A likelihood ratio test for subvector of mean vector with two-step monotone missing data

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Abstract. In this paper, we consider the one-sample problem of testing for the subvector of a mean vector with two-step monotone missing data. When the data set consists of complete data with $p(= p_1 + p_2 + p_3)$ dimensions and incomplete data with $(p_1 + p_2)$ dimensions, we derive the likelihood ratio criterion for testing the $(p_2 + p_3)$ mean vector under the given mean vector of p_1 dimensions. Further, we propose an approximation to the upper percentile of the likelihood ratio test (LRT) statistic. The accuracy and asymptotic behavior of the approximation are investigated by Monte Carlo simulation. An example is given to illustrate the method.

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§1. Introduction

In statistical data analysis, it is important to consider about missing observations. A statistical analysis for such monotone missing data has been discussed by many authors. For example, Anderson (1957) showed an approach to derive the maximum likelihood estimators (MLEs) of the mean vector and covariance matrix by the likelihood equations for monotone missing data. Kanda and Fujikoshi (1998) gave the properties of MLEs based on two-step and three-step monotone missing samples and a general k. We note, among many other papers, Krishnamoorthy and Pannala (1999), Yu, Krishnamoorthy, and Pannala (2006) and Chang and Richards (2009), that the methods of testing mean vectors with monotone missing data were proposed. In particular, for testing the mean vector with two-step monotone missing data, Seko, Yamazaki, and

Seo (2012), and Seko, Kawasaki, and Seo (2011) provide an accurate simple approach to give the approximate upper percentiles of Hotelling's T^2 type statistic and LRT statistic for one-sample and two-sample problems. Moreover, various statistical methods developed to analyze data with nonmonotone missing values have been studied by Srivastava (1985), Srivastava and Carter (1986), and S hutoh, Kusumi, Morinaga, Yamada, and Seo (2010), among others. In this paper, we consider the one-sample problem of testing for the mean vector with two-step monotone missing data. In particular, the test for the subvector of a mean vector is discussed.

We first describe the case of nonmissing data. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be distributed as $N_p(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$ and Σ are unknown. Let $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_{(23)}')'$, where $\boldsymbol{\mu}_1 = (\mu_1, \mu_2, \ldots, \mu_{p_1})'$ and $\boldsymbol{\mu}_{(23)} = (\mu_{p_1+1}, \mu_{p_1+2}, \ldots, \mu_p)'$, $p_1 < p < n$. Then, the sample mean vector and unbiased covariance matrix are defined as

$$\overline{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i = \begin{pmatrix} \overline{\boldsymbol{x}}_1 \\ \overline{\boldsymbol{x}}_{(23)} \end{pmatrix}, S = \frac{1}{n-1}\sum_{i=1}^{n} (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})' = \begin{pmatrix} S_{11} & S_{1(23)} \\ S_{(23)1} & S_{(23)(23)} \end{pmatrix},$$

respectively, where $\overline{\boldsymbol{x}}_1$ is a p_1-vector and S_{11} is a $p_1 \times p_1$ matrix. Then, we consider the following hypothesis:

(1.1)

$$H_0 : \boldsymbol{\mu}_{(23)} = \boldsymbol{\mu}_{(23)0} \text{ given } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{10} \text{ vs. } H_1 : \boldsymbol{\mu}_{(23)} \neq \boldsymbol{\mu}_{(23)0} \text{ given } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{10},$$

where $\boldsymbol{\mu}_{(23)0}$ and $\boldsymbol{\mu}_{10}$ are known. The equivalent criterion to the likelihood ratio can be written as

$$U = \frac{T_p^2 - T_{p_1}^2}{n - 1 + T_{p_1}^2},$$

where $T_p^2 = n(\overline{\boldsymbol{x}} - \boldsymbol{\mu}_0)'S^{-1}(\overline{\boldsymbol{x}} - \boldsymbol{\mu}_0)$, and $T_{p_1}^2 = n(\overline{\boldsymbol{x}}_1 - \boldsymbol{\mu}_{10})'S_{11}^{-1}(\overline{\boldsymbol{x}}_1 - \boldsymbol{\mu}_{10})$. We note that $U = \lambda^{-2/n} - 1$, where λ is the likelihood ratio criterion. Under H_0, it follows that $(n-p)U/(p_2+p_3)$ is distributed as an F distribution with $p_2 + p_3$ and $n - p$ degrees of freedom. This result follows from the one in Siotani, Hayakawa, and Fujikoshi (1985, p. 215). The criterion is called Rao's U statistic (See, Rao (1949) and Giri (1964)). In this paper, we consider this problem for the case of two-step monotone missing data. We first derive the

MLEs of $\boldsymbol{\mu}$ and Σ and the MLE of Σ under H_0. Using these MLEs, we propose the likelihood ratio test statistic and its approximate upper percentile.

In the following section, we describe the definition and some notations for two-step monotone missing data. Then, we describe the definition for two-step monotone missing data and derive the MLEs. In Section 3, we propose the LRT statistic and its approximate upper 100α percentiles. The accuracy of the approximate upper percentiles of the test statistic is investigated by Monte Carlo simulation in Section 4. Section 5 gives a numerical example to illustrate the method using the approximate upper percentiles of the test statistic.

§2. Two-step monotone missing data

Let the data set $\{x_{i,j}\}$ be of the form

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,p_1} & x_{1,p_1+1} & \cdots & x_{1,p_1+p_2} & x_{1,p_1+p_2+1} & \cdots & x_{1,p} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n_1,1} & \cdots & x_{n_1,p_1} & x_{n_1,p_1+1} & \cdots & x_{n_1,p_1+p_2} & x_{n_1,p_1+p_2+1} & \cdots & x_{n_1,p} \\ x_{n_1+1,1} & \cdots & x_{n_1+1,p_1} & x_{n_1+1,p_1+1} & \cdots & x_{n_1+1,p_1+p_2} & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n,1} & \cdots & x_{n,p_1} & x_{n,p_1+1} & \cdots & x_{n,p_1+p_2} & * & \cdots & * \end{pmatrix},$$

where $n_2 = n - n_1$, $p = p_1 + p_2 + p_3$, and $n_1 > p$. "$*$" indicates missing data. That is, we have complete data for n_1 mutually independent observations with p dimensions and incomplete data for n_2 mutually independent observations with $(p_1 + p_2)$ dimensions. Such a data set is called two-step monotone missing data. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n_1}$ be distributed as $N_p(\boldsymbol{\mu}, \Sigma)$ and let $\boldsymbol{x}_{n_1+1}, \boldsymbol{x}_{n_1+2}, \ldots, \boldsymbol{x}_n$ be distributed as $N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)})$, where each $\boldsymbol{x}_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,p})'$, $j = 1, 2, \ldots, n_1$ is $p \times 1$, each $\boldsymbol{x}_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,p_1+p_2})'$, $j = n_1 + 1, n_1 + 2, \ldots, n$ is $(p_1 + p_2) \times 1$, and

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \boldsymbol{\mu}_3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix} = \begin{pmatrix} \Sigma_{(12)(12)} & \Sigma_{(12)3} \\ \Sigma_{3(12)} & \Sigma_{33} \end{pmatrix}.$$

We partition $\boldsymbol{x}_j$ into a $p_1 \times 1$ random vector, a $p_2 \times 1$ random vector, and a $p_3 \times 1$ random vector as $\boldsymbol{x}_j = (\boldsymbol{x}'_{1j}, \boldsymbol{x}'_{2j}, \boldsymbol{x}'_{3j})' = (\boldsymbol{x}'_{(12)j}, \boldsymbol{x}'_{3j})'$, where $\boldsymbol{x}_{ij} : p_i \times 1, i = 1,2,3, j = 1,2,\ldots,n_1$. In addition, $\boldsymbol{x}_{(12)j}$ is partitioned into a $p_1 \times 1$ random vector and a $p_2 \times 1$ random vector as $\boldsymbol{x}_{(12)j}=(\boldsymbol{x}'_{1j}, \boldsymbol{x}'_{2j})'$, where $\boldsymbol{x}_{ij} : p_i \times 1, i = 1,2, j = n_1+1, n_1+2,\ldots,n$. Then the joint density function of the observed data set $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{n_1}, \boldsymbol{x}_{(12)n_1+1}, \boldsymbol{x}_{(12)n_1+2}, \ldots, \boldsymbol{x}_{(12)n}$ can be written as

$$\prod_{j=1}^{n_1} f(\boldsymbol{x}_j, \boldsymbol{\mu}, \Sigma) \times \prod_{j=n_1+1}^{n} f(\boldsymbol{x}_{(12)j}, \boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)}),$$

where $f(\boldsymbol{x}_j, \boldsymbol{\mu}, \Sigma)$ and $f(\boldsymbol{x}_{(12)j}, \boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)})$ are the density functions of $N_p(\boldsymbol{\mu}, \Sigma)$ and $N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)})$, respectively. Then, the likelihood function is

$$L(\boldsymbol{\mu}, \Sigma) = \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}_j - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}_j - \boldsymbol{\mu})\right\}$$
$$\times \prod_{j=n_1+1}^{n} \frac{1}{(2\pi)^{(p_1+p_2)/2}|\Sigma_{(12)(12)}|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)})'\Sigma^{-1}_{(12)(12)}(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)})\right\}.$$

The sample mean vectors are defined as

$$\overline{\boldsymbol{x}}_{1T} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{x}_{1j},\ \overline{\boldsymbol{x}}_{2T} = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{x}_{2j},\ \overline{\boldsymbol{x}}_{(12)F} = \frac{1}{n_1}\sum_{j=1}^{n_1} \boldsymbol{x}_{(12)j},\ \overline{\boldsymbol{x}}_{3F} = \frac{1}{n_1}\sum_{j=1}^{n_1} \boldsymbol{x}_{3j}.$$

In order to obtain the MLEs, we use the decomposition of the density into conditional densities, which is called the conditional method (Kanda and Fujikoshi, 1998). In our situation, multiplying the observation vectors $\boldsymbol{x}_j$ by the transformation matrix

$$\Gamma = \left(\begin{array}{cc|c} I_{p_1} & O & \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} & O \\ \hline \multicolumn{2}{c|}{O} & I_{p_3} \end{array}\right)\Gamma_1$$

on the left side, the transformed observation vectors are

$$\begin{aligned} &\boldsymbol{x}_{1j} \sim N_{p_1}(\boldsymbol{\eta}_1, \Psi_{11}), \quad j = 1,2,\ldots,n, \\ &\boldsymbol{x}_{2j} - \Psi_{21}\boldsymbol{x}_{1j} \sim N_{p_2}(\boldsymbol{\eta}_2, \Psi_{22}), \quad j = 1,2,\ldots,n, \\ &\boldsymbol{x}_{3j} - \Psi_{3(12)}\boldsymbol{x}_{(12)j} \sim N_{p_3}(\boldsymbol{\eta}_3, \Psi_{33}), \quad j = 1,2,\ldots,n_1, \end{aligned}$$

where

$$\Gamma_1 = \left(\begin{array}{cc|c} I_{p1} & O & \multirow{2}{*}{O} \\ O & I_{p2} & \\ \hline \multicolumn{2}{c|}{-\Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}} & I_{p3} \end{array}\right),$$

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\eta}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_2 - \Sigma_{21}\Sigma_{11}^{-1}\boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_3 - \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}\boldsymbol{\mu}_{(12)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_2 - \Psi_{21}\boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_3 - \Psi_{3(12)}\boldsymbol{\mu}_{(12)} \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \Psi_{(12)(12)} & \Psi_{(12)3} \\ \Psi_{3(12)} & \Psi_{33} \end{pmatrix} = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} \end{pmatrix} = \left(\begin{array}{cc|c} \Sigma_{11} & \Sigma_{11}^{-1}\Sigma_{12} & \multirow{2}{*}{$\Sigma_{(12)(12)}^{-1}\Sigma_{(12)3}$} \\ \Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22\cdot 1} & \\ \hline \multicolumn{2}{c|}{\Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}} & \Sigma_{33\cdot(12)} \end{array}\right),$$

$\Sigma_{22\cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}$, $\Sigma_{33\cdot(12)} = \Sigma_{33} - \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}\Sigma_{(12)3}$. It should be noted that $\boldsymbol{x}_{1j}$, $\boldsymbol{x}_{2j} - \Psi_{21}\boldsymbol{x}_{1j}$, and $\boldsymbol{x}_{3j} - \Psi_{3(12)}\boldsymbol{x}_{(12)j}$ are independent. Because $(\boldsymbol{\eta}, \Psi)$ has a one-to-one correspondence to $(\boldsymbol{\mu}, \Sigma)$, we derive the MLEs of $(\boldsymbol{\eta}, \Psi)$ instead of $(\boldsymbol{\mu}, \Sigma)$. For the parameter $\boldsymbol{\eta}$ and Ψ, the likelihood function can be written as

$$\begin{aligned} L(\boldsymbol{\eta}, \Psi) = &\prod_{j=1}^{n} \frac{1}{(2\pi)^{p_1/2}|\Psi_{11}|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{1j} - \boldsymbol{\eta}_1)'\Psi_{11}^{-1}(\boldsymbol{x}_{1j} - \boldsymbol{\eta}_1)\right\} \\ &\times \prod_{j=1}^{n} \frac{1}{(2\pi)^{p_2/2}|\Psi_{22}|^{1/2}} \\ &\quad \times \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{2j} - \Psi_{21}\boldsymbol{x}_{1j} - \boldsymbol{\eta}_2)'\Psi_{22}^{-1}(\boldsymbol{x}_{2j} - \Psi_{21}\boldsymbol{x}_{1j} - \boldsymbol{\eta}_2)\right\} \\ &\times \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p_3/2}|\Psi_{33}|^{1/2}} \\ &\quad \times \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{3j} - \Psi_{3(12)}\boldsymbol{x}_{(12)j} - \boldsymbol{\eta}_3)'\Psi_{33}^{-1}(\boldsymbol{x}_{3j} - \Psi_{3(12)}\boldsymbol{x}_{(12)j} - \boldsymbol{\eta}_3)\right\}. \end{aligned}$$

The partial derivative of $\log L(\boldsymbol{\eta}, \Psi)$ with respect to Ψ_{11} is

$$\frac{\partial \log L(\boldsymbol{\eta}, \Psi)}{\partial \Psi_{11}} = -\frac{n}{2}\Psi_{11}^{-1} + \frac{1}{2}\sum_{j=1}^{n} \Psi_{11}^{-1}(\boldsymbol{x}_{1j} - \boldsymbol{\eta}_1)(\boldsymbol{x}_{1j} - \boldsymbol{\eta}_1)'\Psi_{11}^{-1}.$$

Solving the partial derivative of $\log L(\boldsymbol{\eta}, \Psi) = 0$, we obtain the MLE of Ψ_{11} as

$$\widehat{\Psi}_{11} = \frac{1}{n}\sum_{j=1}^{n}(\boldsymbol{x}_{1j} - \boldsymbol{\mu}_{10})(\boldsymbol{x}_{1j} - \boldsymbol{\mu}_{10})'.$$

Similarly, the partial derivative of $\log L(\boldsymbol{\eta},\Psi)$ with respect to $\boldsymbol{\eta}_2, \Psi_{21}, \Psi_{22}, \boldsymbol{\eta}_3$, $\Psi_{3(12)}$, and Ψ_{33} are

$$\frac{\partial \log L(\boldsymbol{\eta},\Psi)}{\partial \boldsymbol{\eta}_2} = \Psi_{22}^{-1}\sum_{j=1}^{n}(\boldsymbol{x}_{2j} - \Psi_{21}\boldsymbol{x}_{1j} - \boldsymbol{\eta}_2),$$

$$\frac{\partial \log L(\boldsymbol{\eta},\Psi)}{\partial \Psi_{21}} = \sum_{j=1}^{n}\Psi_{22}^{-1}\left\{(\boldsymbol{x}_{2j}-\overline{\boldsymbol{x}}_{2T})(\boldsymbol{x}_{1j}-\overline{\boldsymbol{x}}_{1T})' - \Psi_{21}(\boldsymbol{x}_{1j}-\overline{\boldsymbol{x}}_{1T})(\boldsymbol{x}_{1j}-\overline{\boldsymbol{x}}_{1T})'\right\},$$

$$\frac{\partial \log L(\boldsymbol{\eta},\Psi)}{\partial \Psi_{22}} = -\frac{n}{2}\Psi_{22}^{-1} + \frac{1}{2}\sum_{j=1}^{n}\Psi_{22}^{-1}(\boldsymbol{x}_{2j}-\Psi_{21}\boldsymbol{x}_{1j}-\boldsymbol{\eta}_2)(\boldsymbol{x}_{2j}-\Psi_{21}\boldsymbol{x}_{1j}-\boldsymbol{\eta}_2)'\Psi_{22}^{-1},$$

$$\frac{\partial \log L(\boldsymbol{\eta},\Psi)}{\partial \boldsymbol{\eta}_3} = \Psi_{33}^{-1}\sum_{j=1}^{n_1}(\boldsymbol{x}_{3j} - \Psi_{3(12)}\boldsymbol{x}_{(12)j} - \boldsymbol{\eta}_3),$$

$$\frac{\partial \log L(\boldsymbol{\eta},\Psi)}{\partial \Psi_{3(12)}} = \sum_{j=1}^{n_1}\Psi_{33}^{-1}\left\{(\boldsymbol{x}_{3j}-\overline{\boldsymbol{x}}_{3F})(\boldsymbol{x}_{3j}-\overline{\boldsymbol{x}}_{3F})' - \Psi_{3(12)}(\boldsymbol{x}_{(12)j}-\overline{\boldsymbol{x}}_{(12)F})(\boldsymbol{x}_{(12)j}-\overline{\boldsymbol{x}}_{(12)F})'\right\},$$

and

$$\frac{\partial \log L(\boldsymbol{\eta},\Psi)}{\partial \Psi_{33}} = -\frac{n_1}{2}\Psi_{33}^{-1} + \frac{1}{2}\sum_{j=1}^{n_1}\Psi_{33}^{-1}(\boldsymbol{x}_{3j} - \Psi_{3(12)}\boldsymbol{x}_{(12)j} - \boldsymbol{\eta}_3)(\boldsymbol{x}_{3j} - \Psi_{3(12)}\boldsymbol{x}_{(12)j} - \boldsymbol{\eta}_3)'\Psi_{33}^{-1}.$$

Therefore, we obtain the MLEs of $\boldsymbol{\eta}_2, \boldsymbol{\eta}_3, \Psi_{21}, \Psi_{22}, \Psi_{3(12)}$, and Ψ_{33}:

$$\widehat{\boldsymbol{\eta}}_2 = \overline{\boldsymbol{x}}_{2T} - \widehat{\Psi}_{21}\overline{\boldsymbol{x}}_{1T},\ \widehat{\boldsymbol{\eta}}_3 = \overline{\boldsymbol{x}}_{3F} - \widehat{\Psi}_{3(12)}\overline{\boldsymbol{x}}_{(12)F},$$

$$\widehat{\Psi}_{21} = \left\{\sum_{j=1}^{n}(\boldsymbol{x}_{2j}-\overline{\boldsymbol{x}}_{2T})(\boldsymbol{x}_{1j}-\overline{\boldsymbol{x}}_{1T})'\right\}\left\{\sum_{j=1}^{n}(\boldsymbol{x}_{1j}-\overline{\boldsymbol{x}}_{1T})(\boldsymbol{x}_{1j}-\overline{\boldsymbol{x}}_{1T})'\right\}^{-1},$$

$$\widehat{\Psi}_{22} = \frac{1}{n}\sum_{j=1}^{n}(\boldsymbol{x}_{2j} - \widehat{\Psi}_{21}\boldsymbol{x}_{1j} - \widehat{\boldsymbol{\eta}}_2)(\boldsymbol{x}_{2j} - \widehat{\Psi}_{21}\boldsymbol{x}_{1j} - \widehat{\boldsymbol{\eta}}_2)',$$

$$\widehat{\Psi}_{3(12)} = \left\{\sum_{j=1}^{n_1}(\boldsymbol{x}_{3j}-\overline{\boldsymbol{x}}_{3F})(\boldsymbol{x}_{(12)j}-\overline{\boldsymbol{x}}_{(12)F})'\right\} \times \left\{\sum_{j=1}^{n_1}(\boldsymbol{x}_{(12)j}-\overline{\boldsymbol{x}}_{(12)F})(\boldsymbol{x}_{(12)j}-\overline{\boldsymbol{x}}_{(12)F})'\right\}^{-1},$$

$$\widehat{\Psi}_{33} = \frac{1}{n}\sum_{j=1}^{n_1}(\boldsymbol{x}_{3j} - \widehat{\Psi}_{3(12)}\boldsymbol{x}_{(12)j} - \widehat{\boldsymbol{\eta}}_3)(\boldsymbol{x}_{3j} - \widehat{\Psi}_{3(12)}\boldsymbol{x}_{(12)j} - \widehat{\boldsymbol{\eta}}_3)'.$$

Next we derive the MLE under H_0 to obtain the LRT statistic. The null hypothesis in (1.1) can be written as $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \ (= (\boldsymbol{\mu}_{10}', \boldsymbol{\mu}_{20}', \boldsymbol{\mu}_{30}')' = (\boldsymbol{\mu}_{(12)0}', \boldsymbol{\mu}_{30}')')$. Let $\boldsymbol{x}_j = (\boldsymbol{x}_{(12)j}', \boldsymbol{x}_{3j}')'$ be distributed as $N_p(\boldsymbol{\mu}_0, \Sigma), j = 1, 2, \ldots, n_1$ and $\boldsymbol{x}_{(12)j}$ be distributed as $N_{p_1+p_2}(\boldsymbol{\mu}_{(12)0}, \Sigma_{(12)(12)}),\ j = n_1 + 1, n_1 + 2, \ldots, n$, then, the likelihood function is

$$
\begin{aligned}
L(\boldsymbol{\mu}_0, \Sigma) &= \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}_j - \boldsymbol{\mu}_0)'\Sigma^{-1}(\boldsymbol{x}_j - \boldsymbol{\mu}_0)\right\} \\
&= \prod_{j=n_1+1}^{n} \frac{1}{(2\pi)^{(p_1+p_2)/2}|\Sigma_{(12)(12)}|^{1/2}} \\
&\quad \times \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})'\Sigma_{(12)(12)}^{-1}(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})\right\}.
\end{aligned}
$$

Multiplying the observation vectors by Γ_1 in (2.1) on the left side, we have

$$
\begin{aligned}
&\boldsymbol{x}_{(12)j} \sim N_{p_1+p_2}(\boldsymbol{\xi}_{(12)}, \Phi_{(12)(12)}),\ j = 1, 2, \ldots, n, \\
&\boldsymbol{x}_{3j} - \Phi_{3(12)}\boldsymbol{x}_{(12)j} \sim N_{p_3}(\boldsymbol{\xi}_3, \Phi_{33}),\ j = 1, 2, \ldots, n_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{\xi} &= \begin{pmatrix} \boldsymbol{\xi}_{(12)} \\ \boldsymbol{\xi}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)0} \\ \boldsymbol{\mu}_{30} - \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}\boldsymbol{\mu}_{(12)0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)0} \\ \boldsymbol{\mu}_{30} - \Phi_{3(12)}\boldsymbol{\mu}_{(12)0} \end{pmatrix}, \\
\Phi &= \begin{pmatrix} \Phi_{(12)(12)} & \Phi_{(12)3} \\ \Phi_{3(12)} & \Phi_{33} \end{pmatrix} = \begin{pmatrix} \Sigma_{(12)(12)} & \Sigma_{(12)(12)}^{-1}\Sigma_{(12)3} \\ \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1} & \Sigma_{33\cdot(12)} \end{pmatrix},
\end{aligned}
$$

which have one-to-one correspondence with $\boldsymbol{\mu}_0$ and Σ. For the parameters $\boldsymbol{\xi}, \Phi$, the likelihood function can be written as

$$
\begin{aligned}
L(\boldsymbol{\xi}, \Phi) &= \prod_{j=1}^{n} \frac{1}{(2\pi)^{(p_1+p_2)/2}|\Phi_{(12)(12)}|^{1/2}} \\
&\quad \times \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{(12)j} - \boldsymbol{\xi}_{(12)})'\Phi_{(12)(12)}^{-1}(\boldsymbol{x}_{(12)j} - \boldsymbol{\xi}_{(12)})\right\} \\
&= \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p_3/2}|\Phi_{33}|^{1/2}} \\
&\quad \times \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{3j} - \Phi_{3(12)}\boldsymbol{x}_{(12)j} - \boldsymbol{\xi}_3)'\Phi_{33}^{-1}(\boldsymbol{x}_{3j} - \Phi_{3(12)}\boldsymbol{x}_{(12)j} - \boldsymbol{\xi}_3)\right\}.
\end{aligned}
$$

Similarly, as the MLE, we have the MLEs under the H_0 of $\boldsymbol{\xi}_3, \Phi_{(12)(12)}, \Phi_{3(12)}$, and Φ_{33} are expressed as

$$\widetilde{\boldsymbol{\xi}}_3 = \boldsymbol{\mu}_{30} - \widetilde{\Phi}_{3(12)}\boldsymbol{\mu}_{10},\ \widetilde{\Phi}_{(12)(12)} = \frac{1}{n}\sum_{j=1}^{n}(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})',$$

$$\widetilde{\Phi}_{3(12)} = \left\{\sum_{j=1}^{n_1}(\boldsymbol{x}_{3j} - \boldsymbol{\mu}_{30})(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})'\right\}\left\{\sum_{j=1}^{n_1}(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})(\boldsymbol{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})'\right\}^{-1},$$

$$\widetilde{\Phi}_{33} = \frac{1}{n_1}\sum_{j=1}^{n_1}(\boldsymbol{x}_{3j} - \widetilde{\Phi}_{3(12)}\boldsymbol{x}_{(12)j} - \widetilde{\boldsymbol{\xi}}_3)(\boldsymbol{x}_{3j} - \widetilde{\Phi}_{3(12)}\boldsymbol{x}_{(12)j} - \widetilde{\boldsymbol{\xi}}_3)'.$$

§3. Likelihood ratio test

In this section, we provide the LRT statistic for testing the subvector of a mean vector with two-step monotone missing data. In the hypothesis in (1.1), the parameter space Ω and the subspace ω when H_0 is true are, respectively, as follows:

$$\begin{aligned}\Omega &= \{(\boldsymbol{\mu}, \Sigma) : -\infty < \mu_i < \infty, i = p_1 + 1, p_1 + 2, \ldots, p, \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{10}, \\ &\qquad \Sigma > 0 \text{ and } \Sigma_{(23)(23)} > 0\}, \\ \omega &= \{(\boldsymbol{\mu}, \Sigma) : \boldsymbol{\mu} = \boldsymbol{\mu}_0, \Sigma > 0 \text{ and } \Sigma_{(23)(23)} > 0\},\end{aligned}$$

where $\Sigma > 0$ and $\Sigma_{(23)(23)} > 0$ mean that Σ and $\Sigma_{(23)(23)}$ are positive definite matrices. Using the MLEs in Section 2, the likelihood ratio criterion is given by

$$\lambda_{\mathrm{M}} = \frac{\max_{\omega} L(\boldsymbol{\mu}, \Sigma)}{\max_{\Omega} L(\boldsymbol{\mu}, \Sigma)} = \left(\frac{|\widehat{\Psi}_{11}| \cdot |\widehat{\Psi}_{22}|}{|\widetilde{\Phi}_{(12)(12)}|}\right)^{\frac{n}{2}} \left(\frac{|\widehat{\Psi}_{33}|}{|\widetilde{\Phi}_{33}|}\right)^{\frac{n_1}{2}}.$$

Note that under the null hypothesis, the LRT statistic, $-2\log\lambda_M$, is asymptotically distributed as χ^2 with a degree of freedom of $p_2 + p_3$ when $n_1, n \to \infty$ with $n_1/n \to \delta \in (0, 1]$. However, the upper percentile of the χ^2 distribution is not a good approximation to that of the LRT statistic when the sample size is not large. We consider an approximate upper percentile of the LRT statistic because it is not easy to obtain the exact one of the LRT statistic. In this paper, we give a simplified and good approximation using linear interpolation for the $n_1 \times p$ and $n \times p$ complete data sets. We note that, at least, the proposed approximation is better than the χ^2 approximation and uses the same concept adopted for tests of the mean vector with two-step monotone missing data by Seko, Yamazaki, and Seo (2012) and Yagi and Seo (2015). In our case, as in Section 1, we use the property in the case of complete data. That is, the exact upper 100α percentile of λ is satisfied with $\Pr\{\lambda > q_n(\alpha)\} = \alpha$, where

$$q_n(\alpha) = \left\{1 + \frac{(p_2+p_3)F_{p_2+p_3,n-p}(\alpha)}{n-p}\right\}^{-\frac{n}{2}},$$

and $F_{p_2+p_3,n-p}(\alpha)$ is the upper 100α percentile of the F distribution with p_2+p_3 and $n-p$ degrees of freedom. Thus, we have the following theorem.

Theorem 3.1. *Suppose that the data have a two-step monotone missing data pattern. Then, the approximate upper 100α percentile of the LRT statistic $-2\log\lambda_M$ is given by*

$$q^*_{\rm M}(\alpha) = -2\log\left\{\frac{p_3}{p}\, q_{n_1}(\alpha) + \frac{p_1+p_2}{p} q_n(\alpha)\right\}, \tag{3.1}$$

where

$$q_{n_1}(\alpha) = \left\{1 + \frac{(p_2+p_3)F_{p_2+p_3,n_1-p}(\alpha)}{n_1-p}\right\}^{-\frac{n_1}{2}}, q_n(\alpha) = \left\{1 + \frac{(p_2+p_3)F_{p_2+p_3,n-p}(\alpha)}{n-p}\right\}^{-\frac{n}{2}},$$

and $F_{a,b}(\alpha)$ is the upper 100α percentile of the F distribution with a and b degrees of freedom.

Therefore, we reject H_0 if $-2\log\lambda_{\rm M} > q^*_{\rm M}(\alpha)$. In the following section, the accuracy and asymptotic behavior of the approximation are investigated by Monte Carlo simulation.

§4. Simulation studies

In this section, we compute the upper 100α percentiles of the LRT statistic $q_{\rm sim}(\alpha)$ by Monte Carlo simulation for $\alpha = 0.05, 0.01$. We generate artificial two-step missing data from $N_p(\mathbf{0}, I_p)$ for the various conditions of p_1, p_2, p_3, n_1, and n_2. We simulated the upper percentiles of the LRT statistic given the $q^*_{\rm M}(\alpha)$ values and the type I error rates under the simulated LRT statistic when the null hypothesis is rejected using $q^*_{\mathrm{M}}(\alpha)$ and $\chi^2_{p_2+p_3}$, where

$$P_{q^*} = \Pr\{-2\log\lambda_{\mathrm{M}} > q^*_{\mathrm{M}}(\alpha)\}, \quad P_c = \Pr\{-2\log\lambda_{\mathrm{M}} > \chi^2_{p_2+p_3}(\alpha)\},$$

and $\chi^2_{p_2+p_3}(\alpha)$ is the upper 100α percentile of the χ^2 distribution with a degree of freedom of p_2+p_3. The simulation results are shown in Tables 1-6. Computations are made for the following six cases:

Case I : $(p_1, p_2, p_3) = (2, 2, 4), (2, 3, 3), (2, 4, 2),$
$(n_1, n_2) = (n_1, 2n_1), (n_1, n_1), (n_1, n_1/2), n_1 = 20, 40, 80, 160;$

Case II : $(p_1, p_2, p_3) = (2, 2, 4), (2, 3, 3), (2, 4, 2),$
$(n_1, n_2), n_1 = 20, 40, 80, 160, n_2 = 10, 20, 40;$

Case III : $(p_1, p_2, p_3) = (2, 2, 2), (4, 2, 2), (8, 2, 2),$
$(n_1, n_2) = (n_1, 2n_1), (n_1, n_1), (n_1, n_1/2), n_1 = 20, 40, 80, 160;$

Case IV: $(p_1, p_2, p_3) = (2, 2, 4), (4, 3, 3), (6, 2, 2),$
$(n_1, n_2), n_1 = 20, 40, 80, 160, n_2 = 10, 20, 40.$

Case V : $(p_1, p_2, p_3) = (2, 4, 4), (4, 3, 3), (6, 2, 2),$
$(n_1, n_2) = (n_1, 2n_1), (n_1, n_1), (n_1, n_1/2), n_1 = 20, 40, 80, 160;$

Case VI: $(p_1, p_2, p_3) = (2, 4, 4), (4, 3, 3), (6, 2, 2),$
$(n_1, n_2), n_1 = 20, 40, 80, 160, n_2 = 10, 20, 40.$

We note that the cases for $p = 8$ and $p_1 = 2$ are given in Tables 1 and 2. That is, the values of p and p_1 are fixed. Further, Tables 3 and 4 give the case where $p_2 = p_3$, and p_2 and p_3 are fixed, Tables 5 and 6 give the case where $p = 10$ and $p_2 = p_3$.

From Tables 1 and 2, it is seen that the proposed approximation $q^*_{\mathrm{M}}(\alpha)$ is good for the case when the sample sizes n_1 and n_2 are large or the sample size n_1 is large and n_2 is fixed. This result also shows that the type I error rate is close to α when the sample size n_1 is large. From Tables 3 and 4, we can see that the approximation $q^*_{\mathrm{M}}(\alpha)$ is good for the case of $p_2 = p_3 = 2$ when the sample size n_1 is large. It can be seen from Tables 3 and 4 that the value of $q^*_{\mathrm{M}}(\alpha)$ is close to that of the LRT when p_1 is small. However, we note that the proposed approximation is better than the χ^2 approximation for all cases.

TABLE 1 : p_1 and p are fixed, and $\alpha = 0.05, 0.01$

		α=0.05				α=0.01			
n_1	n_2	$q_{\mathrm{sim}}(\alpha)$	$q^*_{\mathrm{M}}(\alpha)$	P_{q^*}	P_c	$q_{\mathrm{sim}}(\alpha)$	$q^*_{\mathrm{M}}(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 4)$									
20	40	17.69	15.17	.093	.171	23.81	19.99	.028	.062
40	80	14.54	13.90	.061	.091	19.50	18.50	.014	.024
80	160	13.48	13.24	.054	.067	18.06	17.66	.011	.016
160	320	13.01	12.91	.052	.058	17.40	17.24	.011	.012
20	20	17.83	15.92	.080	.176	24.00	21.05	.022	.064
40	40	14.61	14.15	.058	.093	19.52	18.85	.013	.025
80	80	13.52	13.33	.053	.068	18.05	17.80	.011	.016
160	160	13.02	12.95	.051	.058	17.43	17.29	.011	.013
20	10	17.93	16.71	.068	.180	23.98	22.21	.016	.066
40	20	14.69	14.39	.055	.095	19.63	19.20	.012	.025
80	40	13.54	13.43	.052	.069	18.10	17.93	.011	.016
160	80	13.06	13.00	.051	.059	17.43	17.35	.010	.013
$(p_1, p_2, p_3) = (2, 3, 3)$									
20	40	17.09	14.81	.090	.154	23.07	19.58	.026	.054
40	80	14.32	13.72	.060	.086	19.16	18.27	.014	.022
80	160	13.35	13.15	.054	.065	17.82	17.55	.011	.015
160	320	12.96	12.87	.052	.057	17.31	17.18	.010	.012
20	20	17.37	15.59	.079	.163	23.35	20.68	.021	.058
40	40	14.46	14.00	.058	.089	19.31	18.66	.013	.023
80	80	13.43	13.27	.053	.067	17.95	17.71	.011	.015
160	160	13.00	12.92	.051	.058	17.38	17.25	.010	.012
20	10	17.62	16.45	.067	.171	23.66	21.90	.016	.061
40	20	14.56	14.29	.055	.092	19.45	19.06	.011	.024
80	40	13.49	13.39	.052	.068	18.00	17.87	.010	.015
160	80	13.01	12.98	.051	.058	17.35	17.33	.010	.012
$(p_1, p_2, p_3) = (2, 4, 2)$									
20	40	16.34	14.51	.081	.134	22.11	19.25	.022	.044
40	80	14.06	13.55	.059	.080	18.78	18.06	.013	.020
80	160	13.24	13.07	.053	.063	17.67	17.44	.011	.014
160	320	12.89	12.83	.051	.055	17.26	17.13	.011	.012
20	20	16.74	15.32	.073	.146	22.55	20.36	.019	.049
40	40	14.23	13.87	.056	.084	19.02	18.49	.012	.021
80	80	13.34	13.21	.052	.064	17.85	17.63	.011	.015
160	160	12.93	12.89	.051	.056	17.31	17.21	.010	.012
20	10	17.24	16.22	.065	.159	23.21	21.62	.015	.056
40	20	14.44	14.19	.054	.089	19.27	18.94	.011	.023
80	40	13.43	13.34	.051	.067	17.92	17.81	.010	.015
160	80	12.98	12.96	.050	.057	17.33	17.30	.010	.012

Note : $\chi^2_6(0.05) = 12.59$, $\chi^2_6(0.01) = 16.81$

TABLE 2 : p_1, p and n_2 are fixed, and $\alpha = 0.05, 0.01$

		α=0.05				α=0.01			
n_1	n_2	$q_{\rm sim}(\alpha)$	$q^*_{\rm M}(\alpha)$	P_{q^*}	P_c	$q_{\rm sim}(\alpha)$	$q^*_{\rm M}(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 4)$									
20	10	17.93	16.71	.068	.180	23.98	22.21	.016	.066
40	10	14.75	14.58	.053	.097	19.72	19.47	.011	.026
80	10	13.58	13.56	.050	.070	18.13	18.10	.010	.016
160	10	13.08	13.07	.050	.059	17.50	17.45	.010	.013
20	20	17.83	15.92	.080	.176	24.00	21.05	.022	.064
40	20	14.69	14.39	.055	.095	19.63	19.20	.012	.025
80	20	13.56	13.51	.051	.069	18.07	18.04	.010	.016
160	20	13.08	13.06	.050	.059	17.43	17.43	.010	.013
20	40	17.69	15.17	.093	.171	23.81	19.99	.028	.062
40	40	14.61	14.15	.058	.093	19.52	18.85	.013	.025
80	40	13.54	13.43	.052	.069	18.10	17.93	.011	.016
160	40	13.06	13.03	.051	.059	17.42	17.40	.010	.013
$(p_1, p_2, p_3) = (2, 3, 3)$									
20	10	17.62	16.45	.067	.171	23.66	21.90	.016	.061
40	10	14.68	14.51	.053	.094	19.60	19.38	.011	.025
80	10	13.56	13.54	.050	.069	18.11	18.08	.010	.016
160	10	13.07	13.07	.050	.059	17.47	17.44	.010	.013
20	20	17.37	15.59	.079	.163	23.35	20.68	.021	.058
40	20	14.56	14.29	.055	.092	19.45	19.06	.011	.024
80	20	13.54	13.48	.051	.069	18.09	18.00	.010	.016
160	20	13.06	13.05	.050	.059	17.43	17.42	.010	.013
20	40	17.09	14.81	.090	.154	23.07	19.58	.026	.054
40	40	14.46	14.00	.058	.089	19.31	18.66	.013	.023
80	40	13.49	13.39	.052	.068	18.00	17.87	.010	.015
160	40	13.05	13.02	.050	.059	17.40	17.38	.010	.013
$(p_1, p_2, p_3) = (2, 4, 2)$									
20	10	17.24	16.22	.065	.159	23.21	21.62	.015	.056
40	10	14.58	14.45	.052	.092	19.47	19.30	.011	.024
80	10	13.57	13.53	.051	.070	18.12	18.06	.010	.016
160	10	13.09	13.06	.050	.060	17.51	17.44	.010	.013
20	20	16.74	15.32	.073	.146	22.55	20.36	.019	.049
40	20	14.44	14.19	.054	.089	19.27	18.94	.011	.023
80	20	13.50	13.45	.051	.068	18.04	17.96	.010	.016
160	20	13.08	13.04	.051	.059	17.49	17.41	.010	.013
20	40	16.34	14.51	.081	.134	22.11	19.25	.022	.044
40	40	14.23	13.87	.056	.084	19.02	18.49	.012	.021
80	40	13.43	13.34	.051	.067	17.92	17.81	.010	.015
160	40	13.02	13.01	.050	.058	17.41	17.37	.010	.013

Note : $\chi^2_6(0.05) = 12.59$, $\chi^2_6(0.01) = 16.81$

TABLE 3 : $p_2 = p_3$, and $\alpha = 0.05, 0.01$

		α=0.05				α=0.01			
n_1	n_2	$q_{\text{sim}}(\alpha)$	$q^*_{\text{M}}(\alpha)$	P_{q^*}	P_c	$q_{\text{sim}}(\alpha)$	$q^*_{\text{M}}(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 2)$									
20	40	11.93	10.88	.070	.109	16.78	15.12	.018	.032
40	80	10.51	10.17	.057	.073	14.72	14.21	.012	.018
80	160	9.96	9.82	.053	.060	13.96	13.74	.011	.013
160	320	9.70	9.65	.051	.055	13.58	13.51	.010	.011
20	20	12.10	11.31	.064	.113	16.98	15.75	.015	.034
40	40	10.57	10.34	.055	.075	14.78	14.45	.011	.018
80	80	10.01	9.90	.052	.061	14.02	13.85	.011	.013
160	160	9.75	9.69	.051	.056	13.61	13.56	.010	.012
20	10	12.29	11.76	.059	.119	17.30	16.42	.013	.037
40	20	10.66	10.51	.053	.077	14.97	14.70	.011	.019
80	40	10.05	9.97	.052	.062	14.06	13.95	.010	.014
160	80	9.75	9.72	.050	.056	13.60	13.61	.010	.011
$(p_1, p_2, p_3) = (4, 2, 2)$									
20	40	13.35	11.23	.091	.146	18.87	15.57	.026	.051
40	80	10.98	10.35	.062	.084	15.39	14.45	.014	.022
80	160	10.13	9.91	.054	.064	14.19	13.86	.011	.014
160	320	9.81	9.70	.052	.057	13.70	13.57	.011	.012
20	20	13.59	11.96	.079	.155	19.19	16.62	.021	.055
40	40	11.06	10.63	.058	.086	15.49	14.85	.013	.022
80	80	10.20	10.03	.053	.066	14.29	14.03	.011	.015
160	160	9.82	9.75	.051	.057	13.75	13.65	.010	.012
20	10	13.94	12.78	.069	.165	19.60	17.82	.017	.060
40	20	11.22	10.91	.056	.091	15.72	15.26	.012	.024
80	40	10.27	10.15	.052	.067	14.40	14.21	.011	.015
160	80	9.86	9.81	.051	.058	13.78	13.73	.010	.012
$(p_1, p_2, p_3) = (8, 2, 2)$									
20	40	18.24	11.99	.170	.271	26.27	16.63	.068	.132
40	80	12.01	10.71	.076	.111	16.91	14.92	.019	.033
80	160	10.55	10.09	.059	.074	14.79	14.10	.013	.018
160	320	9.99	9.78	.054	.061	13.99	13.69	.011	.013
20	20	18.67	13.47	.137	.289	26.83	18.71	.050	.142
40	40	12.28	11.23	.069	.118	17.28	15.67	.017	.036
80	80	10.64	10.30	.057	.076	14.91	14.41	.012	.018
160	160	10.03	9.88	.053	.062	14.03	13.83	.011	.014
20	10	19.33	15.38	.106	.311	27.60	21.42	.033	.158
40	20	12.47	11.81	.062	.125	17.48	16.50	.014	.039
80	40	10.77	10.53	.055	.079	15.06	14.73	.011	.020
160	80	10.07	9.98	.052	.063	14.14	13.97	.011	.014

Note : $\chi^2_4(0.05) = 9.49$, $\chi^2_4(0.01) = 13.28$

TABLE 4 : n_2 is fixed, $p_2 = p_3$, and $\alpha = 0.05, 0.01$

		α=0.05				α=0.01			
n_1	n_2	$q_{\text{sim}}(\alpha)$	$q^*_{\text{M}}(\alpha)$	P_{q^*}	P_c	$q_{\text{sim}}(\alpha)$	$q^*_{\text{M}}(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 2)$									
20	10	12.29	11.76	.059	.119	17.30	16.42	.013	.037
40	10	10.73	10.65	.052	.078	15.04	14.90	.011	.020
80	10	10.11	10.07	.051	.063	14.16	14.10	.010	.014
160	10	9.79	9.78	.050	.056	13.69	13.69	.010	.012
20	20	12.10	11.31	.064	.113	16.98	15.75	.015	.034
40	20	10.66	10.51	.053	.077	14.97	14.70	.011	.019
80	20	10.06	10.03	.050	.062	14.10	14.04	.010	.014
160	20	9.77	9.77	.050	.056	13.68	13.67	.010	.012
20	40	11.93	10.88	.070	.109	16.78	15.12	.018	.032
40	40	10.57	10.34	.055	.075	14.78	14.45	.011	.018
80	40	10.05	9.97	.052	.062	14.06	13.95	.010	.014
160	40	9.79	9.75	.051	.056	13.70	13.65	.010	.012
$(p_1, p_2, p_3) = (4, 2, 2)$									
20	10	13.94	12.78	.069	.165	19.60	17.82	.017	.060
40	10	11.34	11.15	.054	.094	15.86	15.60	.011	.025
80	10	10.34	10.32	.050	.069	14.49	14.44	.010	.016
160	10	9.90	9.90	.050	.059	13.83	13.86	.010	.013
20	20	13.59	11.96	.079	.155	19.19	16.62	.021	.055
40	20	11.22	10.91	.056	.091	15.72	15.26	.012	.024
80	20	10.32	10.25	.051	.068	14.44	14.34	.010	.016
160	20	9.89	9.88	.050	.059	13.84	13.83	.010	.013
20	40	13.35	11.23	.091	.146	18.87	15.57	.026	.051
40	40	11.06	10.63	.058	.086	15.49	14.85	.013	.022
80	40	10.27	10.15	.052	.067	14.40	14.21	.011	.015
160	40	9.88	9.85	.051	.059	13.82	13.79	.010	.012
$(p_1, p_2, p_3) = (8, 2, 2)$									
20	10	19.33	15.38	.106	.311	27.60	21.42	.033	.158
40	10	12.75	12.30	.057	.132	17.88	17.21	.012	.043
80	10	10.92	10.84	.051	.083	15.30	15.17	.011	.021
160	10	10.16	10.15	.050	.065	14.23	14.21	.010	.015
20	20	18.67	13.47	.137	.289	26.83	18.71	.050	.142
40	20	12.47	11.81	.062	.125	17.48	16.50	.014	.039
80	20	10.84	10.72	.052	.081	15.19	14.99	.011	.020
160	20	10.15	10.12	.051	.064	14.18	14.16	.010	.015
20	40	18.24	11.99	.170	.271	26.27	16.63	.068	.132
40	40	12.28	11.23	.069	.118	17.28	15.67	.017	.036
80	40	10.77	10.53	.055	.079	15.06	14.73	.011	.020
160	40	10.13	10.06	.051	.064	14.21	14.08	.010	.014

Note : $\chi^2_4(0.05) = 9.49$, $\chi^2_4(0.01) = 13.28$

TABLE 5 : $p = 10$, $p_2 = p_3$, and $\alpha = 0.05, 0.01$

		α=0.05				α=0.01			
n_1	n_2	$q_{\text{sim}}(\alpha)$	$q^*_{\text{M}}(\alpha)$	P_{q^*}	P_c	$q_{\text{sim}}(\alpha)$	$q^*_{\text{M}}(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 4, 4)$									
20	40	23.13	18.58	.127	.227	30.50	23.82	.043	.094
40	80	18.17	17.14	.067	.103	23.63	22.11	.016	.029
80	160	16.67	16.33	.056	.071	21.62	21.13	.012	.017
160	320	16.07	15.92	.053	.060	20.84	20.62	.011	.013
20	20	23.47	19.86	.105	.238	30.78	25.52	.032	.100
40	40	18.34	17.58	.061	.107	23.81	22.72	.014	.030
80	80	16.74	16.50	.054	.073	21.73	21.37	.011	.017
160	160	16.09	15.99	.051	.060	20.80	20.72	.010	.013
20	10	23.90	21.35	.084	.251	31.26	27.53	.023	.107
40	20	18.53	18.02	.057	.112	24.09	23.33	.013	.032
80	40	16.84	16.67	.053	.075	21.81	21.60	.011	.018
160	80	16.16	16.07	.051	.061	20.94	20.82	.010	.013
$(p_1, p_2, p_3) = (4, 3, 3)$									
20	40	19.48	15.23	.127	.218	26.43	20.13	.043	.090
40	80	14.98	13.95	.068	.102	20.13	18.56	.017	.029
80	160	13.63	13.27	.056	.071	18.19	17.70	.012	.016
160	320	13.07	12.93	.052	.059	17.44	17.26	.011	.013
20	20	19.85	16.46	.105	.231	26.87	21.80	.032	.097
40	40	15.13	14.39	.063	.106	20.22	19.17	.014	.030
80	80	13.73	13.45	.055	.073	18.37	17.95	.012	.017
160	160	13.12	13.01	.052	.060	17.52	17.37	.011	.013
20	10	20.34	17.90	.084	.246	27.52	23.79	.023	.106
40	20	15.37	14.85	.058	.111	20.57	19.80	.013	.032
80	40	13.83	13.63	.053	.075	18.45	18.20	.011	.018
160	80	13.16	13.09	.051	.061	17.55	17.48	.010	.013
$(p_1, p_2, p_3) = (6, 2, 2)$									
20	40	15.27	11.59	.121	.198	21.80	16.07	.041	.081
40	80	11.47	10.53	.068	.096	16.12	14.68	.017	.027
80	160	10.35	10.00	.057	.069	14.51	13.98	.012	.016
160	320	9.87	9.74	.053	.058	13.85	13.63	.011	.013
20	20	15.70	12.67	.102	.212	22.30	17.59	.032	.089
40	40	11.64	10.92	.063	.101	16.30	15.25	.015	.029
80	80	10.41	10.17	.055	.071	14.58	14.22	.012	.017
160	160	9.95	9.82	.053	.060	13.90	13.74	.011	.013
20	10	16.11	13.97	.082	.226	22.80	19.45	.022	.097
40	20	11.81	11.34	.058	.106	16.54	15.86	.013	.031
80	40	10.49	10.34	.053	.073	14.68	14.46	.011	.017
160	80	9.98	9.90	.052	.061	13.92	13.85	.010	.013

$\chi^2_4(0.05) = 9.49$, $\chi^2_6(0.05) = 12.59$, and $\chi^2_8(0.05) = 15.51$

$\chi^2_4(0.01) = 13.28$, $\chi^2_6(0.01) = 16.81$, and $\chi^2_8(0.01) = 20.09$

TABLE 6 : n_2 is fixed, $p = 10$, $p_2 = p_3$, and $\alpha = 0.05, 0.01$

		α=0.05				α=0.01			
n_1	n_2	$q_{\rm sim}(\alpha)$	$q^*_{\rm M}(\alpha)$	P_{q^*}	P_c	$q_{\rm sim}(\alpha)$	$q^*_{\rm M}(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 4, 4)$									
20	10	23.90	21.35	.084	.251	31.26	27.53	.023	.107
40	10	18.66	18.38	.054	.116	24.24	23.82	.011	.034
80	10	16.94	16.90	.050	.077	21.93	21.90	.010	.018
160	10	16.19	16.19	.050	.062	21.01	20.98	.010	.014
20	20	23.47	19.86	.105	.238	30.78	25.52	.032	.100
40	20	18.53	18.02	.057	.112	24.09	23.33	.013	.032
80	20	16.93	16.81	.052	.076	21.95	21.78	.011	.019
160	20	16.18	16.17	.050	.062	20.96	20.95	.010	.014
20	40	23.13	18.58	.127	.227	30.50	23.82	.043	.094
40	40	18.34	17.58	.061	.107	23.81	22.72	.014	.030
80	40	16.84	16.67	.053	.075	21.81	21.60	.011	.018
160	40	16.16	16.13	.051	.061	20.94	20.90	.010	.013
$(p_1, p_2, p_3) = (4, 3, 3)$									
20	10	20.34	17.90	.084	.246	27.52	23.79	.023	.106
40	10	15.52	15.22	.055	.116	20.76	20.32	.012	.034
80	10	13.94	13.88	.051	.077	18.64	18.53	.010	.019
160	10	13.23	13.23	.050	.062	17.68	17.66	.010	.014
20	20	19.85	16.46	.105	.231	26.87	21.80	.032	.097
40	20	15.37	14.85	.058	.111	20.57	19.80	.013	.032
80	20	13.87	13.78	.052	.076	18.53	18.40	.010	.018
160	20	13.22	13.20	.050	.062	17.66	17.63	.010	.014
20	40	19.48	15.23	.127	.218	26.43	20.13	.043	.090
40	40	15.13	14.39	.063	.106	20.22	19.17	.014	.030
80	40	13.83	13.63	.053	.075	18.45	18.20	.011	.018
160	40	13.20	13.16	.051	.062	17.61	17.57	.010	.013
$(p_1, p_2, p_3) = (6, 2, 2)$									
20	10	16.11	13.97	.082	.226	22.80	19.45	.022	.097
40	10	11.99	11.70	.055	.111	16.80	16.36	.012	.033
80	10	10.63	10.57	.051	.076	14.85	14.80	.010	.019
160	10	10.04	10.03	.050	.062	14.04	14.03	.010	.014
20	20	15.70	12.67	.102	.212	22.30	17.59	.032	.089
40	20	11.81	11.34	.058	.106	16.54	15.86	.013	.031
80	20	10.58	10.48	.052	.074	14.81	14.66	.011	.018
160	20	10.02	10.00	.050	.062	14.03	13.99	.010	.014
20	40	15.27	11.59	.121	.198	21.80	16.07	.041	.081
40	40	11.64	10.92	.063	.101	16.30	15.25	.015	.029
80	40	10.49	10.34	.053	.073	14.68	14.46	.011	.017
160	40	10.00	9.96	.051	.061	14.01	13.94	.010	.014

$\chi^2_4(0.05) = 9.49$, $\chi^2_6(0.05) = 12.59$, and $\chi^2_8(0.05) = 15.51$

$\chi^2_4(0.01) = 13.28$, $\chi^2_6(0.01) = 16.81$, and $\chi^2_8(0.01) = 20.09$

§5. Numerical example

We illustrate the results of this study using an example given in Wei and Lachin (1984). The sample data consist of serum cholesterol values that were measured under treatment at five different time points: the baseline and months 6, 12, 20, and 24. The original data have 36 complete observations for months 20 and 24 to create two-step monotone missing data. We are interested in the change from the baseline at each post-baseline time point. Thus, we have $n = 30$, $n_1 = 20$, $n_2 = 10$, $p = 4$, $p_1 = p_2 = 1$, and $p_3 = 2$. We consider the hypothesis $H : (\mu_2, \mu_3, \mu_4)' = (0, 0, 0)'$ given $\mu_1 = 0$. Then, we compute $-2 \log \lambda_{\mathrm{M}} = 10.95$. Because $q_{\mathrm{sim}}(0.05) = 9.36$ from the simulation study, the null hypothesis is rejected at the 0.05 significance level. When we use $q^*_{\mathrm{M}}(0.05) = 9.15$ and $\chi^2_{3(0.05)} = 7.81$, the null hypothesis is also rejected.

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References

[1] Anderson, T. W. (1957). Maximum likelihood estimates for a multivariate normal distribution where some observations are missing. *J. Amer. Statist. Assoc.*, **52**, 200–203.

[2] Chang, W.-Y. and Richard, D. St. P. (2009). Finite-sample inference with monotone incomplete multivariate normal data I. *J. Multivariate Anal.*, **100**, 1883–1899.

[3] Giri, N. C. (1964). On the likelihood ratio test of a normal multivariate testing problem. *Ann. Math. Stat.*, **35**, 181–189, 1388.

[4] Kanda, T. and Fujikoshi, Y. (1998). Some basic properties of the MLE's for a multivariate normal distribution with monotone missing data. *Amer. J. Math. Management Sci.*, **18**, 161–190.

[5] Krishnamoorthy, K. and Pannala, K. M. (1999). Confidence estimation of a normal mean vector with incomplete data. *Canad. J. Statist.*, **27**, 395–407.

[6] Little, R. J. and Rubin, D. B. (2002). *Statistical Analysis with Missing Data, second ed.* Wiley.

[7] Rao, C. R. (1949). On some problems arising out of discrimination with multiple characters. *Sankhya*, **9**, 343–364.

[8] Seko, N., Kawasaki, T. and Seo, T. (2011). Testing equality of two mean vectors with two-step monotone missing data. *Amer. J. Math. Management Sci.*, **31**, 117–135.

[9] Seko, N., Yamazaki, A. and Seo, T. (2012). Tests for mean vector with two-step monotone missing data. *SUT J. Math.*, **48**, 13–38.

[10] Shutoh, N., Kusumi, M., Morinaga, W., Yamada, S. and Seo, T. (2010). Testing equality of mean vector in two sample problem with missing data. *Comm. Statist. Simulation Comput.*, **39**, 487–500.

[11] Siotani, M., Hayakawa, T. and Fujikoshi, Y. (1985). *Modern Multivariate Statistical Analysis : A Graduate Course and Handbook*, American Science Press, Inc., Ohio.

[12] Srivastava, M. S. (1985). Multivariate data with missing observations. *Comm. Statist. Theory Methods*, **14**, 775–792.

[13] Srivastava, M. S. and Carter, E. M. (1986). The maximum likelihood method for non-response in sample survey. *Survey Methodology*, **12**, 61–72.

[14] Wei, L. J. and Lachin, J. M. (1984). Two-sample asymptotically distribution-free tests for incomplete multivariate observations. *J. Amer. Statist. Assoc.*, **79**, 653–661.

[15] Yagi, A. and Seo, T. (2015). Tests for mean vectors with two-step and three-step monotone samples. *Josai Mathematical Monographs*, **8**, 49–71.

[16] Yu, J., Krishnamoorthy, K. and Pannala, K. M. (2006). Two-sample inference for normal mean vectors based on monotone missing data. *J. Multivariate Anal.*, **97**, 2162–2176.