Testing homogeneity of mean vectors under heteroscedasticity in high-dimension

Takayuki Yamada[1] and Tetsuto Himeno[2]

[1] General Studies, College of Engineering,
Nihon University,
1 Nakagawara, Tokusada, Tamuramachi, Koriyama, Fukushima 963-8642, Japan

[2]Department of Computer and Information Science, Faculty of Science and Technology,
Seikei University,
3-3-1 Kichijoji-Kitamachi, Musashino-shi, Tokyo 180-8633, Japan

Abstract

This paper is concerned with the problem of testing the homogeneity of mean vectors. The testing problem is without assuming common covariance matrix. We proposed a testing statistic based on the variation matrix due to the hypothesis and the unbiased estimator of the covariance matrix. The limiting null and non-null distributions are derived as each sample size and the dimensionality go to infinity together under a general population distribution including normal distribution. It is found that our proposed test has the same limiting power as the one of Dempster's trace statistic for MANOVA proposed in Fujikoshi, Himeno and Wakaki (2004, JJSS) for the case that the population distributions are multivariate normal with common covariance matrix for all groups. A small scale simulation study is performed to compare the actual error probability of the first kind with the nominal. It is seen that our proposed test is little affected by the non-normality.

1 Introduction

Let $\boldsymbol{x}_1^{(i)},\dots,\boldsymbol{x}_{N_i}^{(i)}$ be the p-dimensional observation vectors from the ith population Π_i, $i=1,\dots,g$. Assume that the observation vector has the following model:

$$\boldsymbol{x}_j^{(i)}=\boldsymbol{\mu}_i+\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\varepsilon}_j^{(i)}\quad(j=1,\dots,N_i,i=1,\dots,g),\tag{1}$$

where $\boldsymbol{\varepsilon}_1^{(i)},\dots,\boldsymbol{\varepsilon}_{N_i}^{(i)}$ are independently and identically distributed (i.i.d.) as p-dimensional distribution $F=F_p(\mathbf{0},\boldsymbol{I}_p)$ with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{I}_p$. We concern the problem of testing homogeneity of these mean vectors, i.e., the problem is testing the null hypothesis

$$H_0:\boldsymbol{\mu}_1=\cdots=\boldsymbol{\mu}_g$$

against all alternative hypothesis H_1. Some results are obtained under the assumption that $\boldsymbol{\Sigma}_1=\cdots=\boldsymbol{\Sigma}_g$ and F is p-dimensional normal. Let $\boldsymbol{W}$ and $\boldsymbol{B}$ be the variation matrices due to the errors and due to the hypothesis, respectively, which are defined as follows:

$$\boldsymbol{W}=(N_1-1)\boldsymbol{S}_1+\cdots+(N_g-1)\boldsymbol{S}_g,$$
$$\boldsymbol{B}=\sum_{i=1}^g N_i(\bar{\boldsymbol{x}}^{(i)}-\bar{\boldsymbol{x}})(\bar{\boldsymbol{x}}^{(i)}-\bar{\boldsymbol{x}})',$$

where $\bar{\boldsymbol{x}}^{(i)}=N_i^{-1}\sum_{j=1}^{N_i}\boldsymbol{x}_j^{(i)}$, $\bar{\boldsymbol{x}}=N^{-1}\sum_{j=1}^{g}N_j\bar{\boldsymbol{x}}^{(j)}$, $N=N_1+\cdots+N_g$, $\boldsymbol{S}_i$ is the unbiased estimator of $\boldsymbol{\Sigma}_i$, which is defined as $\boldsymbol{S}_i=(N_i-1)^{-1}\sum_{j=1}^{N_i}(\boldsymbol{x}_j^{(i)}-\bar{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)}-\bar{\boldsymbol{x}}^{(i)})'$. When $n=N-g+1\ge p$, the three tests are classically used, where the three tests are the likelihood ratio test $\Lambda=|\boldsymbol{W}|/|\boldsymbol{W}+\boldsymbol{B}|$, Lawley-Hotelling's trace test $\operatorname{tr}\boldsymbol{B}\boldsymbol{W}^{-1}$ and Bartlett-Nanda-Pillai's trace test $\operatorname{tr}\boldsymbol{B}(\boldsymbol{B}+\boldsymbol{W})^{-1}$. For the case that $p>n$, these three tests cannot be defined by the reason that W becomes singular. Srivastava and Fujikoshi [12] proposed adapted versions of these three tests by using Moore-Penrose

[1]E-mail address: yma801228@gmail.com
[2]E-mail address: t-himeno@st.seikei.ac.jp

inverse matrix. They showed asymptotic normality as the dimension and the sample size go to infinity together. Although these three tests are natural extension of the classical tests, the preciseness of the actual error probability of the first kind is worse, which can be checked by simulation. On the other hand, Dempster [4], [5] proposed non-exact tests for one and two sample problems. Later, Bai and Saranadasa [2] proposed other non-exact test for two sample problem. These two tests are both invariant under transformation $(\bar{x}, \boldsymbol{S}) \to (c\boldsymbol{\Gamma}\bar{x}, c^2\boldsymbol{\Gamma}\boldsymbol{S}\boldsymbol{\Gamma}')$ for an orthogonal matrix $\boldsymbol{\Gamma}$ and a constant c. Fujikoshi et al. [6] generalized Dempster's test for MANOVA problem and Srivastava and Fujikoshi [12] did Bai and Saranadasa's test. Generalization for non-normality has been studied. Bai and Saranadasa [2] has shown that their test is robust for the general population distribution with the condition $\mathrm{C_{BS}}$ of F that $E[\varepsilon_i^4] = 3+\gamma$ for $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_p)' \sim F$ and $E[\prod_{i=1}^p \varepsilon_i^{\nu_i}] = 0$ (and 1) when there is at least one $\nu_i = 1$ (there are two ν_i's equal to 2, correspondingly), whenever $\nu_1 + \cdots + \nu_p = 4$ under the model (1). Chen and Qin [3] proposed a test based on Bai and Saranadasa [2]'s testing statistic for two sample problem. They showed the asymptotic normality under the general population distribution with the condition $\mathrm{C_{CQ}}$ of F that $E[\varepsilon_i^4] = 3+\gamma$ and $E[\prod_{i=1}^q \varepsilon_{\ell_i}^{\nu_i}] = \prod_{i=1}^q E[\varepsilon_{\ell_i}^{\nu_i}]$ for a positive integer q such that $\sum_{i=1}^q \nu_i \le 8$ and $\ell_1 \ne \cdots \ne \ell_q$ without assuming that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$. The condition $\mathrm{C_{CQ}}$ implies $\mathrm{C_{BS}}$, and so $\mathrm{C_{BS}}$ is milder condition than $\mathrm{C_{CQ}}$.

This paper is concerned with the testing H_0 without assuming that $\boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_g$. Let $\mathfrak{m} = \sum_{i=1}^g N_i(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})'(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})$ with $\bar{\boldsymbol{\mu}} = (1/N)\sum_{i=1}^g N_i\boldsymbol{\mu}_i$. Then $\mathfrak{m} \ge 0$, where the strict inequality holds except for the case that $\boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_g$. Hence, the null hypothesis H_0 is equivalent to the hypothesis that $\mathfrak{m} = 0$. Rejection of the null hypothesis H_0 results from evidence that the unbiased estimator $\hat{\mathfrak{m}}$ of $\mathfrak{m}$ is significantly larger than zero. Hence we propose the testing statistic as

$$T = \frac{\hat{\mathfrak{m}}}{\sqrt{p}} = \frac{1}{\sqrt{p}}\left\{\operatorname{tr}\boldsymbol{B} - \sum_{i=1}^g\left(1 - \frac{N_i}{N}\right)\operatorname{tr}\boldsymbol{S}_i\right\}.$$

We derive the asymptotic distribution under asymptotic framework A1:

$$\mathrm{A1}: p \to \infty, N_i \to \infty, N_i/p \to c_i \in (0,\infty), N_i/N \to \gamma_i \in (0,1), i = 1, \ldots, g.$$

In addition, we will assume A2 and A3, which are as the following:

A2 : $\operatorname{tr}\boldsymbol{\Sigma}_i^2/p = O(1)$ as $p \to \infty, i = 1, \ldots, g,$ but at least one of them converges to a positive constant;
A3 : $\operatorname{tr}\boldsymbol{\Sigma}_i^4/p = O(1)$ as $p \to \infty, i = 1, \ldots, g.$

These assumptions are concerned with the structure of the covariance matrices. Instead of using the $\mathrm{C_{BS}}$ or $\mathrm{C_{CQ}}$, we use the assumptions A4, A5 and A6, which are as follows:

$$\mathrm{A4}: \kappa_1 = \sup_{1\le i\le g} E[(\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}_i^2\boldsymbol{\varepsilon} - \operatorname{tr}\boldsymbol{\Sigma}_i^2)^2] = O(p^2) \text{ for } \boldsymbol{\varepsilon} \text{ is distributed as } F;$$
$$\mathrm{A5}: \kappa_2 = \sup_{1\le i,j\le g} E[(\boldsymbol{\varepsilon}_1'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\varepsilon}_2)^4] = o(p^4) \text{ for } \boldsymbol{\varepsilon}_1 \text{ and } \boldsymbol{\varepsilon}_2 \text{ are i.i.d. as } F;$$
$$\mathrm{A6}: \kappa_3 = \sup_{1\le i,j\le g} E[(\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\varepsilon})^2] = O(p^2) \text{ for } \boldsymbol{\varepsilon} \text{ is distributed as } F.$$

For the case that $g = 2$, the statistic T is identical to the Chen and Qin [3]'s testing statistic except for multiple of $N/\sqrt{N_1N_2}$. We will show the asymptotic null distribution of T under the asymptotic framework A1 and the assumptions A2, …, A6. The testing statistic is not invariant under transformation: $\boldsymbol{x}_j^{(i)} \mapsto \boldsymbol{A}_i\boldsymbol{x}_j^{(i)}$ for an non-singular matrix $\boldsymbol{A}_i$. So the asymptotic variance of the testing statistic becomes the function of the nuisance parameters $(\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_g)$, which needs to be estimated for practical use. It is common to use the unbiased estimator. To show the consistency, we use the following assumption A7:

$$\mathrm{A7}: \kappa_{22} = \sup_{1\le i\le g}\{E[(\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}_i^2\boldsymbol{\varepsilon})^2] - 2\operatorname{tr}\boldsymbol{\Sigma}_i^4 - (\operatorname{tr}\boldsymbol{\Sigma}_i^2)^2\} = o(p^3),\quad \sup_{1\le i\le g}\{E[(\boldsymbol{\varepsilon}_1'\boldsymbol{\Sigma}_i\boldsymbol{\varepsilon}_2)^4]\} = o(p^4)$$

for $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are i.i.d. as F.

Under the asymptotic framework A1 and the assumptions for covariance matrices A2 and A3, the assumptions for distribution A4, A5, A6 and A7 hold when F is elliptical distribution, and are implied by C_{BS}. Hence our assumption is milder than C_{BS}.

For the nonnull case, we assume the assumption A8:

$$\mathrm{A8}: \sum_{i=1}^{g} \mathrm{tr}\, \boldsymbol{\Sigma}_i^k \boldsymbol{\Omega}_i = O(\sqrt{p}) \quad (k = 1, 2),$$

where

$$\boldsymbol{\Omega}_i = N_i \boldsymbol{\Sigma}_i^{-1/2}(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})' \boldsymbol{\Sigma}_i^{-1/2}.$$

Under the asymptotic framework A1 and the assumptions A2-A8, we gave asymptotic power, and found that it is the same as the one proposed by Fujikoshi et al. [6] or Srivastava and Fujikoshi [12] when $\boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_g$ and $F = N_p(\mathbf{0}, \boldsymbol{I}_p)$.

Later, we denote "$\xrightarrow{p}$" as the convergence in probability, and "$\stackrel{\mathcal{D}}{=}$" as the equality in distribution. In addition, we use the notation "$\sum_{i \neq j}$" as the sum of all pairs of i and j such that $i \neq j$.

2 Assumptions for multivariate distribution

In this section, we show that the assumptions for distribution A4, A5, A6 and A7 hold when F is elliptical distribution and are implied by C_{BS} under the asymptotic framework A1 and the assumptions for covariance matrices A2 and A3,

Lemma 1. *Assume that F is p-dimensional elliptical distribution with mean vector $\mathbf{0}$ and the covariance matrix $\boldsymbol{I}_p$, and $E[R^4] = O(p^2)$ with $R = \sqrt{\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}}$ for $\boldsymbol{\varepsilon} \sim F$. Then* A4, A5, A6 *and* A7 *hold.*

Proof. First of all, we evaluate $E[(\boldsymbol{\varepsilon}'\boldsymbol{\Lambda}\boldsymbol{\varepsilon})^2]$ with positive semi definite matrix $\boldsymbol{\Lambda}$. Since $\boldsymbol{\varepsilon} \stackrel{\mathcal{D}}{=} \boldsymbol{\Gamma}\boldsymbol{\varepsilon}$ for any orthogonal matrix $\boldsymbol{\Gamma}$, we may assume without loss of generality that $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$. It holds that

$$E[(\boldsymbol{\varepsilon}'\boldsymbol{\Lambda}\boldsymbol{\varepsilon})^2] = \sum_{i=1}^{p} \lambda_i^2 E[\varepsilon_i^4] + \sum_{i \neq j}^{p} \lambda_i \lambda_j E[\varepsilon_i^2 \varepsilon_j^2]$$

with $\boldsymbol{\varepsilon} = (\varepsilon_1 \cdots \varepsilon_p)'$. The moments can be evaluated as the following:

$$E[\varepsilon_i^4] = \frac{3E[R^4]}{p(p+2)}, \quad E[\varepsilon_i^2 \varepsilon_j^2] = \frac{E[R^4]}{p(p+2)}$$

for $i, j = 1, \dots, p$, $i \neq j$ (cf. Anderson [1]). So we have

$$E[(\boldsymbol{\varepsilon}'\boldsymbol{\Lambda}\boldsymbol{\varepsilon})^2] = \frac{2E[R^4]}{p(p+2)} \mathrm{tr}\boldsymbol{\Lambda}^2 + \frac{E[R^4]}{p(p+2)} (\mathrm{tr}\boldsymbol{\Lambda})^2. \tag{2}$$

For $i = 1, \dots, p$,

$$E[(\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}_i^2\boldsymbol{\varepsilon} - \mathrm{tr}\, \boldsymbol{\Sigma}_i^2)^2] = \frac{2E[R^4]}{p(p+2)} \mathrm{tr}\boldsymbol{\Sigma}_i^4 + \left\{ \frac{E[R^4]}{p(p+2)} - 1 \right\} (\mathrm{tr}\boldsymbol{\Sigma}_i^2)^2,$$

which is $O(p^2)$ under A1, A2 and A3, so A4 holds. Letting $\boldsymbol{A} = \boldsymbol{a}\boldsymbol{a}'$ with $\boldsymbol{a} = \boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\varepsilon}_2$, it can be expressed that

$$\begin{aligned} E[(\boldsymbol{\varepsilon}_1'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\varepsilon}_2)^4] &= E[E[(\boldsymbol{a}'\boldsymbol{\varepsilon}_1)^4|\boldsymbol{a}]] \\ &= E[E[(\boldsymbol{\varepsilon}_1'\boldsymbol{A}\boldsymbol{\varepsilon}_1)^2|\boldsymbol{a}]] \\ &= E\left[\frac{2E[R^4]}{p(p+2)} \mathrm{tr}\boldsymbol{A}^2 + \frac{E[R^4]}{p(p+2)} (\mathrm{tr}\boldsymbol{A})^2\right], \end{aligned}$$

where the last equality follows from (2). Note that $\operatorname{tr} \boldsymbol{A}^2 = (\operatorname{tr} \boldsymbol{A})^2 = (\boldsymbol{a}'\boldsymbol{a})^2$. Using the result in (2) again, we have

$$
\begin{aligned}
E[(\boldsymbol{\varepsilon}_1' \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{\Sigma}_j^{1/2} \boldsymbol{\varepsilon}_2)^4] &= E\left[\frac{2E[R^4]}{p(p+2)} \operatorname{tr} \boldsymbol{A}^2 + \frac{E[R^4]}{p(p+2)} (\operatorname{tr} \boldsymbol{A})^2 \right] \\
&= 3 \left\{ \frac{E[R^4]}{p(p+2)} \right\}^2 \left\{ 2 \operatorname{tr} (\boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)^2 + (\operatorname{tr} \boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)^2 \right\}
\end{aligned}
$$

for $i, j = 1, \ldots, p$. From the inequalities in (25), it is found that

$$
\frac{1}{p} \operatorname{tr} (\boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)^2 = O(1) \tag{3}
$$

under the asymptotic framework A1 and assumption A3. Using Cauchy-Schwarz's inequality, it holds that

$$
(\operatorname{tr} \boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)^2 \leq \operatorname{tr} \boldsymbol{\Sigma}_i^2 \operatorname{tr} \boldsymbol{\Sigma}_j^2.
$$

Thus,

$$
\frac{1}{p^2} (\operatorname{tr} \boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j)^2 = O(1). \tag{4}
$$

From (3) and (4), $E[(\boldsymbol{\varepsilon}_1' \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{\Sigma}_j^{1/2} \boldsymbol{\varepsilon}_2)^4]$ is $o(p^4)$ under A1, A2 and A3, so A5 holds. By the result in (2), A6 and A7 can be shown immediately. □

Lemma 2. *Under* A1*,* A2 *and* A3*, the condition* $\mathrm{C_{BS}}$ *implies* A4*,* A5*,* A6 *and* A7.

Proof. Let $\boldsymbol{C} = (c_{ij})$ be $p \times p$ positive semi definite matrix. Under the assumption $\mathrm{C_{BS}}$, the expectation $E[(\boldsymbol{\varepsilon}' \boldsymbol{C} \boldsymbol{\varepsilon})^2]$ can be evaluated that

$$
\begin{aligned}
E[(\boldsymbol{\varepsilon}' \boldsymbol{C} \boldsymbol{\varepsilon})^2] &= E\left[\left(\sum_{i=1}^p c_{ii} \varepsilon_i^2 + \sum_{i \neq j}^p c_{ij} \varepsilon_i \varepsilon_j \right)^2 \right] \\
&= \sum_{i=1}^p c_{ii} E[\varepsilon_i^4] + \sum_{i \neq j}^p c_{ii} c_{jj} E[\varepsilon_i^2 \varepsilon_j^2] + 2 \sum_{i \neq j}^p c_{ij}^2 E[\varepsilon_i^2 \varepsilon_j^2] \\
&= (3 + \gamma) \sum_{i=1}^p c_{ii} + \sum_{i \neq j}^p c_{ii} c_{jj} + 2 \sum_{i \neq j}^p c_{ij}^2.
\end{aligned} \tag{5}
$$

Note that $\sum_{i=1}^p c_{ii}^2 \leq \operatorname{tr} \boldsymbol{C}^2$, $\sum_{i \neq j}^p c_{ii} c_{jj} \leq (\operatorname{tr} \boldsymbol{C})^2$ and $\sum_{i \neq j}^p c_{ij}^2 \leq \operatorname{tr} \boldsymbol{C}^2$. Thus we have

$$
\begin{aligned}
E[(\boldsymbol{\varepsilon}' \boldsymbol{C} \boldsymbol{\varepsilon})^2] &= (3 + \gamma) \sum_{i=1}^p c_{ii}^2 + \sum_{i \neq j}^p c_{ii} c_{jj} + 2 \sum_{i \neq j}^p c_{ij}^2 \\
&\leq (3 + \gamma) \operatorname{tr} \boldsymbol{C}^2 + (\operatorname{tr} \boldsymbol{C})^2 + 2 \operatorname{tr} \boldsymbol{C}^2.
\end{aligned} \tag{6}
$$

This leads that

$$
\kappa_1 \leq (5 + \gamma) \sup_{1 \leq i \leq g} \operatorname{tr} \boldsymbol{\Sigma}_i^2,
$$

which the right-hand side is $O(p)$ under A2, and so $\kappa_1 = O(p)$. Hence we find that A4 holds. For any fixed $i, j \in \{1, \ldots, p\}$ with $i \neq j$, letting $\boldsymbol{A} = \boldsymbol{a}\boldsymbol{a}'$ with $\boldsymbol{a} = \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{\Sigma}_j^{1/2} \boldsymbol{\varepsilon}_2$, it can be expressed that

$$
E[(\boldsymbol{\varepsilon}_1' \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{\Sigma}_j^{1/2} \boldsymbol{\varepsilon}_2)^4] = E[E[(\boldsymbol{\varepsilon}_1' \boldsymbol{A} \boldsymbol{\varepsilon}_1)^2 | \boldsymbol{a}]],
$$

From (6), we find that

$$
E[E[(\boldsymbol{\varepsilon}_1' \boldsymbol{A} \boldsymbol{\varepsilon}_1)^2 | \boldsymbol{a}]] \leq E[(3 + \gamma) \operatorname{tr} \boldsymbol{A}^2 + (\operatorname{tr} \boldsymbol{A})^2 + 2 \operatorname{tr} \boldsymbol{A}^2].
$$

Since $\operatorname{tr} \boldsymbol{A}^2 = (\operatorname{tr} \boldsymbol{A})^2 = (\boldsymbol{a}'\boldsymbol{a})^2$,

$$
\begin{aligned}
E[(\boldsymbol{\varepsilon}_1'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\varepsilon}_2)^4] &\le (6+\gamma)E[(\boldsymbol{\varepsilon}_2'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\varepsilon}_2)^2] \\
&\le (6+\gamma)\left\{(3+\gamma)\operatorname{tr}(\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2 + 2\operatorname{tr}(\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2 + (\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2\right\}, \qquad (7)
\end{aligned}
$$

where the second inequality follows by (6). From (25), the right-hand of the inequality (7) is $O(p^2)$, and so κ_2 is $o(p^4)$, which leads that A5 holds. We can show A6 and A7 by the result in (6), immediately. □

3 Asymptotic null distribution of the proposed testing statistic

Let

$$
\Phi = (\phi_{ij}) = \begin{pmatrix} \boldsymbol{\mathcal{P}}_{N_1} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \ddots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{\mathcal{P}}_{N_g} \end{pmatrix} - \boldsymbol{\mathcal{P}}_N, \qquad (8)
$$

where the matrix $\boldsymbol{\mathcal{P}}_j = j^{-1}\boldsymbol{1}_j\boldsymbol{1}_j'$, $\boldsymbol{1}_j$ is j-dimensional vector which all elements are equal to 1. Then it holds that

$$
\boldsymbol{B} = \boldsymbol{X}\boldsymbol{\Phi}\boldsymbol{X}',
$$

where

$$
\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1^{(1)} & \cdots & \boldsymbol{x}_{N_1}^{(1)} & \boldsymbol{x}_1^{(2)} & \cdots & \boldsymbol{x}_{N_g}^{(g)} \end{pmatrix}.
$$

Define

$$
m_i = \begin{cases} N_1 + \cdots + N_{i-1} & (i = 2, 3, \ldots, g), \\ 0 & (i = 1). \end{cases}
$$

Setting $\boldsymbol{x}_{m_i+j} = \boldsymbol{x}_j^{(i)}$, we can rewrite $\boldsymbol{X}$ as

$$
\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_N \end{pmatrix},
$$

and then expand $\operatorname{tr} \boldsymbol{B}$ as

$$
\operatorname{tr} \boldsymbol{B} = \sum_{i=1}^{N} \phi_{ii}\boldsymbol{x}_i'\boldsymbol{x}_i + 2\sum_{i<j}^{N} \phi_{ij}\boldsymbol{x}_i'\boldsymbol{x}_j.
$$

Recalling the definition of $\boldsymbol{\Phi}$, we have $\phi_{kk} = N_i^{-1} - N^{-1}$ when $k \in I_i = \{m_i+1, \ldots, m_i+N_i\}$, and so

$$
\operatorname{tr} \boldsymbol{B} = \sum_{i=1}^{g} \left(\frac{1}{N_i} - \frac{1}{N}\right) \sum_{k=m_i+1}^{m_i+N_i} \boldsymbol{x}_k'\boldsymbol{x}_k + \sum_{i\neq j}^{N} \phi_{ij}\boldsymbol{x}_i'\boldsymbol{x}_j. \qquad (9)
$$

On the other hand, let $\boldsymbol{X}_i = \begin{pmatrix} \boldsymbol{x}_{m_i+1} & \cdots & \boldsymbol{x}_{m_i+N_i} \end{pmatrix}$. Then

$$
\boldsymbol{S}_i = \frac{1}{N_i - 1}\boldsymbol{X}_i(\boldsymbol{I}_{N_i} - \boldsymbol{\mathcal{P}}_{N_i})\boldsymbol{X}_i',
$$

which can be described as

$$
\frac{1}{N_i}\sum_{k=m_i+1}^{m_i+N_i} \boldsymbol{x}_k\boldsymbol{x}_k' - \frac{1}{N_i(N_i-1)} \sum_{\substack{k\neq \ell \\ k,\ell \ge m_i+1}}^{m_i+N_i} \boldsymbol{x}_k\boldsymbol{x}_\ell'. \qquad (10)
$$

So,

$$
\operatorname{tr} \boldsymbol{S}_i = \frac{1}{N_i}\sum_{k=m_i+1}^{m_i+N_i} \boldsymbol{x}_k'\boldsymbol{x}_k - \frac{1}{N_i(N_i-1)} \sum_{\substack{k\neq \ell \\ k,\ell \ge m_i+1}}^{m_i+N_i} \boldsymbol{x}_k'\boldsymbol{x}_\ell. \qquad (11)
$$

From the expressions (9) and (11), we have

$$T = \frac{1}{\sqrt{p}}\sum_{i\neq j}^{N}\phi_{ij}\boldsymbol{x}_i'\boldsymbol{x}_j + \frac{1}{\sqrt{p}}\sum_{i=1}^{g}\left(1-\frac{N_i}{N}\right)\frac{1}{N_i(N_i-1)}\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\boldsymbol{x}_k'\boldsymbol{x}_\ell.$$

Recalling the definition $\boldsymbol{\Phi}$ again, it holds that for $i\neq j$,

$$\phi_{ij} = \begin{cases} \dfrac{1}{N_k}-\dfrac{1}{N} & (i,j\in I_k),\\ -\dfrac{1}{N} & (\text{otherwise}). \end{cases}$$

Thus,

$$\begin{aligned} T = {}& \frac{1}{\sqrt{p}}\left\{\sum_{i=1}^{g}\left(\frac{1}{N_i}-\frac{1}{N}\right)\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\boldsymbol{x}_k'\boldsymbol{x}_\ell - \frac{1}{N}\sum_{i\neq j}^{g}\left(\sum_{k=m_i+1}^{m_i+N_i}\sum_{\ell=m_j+1}^{m_j+N_j}\boldsymbol{x}_k'\boldsymbol{x}_\ell\right)\right\}\\ &+\frac{1}{\sqrt{p}}\sum_{i=1}^{g}\left(1-\frac{N_i}{N}\right)\frac{1}{N_i(N_i-1)}\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\boldsymbol{x}_k'\boldsymbol{x}_\ell, \end{aligned}$$

which can be coordinate as

$$T = \frac{1}{\sqrt{p}}\sum_{i=1}^{g}\frac{1}{N_i-1}\left(1-\frac{N_i}{N}\right)\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\boldsymbol{x}_k'\boldsymbol{x}_\ell - \frac{1}{\sqrt{p}N}\sum_{i\neq j}^{g}\left(\sum_{k=m_i+1}^{m_i+N_i}\sum_{\ell=m_j+1}^{m_j+N_j}\boldsymbol{x}_k'\boldsymbol{x}_\ell\right).$$

Assuming the model (1), under the null hypothesis H_0, it can be expressed that

$$T = \frac{2}{\sqrt{p}}\sum_{i=1}^{g}\frac{1}{N_i-1}\left(1-\frac{N_i}{N}\right)\sum_{\substack{k<\ell\\ k\geq m_i+1}}^{m_i+N_i}\boldsymbol{z}_k'\boldsymbol{\Sigma}_i\boldsymbol{z}_\ell - \frac{2}{\sqrt{p}N}\sum_{i<j}^{g}\left(\sum_{k=m_i+1}^{m_i+N_i}\sum_{\ell=m_j+1}^{m_j+N_j}\boldsymbol{z}_k'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell\right),$$

where each $\boldsymbol{z}_i$ denotes the error vector corresponding to $\boldsymbol{x}_i$ which satisfies $\boldsymbol{z}_i = \boldsymbol{\varepsilon}_k^{(\ell)}$ for the case that $i = m_\ell + k$. Notice that T is represented as the sum of correlated terms. In order to show asymptotic normality, we use Martingale difference central limit theorem. For the case that $\ell \in I_j$, let

$$\eta_\ell = \frac{2}{\sqrt{p}}\left[-\frac{1}{N}\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j^{1/2}\left\{\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{z}_k\right\} + \frac{1}{N_j-1}\left(1-\frac{N_j}{N}\right)\boldsymbol{z}_\ell'\left(\sum_{k=m_j+1}^{\ell-1}\boldsymbol{\Sigma}_j\boldsymbol{z}_k\right)\right],$$

and let $\mathcal{F}_j$ be the σ-algebra generated by the random vectors $\boldsymbol{z}_1,\ldots,\boldsymbol{z}_j$ and $\mathcal{F}_0 = \{\phi,\Omega\}$, where ϕ denotes the empty set and Ω the whole space. It shall be noticed that $\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_N$. Letting $\boldsymbol{z}_0 = \boldsymbol{0}$, we have

$$T = \sum_{\ell=1}^{N}\eta_\ell.$$

In addition,

$$\begin{aligned} &E[\eta_\ell|\mathcal{F}_{\ell-1}] = 0,\\ &E[\eta_\ell^2|\mathcal{F}_{\ell-1}] = \frac{4}{pN^2}\left\{\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{z}_k\right\}'\left\{\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{z}_k\right\}\\ &\qquad+\frac{1}{p}\left\{\frac{2}{N_j-1}\left(1-\frac{N_j}{N}\right)\right\}^2\left(\sum_{k=m_j+1}^{\ell-1}\boldsymbol{z}_k\right)'\boldsymbol{\Sigma}_j^2\left(\sum_{k=m_j+1}^{\ell-1}\boldsymbol{z}_k\right)\\ &\qquad-2\frac{2}{pN}\left\{\frac{2}{N_j-1}\left(1-\frac{N_j}{N}\right)\right\}\left\{\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{z}_k\right\}'\left(\sum_{k=m_j+1}^{\ell-1}\boldsymbol{\Sigma}_j\boldsymbol{z}_k\right) \qquad (12) \end{aligned}$$

for the case that $\ell \in I_j$. By taking expectation for the conditional expectation, we have

$$E[\eta_\ell^2] = \frac{4}{N^2}\sum_{i=1}^{j-1} N_i \frac{\operatorname{tr} \boldsymbol{\Sigma}_j \boldsymbol{\Sigma}_i}{p} + \left\{ \frac{2}{N_j - 1}\left(1 - \frac{N_j}{N}\right)\right\}^2 \{(\ell - 1) - (m_i + 1) + 1\} \frac{\operatorname{tr} \boldsymbol{\Sigma}_j^2}{p}, \qquad (13)$$

which is finite under the asymptotic framework A1 and the assumption A2. So the sequence $(\eta_\ell, \mathcal{F}_\ell)$ is a squared integrable Martingale difference. In order to show the central limit theorem, it shall be verified that

(I) $C = \sum_{\ell=1}^{N} E[\eta_\ell^2 | \mathcal{F}_{\ell-1}] \xrightarrow{p} \sigma_0^2 < \infty;$

(II) $L = \sum_{\ell=1}^{N} E[\eta_\ell^2 I(|\eta_\ell| > \varepsilon) | \mathcal{F}_{\ell-1}] \xrightarrow{p} 0$ for $\forall \varepsilon > 0,$

where the function $I(.)$ denotes the indicator function. The latter is known as the Lindberg's condition in central limit theorem.

We first show the condition (I). From the definition, it can be described as

$$E[C] = \sum_{\ell=1}^{N} E[\eta_\ell^2].$$

Partition the summing as

$$\sum_{\ell=1}^{N} E[\eta_\ell^2] = \sum_{i=1}^{g} \sum_{\ell=m_i+1}^{m_i+N_i} E[\eta_\ell^2].$$

From (13), we have

$$\begin{aligned} E[C] = \sum_{i=1}^{g} \Bigg[& \left\{ \frac{4}{N^2} \left(\sum_{j=1}^{i-1} N_j \frac{\operatorname{tr} \boldsymbol{\Sigma}_j \boldsymbol{\Sigma}_i}{p} \right) \sum_{\ell=m_i+1}^{m_i+N_i} 1 \right\} \\ & + \left\{ \frac{2}{N_i - 1}\left(1 - \frac{N_i}{N}\right)\right\}^2 \sum_{\ell=m_i+1}^{m_i+N_i} \{(\ell - 1) - (m_i + 1) + 1\} \frac{\operatorname{tr} \boldsymbol{\Sigma}_i^2}{p} \Bigg], \end{aligned}$$

which can be represented as

$$\sum_{i=1}^{g} \left\{ \frac{2}{N_i - 1}\left(1 - \frac{N_i}{N}\right)\right\}^2 \frac{N_i(N_i - 1)}{2} \frac{\operatorname{tr} \boldsymbol{\Sigma}_i^2}{p} + \frac{2}{N^2} \sum_{i \neq j} N_i N_j \frac{\operatorname{tr} \boldsymbol{\Sigma}_i \boldsymbol{\Sigma}_j}{p}.$$

This implies that $E[C]$ converges to a positive constant under the asymptotic framework A1 and assumption A2, say σ_0^2. Thus, to show the probability convergence in (I), we need to show that $\operatorname{Var}(C)$ converges to 0. Partition the summing in C as

$$C = \sum_{i=1}^{g} \sum_{\ell=m_i+1}^{m_i+N_i} E[\eta_\ell^2 | \mathcal{F}_{\ell-1}].$$

From (12) it can be expressed that

$$C = T_1 + T_2 + T_3,$$

where

$$
\begin{aligned}
T_1 &= \frac{1}{p}\sum_{i=1}^{g}\left\{\frac{2}{N_i-1}\left(1-\frac{N_i}{N}\right)\right\}^2 \sum_{\ell=m_i+1}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1}\boldsymbol{z}_k\right)'\boldsymbol{\Sigma}_i^2\left(\sum_{k=m_i+1}^{\ell-1}\boldsymbol{z}_k\right),\\
T_2 &= \frac{4}{pN^2}\sum_{i=1}^{g}\sum_{\ell=m_i+1}^{m_i+N_i}\left\{\sum_{j=1}^{i-1}\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_k\right\}'\left\{\sum_{j=1}^{i-1}\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_k\right\},\\
T_3 &= -2\frac{2}{Np}\sum_{i=1}^{g}\left\{\frac{2}{N_i-1}\left(1-\frac{N_i}{N}\right)\right\}\left(\sum_{j=1}^{i-1}\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_k\right)'\left\{\sum_{\ell=m_i+1}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1}\boldsymbol{\Sigma}_i\boldsymbol{z}_k\right)\right\}.
\end{aligned}
$$

Since $\mathrm{Var}(C) \le 3(\mathrm{Var}(T_1)+\mathrm{Var}(T_2)+\mathrm{Var}(T_3))$, it is sufficient to show that $\mathrm{Var}(T_i)\to 0$, $i=1,2,3$. Firstly, we show that $\mathrm{Var}(T_1)$ converges to 0. Let

$$
T_{1i} = \sum_{\ell=m_i+1}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1}\boldsymbol{z}_k\right)'\boldsymbol{\Sigma}_i^2\left(\sum_{k=m_i+1}^{\ell-1}\boldsymbol{z}_k\right).
$$

From the independency, it holds that

$$
\mathrm{Var}(T_1) = \frac{1}{p^2}\sum_{i=1}^{g}\left\{\frac{2}{N_i-1}\left(1-\frac{N_i}{N}\right)\right\}^4\mathrm{Var}\left(T_{1i}\right).
$$

Note that the random variable T_{1i} can be expanded as

$$
T_{1i} = \sum_{\ell=m_i+1}^{m_i+N_i}\left\{\sum_{k=m_i+1}^{\ell-1}\boldsymbol{z}_k'\boldsymbol{\Sigma}_i^2\boldsymbol{z}_k + \sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\ge m_i+1}}^{\ell-1}\boldsymbol{z}_\alpha'\boldsymbol{\Sigma}_i^2\boldsymbol{z}_\beta\right\},
$$

and so we find that

$$
E[T_{1i}] = \frac{N_i(N_i-1)}{2}\,\mathrm{tr}\,\boldsymbol{\Sigma}_i^2.
$$

This gives that

$$
T_{1i}-E[T_{1i}] = \sum_{\ell=m_i+1}^{m_i+N_i}\left\{\sum_{k=m_i+1}^{\ell-1}(\boldsymbol{z}_k'\boldsymbol{\Sigma}_i^2\boldsymbol{z}_k-\mathrm{tr}\,\boldsymbol{\Sigma}_i^2) + \sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\ge m_i+1}}^{\ell-1}\boldsymbol{z}_\alpha'\boldsymbol{\Sigma}_i^2\boldsymbol{z}_\beta\right\}.
$$

Let

$$
\begin{aligned}
Y_{k,i}^{(\mathrm{DQ})} &= \boldsymbol{z}_k'\boldsymbol{\Sigma}_i^2\boldsymbol{z}_k-\mathrm{tr}\,\boldsymbol{\Sigma}_i^2,\\
Y_{\alpha\beta,i}^{(\mathrm{B1})} &= \boldsymbol{z}_\alpha'\boldsymbol{\Sigma}_i^2\boldsymbol{z}_\beta.
\end{aligned}
$$

The variance $\mathrm{Var}(T_{1i})$ can be expressed that

$$
\begin{aligned}
&E[(T_{1i}-E[T_{1i}])^2]\\
&= E\left[\sum_{\ell=m_i+1}^{m_i+N_i}\left\{\left(\sum_{k=m_i+1}^{\ell-1}Y_{k,i}^{(\mathrm{DQ})}\right)^2+\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\ge m_i+1}}^{\ell-1}Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)^2+2\left(\sum_{k=m_i+1}^{\ell-1}Y_{k,i}^{(\mathrm{DQ})}\right)\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\ge m_i+1}}^{\ell-1}Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\right\}\right.\\
&\quad\left.+2\sum_{m_i+1\le\ell<\ell'}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1}Y_{k,i}^{(\mathrm{DQ})}+\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\ge m_i+1}}^{\ell-1}Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\left(\sum_{k=m_i+1}^{\ell'-1}Y_{k,i}^{(\mathrm{DQ})}+\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\ge m_i+1}}^{\ell'-1}Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\right].
\end{aligned}
$$

For evaluating these expectations, we use the following identities.

$$E\left[\sum_{\ell=m_i+1}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1} Y_{k,i}^{(\mathrm{DQ})}\right)^2\right] = \sum_{\ell=m_i+1}^{m_i+N_i}\{(\ell-1)-(m_i+1)+1\}E[(\boldsymbol{z}'\boldsymbol{\Sigma}_i\boldsymbol{z}-\operatorname{tr}\boldsymbol{\Sigma}_i^2)^2]$$

$$= \frac{N_i(N_i-1)}{2}E[(\boldsymbol{z}'\boldsymbol{\Sigma}_i\boldsymbol{z}-\operatorname{tr}\boldsymbol{\Sigma}_i^2)^2],$$

$$E\left[\sum_{\ell=m_i+1}^{m_i+N_i}\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{\ell-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)^2\right] = \sum_{\ell=m_i+1}^{m_i+N_i}\{(\ell-1)-(m_i+1)\}\{(\ell-1)-(m_i+1)+1\}\operatorname{tr}\boldsymbol{\Sigma}_i^4$$

$$= \frac{2N_i(N_i-1)(N_i-2)}{3}\operatorname{tr}\boldsymbol{\Sigma}_i^4,$$

$$E\left[\sum_{\ell=m_i+1}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1} Y_{k,i}^{(\mathrm{DQ})}\right)\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{\ell-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\right] = 0,$$

$$E\left[\sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1} Y_{k,i}^{(\mathrm{DQ})}\right)\left(\sum_{k=m_i+1}^{\ell'-1} Y_{k,i}^{(\mathrm{DQ})}\right)\right]$$

$$= \sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i} E\left[\sum_{k=m_i+1}^{\ell-1}(Y_{k,i}^{(\mathrm{DQ})})^2\right] = \sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i}\{(\ell-1)-(m_i+1)+1\}E[(\boldsymbol{z}'\boldsymbol{\Sigma}_i^2\boldsymbol{z}-\operatorname{tr}\boldsymbol{\Sigma}_i^2)^2]$$

$$= \frac{(N_i-2)(N_i-1)N_i}{6}E[(\boldsymbol{z}'\boldsymbol{\Sigma}_i^2\boldsymbol{z}-\operatorname{tr}\boldsymbol{\Sigma}_i^2)^2],$$

$$E\left[\sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i}\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{\ell-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{\ell'-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\right] = 4\sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i} E\left[\left(\sum_{\substack{\alpha<\beta\\ \alpha\geq m_i+1}}^{\ell-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\left(\sum_{\substack{\alpha<\beta\\ \alpha\geq m_i+1}}^{\ell'-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\right]$$

$$= 4\sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i} E\left[\left(\sum_{\substack{\alpha<\beta\\ \alpha\geq m_i+1}}^{\ell-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)^2\right] = 4\sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i}\sum_{\substack{\alpha<\beta\\ \alpha\geq m_i+1}}^{\ell-1} E[(Y_{\alpha\beta,i}^{(\mathrm{B1})})^2]$$

$$= 4\sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i}\frac{\{(\ell-1)-(m_i+1)\}\{(\ell-1)-(m_i+1)+1\}}{2}\operatorname{tr}\boldsymbol{\Sigma}_i^4 = \frac{N_i(N_i-1)(N_i-2)(N_i-3)}{6}\operatorname{tr}\boldsymbol{\Sigma}_i^4,$$

$$E\left[\sum_{m_i+1\leq\ell<\ell'}^{m_i+N_i}\left(\sum_{k=m_i+1}^{\ell-1} Y_{k,i}^{(\mathrm{DQ})}\right)\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{\ell'-1} Y_{\alpha\beta,i}^{(\mathrm{B1})}\right)\right] = 0,$$

where $\boldsymbol{z}\sim F$. Combining these results, we have

$$\operatorname{Var}(T_{1i}) = \frac{N_i(N_i-1)(2N_i-1)}{6}E[(\boldsymbol{z}'\boldsymbol{\Sigma}_i^2\boldsymbol{z}-\operatorname{tr}\boldsymbol{\Sigma}_i^2)^2] + \frac{N_i(N_i-1)^2(N_i-2)}{3}\operatorname{tr}\boldsymbol{\Sigma}_i^4,$$

and so $\operatorname{Var}(T_1)$ converges to 0 under the asymptotic framework A1 and the assumptions A3 and A4. Next, we show that $\operatorname{Var}(T_2)$ converges to 0. To do it, we use the following inequality:

$$\operatorname{Var}(X_1+\cdots+X_n) \leq n\sum_{i=1}^{n}\operatorname{Var}(X_i), \tag{14}$$

where the strict inequality holds unless $X_1=\cdots=X_n$. Using the inequality, we have

$$\operatorname{Var}(T_2) \leq \frac{1}{p^2}\frac{16}{N^4}g\sum_{i=1}^{g}N_i^2\operatorname{Var}\left(\left\{\sum_{j=1}^{i-1}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right\}'\left\{\sum_{j=1}^{i-1}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right\}\right),$$

The right-hand side of the inequality can be expressed as

$$\frac{1}{p^2}\frac{16}{N^4}g\sum_{i=1}^{g} N_i^2 \mathrm{Var}\left(\sum_{j=1}^{i-1}\left(\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right)' \boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{1/2}\left(\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right)\right.$$
$$\left. +2\sum_{\alpha<\beta}^{i-1}\left(\sum_{k=m_\alpha+1}^{m_\alpha+N_\alpha}\boldsymbol{z}_k\right)' \boldsymbol{\Sigma}_\alpha^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_\beta^{1/2}\left(\sum_{k=m_\beta+1}^{m_\beta+N_\beta}\boldsymbol{z}_k\right)\right),$$

and from the uncorrelatedness,

$$\frac{1}{p^2}\frac{16}{N^4}g\sum_{i=1}^{g} N_i^2 \left\{\sum_{j=1}^{i-1}\mathrm{Var}\left(\left(\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right)' \boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{1/2}\left(\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right)\right)\right.$$
$$\left. +\ 4\sum_{\alpha<\beta}^{i-1}\mathrm{Var}\left(\left(\sum_{k=m_\alpha+1}^{m_\alpha+N_\alpha}\boldsymbol{z}_k\right)' \boldsymbol{\Sigma}_\alpha^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_\beta^{1/2}\left(\sum_{k=m_\beta+1}^{m_\beta+N_\beta}\boldsymbol{z}_k\right)\right)\right\}.$$

Further, it can be expanded as

$$\frac{1}{p^2}\frac{16}{N^4}g\sum_{i=1}^{g} N_i^2 \left\{\sum_{j=1}^{i-1}\left(\sum_{k=m_j+1}^{m_j+N_j}\mathrm{Var}(\boldsymbol{z}_k'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_k) + 4\sum_{\substack{k<\ell\\ k\geq m_j+1}}^{m_j+N_j}\mathrm{Var}(\boldsymbol{z}_k'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell)\right)\right.$$
$$\left. +4\sum_{\alpha<\beta}^{i-1}\mathrm{Var}\left(\left(\sum_{k=m_\alpha+1}^{m_\alpha+N_\alpha}\boldsymbol{z}_k\right)' \boldsymbol{\Sigma}_\alpha^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_\beta^{1/2}\left(\sum_{k=m_\beta+1}^{m_\beta+N_\beta}\boldsymbol{z}_k\right)\right)\right\}.$$

Evaluating these variances, and coordinating them, we have

$$16g\sum_{j<i}^{g}\left[\frac{N_i^2N_j}{N^4}\frac{1}{p^2}E[(\boldsymbol{z}'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}-\mathrm{tr}\,\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2]+\frac{2}{p}\frac{N_i^2N_j(N_j-1)}{N^4}\frac{1}{p}\,\mathrm{tr}(\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2\right]$$
$$\left. +\ 64\frac{g}{p}\sum_{i=1}^{g}\sum_{\alpha<\beta}^{i-1}\frac{N_i^2N_\alpha N_\beta}{N^4}\frac{1}{p}\,\mathrm{tr}\,\boldsymbol{\Sigma}_\alpha\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_\beta\boldsymbol{\Sigma}_i\right]. \tag{15}$$

From the inequalities in (26), it is found that

$$\frac{1}{p}\,\mathrm{tr}\,\boldsymbol{\Sigma}_\alpha\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_\beta\boldsymbol{\Sigma}_i = O(1)$$

under the asymptotic framework A1 and the assumption A3. With using (3), it is found that (15) converges to 0 under the asymptotic framework A1 and the assumption A3, and so $\mathrm{Var}(T_2)$ also converges to 0. Lastly, we show that $\mathrm{Var}(T_3)$ converges to 0. Making use of the inequality in (14), it holds that

$$\mathrm{Var}(T_3) \leq \frac{16g}{p^2N^2}\sum_{i=1}^{g}\frac{1}{(N_i-1)^2}\left(1-\frac{N_i}{N}\right)^2$$
$$\cdot\ \mathrm{Var}\left(\left\{\sum_{j=1}^{i-1}(\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2})\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right\}'\left\{\boldsymbol{\Sigma}_i\left(\sum_{\ell=m_i+1}^{m_i+N_i}\sum_{k=m_i+1}^{\ell-1}\boldsymbol{z}_k\right)\right\}\right). \tag{16}$$

Note that

$$\begin{aligned}
&\mathrm{Var}\left(\left\{\sum_{j=1}^{i-1}(\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2})\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right\}'\left\{\boldsymbol{\Sigma}_i\left(\sum_{\ell=m_i+1}^{m_i+N_i}\sum_{k=m_i+1}^{\ell-1}\boldsymbol{z}_k\right)\right\}\right)\\
&=\mathrm{tr}\left(E\left[\left\{\boldsymbol{\Sigma}_i\sum_{k=m_i+1}^{m_i+N_i-1}(N_i+m_i-k)\boldsymbol{z}_k\right\}\left\{\boldsymbol{\Sigma}_i\sum_{k=m_i+1}^{m_i+N_i-1}(N_i+m_i-k)\boldsymbol{z}_k\right\}'\right]\right.\\
&\quad\left.\cdot E\left[\left\{\sum_{j=1}^{i-1}(\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2})\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right\}\left\{\sum_{j=1}^{i-1}(\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2})\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{z}_k\right\}'\right]\right),
\end{aligned}$$

which is evaluated as

$$\mathrm{tr}\left[\left\{\frac{N_i(N_i-1)(2N_i-1)}{6}\boldsymbol{\Sigma}_i^2\right\}\left(\sum_{j=1}^{i-1}N_j\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j\boldsymbol{\Sigma}_i^{1/2}\right)\right].$$

Thus, the right-hand side of the inequality (16) is

$$\frac{16g}{6p}\sum_{i=1}^{g}\left(1-\frac{N_i}{N}\right)^2\frac{N_i}{N_i-1}\frac{2N_i-1}{N}\sum_{j=1}^{i-1}\frac{N_j}{N}\frac{\mathrm{tr}\,\boldsymbol{\Sigma}_j\boldsymbol{\Sigma}_i^3}{p}. \tag{17}$$

From the inequalities in (26), it is found that $\mathrm{tr}\,\boldsymbol{\Sigma}_j\boldsymbol{\Sigma}_i^3/p=O(1)$, and so (17) converges to 0 under the asymptotic framework A1 and the assumptions A2 and A3. It implies that $\mathrm{Var}(T_3)$ converges to 0. Since $\mathrm{Var}(T_1)$, $\mathrm{Var}(T_2)$ and $\mathrm{Var}(T_3)$ are all converge to 0, $\mathrm{Var}(C)=\mathrm{Var}(T_1+T_2+T_3)$ converges to 0. Thus C converges in probability to σ_0^2.

To show (II), it is sufficient to show that

$$\sum_{\ell=1}^{N}E[\eta_\ell^4]\to 0$$

under the asymptotic framework A1. From Jensen's inequality, it holds that

$$E[\eta_\ell^4]\le 2^3\left(E\left[\left(-\frac{1}{\sqrt{p}}\frac{2}{N}\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i}Y_{\ell k,ji}^{(\mathrm{B2})}\right)^4\right]+E\left[\left(\frac{1}{\sqrt{p}}\frac{2}{N_j-1}\left(1-\frac{N_j}{N}\right)\sum_{k=m_j+1}^{\ell-1}Y_{\ell k,j}^{(\mathrm{B2})}\right)^4\right]\right) \tag{18}$$

for $\ell\in I_j$, where

$$\begin{aligned}
Y_{\ell k,j}^{(\mathrm{B2})}&=\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j\boldsymbol{z}_k,\\
Y_{\ell k,ji}^{(\mathrm{B2})}&=\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{z}_k.
\end{aligned}$$

Firstly, we evaluate the first expectation in the right-hand side of the inequality. It can be expanded that

$$\left(\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i}Y_{\ell k,ji}^{(\mathrm{B2})}\right)^2=\sum_{i=1}^{j-1}\left\{\sum_{k=m_i+1}^{m_i+N_i}Y_{\ell k,ji}^{(\mathrm{B2})}\right\}^2+\sum_{\alpha\ne\beta}^{j-1}\left\{\sum_{k=m_\alpha+1}^{m_\alpha+N_\alpha}Y_{\ell k,ji}^{(\mathrm{B2})}\right\}\left\{\sum_{k=m_\beta+1}^{m_\beta+N_\beta}Y_{\ell k,ji}^{(\mathrm{B2})}\right\},$$

and so the expectation of the squared is described as

$$E\left[\left(\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i}Y_{\ell k,ji}^{(\mathrm{B2})}\right)^4\right]=\sum_{i=1}^{j-1}E\left[\left(\sum_{k=m_i+1}^{m_i+N_i}Y_{\ell k,ji}^{(\mathrm{B2})}\right)^4\right]+3\sum_{\alpha\ne\beta}^{j-1}E\left[\left(\sum_{k=m_\alpha+1}^{m_\alpha+N_\alpha}Y_{\ell k,j\alpha}^{(\mathrm{B2})}\right)^2\left(\sum_{k=m_\beta+1}^{m_\beta+N_\beta}Y_{\ell k,j\beta}^{(\mathrm{B2})}\right)^2\right]. \tag{19}$$

It can be expressed that

$$E\left[\left(\sum_{k=m_i+1}^{m_i+N_i} Y_{\ell k,ji}^{(\mathrm{B2})}\right)^4\right] = \sum_{k=m_i+1}^{m_i+N_i} E[(\boldsymbol{z}_k'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell)^4]$$
$$+ 3 \sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\geq m_i+1}}^{m_i+N_i} E[(\boldsymbol{z}_\alpha'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell)^2(\boldsymbol{z}_\beta'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell)^2],$$

which is bounded by $N_i\kappa_2 + 3N_i(N_i-1)\kappa_3$. Besides, we can see that

$$E\left[\left\{\sum_{k=m_\alpha+1}^{m_\alpha+N_\alpha} Y_{\ell k,j\alpha}^{(\mathrm{B2})}\right\}^2 \left\{\sum_{k=m_\beta+1}^{m_\beta+N_\beta} Y_{\ell k,j\beta}^{(\mathrm{B2})}\right\}^2\right] = E[N_\alpha \boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_\alpha\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell N_\beta \boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_\beta\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell],$$

where $\alpha \neq \beta$. Using Cauchy-Schwarz's inequality, the right-hand side of the equality is bounded by

$$N_\alpha N_\beta \sqrt{E[(\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_\alpha\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell)^2]E[(\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_\beta\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_\ell)^2]},$$

which is bounded by $N_\alpha N_\beta \kappa_3$. Combining these results, we have

$$E\left[\left(\sum_{i=1}^{j-1}\sum_{k=m_i+1}^{m_i+N_i} Y_{\ell k,ji}^{(\mathrm{B2})}\right)^4\right] \leq \sum_{i=1}^{j-1}\{N_i\kappa_2 + 3N_i(N_i-1)\kappa_3\} + 3\sum_{\alpha\neq\beta}^{j-1} N_\alpha N_\beta \kappa_3. \tag{20}$$

Next, we evaluate the second expectation in the right-hand side of the inequality (18). It can be expanded that

$$\left(\sum_{k=m_j+1}^{\ell-1} Y_{\ell k,j}^{(\mathrm{B2})}\right)^2 = \sum_{k=m_j+1}^{\ell-1} (\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j\boldsymbol{z}_k)^2 + \sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\geq m_j+1}}^{\ell-1} (\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j\boldsymbol{z}_\alpha)(\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j\boldsymbol{z}_\beta),$$

and so the expectation of the squared can be described as

$$E\left[\left(\sum_{k=m_j+1}^{\ell-1} Y_{\ell k,j}^{(\mathrm{B2})}\right)^4\right] = \sum_{k=m_j+1}^{\ell-1} E[(\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j\boldsymbol{z}_k)^4] + 3\sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\geq m_j+1}}^{\ell-1} E[(\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j\boldsymbol{z}_\alpha)^2(\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j\boldsymbol{z}_\beta)^2].$$

From the assumptions A5 and A6, if $\ell \in I_j$, the right-hand side of the equality is bounded by

$$\{(\ell-1)-(m_j+1)+1\}\kappa_2 + 3\{(\ell-1)-(m_j+1)+1\}\{(\ell-1)-(m_j+1)\}\kappa_3.$$

Thus, we have

$$E\left[\left(\sum_{k=m_j+1}^{\ell-1} Y_{\ell k,j}^{(\mathrm{B2})}\right)^4\right] \leq (\ell-m_j-1)\kappa_2 + 3(\ell-m_j-1)(\ell-m_j-2)\kappa_3. \tag{21}$$

From the inequalities (20) and (21), it holds that

$$
\begin{aligned}
\sum_{\ell=1}^{N} E[\eta_\ell^4] &= \sum_{j=1}^{g}\sum_{\ell=m_j+1}^{m_j+N_j} E[\eta_\ell^4] \\
&\leq 8\left(\frac{16}{N}\sum_{j=1}^{g}\sum_{\ell=m_j+1}^{m_j+N_j}\left[\sum_{i=1}^{j-1}\left\{\frac{N_i}{N}\frac{p^2}{N^2}\frac{\kappa_2}{p^4}+\frac{3}{N}\frac{N_i(N_i-1)}{N^2}\frac{\kappa_3}{p^2}\right\}+\frac{3}{N}\sum_{\alpha\neq\beta}^{j-1}\frac{N_\alpha N_\beta}{N^2}\frac{\kappa_3}{p^2}\right]\right. \\
&\quad \left.+\sum_{j=1}^{g}\sum_{\ell=m_j+1}^{m_j+N_j}\frac{16}{p^2(N_j-1)^4}\left(1-\frac{N_j}{N}\right)^2\{(\ell-m_j-1)\kappa_2+3(\ell-m_j-1)(\ell-m_j-2)\kappa_3\}\right) \\
&= 8\left(\frac{16}{N}\sum_{j=1}^{g}\sum_{\ell=m_j+1}^{m_j+N_j}\left[\sum_{i=1}^{j-1}\left\{\frac{N_i}{N}\frac{p^2}{N^2}\frac{\kappa_2}{p^4}+\frac{3}{N}\frac{N_i(N_i-1)}{N^2}\frac{\kappa_3}{p^2}\right\}+\frac{3}{N}\sum_{\alpha\neq\beta}^{j-1}\frac{N_\alpha N_\beta}{N^2}\frac{\kappa_3}{p^2}\right]\right. \\
&\quad \left.+\sum_{j=1}^{g}\frac{16Np^2}{(N_j-1)^3}\left(1-\frac{N_j}{N}\right)^2\left\{\frac{N_j}{2N}\frac{\kappa_2}{p^4}+\frac{1}{N}\frac{N_j(N_j-2)}{p^2}\frac{\kappa_3}{p^2}\right\}\right),
\end{aligned}
$$

which goes to 0 under asymptotic framework A1 and assumptions A2, A3, A5 and A6, and so the condition (II) holds.

Thus it completes the proof of the asymptotic normality of T, which is given in the following theorem.

Theorem 1. *Assume that the observation vectors have the model* (1)*, where these error vectors are independent and identically distributed as F with the mean $\mathbf{0}$ and the covariance matrix $\boldsymbol{I}_p$ for $i = 1,\dots,g$. Under the asymptotic framework* A1 *and assumptions* A2-A6*, the null distribution of T converges in distribution to the normal distribution with mean 0 and variance σ_0^2, where $\sigma_0^2 = \lim \sigma^2$,*

$$
\sigma^2 = 2\sum_{i=1}^{g}\left(1-\frac{N_i}{N}\right)^2\frac{N_i}{N_i-1}\frac{\operatorname{tr}\boldsymbol{\Sigma}_i^2}{p}+2\sum_{i\neq j}^{g}\frac{N_iN_j}{N^2}\frac{\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j}{p}.
$$

For the actual use of Theorem 1, we need to estimate σ_0^2. The unbiased estimator of σ^2 is given by

$$
\hat{\sigma^2} = 2\sum_{i=1}^{g}\left(1-\frac{N_i}{N}\right)^2\frac{N_i}{N_i-1}\frac{\widehat{\operatorname{tr}\boldsymbol{\Sigma}_i^2}}{p}+2\sum_{i\neq j}^{g}\frac{N_iN_j}{N^2}\frac{\widehat{\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j}}{p},
$$

where $\widehat{\operatorname{tr}\boldsymbol{\Sigma}_i^2/p}$ and $\widehat{\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j/p}$ are unbiased estimators of $\operatorname{tr}\boldsymbol{\Sigma}_i^2/p$ and $\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j/p$, respectively, which are defined as

$$
\begin{aligned}
\frac{\widehat{\operatorname{tr}\boldsymbol{\Sigma}_i^2}}{p} &= \frac{N_i-1}{N_i(N_i-2)(N_i-3)p}\{(N_i-1)(N_i-2)\operatorname{tr}\boldsymbol{S}_i^2+(\operatorname{tr}\boldsymbol{S}_i)^2-N_iQ_i\}, \\
\frac{\widehat{\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j}}{p} &= \frac{\operatorname{tr}\boldsymbol{S}_i\boldsymbol{S}_j}{p}.
\end{aligned}
$$

Here,

$$
Q_i = \frac{1}{N_i-1}\sum_{j=1}^{N_i}((\boldsymbol{x}_j^{(i)}-\bar{\boldsymbol{x}}^{(i)})'(\boldsymbol{x}_j^{(i)}-\bar{\boldsymbol{x}}^{(i)}))^2.
$$

The unbiased estimator $\widehat{\operatorname{tr}\boldsymbol{\Sigma}_i^2/p}$ has consistency under the asymptotic framework A1 and the assumptions A2, A3, A5 and A7, which can be checked in Himeno and Yamada [8]. We need to show the consistency of $\operatorname{tr}\widehat{\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j}/p$. From (10) and the invariance property of mean vector,

$$\frac{1}{p}\operatorname{tr}\boldsymbol{S}_i\boldsymbol{S}_j = \frac{1}{p}\operatorname{tr}\left[\left\{\frac{1}{N_i}\sum_{k=m_i+1}^{m_i+N_i}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{z}_k\boldsymbol{z}_k'\boldsymbol{\Sigma}_i^{1/2} - \frac{1}{N_i(N_i-1)}\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{z}_k\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_i^{1/2}\right\}\right.$$
$$\left.\cdot\left\{\frac{1}{N_j}\sum_{k=m_j+1}^{m_j+N_j}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_k\boldsymbol{z}_k'\boldsymbol{\Sigma}_j^{1/2} - \frac{1}{N_j(N_j-1)}\sum_{\substack{k\neq\ell\\ k,\ell\geq m_j+1}}^{m_j+N_j}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{z}_k\boldsymbol{z}_\ell'\boldsymbol{\Sigma}_j^{1/2}\right\}\right],$$

which can be expanded as $U_1 - U_2 - U_3 + U_4$ with

$$U_1 = \frac{1}{pN_iN_j}\sum_{k=m_i+1}^{m_i+N_i}\sum_{\ell=m_j+1}^{m_j+N_j}(Y_{\ell k,ji}^{(\mathrm{B2})})^2,$$
$$U_2 = \frac{1}{pN_iN_j(N_j-1)}\sum_{k=m_i+1}^{m_i+N_i}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j}Y_{\alpha k,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})},$$
$$U_3 = \frac{1}{pN_jN_i(N_i-1)}\sum_{\ell=m_j+1}^{m_j+N_j}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{m_i+N_i}Y_{\ell\alpha,ji}^{(\mathrm{B2})}Y_{\ell\beta,ji}^{(\mathrm{B2})},$$
$$U_4 = \frac{1}{pN_iN_j(N_i-1)(N_j-1)}\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j}Y_{\alpha\ell,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}.$$

Since U_1, U_2, U_3 and U_4 are uncorrelated,

$$\operatorname{Var}\left(\frac{1}{p}\operatorname{tr}\boldsymbol{S}_i\boldsymbol{S}_j\right) = E\left[\{U_1 - (1/p)\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j\}^2\right] + E[U_2^2] + E[U_3^2] + E[U_4^2].$$

Firstly, we treat $E\left[\{U_1 - (1/p)\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j\}^2\right]$. It follows that

$$\begin{aligned}
&E\left[\{U_1 - (1/p)\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j\}^2\right]\\
&= E\left[\frac{1}{p^2N_i^2N_j^2}\left\{\sum_{k=m_i+1}^{m_i+N_i}\sum_{\ell=m_j+1}^{m_j+N_j}(Y_{\ell k,ji}^{(\mathrm{B2})})^4 + \sum_{k=m_i+1}^{m_i+N_i}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j}(Y_{\alpha k,ji}^{(\mathrm{B2})})^2(Y_{\beta k,ji}^{(\mathrm{B2})})^2\right\}\right.\\
&\qquad\left. + \sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{m_i+N_i}\sum_{\ell=m_j+1}^{m_j+N_j}(Y_{\ell\alpha,ji}^{(\mathrm{B2})})^2(Y_{\ell\beta,ji}^{(\mathrm{B2})})^2 + \sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_i+1}}^{m_i+N_i}\sum_{\substack{k\neq\ell\\ k,\ell\geq m_j+1}}^{m_j+N_j}(Y_{k\alpha,ji}^{(\mathrm{B2})})^2(Y_{\ell\beta,ji}^{(\mathrm{B2})})^2\right] - \left(\frac{1}{p}\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j\right)^2\\
&= \frac{1}{p^2N_iN_j}E[(\boldsymbol{\varepsilon}_1'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\varepsilon}_2)^4] + \frac{N_j-1}{p^2N_iN_j}E[(\boldsymbol{\varepsilon}_1'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\varepsilon}_1)^2]\\
&\quad + \frac{N_i-1}{p^2N_iN_j}E[(\boldsymbol{\varepsilon}_1'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\varepsilon}_1)^2] + \left(\frac{1}{N_iN_j} - \frac{1}{N_i} - \frac{1}{N_j}\right)\left(\frac{1}{p}\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j\right)^2,
\end{aligned}$$

which is bounded by

$$\frac{p^2}{N_iN_j}\left(\frac{1}{p^4}\kappa_2\right) + \frac{N_i+N_j-2}{N_iN_j}\left(\frac{1}{p^2}\kappa_3\right) + \left(\frac{1}{N_iN_j} - \frac{1}{N_i} - \frac{1}{N_j}\right)\left(\frac{1}{p}\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j\right)^2.$$

Under the asymptotic framework A1 and the assumptions A2, A5 and A6, the boundary converges to 0, and so $E[\{U_1 - (1/p)\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j\}^2]$ converges to 0. Next, we treat $E[U_2^2]$. It follows that

$$
\begin{aligned}
E[U_2^2] &= E\left[\frac{1}{p^2N_i^2N_j^2(N_j-1)^2}\left\{\sum_{k=m_i+1}^{m_j+N_i}\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha k,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}\right)^2\right.\right.\\
&\quad\left.\left.+\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha k,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}\right)\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha \ell,ji}^{(\mathrm{B2})}Y_{\beta \ell,ji}^{(\mathrm{B2})}\right)\right\}\right]\\
&= E\left[\frac{1}{p^2N_i^2N_j^2(N_j-1)^2}\left\{2\sum_{k=m_i+1}^{m_i+N_i}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j}(Y_{\alpha k,ji}^{(\mathrm{B2})})^2(Y_{\beta k,ji}^{(\mathrm{B2})})^2\right.\right.\\
&\quad\left.\left.+2\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha k,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}Y_{\alpha \ell,ji}^{(\mathrm{B2})}Y_{\beta \ell,ji}^{(\mathrm{B2})}\right\}\right]\\
&= \frac{2}{p^2N_iN_j(N_j-1)}E[(\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\Sigma}_j\boldsymbol{\Sigma}_i^{1/2}\boldsymbol{\varepsilon})^2]+\frac{2(N_i-1)}{p^2N_iN_j(N_j-1)}\operatorname{tr}(\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2,
\end{aligned}
$$

which is bounded by

$$
\frac{2}{N_iN_j(N_j-1)}\left(\frac{1}{p^2}\kappa_3\right)+\frac{2(N_i-1)}{pN_iN_j(N_j-1)}\left\{\frac{1}{p}\operatorname{tr}(\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2\right\}.
$$

Using (3), it is checked that the boundary converges to 0 under the asymptotic framework A1 and the assumptions A3 and A6, and so $E[U_2^2]$ converges to 0. By similar derivation, it can be shown that $E[U_3^2]$ converges to 0. Lastly, we treat $E[U_4^2]$. It follows that

$$
\begin{aligned}
E[U_4^2] &= E\left[\frac{1}{p^2N_i^2N_j^2(N_i-1)^2(N_j-1)^2}\left(\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha \ell,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}\right)^2\right]\\
&= \frac{1}{p^2N_i^2N_j^2(N_i-1)^2(N_j-1)^2}E\left[\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\left\{\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha \ell,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}\right)^2\right.\right.\\
&\quad\left.\left.+\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha \ell,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}\right)\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j} Y_{\alpha k,ji}^{(\mathrm{B2})}Y_{\beta \ell,ji}^{(\mathrm{B2})}\right)\right\}\right]\\
&= \frac{2}{p^2N_i^2N_j^2(N_i-1)^2(N_j-1)^2}\sum_{\substack{k\neq\ell\\ k,\ell\geq m_i+1}}^{m_i+N_i}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\geq m_j+1}}^{m_j+N_j}E\left[(Y_{\alpha \ell,ji}^{(\mathrm{B2})})^2(Y_{\beta k,ji}^{(\mathrm{B2})})^2+Y_{\alpha \ell,ji}^{(\mathrm{B2})}Y_{\beta \ell,ji}^{(\mathrm{B2})}Y_{\alpha k,ji}^{(\mathrm{B2})}Y_{\beta k,ji}^{(\mathrm{B2})}\right]\\
&= \frac{2}{N_iN_j(N_i-1)(N_j-1)}\left\{\left(\frac{\operatorname{tr}\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j}{p}\right)^2+\frac{1}{p}\frac{\operatorname{tr}(\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j)^2}{p}\right\}.
\end{aligned}
$$

From (3) and (4), it is found that $E[U_4^2]$ converges to 0 under asymptotic framework A1 and the assumptions A2 and A3. Thus, the consistency of $\operatorname{tr}\widehat{\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_j}/p$ is shown under the asymptotic framework A1 and the assumptions A2, A3, A5 and A6. By Slutsky's theorem (cf. Rao [9]), $T/\sqrt{\hat{\sigma}^2}$ converges in distribution to the standard normal distribution.

4 Asymptotic non-null distribution of the proposed statistic

Under the alternative hypothesis H_1,

$$T \stackrel{\mathcal{D}}{=} T_{H_0} + 2T_{\mathrm{C}} + \frac{1}{\sqrt{p}} \sum_{i=1}^{g} \operatorname{tr} \boldsymbol{\Omega}_i \boldsymbol{\Sigma}_i,$$

where T_{H_0} is the T under H_0, $T_{\mathrm{C}} = T_{\mathrm{C1}} - T_{\mathrm{C2}}$,

$$T_{\mathrm{C1}} = \frac{1}{\sqrt{p}} \sum_{i=1}^{g} \left(1 - \frac{N_i}{N}\right) \sum_{k=m_i+1}^{m_i+N_i} Y_{k,i}^{(\mathrm{C1})},$$

$$T_{\mathrm{C2}} = \frac{1}{\sqrt{p}} \sum_{i \neq j}^{g} \frac{N_j}{N} \sum_{k=m_i+1}^{m_i+N_i} Y_{k.i;j}^{(\mathrm{C2})},$$

with

$$\begin{aligned} Y_{k,i}^{(\mathrm{C1})} &= \boldsymbol{\mu}_i' \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{z}_k \quad (k = m_i+1, \ldots, m_i+N_i, i = 1, \ldots, g), \\ Y_{k,i;j}^{(\mathrm{C2})} &= \boldsymbol{\mu}_j' \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{z}_k \quad (k = m_i+1, \ldots, m_i+N_i, i, j = 1, \ldots, g, i \neq j). \end{aligned}$$

It can be found that $E[T_{\mathrm{C}}] = 0$. To show that T_{C} converges to 0 in probability, it is sufficient to show that $\mathrm{Var}(T_{\mathrm{C}})$ converges to 0. Firstly, we treat $\mathrm{Var}(T_{\mathrm{C1}})$. It follows that

$$\begin{aligned} \mathrm{Var}(T_{\mathrm{C1}}) &= E\left[\frac{1}{p} \sum_{i=1}^{g} \left(1 - \frac{N_i}{N}\right)^2 \left(\sum_{k=m_i+1}^{m_i+N_i} Y_{k,i}^{(\mathrm{C1})}\right)^2\right] \\ &= \frac{1}{p} \sum_{i=1}^{g} \left(1 - \frac{N_i}{N}\right)^2 \sum_{k=m_i+1}^{m_i+N_i} E\left[(Y_{k,i}^{(\mathrm{C1})})^2\right] \\ &= \frac{1}{p} \sum_{i=1}^{g} N_i \left(1 - \frac{N_i}{N}\right)^2 \boldsymbol{\mu}_i' \boldsymbol{\Sigma}_i \boldsymbol{\mu}_i. \end{aligned}$$

Next, we evaluate $\mathrm{Var}(T_{\mathrm{C2}})$. It follows that

$$\begin{aligned} \mathrm{Var}(T_{\mathrm{C2}}) &= \frac{1}{p} \sum_{i=1}^{g} \sum_{\substack{j_1=1 \\ j_1 \neq i}}^{g} \sum_{\substack{j_2=1 \\ j_2 \neq i}}^{g} E\left[\left(\frac{N_{j_1}}{N} \sum_{k=m_i+1}^{m_i+N_i} \boldsymbol{\mu}_{j_1}' \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{z}_k\right)\left(\frac{N_{j_2}}{N} \sum_{k=m_i+1}^{m_i+N_i} \boldsymbol{\mu}_{j_2}' \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{z}_k\right)\right] \\ &= \frac{1}{p} \sum_{i=1}^{g} \sum_{\substack{j_1=1 \\ j_1 \neq i}}^{g} \sum_{\substack{j_2=1 \\ j_2 \neq i}}^{g} N_i \frac{N_{j_1} N_{j_2}}{N^2} \boldsymbol{\mu}_{j_1}' \boldsymbol{\Sigma}_i \boldsymbol{\mu}_{j_2} \\ &= \frac{1}{p} \sum_{i=1}^{g} N_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{g} \frac{N_j}{N} \boldsymbol{\mu}_j\right)' \boldsymbol{\Sigma}_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{g} \frac{N_j}{N} \boldsymbol{\mu}_j\right)'. \end{aligned}$$

Lastly, we evaluate $\mathrm{Cov}(T_{\mathrm{C1}}, T_{\mathrm{C2}})$. It follows that

$$\begin{aligned} \mathrm{Cov}(T_{\mathrm{C1}}, T_{\mathrm{C2}}) &= E\left[\frac{1}{p} \sum_{i \neq j}^{g} \left(1 - \frac{N_i}{N}\right) \frac{N_j}{N} \left(\sum_{k=m_i+1}^{m_i+N_i} Y_{k,i}^{(\mathrm{C1})}\right) \left(\sum_{k=m_i+1}^{m_i+N_i} Y_{k,i;j}^{(\mathrm{C2})}\right)\right] \\ &= \frac{1}{p} \sum_{i \neq j}^{g} \left(1 - \frac{N_i}{N}\right) \frac{N_j}{N} N_i \boldsymbol{\mu}_j' \boldsymbol{\Sigma}_i \boldsymbol{\mu}_i \\ &= \sum_{i=1}^{g} \frac{N_i}{p} \left(1 - \frac{N_i}{N}\right) \sum_{\substack{j=1 \\ j \neq i}}^{g} \frac{N_j}{N} \boldsymbol{\mu}_i' \boldsymbol{\Sigma}_i \boldsymbol{\mu}_j. \end{aligned}$$

From these results, it can be shown that

$$\begin{aligned}\mathrm{Var}(T_{\mathrm{C}}) &= \mathrm{Var}(T_{\mathrm{C1}}) - 2\mathrm{Cov}(T_{\mathrm{C1}}, T_{\mathrm{C2}}) + \mathrm{Var}(T_{\mathrm{C2}}) \\ &= \sum_{i=1}^{g} \frac{N_i}{p} \left\{ \left(1 - \frac{N_i}{N}\right) \boldsymbol{\mu}_i - \sum_{\substack{j=1 \\ j\neq i}}^{g} \frac{N_j}{N} \boldsymbol{\mu}_j \right\}' \boldsymbol{\Sigma}_i \left\{ \left(1 - \frac{N_i}{N}\right) \boldsymbol{\mu}_i - \sum_{\substack{j=1 \\ j\neq i}}^{g} \frac{N_j}{N} \boldsymbol{\mu}_j \right\}' \\ &= \frac{1}{p} \sum_{i=1}^{g} \mathrm{tr}\, \boldsymbol{\Sigma}_i^2 \boldsymbol{\Omega}_i.\end{aligned}$$

Thus under the asymptotic framework A1 and the assumption A8, $\mathrm{Var}(T_{\mathrm{C}})$ converges to 0, and by Chebyshef's inequality, it can be shown that T_{C} converges to 0 in probability.

Theorem 2. *Assume the same model as in* Theorem 1*. Under the asymptotic framework* A1 *and the assumptions* A2-A8*,*

$$\lim_{\mathrm{A1}} P(T/\sqrt{\hat{\sigma}} > x) = \Phi\left(-x + \frac{\lim_{p\to\infty} (1/\sqrt{p}) \sum_{i=1}^{g} \mathrm{tr}\, \boldsymbol{\Sigma}_i \boldsymbol{\Omega}_i}{\sigma_0} \right).$$

For the special case that $F = N_p(\mathbf{0}, \boldsymbol{I}_p)$ and $\boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_g = \boldsymbol{\Sigma}$, the limiting power for the significance level α is given by

$$\lim_{\mathrm{A1}} P(T/\sqrt{\hat{\sigma}} > z_{1-\alpha}) = \Phi\left(-z_{1-\alpha} + \lim_{p\to\infty} \frac{\mathrm{tr}\, \boldsymbol{\Sigma}\boldsymbol{\Omega}}{\sqrt{2(g-1)\,\mathrm{tr}\, \boldsymbol{\Sigma}^2}} \right),$$

where z_α is the $1-\alpha$ point of the standard normal distribution and

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1/2} \left\{ \sum_{i=1}^{g} (\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})' \right\} \boldsymbol{\Sigma}^{-1/2}.$$

The asymptotic power is the same as the one of Fujikoshi et al. [6]'s test, which is given as the following corollary.

Corollary 1. *Assume that* $F = N_p(\mathbf{0}, \boldsymbol{I}_p)$ *and* $\boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_g = \boldsymbol{\Sigma}$ *on the model* (1)*. Under the asymptotic framework* A1 *and the assumptions* A2*,* A3 *and* A8*,*

$$\lim_{\mathrm{A1}} P(T/\hat{\sigma} > x) = \lim_{\mathrm{A1}} P(\tilde{T}_{\mathrm{FHW}}/\widehat{\sigma_{\mathrm{FHW}}} > x) = \Phi\left(-x + \lim_{p\to\infty} \frac{\mathrm{tr}\, \boldsymbol{\Sigma}\boldsymbol{\Omega}}{\sqrt{2(g-1)\,\mathrm{tr}\, \boldsymbol{\Sigma}^2}} \right),$$

where $n = N - g$*,*

$$\tilde{T}_{\mathrm{FHW}} = \sqrt{p} \left\{ n \frac{\mathrm{tr}\, \boldsymbol{B}}{\mathrm{tr}\, \boldsymbol{W}} - (g-1) \right\},$$

and $\widehat{\sigma^2_{\mathrm{FHW}}}$ *is consistent estimator of the asymptotic variance for* $\tilde{T}_{\mathrm{FHW}}$*, which is given as follows:*

$$\widehat{\sigma^2_{\mathrm{FHW}}} = \frac{2(g-1)\{\mathrm{tr}\, \boldsymbol{W}^2/n^2 - (\mathrm{tr}\, \boldsymbol{W})^2/n^3\}/p}{\{\mathrm{tr}\, \boldsymbol{W}/(np)\}^2}.$$

5 Numerical results

In this section, we did some simulations to check the precision of the proposed test. The proposed testing criterion with the significance level α is that the null hypothesis is rejected if

$$T_p = T/\hat{\sigma} > z_{1-\alpha}, \tag{22}$$

where $z_{1-\alpha}$ denotes the $100(1-\alpha)$ percentile point of the standard normal distribution. Firstly, we treated the two sample problem, i.e., $g = 2$. Since the proposed test can also be defined for the case that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}$, we compare with the test proposed in Fujikoshi et al. [6], which the test rejects H_0 when

$$T_{\rm FHW} = \tilde{T}_{\rm FHW}/\widehat{\sigma_{\rm FHW}} > z_{1-\alpha}. \tag{23}$$

By Monte-Carlo simulation, the actual error probabilities of the first kind (α error) of the proposed test (22) with the nominal α and the Fujikoshi et al. [6]'s test (23) are estimated by the proportions

$$\widehat{\alpha_{\rm p}} = \widehat{\alpha_{\rm p}}(\alpha) = \frac{\#\{T_{\rm p} > z_{1-\alpha}\}}{m}, \quad \widehat{\alpha_{\rm FHW}} = \widehat{\alpha_{\rm FHW}}(\alpha) = \frac{\#\{T_{\rm FHW} > z_{1-\alpha}\}}{m},$$

respectively, where m denotes the number of the replication. We carried out the simulation with 1,000,000 replications of random samples having the model (1) with

$$\boldsymbol{\Sigma} = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_p \end{pmatrix} \begin{pmatrix} \rho^{|1-1|} & \rho^{|1-2|} & \cdots & \rho^{|1-p|} \\ \rho^{|2-1|} & \rho^{|2-2|} & \cdots & \rho^{|2-p|} \\ & \cdots & & \\ \rho^{|p-1|} & \rho^{|p-2|} & \cdots & \rho^{|p-p|} \end{pmatrix} \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_p \end{pmatrix}, \tag{24}$$

where $d_i = 5 + (-1)^{i-1} * (p - i + 1)/p$ and $\rho = 0.1$, which the results were given in Table 1. We chose the total sample size as $N = 50$ and 100 and the dimensions as $p = 50$, 100, 200, 500 and 1000. The ratios of sample sizes $N_1 : N_2$ are $7:3$, $6:4$ and $5:5$, i.e., $(N_1, N_2) = (35, 15)$, $(30, 20)$ and $(25, 25)$ for $N = 50$, and $(70, 30)$, $(60, 40)$ and $(50, 50)$ for $N = 100$. We treated the following 3 cases as the distribution F of the error vector on the model (1):

Case 1: F is the multivariate normal distribution with the mean $\mathbf{0}$ and the covariance matrix $\boldsymbol{I}_p$.

Case 2: For $w_1, \ldots, w_p$ are i.i.d. as the chi-squared distribution with 5 degrees of freedom, $z_i = \sqrt{5}(w_i/5 - 1)/\sqrt{2}$, $i = 1, \ldots, p$.

Case 3: F is the scaled multivariate t distribution with 10 degrees of freedom, the mean $\mathbf{0}$ and the covariance matrix $\boldsymbol{I}_p$.

From Table 1, we found that the actual error probabilities of the first kind for $T_{\rm p}$ are almost the same in all cases, larger than 0.05 and almost monotone decreasing for n and p. The actual error probabilities of the first kind for $T_{\rm FHW}$ are also the same tendency as the ones for T_p in Case 1 and 2, but are smaller than 0.05 when $p \geq 100$ and come cross to 0 as p becomes large in Case 3.

Next, we confirmed Corollary 1 for 2-sample case. For ease, the common covariance matrix is set to be the identical matrix $\boldsymbol{I}_p$. For the alternative hypothesis with satisfying A8, we chose as $\boldsymbol{\mu}_1 = \mathbf{0}_p$ and $\boldsymbol{\mu}_2 = (p^{\delta/256}, 0, \ldots, 0)'$, $\delta = 4, 8, 12, 16, 20, 24, 64$. The empirical powers of the proposed test (22) for the significance level α and the Fujikoshi et al. [6]'s test (23) are calculated, which are defined as

$$\hat{\beta_{\rm p}} = \#(T_{\rm p} > z_{1-\alpha})/m,$$
$$\hat{\beta_{\rm FHW}} = \#(T_{\rm FHW} > z_{1-\alpha})/m.$$

Table 2 gave these values for the case that $F = N_p(\mathbf{0}, \boldsymbol{I}_p)$, $p = 200$, $N_1 = N_2 = 50$, $\alpha = 0.05$ and $m = 10,000$. It can be confirmed that the proposed test (22) is almost the same power as the Fujikoshi et al. [6]'s test (23) under normal population distribution.

Lastly, we checked the actual error probabilities of the first kind for the proposed test (22) when the covariance matrices are not common. We checked when $g = 2$ (Table 3) and $g = 3$ (Table 4). As covariance matrices, we set $\boldsymbol{\Sigma}_1$ as the matrix (24), $\boldsymbol{\Sigma}_2$ as the identity matrix and $\boldsymbol{\Sigma}_3$ as the diagonal matrix $\mathrm{diag}(b_1, \ldots, b_p)$ where $b_1, \ldots, b_p$ are i.i.d. as $\chi^2(3)$. For $g = 2$, generate the observation vectors $\boldsymbol{x}_1^{(1)}, \ldots, \boldsymbol{x}_{N_1}^{(1)}$ which are i.i.d. as $F_p(\mathbf{0}, \boldsymbol{\Sigma}_1)$ and $\boldsymbol{x}_1^{(2)}, \ldots, \boldsymbol{x}_{N_2}^{(2)}$ which are i.i.d. as $F_p(\mathbf{0}, \boldsymbol{\Sigma}_2)$. Table 3 are listed the values of $\widehat{\alpha_{\rm p}} = \widehat{\alpha_{\rm p}}(0.05)$ based on 1,000,000 replications for the case that $p = 50, 100, 200, 500, 1000$ and

$$(N_1, N_2) = \begin{cases} (35, 15), (30, 20), (25, 25), (20, 30), (15, 35) & (N = 50), \\ (70, 30), (60, 40), (50, 50), (40, 60), (30, 70) & (N = 100). \end{cases}$$

Table 1: Actual error probabilities of the first kind when $g = 2$ and $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \mathbf{\Sigma}$.

N	N_1	N_2	p	Case 1		Case 2		Case 3	
				$\widehat{\alpha_{\mathrm{p}}}$	$\widehat{\alpha_{\mathrm{FHW}}}$	$\widehat{\alpha_{\mathrm{p}}}$	$\widehat{\alpha_{\mathrm{FHW}}}$	$\widehat{\alpha_{\mathrm{p}}}$	$\widehat{\alpha_{\mathrm{FHW}}}$
50	35	15	50	0.063	0.060	0.063	0.059	0.062	0.051
			100	0.059	0.057	0.059	0.056	0.059	0.042
			200	0.057	0.055	0.057	0.053	0.057	0.032
			500	0.055	0.052	0.054	0.051	0.055	0.021
			1000	0.054	0.052	0.054	0.051	0.053	0.014
	30	20	50	0.062	0.060	0.062	0.058	0.062	0.044
			100	0.058	0.056	0.058	0.055	0.059	0.031
			200	0.056	0.054	0.057	0.053	0.056	0.017
			500	0.054	0.052	0.054	0.051	0.054	0.005
			1000	0.053	0.051	0.053	0.050	0.053	0.001
	25	25	50	0.061	0.059	0.061	0.058	0.062	0.042
			100	0.059	0.057	0.059	0.055	0.058	0.028
			200	0.056	0.055	0.056	0.053	0.056	0.014
			500	0.054	0.053	0.054	0.051	0.055	0.002
			1000	0.053	0.052	0.053	0.050	0.053	0.000
100	70	30	50	0.061	0.060	0.061	0.060	0.061	0.055
			100	0.058	0.057	0.058	0.057	0.058	0.047
			200	0.056	0.055	0.056	0.055	0.056	0.039
			500	0.054	0.053	0.054	0.052	0.054	0.026
			1000	0.053	0.052	0.053	0.052	0.053	0.018
	60	40	50	0.061	0.060	0.061	0.059	0.060	0.050
			100	0.059	0.058	0.058	0.056	0.058	0.040
			200	0.056	0.055	0.056	0.054	0.056	0.028
			500	0.054	0.053	0.054	0.052	0.054	0.011
			1000	0.053	0.052	0.053	0.051	0.053	0.003
	50	50	50	0.061	0.060	0.061	0.059	0.061	0.050
			100	0.058	0.057	0.058	0.056	0.058	0.039
			200	0.056	0.055	0.056	0.054	0.056	0.025
			500	0.054	0.053	0.054	0.052	0.054	0.008
			1000	0.053	0.052	0.053	0.051	0.053	0.001

Table 2: Simulation results for $\hat{\beta_{\mathrm{p}}}$ and $\hat{\beta_{\mathrm{FHW}}}$ when $\alpha = 0.05$, $p = 200$ and $N_1 = N_2 = 50$

		δ						
		4	8	12	16	20	24	64
Case 1	T_{p}	0.61	0.70	0.78	0.87	0.92	0.96	1.00
	T_{FHW}	0.60	0.69	0.78	0.86	0.92	0.96	1.00

For $g = 3$, generate the observation vectors $\boldsymbol{x}_1^{(i)}, \ldots, \boldsymbol{x}_{N_i}^{(i)}$ which are i.i.d. as $F_p(\mathbf{0}, \mathbf{\Sigma}_i)$ for $i = 1, 2, 3$. Table 4 are listed the values of $\widehat{\alpha_{\rm p}} = \widehat{\alpha_{\rm p}}(0.05)$ based on 1,000,000 replications for the case that $p = 50, 100, 200$ and $(N_1, N_2, N_3) = (50, 25, 25)$, $(25, 50, 25)$ and $(25, 25, 50)$. The settings of the multivariate distributions F are the same as the ones of Table 1. From Table 3 and 4, we found that the actual error probabilities of the first kind for $T_{\rm p}$ are almost the same in all cases, larger than 0.05 and almost monotone decreasing for n and p. We can see that the value of $\widehat{\alpha}_{\rm p}$ for $(N_1, N_2) = (a, b)$ with $a > b$ is smaller than the one for $(N_1, N_2) = (b, a)$ in Table 3. It is conjectured that the precision of the approximation becomes good when the size of sample with complicated structure of the covariance matrix is relatively large. We can also check it from Table 4.

Table 3: Actual error probabilities of the first kind when $g = 2$ under heteroscedasticity.

N	N_1	N_2	p	Case 1	Case 2	Case 3	N	N_1	N_2	Case 1	Case 2	Case 3
			50	0.062	0.063	0.062				0.061	0.061	0.061
			100	0.059	0.060	0.059				0.058	0.058	0.058
	35	15	200	0.057	0.058	0.057		70	30	0.056	0.057	0.056
			500	0.054	0.054	0.054				0.054	0.054	0.054
			1000	0.053	0.053	0.053				0.053	0.053	0.053
			50	0.063	0.063	0.062				0.061	0.061	0.060
			100	0.059	0.060	0.058				0.059	0.058	0.058
	30	20	200	0.057	0.057	0.057		60	40	0.056	0.056	0.056
			500	0.054	0.054	0.055				0.054	0.054	0.054
			1000	0.053	0.053	0.053				0.053	0.053	0.053
			50	0.062	0.062	0.063				0.062	0.060	0.061
			100	0.060	0.059	0.060				0.059	0.059	0.058
50	25	25	200	0.057	0.057	0.057	100	50	50	0.056	0.056	0.056
			500	0.055	0.055	0.056				0.054	0.054	0.054
			1000	0.054	0.053	0.054				0.053	0.053	0.053
			50	0.063	0.063	0.063				0.062	0.061	0.061
			100	0.060	0.059	0.060				0.059	0.059	0.059
	20	30	200	0.058	0.057	0.058		40	60	0.057	0.057	0.057
			500	0.055	0.055	0.055				0.054	0.054	0.055
			1000	0.054	0.054	0.055				0.053	0.053	0.054
			50	0.065	0.064	0.065				0.062	0.061	0.062
			100	0.062	0.061	0.061				0.059	0.059	0.059
	15	35	200	0.059	0.058	0.059		30	70	0.057	0.056	0.057
			500	0.056	0.056	0.057				0.055	0.054	0.055
			1000	0.055	0.055	0.055				0.054	0.054	0.054

Table 4: Actual error probabilities of the first kind when $g = 3$ under heteroscedasticity.

N_1	N_2	N_3	p	Case1	Case2	Case 3
			50	0.060	0.060	0.060
50	25	25	100	0.058	0.058	0.058
			200	0.056	0.056	0.056
			50	0.062	0.061	0.062
25	50	25	100	0.059	0.059	0.059
			200	0.057	0.057	0.057
			50	0.062	0.062	0.062
25	25	50	100	0.059	0.059	0.059
			200	0.057	0.057	0.057

6 Concluding remarks

This article is considered to test the homogeneity of mean vectors under heteroscedasticity for some groups. We have proposed a test based on the unbiased estimator of the measure from the null hypothesis. It has been shown to perform for wide range of the population distribution which includes elliptical distribution, theoretically and numerically. As a special case that the population distribution is multivariate normal and assuming common covariance matrix, our proposed test has the same asymptotic power as the one proposed in Fujikoshi et al [6] or Srivastava and Fujikoshi [12] when the sample sizes and the dimension are large.

A Results on matrix algebra

We here show some results on matrix algebra

Lemma 3. *Let $\mathbf{\Sigma}_1, \mathbf{\Sigma}_2, \mathbf{\Sigma}_3$ be positive semi definite matrices. Then the following inequalities hold.*

$$\mathrm{tr}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2)^2 \le \mathrm{tr}(\mathbf{\Sigma}_1^2\mathbf{\Sigma}_2^2) \le \sqrt{\mathrm{tr}\,\mathbf{\Sigma}_1^4\,\mathrm{tr}\,\mathbf{\Sigma}_2^4}, \tag{25}$$

$$\mathrm{tr}\,\mathbf{\Sigma}_1\mathbf{\Sigma}_2\mathbf{\Sigma}_3\mathbf{\Sigma}_2 \le \sqrt{\mathrm{tr}(\mathbf{\Sigma}_1^2\mathbf{\Sigma}_2^2)\,\mathrm{tr}(\mathbf{\Sigma}_2^2\mathbf{\Sigma}_3^2)} \le \left(\mathrm{tr}\,\mathbf{\Sigma}_1^4\,\mathrm{tr}\,\mathbf{\Sigma}_3^4\right)^{1/4}\left(\mathrm{tr}\,\mathbf{\Sigma}_2^4\right)^{1/2}. \tag{26}$$

Proof. For $p \times q$ matrix T, let $\mathrm{vec}(T)$ be the $pq \times 1$ vector formed by stacking the columns of T under each other; that is, if $T = (\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$, where $\boldsymbol{t}_i$ is $p \times 1$ for $i = 1, \dots, q$, then $\mathrm{vec}(T) = (\boldsymbol{t}_1' \cdots \boldsymbol{t}_q')'$. It holds that

$$\mathrm{tr}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2)^2 = (\mathrm{vec}(\mathbf{\Sigma}_2\mathbf{\Sigma}_1))'\mathrm{vec}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2).$$

By Cauchy-Schwarz's inequality,

$$(\mathrm{vec}(\mathbf{\Sigma}_2\mathbf{\Sigma}_1))'\mathrm{vec}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2) \le \sqrt{(\mathrm{vec}(\mathbf{\Sigma}_2\mathbf{\Sigma}_1))'\mathrm{vec}(\mathbf{\Sigma}_2\mathbf{\Sigma}_1) \cdot (\mathrm{vec}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2))'\mathrm{vec}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2)}.$$

Since the right-hand side of the inequality equals to $\sqrt{\mathrm{tr}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^2\mathbf{\Sigma}_1)\,\mathrm{tr}(\mathbf{\Sigma}_2\mathbf{\Sigma}_1^2\mathbf{\Sigma}_2)}$, we have the first inequality in (25). The second inequality in (25) also can be shown by using Cauchy-Schwarz's inequality again. Using similar derivation method, we can also prove the inequalities in (26). □

References

[1] T.W. Anderson, An Introduction to Multivariate Statistical Analysis, 3rd ed., Wiley, Hoboken, NJ (2003).

[2] Z. Bai, H. Saranadasa, Effect of high dimension: by an example of a two sample problem, Statist. Sinica, 6 (1996) 311–329.

[3] S.X. Chen and Y.L. Qin, A two-sample test for high-dimensional data with applications to gean-set testing, Ann. Statist., 38 (2010) 808–835.

[4] A.P. Dempster, A high dimensional two sample significance test, Ann. Math, Statist., 29 (1958) 995–1010.

[5] A.P. Dempster, A significance test for the separation of two highly multivariate small samples, Biometrics, 16 (1960) 41–50.

[6] Y. Fujikoshi, T. Himeno, H. Wakaki, Asymptotic results of a high dimensional MANOVA test and power comparison when the dimension is large compared to the sample size, J. Japan Statist. Soc., 34 (2004) 19–26.

[7] T. Himeno, Asymptotic expansions of the null distributions for the Dempster trace criterion, Hiroshima Math. J., 37 (2007) 431–454.

[8] T. Himeno, T. Yamada, Estimations for some functions of covariance matrix in high dimension under non-normality, Submitted.

[9] C.R. Rao, Linear Statistical Inference and It's Applications, 2nd ed., Wiley, New York (1973).

[10] J.R. Shiryayev, Probability, Springer-Verlag, New York (1984).

[11] M.S. Srivastava, Some tests concerning the covariance matrix in high-dimensional data, J. Japan Statist. Soc., 35 (2005) 251–272.

[12] M.S. Srivastava, Y. Fujikoshi, Multivariate analysis of variance with fewer observations than the dimension, J. Multivariate Anal., 97 (2006) 1927–1940.

[13] M.S. Srivastava, Multivariate theory for analyzing high dimensional data, J. Japan Statist. Soc., 37 (2007) 53–86.