^{(1)}$ and $T^{(1)} A T^{(1)} B T^{(1)} C T^{(1)}$ can be represented as

$$T^{(1)} A T^{(1)} B T^{(1)} = \frac{1}{n_1 \sqrt{n_1}} (n_1 S) A (n_1 S) B (n_1 S) - \frac{1}{\sqrt{n_1}} (n_1 S) A (n_1 S) B$$

$$
\begin{aligned}
&- \frac{1}{\sqrt{n_1}}(n_1S)AB(n_1S) + \sqrt{n_1}(n_1S)AB \\
&- \frac{1}{\sqrt{n_1}}A(n_1S)B(n_1S) + \sqrt{n_1}A(n_1S)B \\
&+ \sqrt{n_1}AB(n_1S) - n_1\sqrt{n_1}AB, \\
T^{(1)}AT^{(1)}BT^{(1)}CT^{(1)} =& \frac{1}{n_1^2}(n_1S)A(n_1S)B(n_1S)C(n_1S) - \frac{1}{n_1}(n_1S)A(n_1S)B(n_1S)C \\
&- \frac{1}{n_1}(n_1S)A(n_1S)BC(n_1S) + (n_1S)A(n_1S)BC \\
&- \frac{1}{n_1}(n_1S)AB(n_1S)C(n_1S) + (n_1S)AB(n_1S)C \\
&+ (n_1S)ABC(n_1S) - n_1(n_1S)ABC \\
&- \frac{1}{n_1}A(n_1S)B(n_1S)C(n_1S) + A(n_1S)B(n_1S)C \\
&+ A(n_1S)BC(n_1S) - n_1A(n_1S)BC \\
&+ AB(n_1S)C(n_1S) - n_1AB(n_1S)C \\
&- n_1ABC(n_1S) + n_1^2ABC.
\end{aligned}
$$

Then, we calculate up to the fourth moment using the expectations of Wishart matrices and add each expectation, which completes the proof.

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Table 1
The comparisons of the accuracy of $\widehat{e}_K(2|1)$ and $\widehat{e}_O(2|1)$ when $\delta = 0.42$

(p_1, p_2)	(M_1, M_2)	$e_1(2\|1)$	$\widehat{e}_O(2\|1)$	$e_1(2\|1) - \widehat{e}_L(2\|1)$	$e_2(2\|1)$	$\widehat{e}_K(2\|1)$	$e_2(2\|1) - \widehat{e}_K(2\|1)$
(2, 1)	(10, 10)	0.345354	0.319227	0.026127	0.333407	0.411930	–0.078523
	(20, 20)	0.323210	0.310219	0.012991	0.315529	0.313745	0.001784
	(30, 30)	0.316426	0.306775	0.009651	0.311150	0.303559	0.007591
	(40, 40)	0.311045	0.305084	0.005961	0.307088	0.301464	0.005624
	(50, 50)	0.308153	0.303982	0.004171	0.305070	0.301061	0.004009
(3, 2)	(10, 10)	0.370663	0.301135	0.069528	0.359814	0.372791	–0.012977
	(20, 20)	0.339207	0.307855	0.031352	0.330774	0.308850	0.021924
	(30, 30)	0.326921	0.306821	0.020100	0.319958	0.301202	0.018756
	(40, 40)	0.320251	0.305709	0.014542	0.315276	0.300812	0.014464
	(50, 50)	0.317238	0.304783	0.012455	0.312765	0.300241	0.012524
(4, 2)	(10, 10)	0.380507	0.294575	0.085932	0.367689	0.382524	–0.014835
	(20, 20)	0.347129	0.306044	0.041085	0.335615	0.308860	0.026755
	(30, 30)	0.332856	0.306445	0.026411	0.324202	0.303738	0.020464
	(40, 40)	0.325367	0.305635	0.019732	0.318508	0.300409	0.018099
	(50, 50)	0.320455	0.304968	0.015487	0.314789	0.299307	0.015482
(4, 3)	(10, 10)	0.390690	0.288681	0.102009	0.381335	0.384429	–0.003094
	(20, 20)	0.353724	0.304364	0.049360	0.344795	0.313246	0.031549
	(30, 30)	0.338402	0.305800	0.032602	0.330668	0.300438	0.030230
	(40, 40)	0.330075	0.305515	0.024560	0.323583	0.299473	0.024110
	(50, 50)	0.323813	0.304940	0.018873	0.317719	0.298623	0.019096

Table 2
The comparisons of the accuracy of $\widehat{e}_K(2|1)$ and $\widehat{e}_S(2|1)$ when $\delta = 0.42$

(p_1, p_2)	(M_1, M_2)	$e_2(2\|1)$	$\widehat{e}_K(2\|1)$	$\widehat{e}_S(2\|1)$	$e_2(2\|1) - \widehat{e}_K(2\|1)$	$e_2(2\|1) - \widehat{e}_S(2\|1)$
(2, 1)	(10, 10)	0.333407	0.411930	0.300325	–0.078523	0.033082
	(20, 20)	0.315529	0.313745	0.299239	0.001784	0.016290
	(30, 30)	0.311150	0.303559	0.299168	0.007591	0.011982
	(40, 40)	0.307088	0.301464	0.299290	0.005624	0.007798
	(50, 50)	0.305070	0.301061	0.299315	0.004009	0.005755
(3, 2)	(10, 10)	0.359814	0.372791	0.291911	–0.012977	0.067903
	(20, 20)	0.330774	0.308850	0.293937	0.021924	0.036837
	(30, 30)	0.319958	0.301202	0.295507	0.018756	0.024451
	(40, 40)	0.315276	0.300812	0.296466	0.014464	0.018810
	(50, 50)	0.312765	0.300241	0.297054	0.012524	0.015711
(4, 2)	(10, 10)	0.367689	0.382524	0.288578	–0.014835	0.079111
	(20, 20)	0.335615	0.308860	0.291482	0.026755	0.044133
	(30, 30)	0.324202	0.303738	0.293725	0.020464	0.030477
	(40, 40)	0.318508	0.300409	0.294967	0.018099	0.023541
	(50, 50)	0.314789	0.299307	0.295898	0.015482	0.018891
(4, 3)	(10, 10)	0.381335	0.384429	0.284609	–0.003094	0.096726
	(20, 20)	0.344795	0.313246	0.288934	0.031549	0.055861
	(30, 30)	0.330668	0.300438	0.292123	0.030230	0.038545
	(40, 40)	0.323583	0.299473	0.293805	0.024110	0.029778
	(50, 50)	0.317719	0.298623	0.294856	0.019096	0.022863

Table 3
The comparisons of the accuracy of $\widehat{e}_K(2|1)$ and $\widehat{e}_O(2|1)$ when $\delta = 0.63$

(p_1, p_2)	(M_1, M_2)	$e_1(2\|1)$	$\widehat{e}_O(2\|1)$	$e_1(2\|1) - \widehat{e}_L(2\|1)$	$e_2(2\|1)$	$\widehat{e}_K(2\|1)$	$e_2(2\|1) - \widehat{e}_K(2\|1)$
(2, 1)	(10, 10)	0.345376	0.319216	0.026160	0.333453	0.401176	–0.067723
	(20, 20)	0.323116	0.310194	0.012922	0.316171	0.316184	–0.000013
	(30, 30)	0.316388	0.306759	0.009629	0.311141	0.305642	0.005499
	(40, 40)	0.311153	0.305076	0.006077	0.307473	0.302682	0.004791
	(50, 50)	0.308379	0.303962	0.004417	0.305032	0.301377	0.003655
(3, 2)	(10, 10)	0.370761	0.301031	0.069730	0.359543	0.382418	–0.022875
	(20, 20)	0.339401	0.307843	0.031558	0.330774	0.313614	0.017160
	(30, 30)	0.326806	0.306843	0.019963	0.320288	0.303739	0.016549
	(40, 40)	0.320942	0.305713	0.015229	0.315705	0.300915	0.014790
	(50, 50)	0.317379	0.304780	0.012599	0.312883	0.300664	0.012219
(4, 2)	(10, 10)	0.380427	0.294517	0.085910	0.367812	0.397328	–0.029516
	(20, 20)	0.347168	0.306047	0.041121	0.335361	0.315190	0.020171
	(30, 30)	0.332856	0.306445	0.026411	0.324202	0.303738	0.020464
	(40, 40)	0.325367	0.305635	0.019732	0.318508	0.300409	0.018099
	(50, 50)	0.320455	0.304968	0.015487	0.314789	0.299307	0.015482
(4, 3)	(10, 10)	0.390690	0.288681	0.102009	0.381335	0.384429	–0.003094
	(20, 20)	0.353724	0.304364	0.049360	0.344795	0.313246	0.031549
	(30, 30)	0.339085	0.305764	0.033321	0.330982	0.306670	0.024312
	(40, 40)	0.329931	0.305513	0.024418	0.323527	0.299588	0.023939
	(50, 50)	0.323286	0.304939	0.018347	0.317521	0.299212	0.018309

Table 4
The comparisons of the accuracy of $\widehat{e}_K(2|1)$ and $\widehat{e}_S(2|1)$ when $\delta = 0.63$

(p_1, p_2)	(M_1, M_2)	$e_2(2\|1)$	$\widehat{e}_K(2\|1)$	$\widehat{e}_S(2\|1)$	$e_2(2\|1) - \widehat{e}_K(2\|1)$	$e_2(2\|1) - \widehat{e}_S(2\|1)$
(2, 1)	(10, 10)	0.333453	0.401176	0.300138	–0.067723	0.033315
	(20, 20)	0.316171	0.316184	0.299083	–0.000013	0.017088
	(30, 30)	0.311141	0.305642	0.299047	0.005499	0.012094
	(40, 40)	0.307473	0.302682	0.299189	0.004791	0.008284
	(50, 50)	0.305032	0.301377	0.299219	0.003655	0.005813
(3, 2)	(10, 10)	0.359543	0.382418	0.291695	–0.022875	0.067848
	(20, 20)	0.330774	0.313614	0.293755	0.017160	0.037019
	(30, 30)	0.320288	0.303739	0.295394	0.016549	0.024894
	(40, 40)	0.315705	0.300915	0.296357	0.014790	0.019348
	(50, 50)	0.312883	0.300664	0.296962	0.012219	0.015921
(4, 2)	(10, 10)	0.367812	0.397328	0.288396	–0.029516	0.079416
	(20, 20)	0.335361	0.315190	0.291313	0.020171	0.044048
	(30, 30)	0.324202	0.303738	0.293725	0.020464	0.030477
	(40, 40)	0.318508	0.300409	0.294967	0.018099	0.023541
	(50, 50)	0.314789	0.299307	0.295898	0.015482	0.018891
(4, 3)	(10, 10)	0.381335	0.384429	0.284609	–0.003094	0.096726
	(20, 20)	0.344795	0.313246	0.288934	0.031549	0.055861
	(30, 30)	0.330982	0.306670	0.291690	0.024312	0.039292
	(40, 40)	0.323527	0.299588	0.293675	0.023939	0.029852
	(50, 50)	0.317521	0.299212	0.294759	0.018309	0.022762

Table 5
The comparisons of the accuracy of $\widehat{e}_K(2|1)$ and $\widehat{e}_O(2|1)$ when $\delta = 1.00$

(p_1, p_2)	(M_1, M_2)	$e_1(2\|1)$	$\widehat{e}_O(2\|1)$	$e_1(2\|1) - \widehat{e}_L(2\|1)$	$e_2(2\|1)$	$\widehat{e}_K(2\|1)$	$e_2(2\|1) - \widehat{e}_K(2\|1)$
(2, 1)	(10, 10)	0.346416	0.319110	0.027306	0.334412	0.397841	–0.063429
	(20, 20)	0.323429	0.310090	0.013339	0.316270	0.324347	–0.008077
	(30, 30)	0.316291	0.306702	0.009589	0.311765	0.309478	0.002287
	(40, 40)	0.311266	0.305040	0.006226	0.308137	0.304948	0.003189
	(50, 50)	0.308554	0.303928	0.004626	0.306037	0.302862	0.003175
(3, 2)	(10, 10)	0.370297	0.301053	0.069244	0.358527	0.396118	–0.037591
	(20, 20)	0.339631	0.307815	0.031816	0.331150	0.320346	0.010804
	(30, 30)	0.327573	0.306867	0.020706	0.320406	0.307301	0.013105
	(40, 40)	0.321198	0.305717	0.015481	0.315904	0.303045	0.012859
	(50, 50)	0.316789	0.304777	0.012012	0.312362	0.301235	0.011127
(4, 2)	(10, 10)	0.380929	0.294657	0.086272	0.367667	0.399946	–0.032279
	(20, 20)	0.347431	0.306038	0.041393	0.335778	0.325509	0.010269
	(30, 30)	0.332899	0.306420	0.026479	0.324266	0.309024	0.015242
	(40, 40)	0.325352	0.305639	0.019713	0.318426	0.303618	0.014808
	(50, 50)	0.321614	0.304957	0.016657	0.315900	0.301446	0.014454
(4, 3)	(10, 10)	0.390455	0.288767	0.101688	0.381371	0.394054	–0.012683
	(20, 20)	0.353378	0.304305	0.049073	0.343806	0.320777	0.023029
	(30, 30)	0.339085	0.305764	0.033321	0.330982	0.306670	0.024312
	(40, 40)	0.329283	0.305506	0.023777	0.323098	0.302071	0.021027
	(50, 50)	0.322743	0.304935	0.017808	0.317221	0.300195	0.017026

Table 6
The comparisons of the accuracy of $\widehat{e}_K(2|1)$ and $\widehat{e}_S(2|1)$ when $\delta = 1.00$

(p_1, p_2)	(M_1, M_2)	$e_2(2\|1)$	$\widehat{e}_K(2\|1)$	$\widehat{e}_S(2\|1)$	$e_2(2\|1) - \widehat{e}_K(2\|1)$	$e_2(2\|1) - \widehat{e}_S(2\|1)$
(2, 1)	(10, 10)	0.334412	0.397841	0.299661	–0.063429	0.034751
	(20, 20)	0.316270	0.324347	0.298735	–0.008077	0.017535
	(30, 30)	0.311765	0.309478	0.298847	0.002287	0.012918
	(40, 40)	0.308137	0.304948	0.299044	0.003189	0.009093
	(50, 50)	0.306037	0.302862	0.299112	0.003175	0.006925
(3, 2)	(10, 10)	0.358527	0.396118	0.291221	–0.037591	0.067306
	(20, 20)	0.331150	0.320346	0.293413	0.010804	0.037737
	(30, 30)	0.320406	0.307301	0.295213	0.013105	0.025193
	(40, 40)	0.315904	0.303045	0.296213	0.012859	0.019691
	(50, 50)	0.312362	0.301235	0.296849	0.011127	0.015513
(4, 2)	(10, 10)	0.367667	0.399946	0.288003	–0.032279	0.079664
	(20, 20)	0.335778	0.325509	0.291017	0.010269	0.044761
	(30, 30)	0.324266	0.309024	0.293500	0.015242	0.030766
	(40, 40)	0.318426	0.303618	0.294833	0.014808	0.023593
	(50, 50)	0.315900	0.301446	0.295799	0.014454	0.020101
(4, 3)	(10, 10)	0.381371	0.394054	0.284038	–0.012683	0.097333
	(20, 20)	0.343806	0.320777	0.288512	0.023029	0.055294
	(30, 30)	0.330982	0.306670	0.291690	0.024312	0.039292
	(40, 40)	0.323098	0.302071	0.293465	0.021027	0.029633
	(50, 50)	0.317221	0.300195	0.294617	0.017026	0.022604