Computable error bounds for high–dimensional approximations of LR test for additional information in canonical correlation analysis

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Abstract

This paper concerns with high–dimensional approximations of a LR criterion in canonical correlation analysis of p–variate random vector $\boldsymbol{x}$ and q–variate random vector $\boldsymbol{y}$. Let λ be the LR criterion for testing an additional information hypothesis on a subvector of $\boldsymbol{x}$ and a subvector of $\boldsymbol{y}$, based on a sample of size $N = n + 1$. Using the fact that the null distribution of $-(2/N)\log\lambda$ can be expressed a product of two independent Λ distributions, first we derive high–dimensional asymptotic distributions of λ under a high–dimensional framework when the sample size and the dimensions are large. Next we derive a high-dimensional asymptotic expansion. Furthermore, we derive computable error bounds for the high-dimensional approximations. Through numerical experiments it is noted that our error bounds are useful in a wide range of p, q and n.

1 Introduction

This paper concerns with high–dimensional approximations of the LR criterion in canonical correlation analysis of p–variate random vector $\boldsymbol{x}$ and q–variate random vector $\boldsymbol{y}$. Let λ be the likelihood ratio criterion for testing additional information on $p_2(= p - p_1)$–subvector $\boldsymbol{x}_2$ of $\boldsymbol{x}$ and $q_2(= q - q_1)$–subvector $\boldsymbol{y}_2$ of $\boldsymbol{y}$, based on a sample of size $N = n+1$. Then it is known (see Fujikoshi (1982)) that λ is expressed as

$$-(2/N)\log\lambda = -\log L_{(1)}L_{(2)}, \tag{1.1}$$

where under the hypothesis, $L_{(1)}$ and $L_{(2)}$ are independently distributed as $\Lambda_{p_2}(q, n-p_1-q)$ and $\Lambda_{q_2}(p_1, n-p_1-q_1)$, respectively. Here $\Lambda_p(m, n)$ denotes the p–dimensional lambda distribution with (m, n)–degrees freedom. The lambda distribution may be defined as $|A|/|A + B|$, where A and B are independently distributed as Wishart distributions $W_p(n, \Sigma)$ and $W_p(m, \Sigma)$, respectively.

Under a large sample framework such that p and q are fixed and $n \to \infty$, $-2\log\lambda$ tends to a chi-square distribution with $p_1q_2 + p_2q$ degrees of freedom. Note that the λ belongs to Box class of likelihood ratio criteria. Therefore, based on Box (1949), it is also possible to give an asymptotic expansion. However, such large-sample approximations will not work well as the dimensions p and q are large. In order to overcome this weakness, high-dimensional approximations have been studied. For a statistic $T = -\log\Lambda$, where $\Lambda \sim \Lambda_p(m, n)$, we have some approximations under high-dimensional frameworks such that "m; fix and $p/n \to c \in (0, 1)$" or "$p/n \to c \in (0, 1)$ and $m/n \to d \in (0, 1)$". For these results, see Mudholkar (1980), Tonda and Fujikoshi (2004), Wakaki (2006), etc. Sakurai (2009) attempted to extend the approximations due to Tonda and Fujikoshi (2004) and Wakaki (2006) to $L = L_{(1)}L_{(2)}$ without discussing rigorous validity of the extended approximations.

In this paper we derive high-dimensional approximations for the null distribution of $L = L_{(1)}L_{(2)}$ in (1.1). It is shown that our approximations are valid under a mild high-dimensional framework such that

$$n - p - q \to \infty \tag{1.2}$$

with the condition

$$\frac{(n-p)(n-q)(n-p_1-q_1)}{(n-p_1)(n-q_1)(n-p-q)} > c \quad \text{some constant } c > 1. \tag{1.3}$$

We note that the condition (1.3) means that p, q and $p_2 + q_2$ should tend to infinity under the framework (1.2) because the left side of the inequality can be represented as

$$\begin{aligned}\frac{(n-p)(n-q)(n-p_1-q_1)}{(n-p_1)(n-q_1)(n-p-q)} &= \left\{1 + \frac{pq}{n(n-p-q)}\right\}\left\{1 + \frac{p_1q_1}{n(n-p_1-q_1)}\right\}^{-1} \\ &= \left(1 + \frac{p_2+q_2}{n-p-q}\right)\frac{(n-p)(n-q)}{(n-p_1)(n-q_1)}.\end{aligned} \tag{1.4}$$

A simple high-dimensional approximation is also derived. Furthermore, we derive a computable error bound for the high-dimensional approximations. This result is obtained by extending the error bounds (see Wakaki (2009), Akita, Jin and Wakaki (2010)) for $\Lambda_p(m,n)$ to a product of Lambda distributions. Through numerical experiments it is noted that our error bounds are useful in a wide range of p, q and n.

2 Test for additional information hypothesis

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be p–variate and q–variate random vectors, respectively. We denote the mean vectors and the covariance matrices as follows.

$$\begin{aligned}&\boldsymbol{\mu}_x = \mathrm{E}[\boldsymbol{x}],\ \boldsymbol{\mu}_y = \mathrm{E}[\boldsymbol{y}],\\ &\Sigma_{xx} = \mathrm{Cov}[\boldsymbol{x}],\ \Sigma_{yy} = \mathrm{Cov}[\boldsymbol{y}],\ \Sigma_{xy} = \mathrm{Cov}[\boldsymbol{x},\boldsymbol{y}].\end{aligned}$$

We assume without loss of generality that $p \le q$. Let $\rho_j^2,\ j = 1, 2, \ldots, r$ be the jth largest root of the equation

$$\det(\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} - \rho^2\Sigma_{xx}) = 0,$$

where $r = \mathrm{rank}(\Sigma_{xy})$. Then ρ_j is called as the jth canonical correlation coefficient of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are partitioned as

$$\boldsymbol{x} = \begin{pmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{pmatrix},\ \boldsymbol{y} = \begin{pmatrix}\boldsymbol{y}_1\\ \boldsymbol{y}_2\end{pmatrix},$$

where $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ are p_1 and q_1 dimensional subvectors, respectively. Along the above partition, we partition Σ_{xx}, Σ_{xy} and Σ_{yy} as

$$\Sigma_{xx} = \begin{pmatrix}\Sigma_{x_1x_1} & \Sigma_{x_1x_2}\\ \Sigma_{x_2x_1} & \Sigma_{x_2x_2}\end{pmatrix},\ \Sigma_{xy} = \begin{pmatrix}\Sigma_{x_1y_1} & \Sigma_{x_1y_2}\\ \Sigma_{x_2y_1} & \Sigma_{x_2y_2}\end{pmatrix},\ \Sigma_{yy} = \begin{pmatrix}\Sigma_{y_1y_1} & \Sigma_{y_1y_2}\\ \Sigma_{y_2y_1} & \Sigma_{y_2y_2}\end{pmatrix}.$$

In this section we summarize an additional information hypothesis (or redundancy hypothesis) of $p_2(= p - p_1)$–subvector on $\boldsymbol{x}$ and $q_2(= q - q_1)$–subvector $\boldsymbol{y}_2$ of $\boldsymbol{y}$ and its likelihood ratio criterion, based on Fujikoshi (1983) and Fujikoshi et al. (2010). A natural measure $\delta^2(\boldsymbol{x},\boldsymbol{y})$ of the association between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the sum of squares of canonical correlations:

$$\delta^2(\boldsymbol{x},\boldsymbol{y}) \equiv \sum_{j=1}^{r}\rho_j^2 = \mathrm{tr}(\Sigma_{xx}^{-1}\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}).$$

So a redundancy hypothesis of $\boldsymbol{x}_2$ and $\boldsymbol{y}_2$ can be formulated as

$$\delta^2(\boldsymbol{x},\boldsymbol{y}) = \delta^2(\boldsymbol{x}_1,\boldsymbol{y}_1),$$

which is equivalent to

$$H_0 : \Sigma_{x_2y_2\cdot x_1x_3} = O,\ \Sigma_{x_2y_1\cdot x_1} = O,\ \Sigma_{x_1y_2\cdot y_1} = O, \tag{2.1}$$

where

$$\Sigma_{x_2y_2\cdot x_1y_1} = \Sigma_{x_2y_2} - (\Sigma_{x_2x_1}\ \Sigma_{x_2y_1}) \begin{pmatrix} \Sigma_{x_1x_1} & \Sigma_{x_1y_1} \\ \Sigma_{y_1x_1} & \Sigma_{y_1y_1} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{x_1y_2} \\ \Sigma_{y_1y_2} \end{pmatrix},$$

$$\Sigma_{x_2y_1\cdot x_1} = \Sigma_{x_2y_1} - \Sigma_{x_2x_1}\Sigma_{x_1x_1}^{-1}\Sigma_{x_1y_1},\ \Sigma_{x_1y_2\cdot y_1} = \Sigma_{x_1y_2} - \Sigma_{x_1y_1}\Sigma_{y_1y_1}^{-1}\Sigma_{y_1y_2}.$$

Suppose that $(\boldsymbol{x}_1', \boldsymbol{y}_1')', \ldots, (\boldsymbol{x}_N', \boldsymbol{y}_N')'$ be a sample of size $N = n+1$ from $N_{p+q}(\boldsymbol{\mu}, \Sigma)$, where

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_x \\ \boldsymbol{\mu}_y \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}.$$

Let

$$S = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{N} \begin{pmatrix} \boldsymbol{x}_i - \bar{\boldsymbol{x}} \\ \boldsymbol{y}_i - \bar{\boldsymbol{y}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_i - \bar{\boldsymbol{x}} \\ \boldsymbol{y}_i - \bar{\boldsymbol{y}} \end{pmatrix}',$$

where $\bar{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^N \boldsymbol{x}_i$, and $\bar{\boldsymbol{y}} = \frac{1}{N}\sum_{i=1}^N \boldsymbol{y}_i$. We partition S_{xx}, S_{xy} and S_{yy} similarly as the partitions of Σ_{xx}, Σ_{xy} and Σ_{yy}.

Then the LR criterion λ for testing H_0 is given by

$$\lambda^{2/N} = L = \frac{|S_{x_2x_2\cdot x_1y}|}{|S_{x_2x_2\cdot x_1}|} \cdot \frac{|S_{y_2y_2\cdot x_1y_1}|}{|S_{y_2y_2\cdot y_1}|} = L_{(1)} \cdot L_{(2)} \text{ (say)}, \tag{2.2}$$

where

$$S_{x_2x_2\cdot x_1} = S_{x_2x_2} - S_{x_2x_1}S_{x_1x_1}^{-1}S_{x_1x_2}, \quad S_{y_2y_2\cdot y_1} = S_{y_2y_2} - S_{y_2y_1}S_{y_1y_1}^{-1}S_{y_1y_2},$$

$$S_{x_2y_2\cdot x_1y} = S_{x_2y_2} - (S_{x_2x_1}\ S_{x_2y_1}\ S_{x_2y_2}) \begin{pmatrix} S_{x_1x_1} & S_{x_1y_1} & S_{x_1y_2} \\ S_{y_1x_1} & S_{y_1y_1} & S_{y_1y_2} \\ S_{y_2x_1} & S_{y_2y_1} & S_{y_2y_2} \end{pmatrix}^{-1} \begin{pmatrix} S_{x_1y_2} \\ S_{y_1y_2} \\ S_{y_2y_2} \end{pmatrix},$$

$$S_{y_2y_2\cdot x_1y_1} = S_{y_2y_2} - (S_{y_2x_1}\ S_{y_2y_1}) \begin{pmatrix} S_{x_1x_1} & S_{x_1y_1} \\ S_{y_1x_1} & S_{y_1y_1} \end{pmatrix}^{-1} \begin{pmatrix} S_{x_1y_2} \\ S_{y_1y_2} \end{pmatrix}.$$

3 The cumulants of $-(2/N)\log\lambda$

It is known that under the hypothesis H_0, $L_{(1)}$ and $L_{(2)}$ are independently distributed as $\Lambda_{p_2}(q, n-p_1-q)$ and $\Lambda_{q_2}(p_1, n-p_1-q_1)$, respectively (see, e.g., Theorem 11.5.2 in Fujikoshi et al., 2010).

The moments and the distributional results on the lambda distribution $\Lambda_p(m,n)$ can be found in Anderson (2003, chapter 8), Muirhead (1982, chapter 10) and Siotani et al. (1985, chapter 7). If $\Lambda \sim \Lambda_p(m,n)$,

$$\mathrm{E}[\Lambda^h] = \prod_{j=1}^{p} \frac{\Gamma[\frac{1}{2}(n-p+j)+h]\Gamma[\frac{1}{2}(m+n-p+j)]}{\Gamma[\frac{1}{2}(n-p+j)]\Gamma[\frac{1}{2}(m+n-p+j)+h]},$$

and hence the characteristic function of $-\log\Lambda$ is given by

$$\begin{aligned} ch_p(t;m,n) &\equiv \mathrm{E}[e^{it(-\log\Lambda)} = \mathrm{E}[\Lambda^{-it}] \\ &= \prod_{j=1}^{p} \frac{\Gamma[\frac{1}{2}(n-p+j)-it]\Gamma[\frac{1}{2}(m+n-p+j)]}{\Gamma[\frac{1}{2}(n-p+j)]\Gamma[\frac{1}{2}(m+n-p+j)-it]}. \end{aligned} \tag{3.1}$$

Since the gamma function is analytic except at the non–positive integers, using (3.1) the cumulant generating function of $-\log\Lambda$ can be expanded in a neighborhood of $t=0$ as

$$\log ch_p(t;m,n) = \sum_{s=1}^{\infty}\frac{(-it)^s}{s!}\Big\{\psi_p^{(s-1)}\big(\frac{n}{2}\big) - \psi_p^{(s-1)}\big(\frac{n+m}{2}\big)\Big\},$$

where

$$\psi_p^{(s)}(a) = \sum_{j=1}^{p}\psi^{(s)}\big(\frac{2a-p+j}{2}\big)$$

and $\psi^{(s)}(a)$ is the polygamma function defined as

$$\psi^{(s)}(a) = \big(\frac{d}{da}\big)^{s+1}\log\Gamma[a] = \begin{cases} -C + \sum_{k=0}^{\infty}\Big(\frac{1}{1+k} - \frac{1}{k+a}\Big) & (s=0), \\ \sum_{k=0}^{\infty}\frac{(-1)^{s+1}s!}{(k+a)^{s+1}} & (s=1,2,\cdots). \end{cases} \tag{3.2}$$

Here C is the Euler's constant.

Hence the characteristic function and the s–th cumulant of the null distribution of $-(2/N)\log\lambda$ are given by

$$\phi_\lambda(t) = ch_{p_2}(t;q,n-p_1-q)ch_{q_2}(t;p_1,n-p_1-q_1)$$

and

$$\begin{aligned}\kappa_\lambda^{(s)} = (-1)^s\Big\{&\psi_{p_2}^{(s-1)}\big(\frac{n-p_1-q}{2}\big) - \psi_{p_2}^{(s-1)}\big(\frac{n-p_1}{2}\big) \\ &+ \psi_{q_2}^{(s-1)}\big(\frac{n-p_1-q_1}{2}\big) - \psi_{q_2}^{(s-1)}\big(\frac{n-q_1}{2}\big)\Big\}.\end{aligned} \tag{3.3}$$

Let

$$T = \frac{-\frac{2}{N}\log\lambda - \kappa_\lambda^{(1)}}{\sqrt{\kappa_\lambda^{(2)}}}, \tag{3.4}$$

and denote the standardized cumulants as

$$\kappa^{(s)} = \frac{\kappa_\lambda^{(s)}}{(\kappa_\lambda^{(2)})^{s/2}} \quad (s=3,4,\ldots).$$

Then, $\mathrm{E}[T]=0$, $\mathrm{Var}[T]=1$, and the $s\,(s\geq 3)$–th cumulant of T is $\kappa^{(s)}$. Therefore, writing the characteristic function of T as $\varphi(t)$ we have

$$\log\varphi(t) = -\frac{1}{2}t^2 + \sum_{s=3}^{\infty}\frac{\kappa^{(s)}}{s!}(it)^s. \tag{3.5}$$

Using (3.5) we can obtain formally the Edgeworth expansion of the null distribution of standardized test statistic T. The purpose of this paper is to give a sufficient condition for the validity of the Edgeworth expansion and to give a computable error bound for the approximated distribution function by using the asymptotic expansion of it.

4 Edgeworth expansion

Bounds for the cumulants

From (3.3), the s–th cumulant of $-(2/N)\log\lambda$ is given by

$$\begin{aligned}\kappa_\lambda^{(s)} = \sum_{k=0}^{\infty}\Big\{\sum_{j=1}^{p-p_1}\Big(\frac{(s-1)!}{(k+\frac{n-p-q+j}{2})^s}-\frac{(s-1)!}{(k+\frac{n-p+j}{2})^s}\Big)\\ +\sum_{j=1}^{q-q_1}\Big(\frac{(s-1)!}{(k+\frac{n-p_1-q+j}{2})^s}-\frac{(s-1)!}{(k+\frac{n-q+j}{2})^s}\Big)\Big\}>0.\end{aligned} \tag{4.1}$$

First we give a lower bound for the variance. Consider a function defined by

$$\begin{aligned} f(a) &= \frac{1}{a^2}-\int_0^1\Big\{\int_{-1}^0\frac{1}{(a+y+\frac{x}{2})^2}dx\Big\}dy \\ &= \frac{1}{a^2}+2\log\frac{a+1}{a}+2\log\frac{a-1/2}{a+1/2}.\end{aligned}$$

It is easily seen that $f(a)$ is a decreasing and positive function of a in $(\frac{1+\sqrt{13}}{6},\infty)$. This implies that

$$\begin{aligned}&\frac{1}{(k+\frac{n-p-q+j}{2})^2}-\frac{1}{(k+\frac{n-p+j}{2})^2}\\ &\quad\geq\int_0^1\Big\{\int_{-1}^0\Big(\frac{1}{(k+y+\frac{n-p-q+j+x}{2})^2}-\frac{1}{(k+y+\frac{n-p+j+x}{2})^2}\Big)dx\Big\}dy,\\ &\frac{1}{(k+\frac{n-p_1-q+j}{2})^2}-\frac{1}{(k+\frac{n-q+j}{2})^2}\\ &\quad\geq\int_0^1\Big\{\int_{-1}^0\Big(\frac{1}{(k+y+\frac{n-p_1-q+j+x}{2})^2}-\frac{1}{(k+y+\frac{n-q+j+x}{2})^2}\Big)dx\Big\}dy\end{aligned}$$

for all $k\geq 0$ and $j\geq 1$ if $n-p-q\geq 1$. All through this paper, we assume that $n-p-q\geq 1$. Then we have

$$\begin{aligned}\kappa_\lambda^{(2)} &\geq \int_0^\infty\Big\{\int_0^{p-p_1}\Big(\frac{1}{(y+\frac{n-p-q+x}{2})^2}-\frac{1}{(y+\frac{n-p+x}{2})^2}\Big)dx\\ &\qquad+\int_0^{q-q_1}\Big(\frac{1}{(y+\frac{n-p_1-q+x}{2})^2}-\frac{1}{(y+\frac{n-q+x}{2})^2}\Big)dx\Big\}dy \\ &= 2\log\frac{(n-p)(n-q)(n-p_1-q_1)}{(n-p_1)(n-q_1)(n-p-q)}.\end{aligned} \tag{4.2}$$

From the above lower bound we can see that the limit (or the limitinf) of $\kappa_\lambda^{(2)}$ under (1.2) is positive if the condition (1.3) is satisfied.

Next we give an upper bound for the s–th cumulant for $s\geq 3$. Consider a function

$$f(x,y)=\frac{1}{(y+\frac{a+x}{2})^s}-\frac{1}{(y+\frac{b+x}{2})^s}.$$

Since $f(x,y)$ is convex in $[0,\infty)\times[0,\infty)$ for $0<a<b$, we have the inequality

$$f(k,j)\le\int_{k-1/2}^{k+1/2}\left\{\int_{j-1/2}^{j+1/2}f(x,y)dx\right\}dy$$

for any nonnegative integer k and any positive integer j when $a>1/2$. Hence

$$\begin{aligned}\kappa_\lambda^{(s)}&\le\int_{-1/2}^{\infty}\Bigg\{\int_{1/2}^{p-p_1+1/2}\Big(\frac{(s-1)!}{(y+\frac{n-p-q+x}{2})^s}-\frac{(s-1)!}{(y+\frac{n-p+x}{2})^s}\Big)dx\\&\quad+\int_{1/2}^{q-q_1+1/2}\Big(\frac{(s-1)!}{(y+\frac{n-p_1-q+x}{2})^s}-\frac{(s-1)!}{(y+\frac{n-q+x}{2})^s}\Big)dx\Bigg\}dy\\&=\frac{(s-3)!2^{s-1}}{(n-p-q-\frac{1}{2})^{s-2}}-\frac{(s-3)!2^{s-1}}{(n-p-\frac{1}{2})^{s-2}}-\frac{(s-3)!2^{s-1}}{(n-q-\frac{1}{2})^{s-2}}\\&\quad-\frac{(s-3)!2^{s-1}}{(n-p_1-q_1-\frac{1}{2})^{s-2}}+\frac{(s-3)!2^{s-1}}{(n-p_1-\frac{1}{2})^{s-2}}+\frac{(s-3)!2^{s-1}}{(n-q_1-\frac{1}{2})^{s-2}}.\end{aligned}$$

The inequality can be expressed as

$$\begin{aligned}\frac{\kappa^{(s)}}{s!}&<\left\{\frac{n-p-q-\frac{1}{2}}{2}(\kappa_\lambda^{(2)})^{1/2}\right\}^{-(s-2)}\frac{2}{\kappa_\lambda^{(2)}(s-2)(s-1)s}\\&\quad\cdot\left\{1-\Big(\frac{n-p-q-\frac{1}{2}}{n-p-\frac{1}{2}}\Big)^{s-2}-\Big(\frac{n-p-q-\frac{1}{2}}{n-q-\frac{1}{2}}\Big)^{s-2}\right.\\&\quad\left.-\Big(\frac{n-p-q-\frac{1}{2}}{n-p_1-q_1-\frac{1}{2}}\Big)^{s-2}+\Big(\frac{n-p-q-\frac{1}{2}}{n-p_1-\frac{1}{2}}\Big)^{s-2}+\Big(\frac{n-p-q-\frac{1}{2}}{n-q_1-\frac{1}{2}}\Big)^{s-2}\right\}\\&=M^{-(s-2)}b_{s-3},\end{aligned}\tag{4.3}$$

where

$$\begin{aligned}M&=\frac{n-p-q-\frac{1}{2}}{2}(\kappa_\lambda^{(2)})^{1/2},\\b_s&=\frac{2}{\kappa_\lambda^{(2)}(s+1)(s+2)(s+3)}\left\{1-\Big(\frac{n-p-q-\frac{1}{2}}{n-p-\frac{1}{2}}\Big)^{s+1}\right.\\&\quad-\Big(\frac{n-p-q-\frac{1}{2}}{n-q-\frac{1}{2}}\Big)^{s+1}-\Big(\frac{n-p-q-\frac{1}{2}}{n-p_1-q_1-\frac{1}{2}}\Big)^{s+1}\\&\quad\left.+\Big(\frac{n-p-q-\frac{1}{2}}{n-p_1-\frac{1}{2}}\Big)^{s+1}+\Big(\frac{n-p-q-\frac{1}{2}}{n-q_1-\frac{1}{2}}\Big)^{s+1}\right\}.\end{aligned}\tag{4.4}$$

The following theorem gives a sufficient condition of the asymptotic normality of the standardized test statistic T.

Theorem 4.1 *Under the asymptotic framework* (1.2) *with the condition* (1.3), *it holds that under* H_0

$$T\xrightarrow{d}N(0,1).$$

Proof First we note that $\kappa_\lambda^{(s)} > 0$ (see (4.1)), and the limit of $\kappa_\lambda^{(2)}$ is positive. Since $\sum_{s=0}^\infty b_s v^s$ is finite for $|v| < 1$, using (4.3) we have for any fixed t

$$\begin{aligned}\left|\sum_{s=3}^\infty \frac{\kappa^{(s)}}{s!} t^s\right| &\leq \sum_{s=0}^\infty \frac{\kappa^{(s+3)}}{(s+3)!}|t|^{s+3} \\ &\leq \frac{|t|^3}{M}\sum_{s=0}^\infty b_s v^s \to 0 \quad (M \to \infty),\end{aligned}$$

if $|t|/M < v < 1$. Hence the cumulant generating function of T converges to $-t^2/2$ $(M \to \infty)$, which shows the asymptotic normality of T. □

Corollary 4.1 *Let*

$$\tilde{T} = \frac{-\frac{2}{N}\log\lambda - \tilde{\kappa}_\lambda^{(1)}}{\sqrt{\tilde{\kappa}_\lambda^{(2)}}},$$

where

$$\begin{aligned}\tilde{\kappa}_\lambda^{(1)} &= \left(n-p-q-\frac{1}{2}\right)\log(n-p-q+1) - \left(n-q-\frac{1}{2}\right)\log(n-q+1) \\ &\quad + \left(n-p_1-\frac{1}{2}\right)\log(n-p_1+1) - \left(n-p-\frac{1}{2}\right)\log(n-p+1) \\ &\quad + \left(n-q_1-\frac{1}{2}\right)\log(n-q_1+1) - \left(n-q_1-p_1-\frac{1}{2}\right)\log(n-q_1-p_1+1), \\ \tilde{\kappa}_\lambda^{(2)} &= 2\log\frac{(n-p-\frac{1}{2})(n-q-\frac{1}{2})(n-p_1-q_1-\frac{1}{2})}{(n-p_1-\frac{1}{2})(n-q_1-\frac{1}{2})(n-p-q-\frac{1}{2})}.\end{aligned}$$

Then under the same assumption as in Theorem 4.1 *it holds that under* H_0

$$\tilde{T} \xrightarrow{d} N(0,1).$$

Proof Applying Lemma A.1 in Appendix to (4.1) we can see

$$\begin{aligned}\kappa_\lambda^{(1)} &= \tilde{\kappa}_\lambda^{(1)} + O((n-p-q)^{-1}), \\ \kappa_\lambda^{(2)} &= \tilde{\kappa}_\lambda^{(2)} + O((n-p-q)^{-2}).\end{aligned}$$

Therefore we have $\tilde{T} = T + O((n-p-q)^{-1})$, which implies the required result. □

Edgeworth expansion

Let $\varphi(t)$ be the characteristic function of T given by (3.4). Then we can expand

$\varphi(t)$ as follows:

$$\begin{aligned}
\varphi(t) &= \exp\Big\{-\frac{t^2}{2} + \sum_{s=3}^{\infty} \frac{\kappa^{(s)}}{s!}(it)^s\Big\} \\
&= \exp\Big(-\frac{t^2}{2}\Big)\Big\{1 + \sum_{k=1}^{\infty} \frac{(it)^{3k}}{k!}\Big(\sum_{s=0}^{\infty} \frac{\kappa^{(s+3)}}{(s+3)!}(it)^s\Big)^k\Big\} \\
&= \exp\big(-\frac{t^2}{2}\big)\Big\{1 + \sum_{k=1}^{\infty} \frac{(it)^{3k}}{k!} \sum_{j=0}^{\infty} \gamma_{k,j}(it)^j\Big\},
\end{aligned} \tag{4.5}$$

where

$$\gamma_{k,j} = \sum_{s_1+\cdots+s_k=j} \frac{\kappa^{(s_1+3)} \cdots \kappa^{(s_k+3)}}{(s_1+3)! \cdots (s_k+3)!}. \tag{4.6}$$

Under the asymptotic framework (1.2) with the condition (1.3) $M \to \infty$. From (4.3) we can see that $\gamma_{k,j} = O(M^{-(j+k)})$. Let

$$\varphi_s(t) = \exp\Big(-\frac{t^2}{2}\Big)\Big\{1 + \sum_{k=1}^{s} \frac{(it)^{3k}}{k!} \sum_{j=0}^{s-k} \gamma_{k,j}(it)^j\Big\}. \tag{4.7}$$

Define $B[v]$ and $R_{k,l}[v]$ by

$$\begin{aligned}
B[v] &= \sum_{s=0}^{\infty} b_s v^s, \\
R_{k,l}[v] &= v^{-l}\Big\{(B[v])^k - \sum_{j=0}^{l-1}\Big(\sum_{s_1+\cdots+s_k=j} b_{s_1}\cdots b_{s_k}\Big)v^j\Big\} \\
&= \sum_{j=l}^{\infty}\Big(\sum_{s_1+\cdots+s_k=j} b_{s_1}\cdots b_{s_k}\Big)v^{j-l}.
\end{aligned} \tag{4.8}$$

If $|t| \le Mv$ and $0 < v < 1$, then using (4.3) we can see

$$
\begin{aligned}
\frac{1}{|t|}|\varphi(t) - \varphi_s(t)| &= \frac{1}{|t|}\exp\big(-\frac{t^2}{2}\big) \\
&\quad \cdot \Big|\sum_{k=1}^{s}\frac{(it)^{3k}}{k!}\sum_{j=s-k+1}^{\infty}\gamma_{k,j}(it)^j + \sum_{k=s+1}^{\infty}\frac{(it)^{3k}}{k!}\Big(\sum_{j=0}^{\infty}\frac{\kappa^{(j+3)}}{(j+3)!}(it)^j\Big)^k\Big| \\
&\le \exp\Big(-\frac{t^2}{2}\Big)\Big\{\sum_{k=1}^{s}\frac{1}{k!}|t|^{3k-1}\sum_{j=s-k+1}^{\infty}\Big(\sum_{s_1+\cdots+s_k=j} b_{s_1}\cdots b_{s_k}\Big)M^{-(j+k)}|t|^j \\
&\qquad + \sum_{k=s+1}^{\infty}\frac{1}{k!}|t|^{3k-1}\Big(\sum_{j=0}^{\infty} b_j M^{-(j+1)}|t|^j\Big)^k\Big\} \\
&= \exp\Big(-\frac{t^2}{2}\Big)\Big\{\sum_{k=1}^{s}\frac{1}{k!}|t|^{3k-1}M^{-k}\Big(\frac{|t|}{M}\Big)^{s-k+1}R_{k,s-k+1}\Big[\frac{|t|}{M}\Big] \\
&\qquad + \sum_{k=s+1}^{\infty}\frac{1}{k!}|t|^{3k-1}M^{-k}\Big(B\Big[\frac{|t|}{M}\Big]\Big)^k\Big\} \\
&\le M^{-(s+1)}\exp\Big(-\frac{t^2}{2}\Big)\Big\{\sum_{k=1}^{s}\frac{1}{k!}|t|^{s+2k}R_{k,s-k+1}[v] \\
&\qquad + \frac{1}{(s+1)!}|t|^{3s+2}(B[v])^{s+1}\exp\big(t^2 vB[v]\big)\Big\}.
\end{aligned}
\tag{4.9}
$$

Hence

$$\varphi(t) = \varphi_s(t) + O(M^{-(s+1)}).$$

Inverting (4.7), we obtain an Edgeworth expansion of the null distribution of the standardized test statistic T up to the order $O(M^{-s})$ as

$$Q_s(x) = \Phi(x) - \phi(x)\Big\{\sum_{k=1}^{s}\frac{1}{k!}\sum_{j=0}^{s-k}\gamma_{k,j}h_{3k+j-1}(x)\Big\}, \tag{4.10}$$

where Φ and ϕ are the distribution function and the probability density function of the standard normal distribution, respectively, $\gamma_{k,j}$ is given by (4.6), and $h_r(x)$ is the r–th order Hermite polynomial defined by

$$\Big(\frac{d}{dx}\Big)^r\exp\Big(-\frac{x^2}{2}\Big) = (-1)^r h_r(x)\exp\Big(-\frac{x^2}{2}\Big).$$

5 The validity and an error bound

Using the inverse Fourier transformation we obtain a uniform bound for the error of the Edgeworth expansion as

$$
\begin{aligned}
\sup_x |\mathrm{P}(T \le x) - Q_s(x)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|t|}\,|\varphi(t) - \varphi_s(t)|\,dt \\
&= \frac{1}{2\pi}(I_1[v] + I_2[v] + I_3[v]),
\end{aligned}
\tag{5.1}
$$

where

$$I_1[v] = \int_{-Mv}^{Mv} \frac{1}{|t|} |\varphi(t) - \varphi_s(t)| \, dt,$$

$$I_2[v] = \int_{|t|>Mv} \frac{1}{|t|} |\varphi_s(t)| \, dt \quad \text{and} \quad I_3[v] = \int_{|t|>Mv} \frac{1}{|t|} |\varphi(t)| \, dt$$

with some positive constant $v < 1$.

First we derive a bound for $I_1[v]$. Let

$$L[v] = \begin{cases} \dfrac{3v-2}{4v} - \dfrac{(1-v)^2}{2v^2} \log(1-v) & (0 < |v| < 1) \\ 0 & (v = 0). \end{cases} \tag{5.2}$$

Using $-\log(1-v) = v + \frac{1}{2}v^2 + \frac{1}{3}v^3 + \cdots$, it is easily checked that $L[v]$ can be expanded as

$$L[v] = \sum_{s=1}^{\infty} \frac{1}{s(s+1)(s+2)} v^s.$$

So $B[v]$ can be expressed as

$$\begin{aligned} B[v] = \frac{2}{v\kappa_\lambda^{(2)}} \Big\{ & L[v] - L\Big[\frac{n-p-q-\frac{1}{2}}{n-p-\frac{1}{2}} v\Big] - L\Big[\frac{n-p-q-\frac{1}{2}}{n-q-\frac{1}{2}} v\Big] \\ & - L\Big[\frac{n-p-q-\frac{1}{2}}{n-p_1-q_1-\frac{1}{2}} v\Big] + L\Big[\frac{n-p-q-\frac{1}{2}}{n-p_1-\frac{1}{2}} v\Big] + L\Big[\frac{n-p-q-\frac{1}{2}}{n-q_1-\frac{1}{2}} v\Big] \Big\}, \end{aligned} \tag{5.3}$$

which is bounded if $0 < v < 1$.

From (4.9) we obtain $I_1[v] \le U_1[v]$, where

$$\begin{aligned} U_1[v] = \frac{2}{M^{s+1}} \Big\{ & \sum_{k=1}^{s} \frac{1}{k!} R_{k,s-k+1}[v] \int_0^{Mv} t^{s+2k} \exp\Big(-\frac{t^2}{2}\Big) dt \\ & + \frac{1}{(s+1)!} (B[v])^{s+1} \int_0^{Mv} t^{3s+2} \exp\Big(-\frac{t^2}{2} c_v\Big) dt \Big\}, \end{aligned} \tag{5.4}$$

and $c_v = 1 - 2vB[v]$.

The calculation of integral I_2 is not difficult. From (4.7)

$$I_2[v] = \int_{Mv}^{\infty} \frac{2}{t} \exp\big(-\frac{t^2}{2}\big) dt + \sum_{k=1}^{s} \frac{2}{k!} \sum_{j=0}^{s-k} \gamma_{k,j} \int_{Mv}^{\infty} t^{3k+j-1} \exp\big(-\frac{t^2}{2}\big) dt. \tag{5.5}$$

In order to give a bound for I_3 we prepare the following lemma.

Lemma 5.1 *If $p < n < m + n$, then*

$$\begin{aligned} \log|ch_p(t;m,n)| &< -\frac{1}{4} \sum_{j=1}^{p} \int_{n-p}^{m+n-p} \log\Big\{1 + \frac{4t^2}{(j+x)^2}\Big\} dx \\ &< -\frac{mp}{4} \log\Big\{1 + \frac{16t^2}{(m+2n-p+1)^2}\Big\}, \end{aligned} \tag{5.6}$$

where $ch_p(t;m,n)$ is the characteristic function of $\Lambda_p(m,n)$ given by (3.1).

Proof It is known that

$$\left|\frac{\Gamma[x+yi]}{\Gamma[x]}\right|^2 = \prod_{k=0}^{\infty}\left\{1+\frac{y^2}{(x+k)^2}\right\}^{-1}$$

for any real number $x, y; (x > 0)$. Since if $A < B$

$$\log\left\{1+\frac{t^2}{(A+x)^2}\right\} - \log\left\{1+\frac{t^2}{(B+x)^2}\right\}$$

is a decreasing function of $x > 0$, we have

$$\begin{aligned}
&\log|ch_p(t;m,n)| \\
&= -\frac{1}{2}\sum_{k=0}^{\infty}\sum_{j=1}^{p}\left\{\log\left[1+\frac{t^2}{(\frac{n-p+j}{2}+k)^2}\right] - \log\left[1+\frac{t^2}{(\frac{m+n-p+j}{2}+k)^2}\right]\right\} \\
&\leq -\frac{1}{2}\sum_{j=1}^{p}\int_0^{\infty}\left\{\log\left[1+\frac{t^2}{(\frac{n-p+j}{2}+x)^2}\right] - \log\left[1+\frac{t^2}{(\frac{m+n-p+j}{2}+x)^2}\right]\right\}dx \\
&= -\frac{1}{2}\sum_{j=1}^{p}\int_{(n-p+j)/2}^{(m+n-p+j)/2}\log\left\{1+\frac{t^2}{x^2}\right\}dx \\
&= -\frac{1}{4}\sum_{j=1}^{p}\int_{n-p}^{m+n-p}\log\left\{1+\frac{4t^2}{(x+j)^2}\right\}dx.
\end{aligned} \tag{5.7}$$

The last inequality of (5.6) is obtained by using the inequality

$$\sum_{j=1}^{p}\int_a^b f(x,j)dx \geq \sum_{j=1}^{p}(b-a)f\left(\frac{a+b}{2},j\right) \geq p(b-a)f\left(\frac{a+b}{2},\frac{p+1}{2}\right)$$

for any convex function $f(x, y)$. □

Using the above lemma

$$\begin{aligned}
\log|\phi_\lambda(t)| &= \log|ch_{p-p_1}(t;q,n-p_1-q)| + \log|ch_{q-q_1}(t;p_1,n-p_1-q_1)| \\
&< -\frac{q(p-p_1)}{4}\log\left\{1+\frac{16t^2}{(2n-q-p_1-p+1)^2}\right\} \\
&\quad -\frac{p_1(q-q_1)}{4}\log\left\{1+\frac{16t^2}{(2n-q-p_1-q_1+1)^2}\right\} \\
&=: \log G(t) \text{ (say) }.
\end{aligned}$$

Hence

$$I_3(v) \leq \int_{Mv}^{\infty}\frac{2}{|t|}G(t(\kappa_\lambda^{(2)})^{-1/2})dt = \int_{M_0 v}^{\infty}\frac{2}{|t|}G(t)dt,$$

where

$$M_0 = \frac{n-p-q-\frac{1}{2}}{2}.$$

Refinement of the bound for I_3

From (5.7)

$$\begin{aligned}\log|ch_p(t;m,n)| &< -\frac{1}{4}\sum_{j=1}^{p}\int_{n-p}^{m+n-p}\log\Big\{1+\frac{4t^2}{(x+j)^2}\Big\}dx\\ &< -\frac{1}{4}\int_1^{p+1}\Big[\int_{n-p}^{m+n-p}\log\Big(1+\frac{4t^2}{(y+x)^2}\Big)dx\Big]dy\\ &= -t^2\Big\{F\Big(\frac{m+n+1}{2|t|}\Big)-F\Big(\frac{n+1}{2|t|}\Big)-F\Big(\frac{m+n-p+1}{2|t|}\Big)+F\Big(\frac{n-p+1}{2|t|}\Big)\Big\}.\end{aligned}$$

where

$$\begin{aligned}F(z) &= \int_0^z\int_0^y \log(1+\frac{1}{x^2})dxdy\\ &= \frac{z^2}{2}\log\Big(1+\frac{1}{z^2}\Big)+2z\arctan(z)-\frac{1}{2}\log(1+z^2).\end{aligned}$$

Hence

$$|\varphi(t)| \le \exp\Big\{-\frac{t^2}{\kappa_\lambda^{(2)}}G_0(t(\kappa_\lambda^{(2)})^{-1/2})\Big\},$$

where

$$\begin{aligned}G_0(t) = \Big\{&F\Big(\frac{n-p_1+1}{2|t|}\Big)-F\Big(\frac{n-p+1}{2|t|}\Big)+F\Big(\frac{n-p-q+1}{2|t|}\Big)\\ &+F\Big(\frac{n-q_1+1}{2|t|}\Big)-F\Big(\frac{n-q_1-p_1+1}{2|t|}\Big)-F\Big(\frac{n-q+1}{2|t|}\Big)\Big\}.\end{aligned}$$

Therefore

$$I_3(v) \le \int_{M_0 v}^{\infty}\frac{2}{|t|}\exp\Big\{-t^2G_0(t)\Big\}dt =: U_3[v] \text{ (say)}. \tag{5.8}$$

Summarizing the above we obtain an error bound of the Edgeworth expansion as in the following theorem.

Theorem 5.1 *Let T be the standardized test statistic given by* (3.4) *and $Q_s(x)$ be the Edgeworth expansion of the null distribution function of T given by* (4.10). *Then we have a uniform bound for the error given by*

$$\sup_x |\mathrm{P}(T\le x)-Q_s(x)| \le \frac{1}{2\pi}|U_1[v]+I_2[v]+U_3[v]|,$$

where U_1, I_2 and U_3 are given by (5.4), (5.5) *and* (5.8), *respectively.*

Corollary 5.1 *Under the asymptotic framework* (1.2) *with the condition* (1.3) *and*

$$\frac{p+q}{n-p-q} \le C \quad \textit{some constant}, \tag{5.9}$$

it holds that

$$\sup_x |\mathrm{P}(T\le x)-Q_s(x)| = O(M^{-(s+1)}).$$

Proof If we choose v such that $c_v > 0$, $U_1[v] = O(M^{-(s+1)})$. It is easy to show $I_2[v] = O(\exp(-M^2v^2c))$ for any positive number $c < 1/2$. From the first equality in (1.4), the condition (1.3) imposes that $q/(n-p-q) \geq d$ for some positive constant d. Hence

$$\begin{aligned}\log G(t) &< -\frac{q(p-p_1)}{4}\log\Big\{1+\frac{16t^2}{(2n-q-p_1-p+1)^2}\Big\} \\ &< -\frac{dp_2}{4}(n-p-q)\log\Big\{1+\frac{16t^2}{(2+C)(n-p-q)}\Big\}\end{aligned}$$

Therefore

$$\begin{aligned}I_3(v) &\leq \int_{M_0v}^{\infty}\frac{2}{t}\Big\{1+\frac{16t^2}{(2+C)(n-p-q)}\Big\}^{-dp_2(n-p-q)/4}dt \\ &= \int_{C_v}^{\infty}\frac{1}{u}(1+u)^{-dp_2(n-p-q)/4}du = C_v^{-1}(1+C_v)^{1-dp_2(n-p-q)/4} \\ &= O((1+C_v)^{-dp_2M_0/4})\end{aligned}$$

where $C_v = \{4M_0v/((2+C)(n-p-q))\}^2$. □

The condition (5.9) is a suffucient condition and will be weakened since $I_3[v] = o(M^{-s})$ for any s.

6 Numerical experiments

The upper bound given in the previous section depends on v. Some numerical calculations show that it will be sufficient to calculate the upper bounds with $v = 0.05, 0.01, \ldots, 0.95$ and choose the minimum.

Table 1 gives such minimums in the case that $s = 1, n = 50$ and several combinations of p, q, p_1, q_1. If p and q is less than 20 the bounds are not so small. We can see that the bounds get large as p_1 and q_1 get large. The reason is that $\kappa_\lambda^{(2)}$ becomes large which leads that b_s becomes large and M becomes small.

Table 2 shows the similar result in the case of $s = 2$. If p and q are larger than 10, the bounds are sufficiently small.

Table 1: Error bounds in the case of $n = 50$ and $s = 1$

p	q	p_1	q_1	bound	v	m	$\kappa_\lambda^{(2)}$
5	5	1	1	0.0763	0.6	3.1	0.16
5	5	2	2	0.0868	0.6	2.9	0.15
5	5	3	3	0.1116	0.65	2.6	0.13
5	5	4	4	0.1859	0.7	2.	0.1
10	10	2	2	0.0186	0.5	5.3	0.36
10	10	4	4	0.0211	0.5	5.	0.34
10	10	6	6	0.0285	0.5	4.5	0.31
10	10	8	8	0.0534	0.6	3.5	0.24
15	15	3	3	0.0078	0.45	6.2	0.64
15	15	6	6	0.0091	0.5	6.	0.62
15	15	9	9	0.0121	0.5	5.5	0.56
15	15	12	12	0.0237	0.55	4.4	0.45
20	20	4	4	0.0057	0.6	5.2	1.1
20	20	8	8	0.0064	0.6	5.1	1.07
20	20	12	12	0.0085	0.6	4.8	1.01
20	20	16	16	0.0165	0.65	4.	0.84

Appendix

Lemma A.1 *For any positive numbers a, b $(a < b)$ and positive integer p we have the following two asymptotic formulas:*

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{j=1}^{p}&\Big\{\frac{1}{k+\frac{a+j}{2}}-\frac{1}{k+\frac{b+j}{2}}\Big\}\\ &=\big(b+p-\frac{1}{2}\big)\log(b+p+1)-\big(a+p-\frac{1}{2}\big)\log(a+p+1)\\ &\quad-\big(b-\frac{1}{2}\big)\log(b+1)+\big(a-\frac{1}{2}\big)\log(a+1)+O(a^{-1})\quad(a\to\infty).\end{aligned}\tag{A.1}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{j=1}^{p}&\Big\{\frac{1}{(k+\frac{a+j}{2})^2}-\frac{1}{(k+\frac{b+j}{2})^2}\Big\}\\ &=2\log\frac{(b-\frac{1}{2})(a+p-\frac{1}{2})}{(a-\frac{1}{2})(b+p-\frac{1}{2})}+O(a^{-2})\quad(a\to\infty).\end{aligned}\tag{A.2}$$

Here the order of the remainder terms is uniform with respect to b and p.

Table 2: Error bounds in the case of $n = 50$ and $s = 2$

p	q	p_1	q_1	bound	v	m	$\kappa_\lambda^{(2)}$
5	5	1	1	0.0706	0.65	3.1	0.16
5	5	2	2	0.0824	0.65	2.9	0.15
5	5	3	3	0.1138	0.7	2.6	0.13
5	5	4	4	0.2227	0.7	2.	0.1
10	10	2	2	0.0109	0.5	5.3	0.36
10	10	4	4	0.0135	0.55	5.	0.34
10	10	6	6	0.0195	0.55	4.5	0.31
10	10	8	8	0.0441	0.6	3.5	0.24
15	15	3	3	0.0034	0.5	6.2	0.64
15	15	6	6	0.0042	0.5	6.	0.62
15	15	9	9	0.0063	0.55	5.5	0.56
15	15	12	12	0.0154	0.6	4.4	0.45
20	20	4	4	0.0022	0.65	5.2	1.1
20	20	8	8	0.0026	0.65	5.1	1.07
20	20	12	12	0.0039	0.65	4.8	1.01
20	20	16	16	0.0095	0.7	4.	0.84

Proof In order to show (A.1) and (A.2), it is sufficient to show

$$\sum_{k=1}^{\infty}\sum_{j=1}^{p}\frac{1}{k+\frac{a+j}{2}} = -\big(a+p-\frac{1}{2}\big)\log(a+p+1) + \big(a-\frac{1}{2}\big)\log(a+1) + O(a^{-1}). \tag{A.3}$$

$$\sum_{k=0}^{\infty}\sum_{j=1}^{p}\frac{1}{(k+\frac{a+j}{2})^2} = 2\log\frac{a+p-\frac{1}{2}}{a-\frac{1}{2}} + O(a^{-2}). \tag{A.4}$$

We apply Lemma A.2 to each of the following sums:

$$\sum_{k=1}^{K-1}\sum_{j=1}^{p}\frac{1}{k-1+\frac{a+j}{2}} \quad \text{and} \quad \sum_{k=1}^{K-1}\sum_{j=1}^{p}\frac{1}{(k-1+\frac{a+j}{2})^2},$$

and taking the limit $K \to \infty$. Then we can express each of the sums as the sums of six definite integrals. The first three integrals can be exactly computed.

The remainder three definite integrals can be estimated by noting that $B_i(x-[x])$ are finite. It is shown that these are $O(a^{-1})$ for the left side of (A.3) and $O(a^{-2})$ for the left side of (A.4). By proceeding these procedures we can show the required results. □

Lemma A.2 *Let $f(x,y)$ be a C^2–class function. Then*

$$\sum_{k=1}^{m-1}\sum_{l=1}^{n-1} f(k,l) = \int_1^m \left\{ \int_1^n f(x,y)dy \right\} dx - \frac{1}{2}\int_1^n \{f(m,y) - f(1,y)\}dy$$
$$- \frac{1}{2}\int_1^m \{f(x,n) - f(x,1)\}dx + \int_1^m \left\{ \int_1^n \frac{B_2(0) - B_2(x-[x])}{2} f_{xx}(x,y)dy \right\} dx$$
$$+ \int_1^m \left\{ \int_1^n \frac{B_2(0) - B_2(y-[y])}{2} f_{yy}(x,y)dy \right\} dx$$
$$+ \int_1^m \left\{ \int_1^n \{B_1(1) - B_1(x-[x])\}\{B_1(1) - B_1(y-[y])\} f_{xy}(x,y)dy \right\} dx,$$

where f_{xx}, f_{xy} and f_{yy} are the second order partial derivatives, and $B_k(x)$ ($k = 0,1,2,\ldots$) is the k–th order Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}.$$

Proof The Euler–Maclaurin formula can be given with two parts:

$$\int_1^n g(x)dx = \sum_{l=1}^{n-1} g(l) + \frac{g(n) - g(1)}{2} - \int_1^n B_1(x-[x])g'(x)dx \tag{A.5}$$

$$\int_1^n B_j(x-[x])g(x)dx = \int_1^n \frac{B_{j+1}(0) - B_{j+1}(x-[x])}{j+1} g'(x)dx \quad (j \geq 1). \tag{A.6}$$

Noting that $B_1(x) = x - 1/2$ and hence $B_1(1) = 1/2$, the formula (A.5) can be written as

$$\sum_{l=1}^{n-1} g(l) = \int_1^n [g(x) - \{B_1(1) - B_1(x-[x])\}g'(x)]dx. \tag{A.7}$$

Applying (A.7) to $f(x,y)$ twice as the function of x and as the function of y we have

$$\sum_{k=1}^{m-1}\sum_{l=1}^{n-1} f(k,l)$$
$$= \sum_{k=1}^{m-1}\int_1^n f(k,y)dy - \sum_{k=1}^{m-1}\int_1^n \{B_1(1) - B_1(y-[y])\} f_y(k,y)dy$$
$$= \int_1^n \Bigg[\int_1^m \big[f(x,y) - \{B_1(1) - B_1(x-[x])\} f_x(x,y) \big] dx$$
$$- \{B_1(1) - B_1(y-[y])\} \int_1^m \big[f_y(x,y) - \{B_1(1) - B_1(x-[x])\} f_{xy}(x,y) \big] dx \Bigg] dy.$$

Apply (A.6) to $g(x) = f_x(x,y)$ and $g(y) = f_y(x,y)$ with $j = 1$ then we obtain the desired result. □

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