# Testing equality of two mean vectors with unequal sample sizes for populations with correlation

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In order to test the equality of mean vectors with unequal sample sizes when populations have correlation, we propose a new $T^2$ type statistic. In this paper, we consider observed data as missing data as described by Seko, Yamazaki and Seo (2012), and derive the approximate upper percentile for the distribution of the proposed statistic. Furthermore, we evaluate the accuracy of the approximate upper percentile using Monte Carlo simulation for selected values of parameters, namely, sample sizes, dimensions, and covariance matrices.

*Keywords:* Hotelling $T^2$ type statistic; Paired $T^2$ statistic; Upper percentile

## 1 Introduction

Testing equality of mean vectors has been considered by many authors. Hotelling's $T^2$ type statistic is a generalization of Student's $t$ statistic. This statistic is important and is discussed for normal and non-normal populations. Asymptotic distribution of Hotelling's $T^2$ statistic in elliptical population has been discussed by Iwashita (1997), and Iwashita and Seo (2002). The non-normal case has been discussed by Kano (1995), Fujikoshi (1997), Kakizawa and Iwashita (2008), and Kakizawa (2008).

Test for equality of mean vectors in multivariate normal populations that have correlation was introduced by Morrison (2005) and discussed for equal sample sizes. Morrison (2005) introduced a test statistic, usually called paired $T^2$ statistic, which is Hotelling's $T^2$ type statistic. Shinozaki, Okamoto and Seo (2014) discussed testing equality of mean vectors for the case of $k$-sample problem, and derived approximate upper percentiles for the distribution of paired $T^2$ statistic under elliptical populations with correlation. In this paper, we expand this discussion for the case of unequal sample sizes. Even though we consider observed data as two-step monotone missing data, practically this data is not missing data. In the case of two-step monotone missing data, Anderson and Olkin (1985) derived maximum-likelihood estimators (MLEs) for mean vectors $\boldsymbol{\mu}$ and covariance matrix $\Sigma$. Kanda and Fujikoshi (1998) discussed the distributions for the MLEs of $\hat{\boldsymbol{\mu}}$ and $\hat{\Sigma}$ based on monotone missing data. We propose a new $T^2$ type statistic using sample mean vectors and MLE of covariance matrix, and derive the approximate upper percentile for the distribution of $T^2$ type statistic using the method described by Seko, Yamazaki and Seo (2012). Finally, the accuracy of approximate

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upper percentile and type I error for the new test statistic is evaluated using Monte Carlo simulation.

## 2 Paired $T^2$ statistic for equal sample sizes

Let $\boldsymbol{x}_1^{(i)}$, $\boldsymbol{x}_2^{(i)}, \ldots, \boldsymbol{x}_N^{(i)} (i = 1, 2)$ represent $N$ independent random sample vectors distributed as the $i$-th $p$-dimensional normal population with mean vector $\boldsymbol{\mu}^{(i)}$ and covariance matrix $\Sigma_{ii}$, where the first and second populations have correlation. Let $\boldsymbol{x}_j = (\boldsymbol{x}_j^{(1)'}, \boldsymbol{x}_j^{(2)'})'$ $(j = 1, 2, \ldots, N)$ represent $N$ independent random sample vectors distributed as a $2p$-dimensional normal population with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma$ given by

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(l)} \\ \boldsymbol{\mu}^{(m)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{ll} & \Sigma_{lm} \\ \Sigma_{lm}' & \Sigma_{mm} \end{pmatrix},$$

respectively. The sample mean vector and sample covariance matrix are defined as

$$\begin{aligned} \overline{\boldsymbol{x}} &= \frac{1}{N}\sum_{j=1}^{N} \boldsymbol{x}_j \\ &= \begin{pmatrix} \overline{\boldsymbol{x}}^{(1)} \\ \overline{\boldsymbol{x}}^{(2)} \end{pmatrix}, \\ S &= \frac{1}{N-1}\sum_{j=1}^{N} (\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})' \\ &= \begin{pmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{pmatrix}, \end{aligned}$$

respectively.

In order to test the equality of mean vectors, we set a hypothesis as

$$H_0 : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} \quad \text{vs.} \quad H_1 : \boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)}.$$

The test statistic for testing the hypothesis $H_0$ is given by

$$T_{\mathrm{pd}}^2 = N(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'(S_{11} + S_{22} - S_{12} - S_{12}')^{-1}(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}).$$

Under $H_0$,

$$F_{p,N-p} = \frac{N-p}{(N-1)p} T_{\mathrm{pd}}^2,$$

where $F_{p,N-p}$ is an $F$ distribution with $p$ and $N - p$ degrees of freedom.

## 3 Paired $T^2$ statistic for unequal sample sizes

In this section, we expand the discussion for the case of unequal sample sizes. Here, the first and second populations have correlation.

| Subject | Condition 1 | Condition 2 |
|---|---|---|
| 1 | $\boldsymbol{x}_1^{(1)'}$ | $\boldsymbol{x}_1^{(2)'}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $N_2$ | $\boldsymbol{x}_{N_2}^{(1)'}$ | $\boldsymbol{x}_{N_2}^{(2)'}$ |
| $\vdots$ | $\vdots$ | |
| $N_1$ | $\boldsymbol{x}_{N_1}^{(1)'}$ | |

Let $\boldsymbol{x}_j^{(i)'}(i = 1, 2, j = 1, 2, \ldots, N_i)$ represent $N_i$ independent random sample vectors distributed as a $p$-dimensional multivariate normal population with mean vector $\boldsymbol{\mu}^{(i)}$ and covariance matrix $\Sigma_{ii}$. Then, $\boldsymbol{x}_j = (\boldsymbol{x}_j^{(1)'}, \boldsymbol{x}_j^{(2)'})'(j = 1, 2, \ldots, N_2)$ represent $N_2$ independent random sample vectors distributed as a $2p$-dimensional multivariate normal population with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma$, where

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix},$$

respectively. The sample mean vector and sample covariance matrix are defined as

$$\begin{aligned}
\overline{\boldsymbol{x}}_1^{(1)} &= \frac{1}{N_2}\sum_{j=1}^{N_2} x_j^{(1)}, \\
\overline{\boldsymbol{x}}_2^{(1)} &= \frac{1}{N_1 - N_2}\sum_{j=N_2+1}^{N_1} x_j^{(1)}, \\
\overline{\boldsymbol{x}}^{(2)} &= \frac{1}{N_2}\sum_{j=1}^{N_2} x_j^{(2)}, \\
\overline{\boldsymbol{x}} &= \frac{1}{N_2}\sum_{j=1}^{N_2} \boldsymbol{x}_j, \\
S^{(1)} &= \frac{1}{N_2 - 1}\sum_{j=1}^{N_2}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})', \\
S^{(2)} &= \frac{1}{N_1 - N_2 - 1}\sum_{j=N_2+1}^{N_1}(\boldsymbol{x}_j^{(1)} - \overline{\boldsymbol{x}}_2^{(1)})(\boldsymbol{x}_j^{(1)} - \overline{\boldsymbol{x}}_2^{(1)})',
\end{aligned}$$

respectively.

### 3.1 $T^2$ type test statistic and its upper percentile

In order to test the equality of mean vectors, we set a hypothesis as follows:

$$H_{20} : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} \quad \text{vs.} \quad H_{21} : \boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)}.$$

The test statistic for testing the hypothesis $H_{20}$ is given by

$$T^{*2}_{\rm pd} = \boldsymbol{y}'\{\widehat{\rm Cov}(\boldsymbol{y})\}^{-1}\boldsymbol{y}, \tag{1}$$

where

$$\begin{aligned}
\boldsymbol{y} &= \frac{1}{N_1}\left\{N_2\overline{\boldsymbol{x}}_1^{(1)} + (N_1 - N_2)\overline{\boldsymbol{x}}_2^{(1)}\right\} - \overline{\boldsymbol{x}}^{(2)}, \\
{\rm E}\,(\boldsymbol{y}) &= \frac{1}{N_1}\left\{N_2\boldsymbol{\mu}^{(1)} + (N_1 - N_2)\boldsymbol{\mu}^{(1)}\right\} - \boldsymbol{\mu}^{(2)} = \boldsymbol{\mu}^{(1)} - \boldsymbol{\mu}^{(2)}, \\
{\rm Cov}\,(\boldsymbol{y}) &= \frac{1}{N_1}\Sigma_{11} - \frac{1}{N_1}\Sigma_{12} - \frac{1}{N_1}\Sigma_{21} + \frac{1}{N_2}\Sigma_{22}.
\end{aligned}$$

Hence, the MLE for covariance matrix of $\boldsymbol{y}$ is obtained using the method proposed by Kanda and Fujikoshi (1998) as

$$\widehat{\rm Cov}(\boldsymbol{y}) = \frac{1}{N_1}\hat{\Sigma}_{11} - \frac{1}{N_1}\hat{\Sigma}_{12} - \frac{1}{N_1}\hat{\Sigma}_{21} + \frac{1}{N_2}\hat{\Sigma}_{22},$$

where

$$\begin{aligned}
\hat{\Sigma}_{11} &= \frac{1}{N_1}(W_{11}^{(1)} + W^{(2)}), \\
\hat{\Sigma}_{12} &= \hat{\Sigma}_{11}(W_{11}^{(1)})^{-1}W_{12}^{(1)}, \\
\hat{\Sigma}_{22} &= \frac{1}{N_2}W_{22\cdot1}^{(1)} + \hat{\Sigma}_{21}\hat{\Sigma}_{11}^{-1}\hat{\Sigma}_{12}, \\
W^{(1)} &= (N_2 - 1)S^{(1)} = \begin{pmatrix} W_{11}^{(1)} & W_{12}^{(1)} \\ W_{21}^{(1)} & W_{22}^{(1)} \end{pmatrix}, \\
W^{(2)} &= (N_1 - N_2 - 1)S^{(2)} + \frac{N_2(N_1 - N_2)}{N_1}(\overline{\boldsymbol{x}}_1^{(1)} - \overline{\boldsymbol{x}}^{(2)})(\overline{\boldsymbol{x}}_1^{(1)} - \overline{\boldsymbol{x}}^{(2)})', \\
W_{22\cdot1}^{(1)} &= W_{22}^{(1)} - W_{21}^{(1)}(W_{11}^{(1)})^{-1}W_{12}^{(1)}.
\end{aligned}$$

We consider that the upper percentile of $T^{*2}_{\rm pd}$ lies between the upper percentiles of $2p$-dimensional data of $N_1$ and $N_2$. Therefore, we propose a formula to calculate the approximate upper percentile of $T^{*2}_{\rm pd}$ given by

$$t^{*2}(\alpha) = \frac{1}{2}\left(t^2_{{\rm pd},N_2}(\alpha) + t^2_{{\rm pd},N_1}(\alpha)\right), \tag{2}$$

where $t^2_{{\rm pd},N_2}(\alpha)$ is the upper percentile of $T^2_{\rm pd}$ for $N = N_2$, and $t^2_{{\rm pd},N_1}(\alpha)$ is the upper percentile of $T^2_{\rm pd}$ for $N = N_1$.

# 4 Simulation study and conclusion

In order to evaluate the accuracy of the obtained approximation, Monte Carlo simulation for the upper percentile $t^{*2}(\alpha)$ as shown in (2) was implemented for varied parameters. Some of the results are listed in this paper. As a numerical experiment, we carried out 1,000,000 replications. The asymptotic distribution of $T^{*2}_{\rm pd}$ is $\chi^2$ distribution with $p$ degrees of freedom for $N_1 \to \infty$ and a fixed $N_2$. These values are listed in the Tables.

For considering the correlation between the two populations, we put the covariance matrix in two patterns, i. e., pattern 1:

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22}, \end{pmatrix},$$

where

$$\Sigma_{11} = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \cdots & \ddots & \vdots \\ \rho & \cdots & \rho & 1 \end{pmatrix}, \Sigma_{12} = \begin{pmatrix} \rho & \rho & \cdots & \rho \\ \rho & \rho & \cdots & \rho \\ \vdots & \cdots & \ddots & \vdots \\ \rho & \cdots & \rho & \rho \end{pmatrix}, \Sigma_{22} = \beta\Sigma_{11},$$

and pattern 2:

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{2p-1} \\ \rho & 1 & \rho & \cdots & \rho^{2p-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{2p-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \rho^{2p-1} & \rho^{2p-2} & \cdots & \rho & 1 \end{pmatrix}.$$

Tables 1-4 give the simulated and approximated values for the upper percentile of $T^{*2}_{\rm pd}$ as shown in (1) for the following parameters: $p = 2, 5, 10$; $\alpha = 0.05$; $N_1 : N_2 = 5 : 1$, $N_1 : N_2 = 2 : 1$, $N_1 : N_2 = 5 : 4$. Tables 1, 2, 3, and 4 illustrate the cases $\Sigma$: pattern 1, $\rho = 0$, and $\beta = 1$; $\Sigma$: pattern 1, $\rho = 0.5$, and $\beta = 1$; $\Sigma$: pattern 1, $\rho = 0.5$, and $\beta = 2$; and $\Sigma$: pattern 2, $\rho = 0.5$, respectively. $t$ and $\chi^2$ represent upper percentiles of the distributions for $T^{*2}_{\rm pd}$ and $\chi^2_p$, respectively. $t^{*2}$ represents the approximate upper percentile $t^{*2}_{\rm pd}(\alpha)$. "type I error ($\chi^2$)" and "type I error ($t^{*2}$)" represent the actual type I error for $\chi^2$ and $t^{*2}$, respectively.

Tables 1-4 show that the simulated value $t^2$ is closer to the upper percentile of the $\chi^2$ distribution with $p$ degrees of freedom and the approximate accuracy increases as $N_1$ and $N_2$ increase. Further, the upper percentile $t^{*2}$ is better than the upper percentile of $\chi^2$ distribution with $p$ degrees of freedom when $N_1$ and $N_2$ are small, and the value of $t^{*2}$ rapidly converges with the simulated value $t$. When $p$ is small, the approximate accuracy is better than when $p$ is large. When $p = 2$, there is not much difference in the unbalance of the sample size. However, approximate accuracy is affected by the unbalance of the sample size for a large $p$. We also simulated a case where $\rho = 0.3$ and $\rho = 0.8$; however, the approximate accuracy is not affected to a great extent by the difference in correlation $\rho$.

In conclusion, the upper percentile $t^{*2}(\alpha)$ for the test statistic $T^{*2}_{\rm pd}$ proposed in this paper is a good approximation even for a small sample size, and it is useful for the test for equality of mean vectors with unequal sample sizes for populations with correlation.

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Table 1: $\Sigma$: pattern 1, $\rho = 0$ and $\beta = 1$

| $p = 2$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
|---|---|---|---|---|---|---|---|
| | 40 | 8 | 9.12 | 5.99 | 9.33 | 0.117 | 0.047 |
| | 80 | 16 | 7.45 | 5.99 | 7.16 | 0.085 | 0.056 |
| | 160 | 32 | 6.70 | 5.99 | 6.50 | 0.068 | 0.054 |
| | 320 | 64 | 6.32 | 5.99 | 6.23 | 0.058 | 0.052 |
| | 20 | 10 | 8.19 | 5.99 | 8.77 | 0.101 | 0.042 |
| | 40 | 20 | 6.99 | 5.99 | 7.08 | 0.074 | 0.048 |
| | 80 | 40 | 6.46 | 5.99 | 6.48 | 0.062 | 0.050 |
| | 160 | 80 | 6.22 | 5.99 | 6.23 | 0.056 | 0.050 |
| | 320 | 160 | 6.10 | 5.99 | 6.11 | 0.053 | 0.050 |
| | 20 | 16 | 7.86 | 5.99 | 7.76 | 0.095 | 0.052 |
| | 40 | 32 | 6.81 | 5.99 | 6.76 | 0.070 | 0.051 |
| | 80 | 64 | 6.38 | 5.99 | 6.35 | 0.060 | 0.051 |
| | 160 | 128 | 6.18 | 5.99 | 6.17 | 0.055 | 0.050 |
| | 320 | 256 | 6.08 | 5.99 | 6.08 | 0.052 | 0.050 |
| $p = 5$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 40 | 8 | 23.43 | 11.07 | 59.50 | 0.247 | 0.001 |
| | 80 | 16 | 15.88 | 11.07 | 17.08 | 0.142 | 0.039 |
| | 160 | 32 | 13.36 | 11.07 | 13.21 | 0.095 | 0.052 |
| | 320 | 64 | 12.17 | 11.07 | 12.01 | 0.072 | 0.053 |
| | 20 | 10 | 19.59 | 11.07 | 31.91 | 0.201 | 0.009 |
| | 40 | 20 | 14.57 | 11.07 | 16.11 | 0.118 | 0.034 |
| | 80 | 40 | 12.66 | 11.07 | 13.08 | 0.082 | 0.044 |
| | 160 | 80 | 11.81 | 11.07 | 11.98 | 0.065 | 0.047 |
| | 320 | 160 | 11.42 | 11.07 | 11.51 | 0.057 | 0.048 |
| | 20 | 16 | 18.91 | 11.07 | 20.11 | 0.198 | 0.041 |
| | 40 | 32 | 14.08 | 11.07 | 14.31 | 0.111 | 0.047 |
| | 80 | 64 | 12.37 | 11.07 | 12.48 | 0.076 | 0.048 |
| | 160 | 128 | 11.68 | 11.07 | 11.73 | 0.062 | 0.049 |
| | 320 | 256 | 11.34 | 11.07 | 11.39 | 0.055 | 0.049 |
| $p = 10$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 80 | 16 | 34.59 | 18.31 | 61.86 | 0.301 | 0.003 |
| | 160 | 32 | 24.66 | 18.31 | 26.22 | 0.159 | 0.037 |
| | 320 | 64 | 21.35 | 18.31 | 21.31 | 0.102 | 0.051 |
| | 40 | 20 | 29.87 | 18.31 | 42.36 | 0.243 | 0.009 |
| | 80 | 40 | 23.04 | 18.31 | 25.18 | 0.132 | 0.032 |
| | 160 | 80 | 20.45 | 18.31 | 21.15 | 0.086 | 0.042 |
| | 320 | 160 | 19.33 | 18.31 | 19.62 | 0.067 | 0.046 |
| | 20 | 16 | 52.30 | 18.31 | 79.04 | 0.511 | 0.010 |
| | 40 | 32 | 28.46 | 18.31 | 30.25 | 0.231 | 0.038 |
| | 80 | 64 | 22.30 | 18.31 | 22.84 | 0.120 | 0.044 |
| | 160 | 128 | 20.05 | 18.31 | 20.33 | 0.079 | 0.046 |
| | 320 | 256 | 19.14 | 18.31 | 19.27 | 0.064 | 0.048 |

Table 2: $\Sigma$: pattern 1, $\rho = 0.5$ and $\beta = 1$

| $p = 2$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
|---|---|---|---|---|---|---|---|
| | 40 | 8 | 9.31 | 5.99 | 9.33 | 0.122 | 0.050 |
| | 80 | 16 | 7.50 | 5.99 | 7.16 | 0.086 | 0.056 |
| | 160 | 32 | 6.70 | 5.99 | 6.50 | 0.067 | 0.054 |
| | 320 | 64 | 6.32 | 5.99 | 6.23 | 0.058 | 0.052 |
| | 20 | 10 | 8.31 | 5.99 | 8.77 | 0.104 | 0.044 |
| | 40 | 20 | 7.05 | 5.99 | 7.08 | 0.076 | 0.049 |
| | 80 | 40 | 6.51 | 5.99 | 6.48 | 0.063 | 0.051 |
| | 160 | 80 | 6.24 | 5.99 | 6.23 | 0.056 | 0.050 |
| | 320 | 160 | 6.10 | 5.99 | 6.11 | 0.053 | 0.050 |
| | 20 | 16 | 7.88 | 5.99 | 7.76 | 0.096 | 0.052 |
| | 40 | 32 | 6.85 | 5.99 | 6.76 | 0.071 | 0.052 |
| | 80 | 64 | 6.39 | 5.99 | 6.35 | 0.060 | 0.051 |
| | 160 | 128 | 6.19 | 5.99 | 6.17 | 0.055 | 0.050 |
| | 320 | 256 | 6.08 | 5.99 | 6.08 | 0.052 | 0.050 |
| $p = 5$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 40 | 8 | 25.35 | 11.07 | 59.50 | 0.274 | 0.002 |
| | 80 | 16 | 16.55 | 11.07 | 17.08 | 0.155 | 0.045 |
| | 160 | 32 | 13.53 | 11.07 | 13.21 | 0.099 | 0.055 |
| | 320 | 64 | 12.22 | 11.07 | 12.01 | 0.073 | 0.054 |
| | 20 | 10 | 20.75 | 11.07 | 31.91 | 0.222 | 0.011 |
| | 40 | 20 | 14.99 | 11.07 | 16.11 | 0.129 | 0.038 |
| | 80 | 40 | 12.86 | 11.07 | 13.08 | 0.086 | 0.047 |
| | 160 | 80 | 11.93 | 11.07 | 11.98 | 0.067 | 0.049 |
| | 320 | 160 | 11.49 | 11.07 | 11.51 | 0.058 | 0.050 |
| | 20 | 16 | 18.87 | 11.07 | 20.11 | 0.201 | 0.041 |
| | 40 | 32 | 14.20 | 11.07 | 14.31 | 0.114 | 0.049 |
| | 80 | 64 | 12.46 | 11.07 | 12.48 | 0.078 | 0.050 |
| | 160 | 128 | 11.72 | 11.07 | 11.73 | 0.063 | 0.050 |
| | 320 | 256 | 11.37 | 11.07 | 11.39 | 0.056 | 0.050 |
| $p = 10$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 80 | 16 | 37.00 | 18.31 | 61.86 | 0.340 | 0.004 |
| | 160 | 32 | 25.58 | 18.31 | 26.22 | 0.177 | 0.045 |
| | 320 | 64 | 21.65 | 18.31 | 21.31 | 0.107 | 0.054 |
| | 40 | 20 | 31.37 | 18.31 | 42.36 | 0.274 | 0.012 |
| | 80 | 40 | 23.77 | 18.31 | 25.18 | 0.147 | 0.037 |
| | 160 | 80 | 20.78 | 18.31 | 21.15 | 0.092 | 0.046 |
| | 320 | 160 | 19.51 | 18.31 | 19.62 | 0.070 | 0.049 |
| | 20 | 16 | 52.42 | 18.31 | 79.04 | 0.517 | 0.011 |
| | 40 | 32 | 28.59 | 18.31 | 30.25 | 0.237 | 0.039 |
| | 80 | 64 | 22.50 | 18.31 | 22.84 | 0.124 | 0.046 |
| | 160 | 128 | 20.19 | 18.31 | 20.33 | 0.082 | 0.048 |
| | 320 | 256 | 19.20 | 18.31 | 19.27 | 0.065 | 0.049 |

Table 3: $\Sigma$: pattern 1, $\rho = 0.5$ and $\beta = 2$

| $p=2$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
|---|---|---|---|---|---|---|---|
| | 40 | 8 | 10.14 | 5.99 | 9.33 | 0.134 | 0.060 |
| | 80 | 16 | 7.78 | 5.99 | 7.16 | 0.092 | 0.061 |
| | 160 | 32 | 6.82 | 5.99 | 6.50 | 0.070 | 0.057 |
| | 320 | 64 | 6.38 | 5.99 | 6.23 | 0.060 | 0.053 |
| | 20 | 10 | 8.77 | 5.99 | 8.77 | 0.113 | 0.050 |
| | 40 | 20 | 7.23 | 5.99 | 7.08 | 0.080 | 0.053 |
| | 80 | 40 | 6.61 | 5.99 | 6.48 | 0.065 | 0.053 |
| | 160 | 80 | 6.28 | 5.99 | 6.23 | 0.057 | 0.051 |
| | 320 | 160 | 6.12 | 5.99 | 6.11 | 0.053 | 0.050 |
| | 20 | 16 | 8.06 | 5.99 | 7.76 | 0.099 | 0.055 |
| | 40 | 32 | 6.91 | 5.99 | 6.76 | 0.073 | 0.053 |
| | 80 | 64 | 6.43 | 5.99 | 6.35 | 0.061 | 0.052 |
| | 160 | 128 | 6.19 | 5.99 | 6.17 | 0.055 | 0.051 |
| | 320 | 256 | 6.09 | 5.99 | 6.08 | 0.053 | 0.050 |
| $p=5$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 40 | 8 | 33.27 | 11.07 | 59.50 | 0.346 | 0.009 |
| | 80 | 16 | 18.03 | 11.07 | 17.08 | 0.177 | 0.059 |
| | 160 | 32 | 14.00 | 11.07 | 13.21 | 0.108 | 0.061 |
| | 320 | 64 | 12.41 | 11.07 | 12.01 | 0.077 | 0.057 |
| | 20 | 10 | 23.99 | 11.07 | 31.91 | 0.265 | 0.021 |
| | 40 | 20 | 15.85 | 11.07 | 16.11 | 0.145 | 0.047 |
| | 80 | 40 | 13.21 | 11.07 | 13.08 | 0.093 | 0.052 |
| | 160 | 80 | 12.08 | 11.07 | 11.98 | 0.070 | 0.052 |
| | 320 | 160 | 11.55 | 11.07 | 11.51 | 0.060 | 0.051 |
| | 20 | 16 | 19.94 | 11.07 | 20.11 | 0.217 | 0.049 |
| | 40 | 32 | 14.54 | 11.07 | 14.31 | 0.120 | 0.053 |
| | 80 | 64 | 12.60 | 11.07 | 12.48 | 0.081 | 0.052 |
| | 160 | 128 | 11.79 | 11.07 | 11.73 | 0.064 | 0.051 |
| | 320 | 256 | 11.41 | 11.07 | 11.39 | 0.057 | 0.050 |
| $p=10$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 80 | 16 | 46.55 | 18.31 | 61.86 | 0.435 | 0.017 |
| | 160 | 32 | 27.44 | 18.31 | 26.22 | 0.208 | 0.061 |
| | 320 | 64 | 22.27 | 18.31 | 21.31 | 0.119 | 0.062 |
| | 40 | 20 | 35.52 | 18.31 | 42.36 | 0.332 | 0.024 |
| | 80 | 40 | 24.92 | 18.31 | 25.18 | 0.168 | 0.048 |
| | 160 | 80 | 21.25 | 18.31 | 21.15 | 0.101 | 0.051 |
| | 320 | 160 | 19.73 | 18.31 | 19.62 | 0.074 | 0.052 |
| | 20 | 16 | 59.46 | 18.31 | 79.04 | 0.569 | 0.019 |
| | 40 | 32 | 29.87 | 18.31 | 30.25 | 0.258 | 0.047 |
| | 80 | 64 | 22.96 | 18.31 | 22.84 | 0.133 | 0.051 |
| | 160 | 128 | 20.41 | 18.31 | 20.33 | 0.085 | 0.051 |
| | 320 | 256 | 19.32 | 18.31 | 19.27 | 0.066 | 0.051 |

Table 4: $\Sigma$: pattern 2, $\rho = 0.5$

| $p = 2$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
|---|---|---|---|---|---|---|---|
| | 40 | 8 | 9.31 | 5.99 | 9.33 | 0.122 | 0.050 |
| | 80 | 16 | 7.51 | 5.99 | 7.16 | 0.086 | 0.057 |
| | 160 | 32 | 6.71 | 5.99 | 6.50 | 0.068 | 0.055 |
| | 320 | 64 | 6.32 | 5.99 | 6.23 | 0.058 | 0.052 |
| | 20 | 10 | 8.43 | 5.99 | 8.77 | 0.107 | 0.045 |
| | 40 | 20 | 7.09 | 5.99 | 7.08 | 0.077 | 0.050 |
| | 80 | 40 | 6.51 | 5.99 | 6.48 | 0.063 | 0.051 |
| | 160 | 80 | 6.24 | 5.99 | 6.23 | 0.056 | 0.050 |
| | 320 | 160 | 6.12 | 5.99 | 6.11 | 0.053 | 0.050 |
| | 20 | 16 | 8.00 | 5.99 | 7.76 | 0.098 | 0.054 |
| | 40 | 32 | 6.88 | 5.99 | 6.76 | 0.072 | 0.053 |
| | 80 | 64 | 6.42 | 5.99 | 6.35 | 0.061 | 0.052 |
| | 160 | 128 | 6.19 | 5.99 | 6.17 | 0.055 | 0.051 |
| | 320 | 256 | 6.08 | 5.99 | 6.08 | 0.052 | 0.050 |
| $p = 5$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 40 | 8 | 24.84 | 11.07 | 59.50 | 0.268 | 0.002 |
| | 80 | 16 | 16.38 | 11.07 | 17.08 | 0.151 | 0.043 |
| | 160 | 32 | 13.51 | 11.07 | 13.21 | 0.099 | 0.054 |
| | 320 | 64 | 12.23 | 11.07 | 12.01 | 0.073 | 0.054 |
| | 20 | 10 | 20.89 | 11.07 | 31.91 | 0.225 | 0.012 |
| | 40 | 20 | 15.07 | 11.07 | 16.11 | 0.130 | 0.039 |
| | 80 | 40 | 12.87 | 11.07 | 13.08 | 0.086 | 0.047 |
| | 160 | 80 | 11.94 | 11.07 | 11.98 | 0.067 | 0.049 |
| | 320 | 160 | 11.47 | 11.07 | 11.51 | 0.058 | 0.049 |
| | 20 | 16 | 19.51 | 11.07 | 20.11 | 0.211 | 0.045 |
| | 40 | 32 | 14.35 | 11.07 | 14.31 | 0.117 | 0.051 |
| | 80 | 64 | 12.50 | 11.07 | 12.48 | 0.079 | 0.050 |
| | 160 | 128 | 11.75 | 11.07 | 11.73 | 0.063 | 0.050 |
| | 320 | 256 | 11.38 | 11.07 | 11.39 | 0.056 | 0.050 |
| $p = 10$ | $N_1$ | $N_2$ | $t^2$ | $\chi^2$ | $t^{*2}$ | type I error ($\chi^2$) | type I error ($t^{*2}$) |
| | 80 | 16 | 36.11 | 18.31 | 61.86 | 0.327 | 0.004 |
| | 160 | 32 | 25.30 | 18.31 | 26.22 | 0.171 | 0.042 |
| | 320 | 64 | 21.58 | 18.31 | 21.31 | 0.106 | 0.053 |
| | 40 | 20 | 31.12 | 18.31 | 42.36 | 0.268 | 0.012 |
| | 80 | 40 | 23.64 | 18.31 | 25.18 | 0.144 | 0.036 |
| | 160 | 80 | 20.75 | 18.31 | 21.15 | 0.092 | 0.045 |
| | 320 | 160 | 19.50 | 18.31 | 19.62 | 0.070 | 0.048 |
| | 20 | 16 | 54.97 | 18.31 | 79.04 | 0.537 | 0.013 |
| | 40 | 32 | 29.09 | 18.31 | 30.25 | 0.244 | 0.042 |
| | 80 | 64 | 22.61 | 18.31 | 22.84 | 0.127 | 0.048 |
| | 160 | 128 | 20.24 | 18.31 | 20.33 | 0.082 | 0.049 |
| | 320 | 256 | 19.24 | 18.31 | 19.27 | 0.065 | 0.050 |