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Illustration of the Varying Coefficient Model for Analyses the Tree Growth from the Age and Space Perspectives

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Abstract

The one of reasons that trees grow over the year but may grow differently could be relative spatial conditions where trees have grown. From this, we introduce a statistical model called the varying coefficient model used in Satoh and Yanagihara (2010) and Tonda *et al.* (2010) to estimate effects on the tree growth from the age and relative space perspectives. This model is an appealing way to understand the tree growth, which is because 1) the space effects can be added in the growth curve model and visually checked by drawing contours and cubic diagrams, and 2) both the space and the age effects are with their confidence interval values, and also a hypothesis test can be conducted, which attains the confidence of the estimation result. The model is applied to a data of *Cryptomeria japonica* in Yoshimoto *et al.* (2012), the longitude and latitude are used as the space information, and the significant spatial effect of the DBH on the stem volume regarded as the tree growth was seen from the estimation result.

Key words: Varying coefficient model, growth curve model, spatial data, tree growth

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1. Introduction

It is known that trees grow over the year but may grow differently, even though they have planted at the same time, which is because they have different growth capacity, relative spatial conditions, and other environmental factors. From this knowledge, by introducing a statistical method, we explain the state of the tree growth from two perspectives; the one is the age of the tree and the other is relative spatial conditions.

The growth curve model is useful for finding the age effect on the tree growth, and it could be very suitable for our case if the spatial conditions could be added in the model and their effects are examined from the estimation result. Satoh and Yanagihara (2010) suggested a new approach to a growth curve model by using a varying coefficient model, which was originally proposed by Hastie and Tibshirani (1993), and applied the model to the dental fissures growth data in Potthoff and Roy (1964) which consists of measurements of the distance (*mm*) from the center of the pituitary to the pterygomaxillary fissure for 11 girls and 16 boys at their age 8, 10, 12, and 14. Tonda *et al.* (2010) applied the varying coefficient model to a spatial data; Columbus Ohio crime data in the R for statistical computing and graphics, which contains spatial information (easting and northing), the mean housing value, and the household income, and they showed the effect of area on the crime rate; residential burglaries and auto thefts per 1,000 households.

The varying coefficient model used in Satoh and Yanagihara (2010) and Tonda *et al.* are basically the same model, but the number of variables used for estimating varying coefficients are different, Satoh and Yanagihara (2010) used one variable; age, on the other hand, Tonda *et al.* (2010) used two; easting and northing. The number of used variables comes out different graphical results of varying coefficients, a curve in the former case and a surface in the latter case, which theoretical background is discussed in the next section.

Satoh and Yanagihara (2010) and Tonda *et al.* (2010) did not use the age and special conditions variables together in the model, however, it is theoretically possible. We use both the age and spatial conditions variable together in the varying coefficient model for finding these effects on the tree growth.

Besides enabling adding spatial information in the growth curve model, the most attractive achievement from Satoh and Yanagihara (2010) and Tonda *et al.* (2010) is enabling calculating confidence intervals of varying coefficients. The estimator of the varying coefficient curve or surface is usually obtained by kernel smoothing methods which is essentially the linear regression around fixed time or location that makes difficult to construct a confidence interval as a function of time or location. Hence, the point wise confidence intervals had been conducted. Satoh and Yanagihara (2010) suggested to use a semiparametric estimation for varying coefficient model to obtain the confidence interval of varying coefficients.

We apply a data of *Cryptomeria japonica* in Yoshimoto *et al.* (2012) for introducing the varying coefficient model to see the tree growth effects of age and spatial conditions.

This paper is organized as follows: In section 2, the varying coefficient model proposed by Satoh and Yanagihara (2010) and Tonda *et al.* (2010) is explained by adapting in our case. The data and estimation result are presented in section 3. Section 4 contains a conclusion.

2. Method

2.1. Space-Time Varying Coefficient Model

The $y(u, v, t)$ is the response variable observed at the space (u, v) and the time t. The expected value of y; $E[y(u, v, t)]$, is assumed to be,

$$E[y(u, v, t)] = \boldsymbol{a}'\boldsymbol{\beta}(u, v, t) = \sum_{j=1}^{k} a_j \beta_j(u, v, t), \tag{1}$$

where $\boldsymbol{a} = (a_1, \ldots, a_k)'$ is a vector of explanatory variables that are independent from the space

(u, v) and the time t, and $\boldsymbol{\beta}(u, v, t) = (\beta_1(u, v, t), \ldots, \beta_k(u, v, t))'$ is a vector of the surfaces of varying coefficients which values change depending on u, v and t. Here, the notation "$'$" denotes a transpose of a vector or matrix.

Let $\boldsymbol{x}(u, v, t) = (x_1(u, v, t), \ldots, x_q(u, v, t))'$ denote a vector of q basis functions with respect to (u, v, t), and we assume the varying coefficient has a linear structure;

$$\boldsymbol{\beta}(u, v, t) = \boldsymbol{\Theta}\boldsymbol{x}(u, v, t), \tag{2}$$

where $\boldsymbol{\Theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_k)'$ is a $k \times q$ matrix of unknown parameters, and $\boldsymbol{\theta}_j = (\theta_{j1}, \ldots, \theta_{jk})'$ is a q-dimensional vector of unknown parameters. If we fit a polynomial surface in Ripley (1981); section 4 and in Venables and Ripley (2002); section 15, we have $\boldsymbol{x}(u, v) = (1, u, v, t)'$ as a linear equation. Other fittings are nonparametric methods, for example, taking B-spline or Gaussian basis functions, e.g., Satoh *et al.* (2003), Ruppert *et al.* (2003); section 3, etc. Satoh and Yanagihara (2010) and Satoh *et al.* (2009) suggested a semiparametric estimation for the varying coefficient model and assumed the linear structure in the varying coefficient, so that they can make a confidence interval and hypothesis test about the varying coefficient. Historically, in the growth curve model, the single element of $\boldsymbol{\Theta}$ is focused, and a confidence interval and a hypothesis test about $\boldsymbol{\Theta}$ has been studied (see more details in Satoh and Yanagihara (2010)), however our interest is not the effect of a covariate on the coefficient of a polynomial function $\boldsymbol{\Theta}$, but the effects of covariates on the amount of growth $\boldsymbol{\beta}(u, v, t)$.

By substituting (2) into (1), we obtain the following,

$$E[y(u, v, t)] = \boldsymbol{a}'\boldsymbol{\Theta}\boldsymbol{x}(u, v, t),$$

which is an interaction model between explanatory variables in $\boldsymbol{a}$ and basis functions in $\boldsymbol{x}(u, v, t)$ such as,

$$E[y(u, v, t)] = \sum_{j=1}^{k} \sum_{\ell=1}^{q} \theta_{j\ell}\{a_j x_\ell(u, v, t)\}. \tag{3}$$

The kq covariates $a_j x_\ell(u, v, t)$ in (3) are known, as far as the space (u, v) and the time t are given. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \ldots, \boldsymbol{\theta}_k')'$ be a qk-dimensional vector of unknown parameters and $\boldsymbol{z}(u, v, t) = \boldsymbol{a} \otimes \boldsymbol{x}(u, v, t)$ be a qk-dimensional vector of explanatory variables, where the notation "$\otimes$" denotes the Kronecker product (see e.g., Harville (1997), chapter 16), we have,

$$E[y(u, v, t)] = \boldsymbol{z}(u, v, t)'\boldsymbol{\theta}$$

The ith observation, $i = 1, \ldots, n$ has a response variable $y_i = y(u_i, v_i, t_i)$ at the space (u_i, v_i) and the time t_i and a k-dimensional vector of explanatory variables $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{ik})'$. We assume that $y_1, \ldots, y_n$ are mutually independent, and $y_i \sim N(\boldsymbol{a}_i\boldsymbol{\Theta}\boldsymbol{x}(u_i, v_i, t_i), \sigma^2)$. Let $\boldsymbol{y} = (y_1, \ldots, y_n)'$, $\boldsymbol{Z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)'$, and $\boldsymbol{z}_i = \boldsymbol{a}_i \otimes \boldsymbol{x}(u_i, v_i, t_i)$, the estimator of $\boldsymbol{\theta}$ is obtained by the least-square method,

$$\hat{\boldsymbol{\theta}} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\boldsymbol{y}. \tag{4}$$

The covariance matrix of $\hat{\boldsymbol{\theta}}$ is,

$$\boldsymbol{\Omega} = \sigma^2(\boldsymbol{Z}'\boldsymbol{Z})^{-1} = \begin{pmatrix} \boldsymbol{\Omega}_{11} & \cdots & \boldsymbol{\Omega}_{1k} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Omega}_{k1} & \cdots & \boldsymbol{\Omega}_{kk} \end{pmatrix}.$$

The $\hat{\boldsymbol{\theta}}_j,\ j = 1, \dots, k$, is distributed as

$$\hat{\boldsymbol{\theta}}_j \sim N_q(\boldsymbol{\theta}_j, \boldsymbol{\Omega}_{jj}).$$

The estimator of the varying coefficient is obtained by $\hat{\beta}_j(u, v, t) = \hat{\boldsymbol{\theta}}_j' \boldsymbol{x}(u, v, t)$. The estimator of σ^2 is derived by

$$s^2 = \frac{1}{n - kq} \boldsymbol{y}'\{\boldsymbol{I}_n - \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'\}\boldsymbol{y},$$

where $\boldsymbol{I}_n$ is the identity matrix of size n. By using s^2, we derive the estimator of $\boldsymbol{\Omega}$ is obtained by

$$\hat{\boldsymbol{\Omega}} = s^2(\boldsymbol{Z}'\boldsymbol{Z})^{-1} = \begin{pmatrix} \hat{\boldsymbol{\Omega}}_{11} & \cdots & \hat{\boldsymbol{\Omega}}_{1k} \\ \vdots & \ddots & \vdots \\ \hat{\boldsymbol{\Omega}}_{k1} & \cdots & \hat{\boldsymbol{\Omega}}_{kk} \end{pmatrix}.$$

2.2. Model without Space-Time Interaction

In the space-time varying coefficient model, it is common to assume that there is no interaction of space and time, i.e., we assume that the varying coefficient $\beta_j(u, v, t)$ of an explanatory variable a_j ($j = 1, \dots, k$) can be decomposed as

$$\beta_j(u, v, t) = \mu_j + \gamma_{j,1}(u, v) + \gamma_{j,2}(t),$$

where μ_j is an unknown location parameter, $\gamma_{j,1}(u, v)$ and $\gamma_{j,2}(t)$ are a surface and a curve of varying coefficients which value changes depending on (u, v) and t, respectively. In particular, we assume that $\gamma_{j,1}(0, 0) = 0$ and $\gamma_{j,2}(0) = 0$. The space and the time varying coefficients are given as

$$\beta_{j,1}(u, v) = \mu_j + \gamma_{j,1}(u, v), \quad \beta_{j,2}(t) = \mu_j + \gamma_{j,2}(t). \tag{5}$$

Let us decompose $\boldsymbol{\theta}_j$ and $\boldsymbol{x}(u, v, t)$ as

$$\boldsymbol{\theta}_j = (\mu_j, \boldsymbol{\xi}_{j,1}', \boldsymbol{\xi}_{j,2}')', \quad \boldsymbol{x}(u, v, t) = (1, \boldsymbol{w}_1(u, v)', \boldsymbol{w}_2(t)')', \tag{6}$$

where $\boldsymbol{w}_1(u, v)$ and $\boldsymbol{w}_2(t)$ are q_1- and q_2-dimensional vectors of basis functions with respect to (u, v) and t, respectively. Then, the $\beta_j(u, v, t)$ is expressed as

$$\beta_j(u, v, t) = \mu_j + \boldsymbol{\xi}_{j,1}'\boldsymbol{w}_1(u, v) + \boldsymbol{\xi}_{j,2}'\boldsymbol{w}_2(t).$$

Estimators of μ_j, $\boldsymbol{\xi}_{j,1}$ and $\boldsymbol{\xi}_{j,2}$ are derived from $\hat{\boldsymbol{\theta}}_j$ in (4) because of $\hat{\boldsymbol{\theta}}_j = (\hat{\mu}_j, \hat{\boldsymbol{\xi}}_{j,1}', \hat{\boldsymbol{\xi}}_{j,2}')'$, where $\hat{\mu}_j$, $\hat{\boldsymbol{\xi}}_{j,1}$ and $\hat{\boldsymbol{\xi}}_{j,2}$ are estimators of μ_j, $\boldsymbol{\xi}_{j,1}$ and $\boldsymbol{\xi}_{j,2}$, respectively. Let $\hat{\gamma}_{j,1}(u, v) = \hat{\boldsymbol{\xi}}_{j,1}'\boldsymbol{w}_1(u, v)$ and $\hat{\gamma}_{j,2}(t) = \hat{\boldsymbol{\xi}}_{j,2}'\boldsymbol{w}_2(t)$. Estimators of the space and the time varying coefficients are obtained by $\hat{\beta}_{j,1}(u, v) = \hat{\mu}_j + \hat{\gamma}_{j,1}(u, v)$ and $\hat{\beta}_{j,2}(t) = \hat{\mu}_j + \hat{\gamma}_{j,2}(t)$, respectively.

Moreover, we decompose $\hat{\boldsymbol{\Omega}}_{jj}$ in (5) corresponding to the division of $\hat{\boldsymbol{\theta}}_j = (\hat{\mu}_j, \hat{\boldsymbol{\xi}}'_{j,1}, \hat{\boldsymbol{\xi}}'_{j,2})'$ as

$$\hat{\boldsymbol{\Omega}}_{jj} = \begin{pmatrix} \hat{\psi}_{jj} & \hat{\boldsymbol{\psi}}'_{jj,1} & \hat{\boldsymbol{\psi}}'_{jj,2} \\ \hat{\boldsymbol{\psi}}_{jj,1} & \hat{\boldsymbol{\Psi}}_{jj,11} & \hat{\boldsymbol{\Psi}}_{jj,12} \\ \hat{\boldsymbol{\psi}}_{jj,2} & \hat{\boldsymbol{\Psi}}_{jj,12} & \hat{\boldsymbol{\Psi}}_{jj,22} \end{pmatrix}. \tag{7}$$

Then, we define

$$\hat{\boldsymbol{\Omega}}_{jj,1} = \begin{pmatrix} \hat{\psi}_{jj} & \hat{\boldsymbol{\psi}}'_{jj,1} \\ \hat{\boldsymbol{\psi}}_{jj,1} & \hat{\boldsymbol{\Psi}}_{jj,11} \end{pmatrix}, \quad \hat{\boldsymbol{\Omega}}_{jj,2} = \begin{pmatrix} \hat{\psi}_{jj} & \hat{\boldsymbol{\psi}}'_{jj,2} \\ \hat{\boldsymbol{\psi}}_{jj,2} & \hat{\boldsymbol{\Psi}}_{jj,22} \end{pmatrix}. \tag{8}$$

Let $\hat{\boldsymbol{\theta}}_{j,1} = (\hat{\mu}_j, \hat{\boldsymbol{\xi}}'_{j,1})'$ and $\hat{\boldsymbol{\theta}}_{j,2} = (\hat{\mu}_j, \hat{\boldsymbol{\xi}}'_{j,2})'$. It is clear that $\hat{\boldsymbol{\Omega}}_{jj,1}$ and $\hat{\boldsymbol{\Omega}}_{jj,2}$ in (8) are estimators of the covariance matrices of $\hat{\boldsymbol{\theta}}_{j,1}$ and $\hat{\boldsymbol{\theta}}_{j,2}$, respectively.

2.3. Confidence Interval and Hypothesis Tests

We consider confidence intervals and hypothesis tests about $\beta_{j,1}(u,v)$ and $\beta_{j,2}(t)$ in (5) which are the surface and the curve of the varying coefficients for the explanatory variable a_j in $\boldsymbol{a} = (a_1, \ldots, a_k)'$. Let $\boldsymbol{x}_1(u,v) = (1, \boldsymbol{w}_1(u,v)')'$ and $\boldsymbol{x}_2(t) = (1, \boldsymbol{w}_2(t)')'$, where $\boldsymbol{w}_1(u,v)$ and $\boldsymbol{w}_2(t)$ are in (6). Variances of $\hat{\beta}_{j,1}(u,v)$ and $\hat{\beta}_{j,2}(t)$ are obtained by $\hat{\lambda}_{j,1}(u,v) = \boldsymbol{x}_1(u,v)'\hat{\boldsymbol{\Omega}}_{jj,1}\boldsymbol{x}_1(u,v)$ and $\hat{\lambda}_{j,2}(u,v) = \boldsymbol{x}_2(t)'\hat{\boldsymbol{\Omega}}_{jj,2}\boldsymbol{x}_2(t)$, where $\hat{\boldsymbol{\Omega}}_{jj,1}$ and $\hat{\boldsymbol{\Omega}}_{jj,2}$ are given by(8). The $100\alpha\%$ confidence intervals of the varying coefficient surface $\beta_{j,1}(u,v)$ and curve $\beta_{j,2}(t)$ are obtained by

$$\mathcal{I}_{j,1,\alpha}(u,v) = \left[\hat{\beta}_{j,1}(u,v) - \sqrt{\hat{\lambda}_{j,1}(u,v)c_{m_1}(\alpha)},\ \hat{\beta}_{j,1}(u,v) + \sqrt{\hat{\lambda}_{j,1}(u,v)c_{m_1}(\alpha)}\right], \tag{9}$$

$$\mathcal{I}_{j,2,\alpha}(t) = \left[\hat{\beta}_{j,2}(t) - \sqrt{\hat{\lambda}_{j,2}(t)c_{m_2}(\alpha)},\ \hat{\beta}_{j,2}(t) + \sqrt{\hat{\lambda}_{j,2}(t)c_{m_2}(\alpha)}\right], \tag{10}$$

where $c_m(\alpha)$ is the upper $100\alpha\%$ point of the chi-square distribution with m degrees of freedom, i.e., $\Pr(\chi^2_m \geq c_m(\alpha)) = \alpha$, and $m_1 = q_1 + 1$ and $m_2 = q_2 + 1$. Here q_1 and q_2 are dimensions of vectors $\boldsymbol{w}_1(u,v)$ and $\boldsymbol{w}_2(t)$, respectively.

For hypothesis tests, null hypotheses can specify that varying coefficient surface and curve have uniform shapes, i.e., null hypotheses are written as,

$$H_{0,j,1} : \beta_{j,1}(u,v) = 0 \text{ (for any } (u,v)), \quad H_{0,j,2} : \beta_{j,2}(t) = 0 \text{ (for any } t). \tag{11}$$

We have a test statistic for $H_{0,j,d}$ $(d = 1,2)$ as

$$T_{j,d} = \hat{\boldsymbol{\xi}}'_{j,d}\hat{\boldsymbol{\Psi}}^{-1}_{jj,dd}\hat{\boldsymbol{\xi}}_{j,d},$$

where $\hat{\boldsymbol{\xi}}_{j,d}$ is the estimator of $\boldsymbol{\xi}_{j,d}$ given in (6), and $\hat{\boldsymbol{\Psi}}_{jj,dd}$ is given by (7). The test statistic asymptotically follows chi-square distribution with q_d degrees of freedom, and the null hypothesis $H_{0,j,d}$ is rejected at the $100\alpha\%$ significant level. See Tonda *et al.* (2010) for more details about theoretical exposition.

3. Analysis

3.1. Data

We use the growth data of *Cryptomeria japonica* in Yoshimoto *et al.* (2012) obtained from a survey conducted at Hoshino village of Fukuoka Prefecture in Kyushu, Japan. The data consists of growth measurements from 30 trees. Each sample has growth information about "Age (year)", "DBH (*cm*)", "Height (*m*)", and "Volume (m^3)" of the tree, and spatial information about "Longitude", "Latitude" and "Altitude" where the tree had grown. The number of samples are 658, observed from 30 trees at their 23 age points with missing 25, 5, and 2 trees observation at age 1, 2, and 3, respectively. The growth information, "DBH", "Height", and "Volume", are observed by conducting a stem analysis in Philip (1994). The "DBH", which stands for diameter at breast height, is the measurement of tree diameter at 1.3 *m* above ground level. The "Volume" is the stem volume.

In our data of of *Cryptomeria japonica*, the u and v are the "Longitude" and "Latitude", respectively, the t is "Age", the y is "Volume", and the $\boldsymbol{a}$ is a three-dimensional vector which components are 1, "Height", and "DBH"", i.e., $\boldsymbol{a} = (1, \text{“height”}, \text{“DBH”})'$. Excluding samples with 0 value in DBH, because these samples were less than 1.3 *m* and not observed DBH, we have 588 samples for estimation.

3.2. Estimation Result

We fit the liner model to estimate the varying coefficient surface or curve, i.e., $\hat{\beta}_j(u, v, t) = \hat{\boldsymbol{\theta}}_j'\boldsymbol{x}(u, v, t)$. For applying the model to any complicated varying coefficient surface or curve shape, we assume that the $\hat{\boldsymbol{\theta}}_j'\boldsymbol{x}(u, v, t)$ is one of either linear, quadratic or cubic expression with their interaction each. For the linear, quadratic, and cubic expressions by u and v, we have

$$\boldsymbol{w}_1(u, v) = \begin{cases} (u, v)' & (r_1 = 1) \\ (u, v, u^2, v^2, uv)' & (r_1 = 2) \\ (u, v, u^2, v^2, uv, u^3, v^3, u^2v, uv^2)' & (r_1 = 3) \end{cases},$$

and those by t,

$$\boldsymbol{w}_2(t) = \begin{cases} (t) & (r_2 = 1) \\ (t, t^2)' & (r_2 = 2) \\ (t, t, t^3)' & (r_2 = 3) \end{cases},$$

where r_d ($d = 1, 2$) denotes the degree of a polynomial. The following subsets of explanatory variables were considered as the candidate subsets.

$$\boldsymbol{a} = \begin{cases} (1, \text{“Height”}, \text{“DBH”})' & (\text{case 1}) \\ (1, \text{“Height”})' & (\text{case 2}) \\ (1, \text{“DBH”})' & (\text{case 3}) \end{cases},$$

The best expression form of $\boldsymbol{z}$, i.e., $\boldsymbol{a} \otimes \boldsymbol{x}(u, v, t)$, is selected by Bayesian information criterion (BIC) in Schwarz (1978), which results are shown in Table 1. From the result in Table 1, the cubic expression for the space and the quadratic expression for the age are selected as the best model such as

Table 1. Variables selection for $\boldsymbol{z}$ by BIC

Degree of a polynomial	BIC		
(r_1, r_2)	Case 1	Case 2	Case 3
(1 , 1)	-4094.87	-3321.48	-4060.77
(1 , 2)	-4367.03	-3539.00	-4231.29
(1 , 3)	-4352.28	-3559.74	-4229.61
(2 , 1)	-4116.45	-3366.48	-4081.04
(2 , 2)	-4395.87	-3571.30	-4261.03
(2 , 3)	-4381.67	-3591.55	-4262.97
(3 , 1)	-4125.56	-3386.51	-4079.40
(3 , 2)	**-4410.95**	-3589.31	-4235.54
(3 , 3)	-4399.19	-3611.78	-4239.51

$\boldsymbol{x}(u, v, t) = (1, u, v, u^2, v^2, uv, u^3, v^3, u^2v, uv^2, t, t^2)$ in case 1.

From a result of the least square estimation (4), the estimate of $\boldsymbol{\Theta}$ is,

$$\hat{\boldsymbol{\Theta}} = \begin{pmatrix} \hat{\boldsymbol{\theta}}_1' \\ \hat{\boldsymbol{\theta}}_2' \\ \hat{\boldsymbol{\theta}}_3' \end{pmatrix} = \begin{pmatrix} \hat{\mu}_1 & \hat{\boldsymbol{\xi}}_{1,1}' & \hat{\boldsymbol{\xi}}_{1,2}' \\ \hat{\mu}_2 & \hat{\boldsymbol{\xi}}_{2,1}' & \hat{\boldsymbol{\xi}}_{2,2}' \\ \hat{\mu}_3 & \hat{\boldsymbol{\xi}}_{3,1}' & \hat{\boldsymbol{\xi}}_{3,2}' \end{pmatrix},$$

where

$$\begin{aligned} \hat{\boldsymbol{\theta}}_1' = (&-2.1 \times 10^{-2}, -2.6 \times 10^{-3}, -2.0 \times 10^{-3}, -2.9 \times 10^{-4}, \\ &- 1.1 \times 10^{-4}, 3.3 \times 10^{-4}, -8.8 \times 10^{-6}, 6.7 \times 10^{-7}, \\ &1.2 \times 10^{-5}, -6.5 \times 10^{-6}, 7.5 \times 10^{-3}, -9.3 \times 10^{-4}), \end{aligned}$$

$$\begin{aligned} \hat{\boldsymbol{\theta}}_2' = (&8.2 \times 10^{-3}, 1.1 \times 10^{-3}, -2.5 \times 10^{-3}, 2.3 \times 10^{-5}, \\ &- 2.7 \times 10^{-4}, 1.7 \times 10^{-5}, -8.9 \times 10^{-7}, -7.5 \times 10^{-6}, \\ &4.0 \times 10^{-6}, 2.5 \times 10^{-6}, 3.5 \times 10^{-4}, 8.0 \times 10^{-6}), \end{aligned}$$

and

$$\begin{aligned} \hat{\boldsymbol{\theta}}_3' = (&-1.2 \times 10^{-2}, -8.6 \times 10^{-4}, 2.3 \times 10^{-3}, 9.1 \times 10^{-7}, \\ &2.4 \times 10^{-4}, -6.4 \times 10^{-5}, 1.5 \times 10^{-6}, 6.1 \times 10^{-6}, \\ &- 5.3 \times 10^{-6}, -1.2 \times 10^{-6}, 6.6 \times 10^{-4}, 1.9 \times 10^{-5}). \end{aligned}$$

Calculation results of surfaces or curves of varying coefficients and their confidence intervals are shown in figure 1 to 9. The Figure 1 to 6 show estimated varying coefficient surface of u and v. Numbers in contours in Figure 1, 3, and 5 provide values of $\hat{\beta}_{1,1}(u, v) = \hat{\mu}_1 + \hat{\boldsymbol{\xi}}_{1,1}'\boldsymbol{w}_1(u, v)$, $\hat{\beta}_{2,1}(u, v) = \hat{\mu}_2 + \hat{\boldsymbol{\xi}}_{2,1}'\boldsymbol{w}_1(u, v)$, and $\hat{\beta}_{3,1}(u, v) = \hat{\mu}_3 + \hat{\boldsymbol{\xi}}_{3,1}'\boldsymbol{w}_1(u, v)$ at certain level of longitude and latitude, respectively. The $\hat{\beta}_{1,1}(u, v)$, $\hat{\beta}_{2,1}(u, v)$, and $\hat{\beta}_{3,1}(u, v)$ are varying coefficients for the constant term, “Height”, and “DBH”, respectively. Black circles in Figures 1, 3, and 5 indicate spaces where sample trees had grown. The vertical axes in Figures 2, 4, and 6 signify values of $\hat{\beta}_k(u, v)$; the gray colored varying coefficient surfaces are $\hat{\beta}_{j,1}(u, v)$, and upper and lower non-colored surfaces

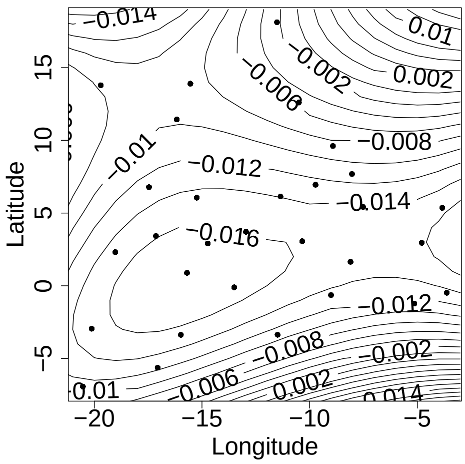

Figure 1. (a) Const. term

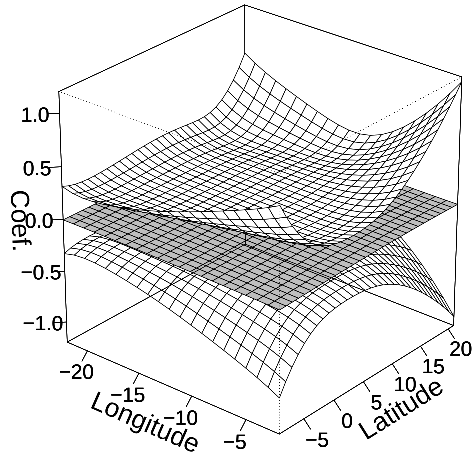

Figure 2. (b) Const. term

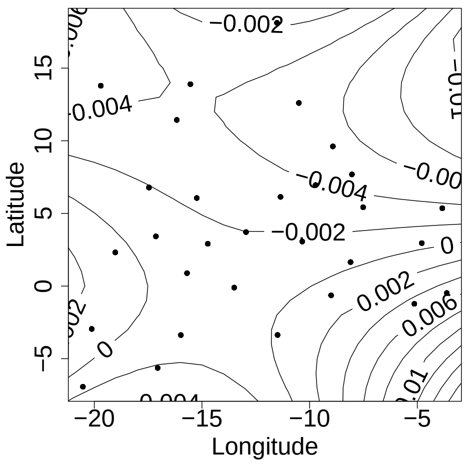

Figure 3. (c) Height

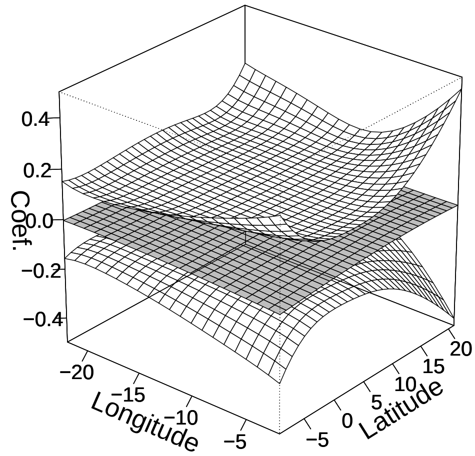

Figure 4. (d) Height

describe the upper and lower boundaries of confidence intervals of $\beta_{j,1}(u,v)$ in (9). From results of the hypothesis test in (11), null hypotheses are rejected in $\beta_{2,1}(u,v)$ and $\beta_{3,1}(u,v)$, which implies that $\beta_{2,1}(u,v)$ and $\beta_{3,1}(u,v)$ are not 0 and those values are significant. From Figure 3 to 6, trees seem to be higher and thinner at the southeast, but confidence intervals are large at this area, which is because there are not may samples in the corner and the $\boldsymbol{\beta}$ may not be well estimated.

In the similar way, Figures 7, 8 and 9 show estimated varying coefficient curves of t; the solid lines describes $\hat{\beta}_{j,2}(t) = \hat{\boldsymbol{\xi}}'_{j,2}\boldsymbol{w}_2(t)$, and the dashed lines describes upper and lower boundaries of confidence intervals of $\beta_{j,2}(t)$ in (10). The null hypotheses in (11) are rejected in $\beta_{1,2}(t)$ and $\beta_{3,2}(t)$, which implies that the height of tree does not explain the stem volume growth. The Figure 9 tells that the DBH positively affects the volume of the tree more with age.

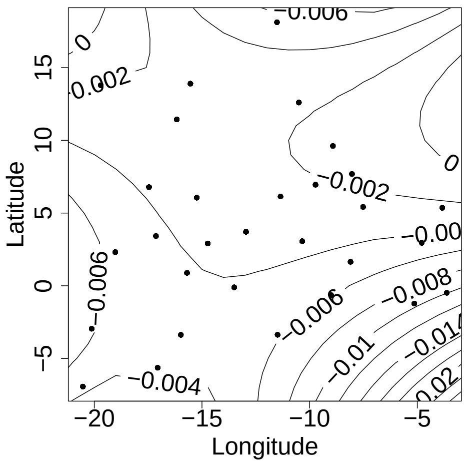

Figure 5. (e) DBH

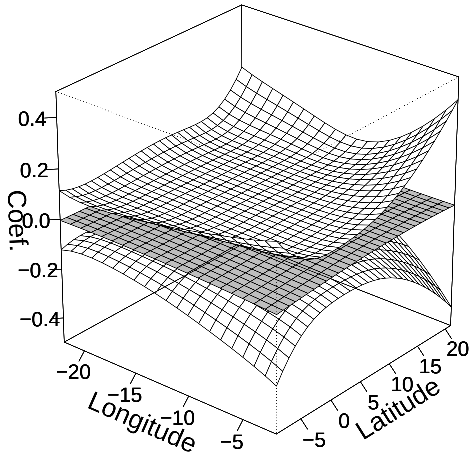

Figure 6. (f) DBH

4. Conclusion

In this paper, we introduced the varying coefficient model used in Satoh and Yanagihara (2010) and Tonda *et al.* (2010) to estimate effects on the tree growth from the tree's age and the space where the tree had grown.

From the estimated results of varying coefficient surfaces through the point of the space, the tree growth state in the relative space was well graphically understood. As an example, from the Figure 3 and 4 showing significant effects on the tree growth, since varying coefficient values are lower at the middle of the sample area than those at the corner especially southeast in Figure 3, trees at the middle area may have thinner stem wood volume than those at the corner, however the state of the growth at the corner is obscure since small sample at the southeast causes large confidence interval result in Figure 4. From results of varying coefficient curve through the point of the age, the tree growth state at each their age points were well described in Figure 7 to 9, however the Figure 8 was not the significant result. The reason that the effect of height was not significant could be that the DBH has a stronger effect on the stem volume than the height.

From the above, the varying coefficient model is an appealing way to estimate a tree growth, which is because 1) space effects can be added in the growth curve model and visually checked by drawing contours and cubic diagrams like Figure 1 to 6, and 2) both the space and the age effects are shown by interval values and a hypothesis test is conducted, which attains a confidence of the estimation result.

In our data, we considered interaction effects about the space but not the space and the age. If the space have some change with time; for example, the day length at a certain space changes with month affects monthly observation values about a tree growth, we can try to add interaction effects about space and time in the model.

The confidence interval calculation (9, 10) proposed by Tonda *et al.* (2010) was used in this paper,

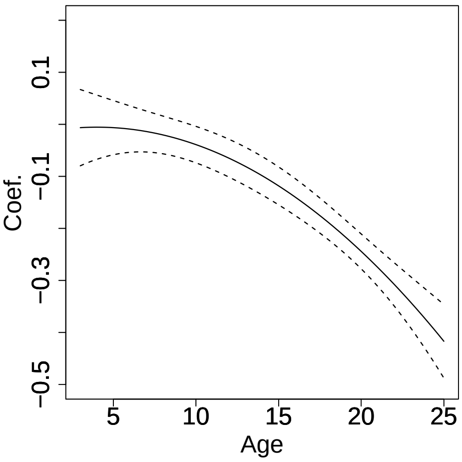

Figure 7. (g) Const. term

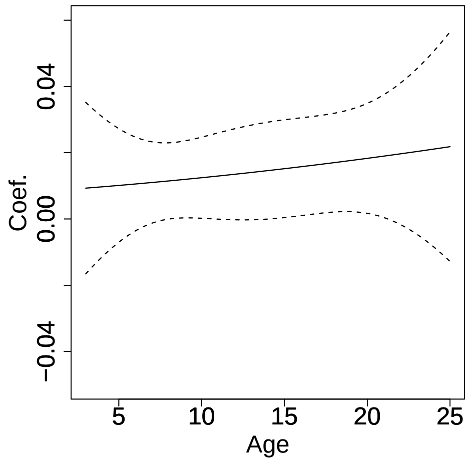

Figure 8. (h) Height

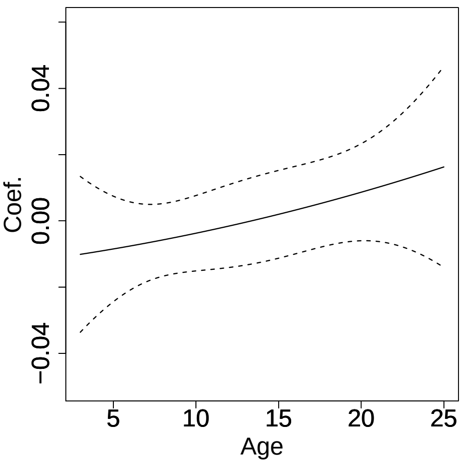

Figure 9. (i) DBH

however Tonda and Satoh (2013) has improved the accuracy of approximation of a coverage probability of the confidence interval. We can use this updated confidence interval for our estimation too.

References

Harville, D. A. (1997). *Matrix Algebra from a Statistician's Perspective*. Springer-Verlag, New York.

Hastie, T. & Tibshirani, R. (1993). Varying-coefficient models. *J. Roy. Statist. Soc. Ser.* **B**, **55**, 757–796.

Philip, M. S. (1994). *Measuring Trees and Forests* (2nd ed.). CAB International, Wallingford, Oxon.

Potthoff, R. F. and Roy, S. N. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems. *Biometrika*, **51**, 313–326.

Ripley, B. D. (1981). *Spatial Statistics*. John Wiley & Sons, Inc., Hoboken, New Jersey.

Ruppert, D., Wand, M. and Carroll, R. (2003). *Semiparametric Regression*. Cambridge University Press, Cambridge.

Satoh, K. and Yanagihara, H. (2010). Estimation of varying coefficients for a growth curve model. *Amer. J. Math. Manage-*

ment Sci., **30**, 243–256.

Satoh, K., Yanagihara, H. and Kamo, K. (2009). Statistical inference on a linear varying coefficient on longitudinal data of discrete distribution. *Japanese J. Appl. Statist.*, **38**, 1–11 (in Japanese).

Satoh, K., Yanagihara, H. and Ohtaki, M. (2003). Bridging the gap between *B*-spline and polynomial regression model. *Comm. Statist. Theory Methods*, **32**, 179–190.

Schwarz, G. (1978). Estimating the dimension of a model. *Ann. Statist.*, **6**, 461–464.

Tonda, T. and Satoh, K. (2013). Improvement of confidence interval for linear varying coefficient. *Japanese J. Appl. Statist.*, **42**, 11–21 (in Japanese).

Tonda, T., Satoh, K. and Yanagihara, H. (2010). Statistical inference on a varying coefficient surface using interaction model for spatial data. *Japanese J. Appl. Statist.*, **39**, 59–70 (in Japanese).

Venables, W. N. and Ripley, B. D. (2002). *Modern Applied Statistics with S* (4th ed.). Springer-Verlag, New York.

Yoshimoto, A., Kamo, K. and Yanagihara, H. (2012). *Environmental Data Analysis by R*. Asakura Syoten, Tokyo (in Japanese).