Asymptotic expansions of the null distribution of the LR test statistic for random-effects covariance structure in a parallel profile model

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ABSTRACT

This paper is concerned with the null distribution of the likelihood ratio test statistic $-2\log\Lambda$ for testing the adequacy of a random-effects covariance structure in a parallel profile model. It is known that the null distribution of $-2\log\Lambda$ converges to χ^2_f or $0.5\chi^2_f + 0.5\chi^2_{f+1}$ when the sample size tends to infinity. In order to extend this result, we derive asymptotic expansions of the null distribution of $-2\log\Lambda$. The accuracy of the approximations based on the limiting distribution and an asymptotic expansion are compared through numerical experiments.

Key Words: asymptotic expansion, parallel profile model, random-effects covariance structure, null distribution, likelihood ratio test.

1. Introduction

Let $\boldsymbol{x}_j^{(g)}$ be a p-dimensional vector of observations on the jth individual in the gth group, where $j = 1, \ldots, N_g$ and $g = 1, \ldots, k$, and let $N = N_1 + \cdots + N_k$. We assume that $\boldsymbol{x}_1^{(1)}, \ldots, \boldsymbol{x}_{N_k}^{(k)}$ are mutually independent and distributed as

$$\boldsymbol{x}_j^{(g)} \sim N_p(\boldsymbol{\mu}^{(g)}, \boldsymbol{\Sigma}),\ \boldsymbol{\mu}^{(g)} = \delta_g \mathbf{1}_p + \boldsymbol{\mu},\ \boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)', \tag{1.1}$$

where $\mathbf{1}_p$ is a p-dimensional vector of ones, $\delta_1, \ldots, \delta_k$ and $\mu_1, \ldots, \mu_p$ are unknown parameters, and $\boldsymbol{\Sigma}$ is an unknown positive definite matrix. Here, we may assume that $\delta_k = 0$, without loss of generality. The model (1.1) is called a parallel profile model. In the parallel profile model, Yokoyama and Fujikoshi (1993) assumed the following random-effects covariance structure:

$$\boldsymbol{\Sigma} = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 \boldsymbol{I}_p,\ \lambda^2 \geq 0,\ \sigma^2 > 0. \tag{1.2}$$

By making the above rather strong assumption for $\boldsymbol{\Sigma}$, they obtained more efficient estimators and more powerful tests. Needless to say, these results are only valid if $\boldsymbol{\Sigma}$ satisfies (1.2). Hence, it is important to test the hypothesis that $\boldsymbol{\Sigma}$ has the random-effects covariance structure.

Yokoyama (1995) proposed a likelihood ratio test for testing the hypothesis

$$\mathrm{H}_0 : \boldsymbol{\Sigma} = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 \boldsymbol{I}_p,\ \lambda^2 \geq 0,\ \sigma^2 > 0 \text{ v.s. } \mathrm{H}_1 : \text{not } \mathrm{H}_0, \tag{1.3}$$

and they derived a restricted likelihood ratio test statistic (R-LRT) for testing (1.3). They showed that the null distribution of the R-LRT converges to χ^2_f or $0.5\chi^2_f + 0.5\chi^2_{f+1}$ when the sample size tends to infinity, where $f = (p^2 + p - 4)/2$. Therefore, the hypothesis (1.3) is tested based on this asymptotic property.

On the other hand, Srivastava and Singull (2012) constructed the test based on the likelihood ratio, without the non-negativity of λ^2 in (1.3), for testing the random-effects covariance structure under the parallel profile model. In other words, they considered the following hypothesis:

$$\mathrm{H}_{0^*} : \boldsymbol{\Sigma} = \rho \mathbf{1}_p \mathbf{1}_p' + \sigma^2 \boldsymbol{I}_p,\ \rho > -\sigma^2/p,\ \sigma^2 > 0 \text{ v.s. } \mathrm{H}_{1^*} : \text{not } \mathrm{H}_{0^*}. \tag{1.4}$$

Here, the covariance structure given by H_{0^*} is called the exchangeable covariance structure. They derived a modified likelihood ratio test statistic (M-LRT) for testing (1.4). Their modification is based on the Box's (1949) asymptotic expansion with Bartlett's correction, and the order of the error term of the null distribution for the M-LRT is $O(N^{-2})$. It is known that under H_{0^*} the limiting distribution of the M-LRT is χ^2_f. Then, the hypothesis (1.4) is tested by using χ^2_f.

Under H_0 Srivastava and Singull (2012) showed that the limiting approximation of the distribution of the M-LRT is better than that of the R-LRT proposed by Yokoyama (1995). Moreover, through numerical experiments, they also showed that the power for the M-LRT is larger than the power for the R-LRT. For these reasons, they suggested using the M-LRT even in the case of testing (1.3).

However, since "the good test" and "the adequacy of the approximation" are different problems, these should be considered separately. In particular, for the adequacy of the approximation, this is not a fair comparison because the R-LRT is not modified. The main purpose of this paper is to derive an asymptotic expansion of the null distribution of the R-LRT up to the order N^{-1} under the parallel profile model.

The remainder of the present paper is organized as follows: In Section 2 and 3 we derive an asymptotic expansion of the null distribution of the R-LRT when $\lambda^2 > 0$ and $\lambda^2 = 0$, respectively. In Section 4 we provide the relevant theorem and corollary. In Section 5 we compare the accuracy of the approximations based on the limiting distribution and an asymptotic expansion through numerical experiments. In Section 6 we conclude our discussion. Technical details are provided in the Appendix.

2. Asymptotic expansion of the null distribution when $\lambda^2 > 0$

In this section we derive an asymptotic expansion of the null distribution of the R-LRT proposed by Yokoyama (1995) under H_0 and $\lambda^2 > 0$. First, we consider the definition and decomposition of the restricted likelihood ratio.

2.1. Definition and decomposition of the restricted likelihood ratio

The restricted likelihood ratio Λ proposed by Yokoyama (1995), is given by

$$\Lambda = \begin{cases} \Lambda_1, & \text{if } s_1/f_1 \geq s_2/f_2, \\ \Lambda_2, & \text{if } s_1/f_1 < s_2/f_2, \end{cases}$$

where, $f_1 = N,\ f_2 = N(p-1)$,

$$\Lambda_1 = \frac{\left(\dfrac{s_3}{f_1}\right)^{f_1/2} \left|\dfrac{1}{f_1}\boldsymbol{S}_4\right|^{f_1/2}}{\left(\dfrac{s_1}{f_1}\right)^{f_1/2}\left(\dfrac{s_2}{f_2}\right)^{f_2/2}},\ \Lambda_2 = \frac{\left(\dfrac{s_3}{f_1}\right)^{f_1/2} \left|\dfrac{1}{f_1}\boldsymbol{S}_4\right|^{f_1/2}}{\left(\dfrac{s_1+s_2}{f_1+f_2}\right)^{(f_1+f_2)/2}},$$

and

$$\boldsymbol{S}_t = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})',\ \boldsymbol{S}_w = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})',$$

$$\bar{\boldsymbol{x}} = \frac{1}{N}\sum_{g=1}^{k}\sum_{j=1}^{N_g}\boldsymbol{x}_j^{(g)},\ \bar{\boldsymbol{x}}^{(g)} = \frac{1}{N_g}\sum_{j=1}^{N_g}\boldsymbol{x}_j^{(g)},$$

$$s_1 = \frac{1}{p}\boldsymbol{1}_p'\boldsymbol{S}_w\boldsymbol{1}_p,\ s_2 = \mathrm{tr}\boldsymbol{S}_t - \frac{1}{p}\boldsymbol{1}_p'\boldsymbol{S}_t\boldsymbol{1}_p,\ s_3 = \left(\frac{1}{p}\boldsymbol{1}_p'\boldsymbol{S}_w^{-1}\boldsymbol{1}_p\right)^{-1},\ |\boldsymbol{S}_4| = \frac{1}{p}\boldsymbol{1}_p'\boldsymbol{S}_t^{-1}\boldsymbol{1}_p|\boldsymbol{S}_t|.$$

Then, the restricted likelihood ratio test statistic is defined by $-2\log\Lambda$.

Under H_{0^*}, from the definitions of $\Lambda_1,\ \Lambda_2,\ s_1$ and s_2, the restricted likelihood ratio is decomposed by using Bartlett decomposition of Wishart distribution (see, e.g., Fujikoshi, *et al.*, 2010) as

$$\Lambda_1 = \left(\prod_{l=0}^{p-2} B_l\right)^{N/2} \times (p-1)^{N(p-1)/2}K_0^{N/2}, \quad \Lambda_2 = \Lambda_1 \times X, \tag{2.1}$$

where

$$K_0 = \prod_{j=1}^{p-1}\frac{K_j}{K}, \quad K = \sum_{k=1}^{p-1}K_k, \quad X = \left\{\frac{p^p}{(p-1)^{p-1}}\frac{s_1}{s_1+s_2}\left(1 - \frac{s_1}{s_1+s_2}\right)^{p-1}\right\}^{N/2}, \tag{2.2}$$

and

$$
\begin{aligned}
&B_0 \sim \ Beta\left(\frac{N-(p-1)-k}{2}, \frac{p-1}{2}\right), \quad B_l \sim \ Beta\left(\frac{N-l-1}{2}, \frac{l}{2}\right), \\
&K_j \sim \chi^2_{N-1}, \quad s_1 \sim \tau^2\chi^2_{N-k}, \quad s_2 \sim \sigma^2\chi^2_{(N-1)(p-1)}, \quad \tau^2 = p\lambda^2 + \sigma^2.
\end{aligned} \tag{2.3}
$$

Here, it is easily checked that $B_0, \ldots, B_{p-2},\ K_1, \ldots, K_{p-1},\ s_1$ and s_2 are mutually independent. Hence, Λ_1 , s_1 and s_2 are also mutually independent. Note that the results obtained in this subsection are derived under H_{0^*}. Therefore, these also hold under H_0.

2.2. Asymptotic expansion under H_0 and $\lambda^2 > 0$

Suppose that the hypothesis H_0 is true and $\lambda^2 > 0$. Then, Λ_1, s_1 and s_2 are mutually independent. Thus, from the definition of Λ, the distribution of $-2\log\Lambda$ can be written as

$$
\begin{aligned}
\mathrm{P}(-2\log\Lambda \le c) &= \mathrm{P}(-2\log\Lambda_1 \le c, s_1/f_1 \ge s_2/f_2) + \mathrm{P}(-2\log\Lambda_2 \le c, s_1/f_1 < s_2/f_2) \\
&= \mathrm{P}(-2\log\Lambda_1 \le c)\mathrm{P}(s_1/f_1 \ge s_2/f_2) + \mathrm{P}(-2\log\Lambda_2 \le c, s_1/f_1 < s_2/f_2) \\
&= \mathrm{P}(-2\log\Lambda_1 \le c) - \mathrm{P}(-2\log\Lambda_1 \le c)\mathrm{P}(s_1/f_1 < s_2/f_2) \\
&\quad + \mathrm{P}(-2\log\Lambda_2 \le c, s_1/f_1 < s_2/f_2).
\end{aligned}
$$

From $\tau^2 > \sigma^2$ and the definitions of s_1 and s_2, for large N, the last two terms satisfy that

$$
\begin{aligned}
&|-\mathrm{P}(-2\log\Lambda_1 \le c)\mathrm{P}(s_1/f_1 < s_2/f_2) + \mathrm{P}(-2\log\Lambda_2 \le c, s_1/f_1 < s_2/f_2)| \\
&\le 2\mathrm{P}(s_1/f_1 < s_2/f_2) \\
&= 2\mathrm{P}\left(\frac{f_2}{f_1}\frac{N-k}{(N-1)(p-1)}\frac{\tau^2}{\sigma^2} < \frac{s_2/\{\sigma^2(N-1)(p-1)\}}{s_1/\{\tau^2(N-k)\}}\right) \le 2\mathrm{P}\left(t < \frac{\chi^2_{l_2}/l_2}{\chi^2_{l_1}/l_1}\right),
\end{aligned}
$$

where t is a constant $(t > 1)$, $l_1 = N-k$ and $l_2 = (N-1)(p-1)$. Here, the last probability is evaluated by the following lemma.

Lemma 2.1. Let $l_1 = N-k,\ l_2 = (N-1)(p-1)$ and $t > 1$. Then, it holds that

$$
\mathrm{P}\left(t < \frac{\chi^2_{l_2}/l_2}{\chi^2_{l_1}/l_1}\right) = O(c_0^{-N}) \quad (c_0 > 1).
$$

Proof. Let c be a constant satisfying $t^{-1} < c < 1$. Then, it holds that

$$
\mathrm{P}\left(\frac{\chi^2_{l_2}}{l_2} > t\frac{\chi^2_{l_1}}{l_1}\right) \le \mathrm{P}\left(\frac{\chi^2_{l_1}}{l_1} < c\right) + \mathrm{P}\left(\frac{\chi^2_{l_2}}{l_2} > tc\right). \tag{2.4}
$$

Using Markov's inequality and the moment generating function of the chi-squared distribution, for any negative number a, the first term of the right hand side in (2.4) can be evaluated as

$$\mathrm{P}\left(\frac{\chi^2_{l_1}}{l_1} < c\right) = \mathrm{P}(a\chi^2_{l_1} > acl_1) = \mathrm{P}(e^{a\chi^2_{l_1}} > e^{acl_1}) \le \frac{\mathrm{E}[e^{a\chi^2_{l_1}}]}{e^{acl_1}} = \left\{\frac{1}{(1-2a)e^{2ac}}\right\}^{\frac{l_1}{2}}.$$

Hence,

$$\mathrm{P}\left(\frac{\chi^2_{l_1}}{l_1} < c\right) \le \inf_{a<0}\left\{\frac{1}{(1-2a)e^{2ac}}\right\}^{\frac{l_1}{2}}.$$

Here, it is easily showed that $(1-2a)e^{2ac}$ is maximized at $a = (2c)^{-1}(c-1)$ (< 0) under $a < 0$, and its maximum value is $e^{c-1-\log c}(> 1)$. Let $c_1 = e^{c-1-\log c}$, and let $c_{11} = c_1^{1/2}$ (> 1). Then, it holds that

$$\mathrm{P}\left(\frac{\chi^2_{l_1}}{l_1} < c\right) \le c_1^{-\frac{l_1}{2}} = c_1^{k/2}(c_1^{1/2})^{-N} = c_1^{k/2}(c_{11})^{-N} = O(c_{11}^{-N}). \tag{2.5}$$

Next, let $tc = d$ (> 1). Using the same argument, for any positive number b satisfying $0 < b < 1/2$, the second term of the right hand side in (2.4) can be evaluated as

$$\mathrm{P}\left(\frac{\chi^2_{l_2}}{l_2} > tc\right) = \mathrm{P}(b\chi^2_{l_2} > bdl_2) = \mathrm{P}(e^{b\chi^2_{l_2}} > e^{bdl_2}) \le \frac{\mathrm{E}[e^{b\chi^2_{l_2}}]}{e^{bdl_2}} = \left\{\frac{1}{(1-2b)e^{2bd}}\right\}^{\frac{l_2}{2}}.$$

Hence,

$$\mathrm{P}\left(\frac{\chi^2_{l_2}}{l_2} > tc\right) \le \inf_{0<b<1/2}\left\{\frac{1}{(1-2b)e^{2bd}}\right\}^{\frac{l_2}{2}}.$$

Similarly, it can be showed that $(1-2b)e^{2bd}$ is maximized at $b = 2^{-1} - (2d)^{-1}$ (> 0) under $0 < b < 1/2$, and its maximum value is $e^{d-1-\log d}$ (> 1). Let $c_2 = e^{d-1-\log d}$, and let $c_{22} = c_2^{(p-1)/2}$ (> 1). Then, it holds that

$$\mathrm{P}\left(\frac{\chi^2_{l_2}}{l_2} > tc\right) \le c_2^{-\frac{l_2}{2}} = c_2^{(p-1)/2}(c_2^{(p-1)/2})^{-N} = c_{22}c_{22}^{-N} = O(c_{22}^{-N}). \tag{2.6}$$

Let $c_0 = \min\{c_{11}, c_{22}\}$ (> 1). Then, substituting (2.5) and (2.6) into (2.4) yields

$$\mathrm{P}\left(\frac{\chi^2_{l_2}/l_2}{\chi^2_{l_1}/l_1} > t\right) = \mathrm{P}\left(\frac{\chi^2_{l_2}}{l_2} > t\frac{\chi^2_{l_1}}{l_1}\right) \le O(c_{11}^{-N}) + O(c_{22}^{-N}) \le O(c_0^{-N}).$$

□

From Lemma 2.1, the distribution function of $-2\log\Lambda$ is

$$\mathrm{P}(-2\log\Lambda \le c) = \mathrm{P}(-2\log\Lambda_1 \le c) + O(c_0^{-N}). \tag{2.7}$$

Thus, we derive an asymptotic expansion of $\mathrm{P}(-2\log\Lambda_1 \le c)$. From (2.2) and (2.3), it is easily checked that the hth moment of B_0, B_l and K_0 are given by

$$\begin{aligned}
\mathrm{E}[B_0^h] &= \frac{\Gamma(\frac{N-(p-1)-k}{2}+h)}{\Gamma(\frac{N-(p-1)-k}{2})}\frac{\Gamma(\frac{N-k}{2})}{\Gamma(\frac{N-k}{2}+h)}, \ \mathrm{E}[B_l^h] = \frac{\Gamma(\frac{N-l-1}{2}+h)}{\Gamma(\frac{N-l-1}{2})}\frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N-1}{2}+h)}, \\
\mathrm{E}[K_0^h] &= \left\{\frac{\Gamma(\frac{N-1}{2}+h)}{\Gamma(\frac{N-1}{2})}\right\}^{p-1}\frac{\Gamma(\frac{(N-1)(p-1)}{2})}{\Gamma(\frac{(N-1)(p-1)}{2}+(p-1)h)},
\end{aligned} \tag{2.8}$$

respectively. Hence, from (2.1) and (2.8), the log-characteristic function of $-2\log\Lambda_1$ can be expressed as

$$\begin{aligned}
\log\varphi(t) &= \log\mathrm{E}[e^{-2it\log\Lambda_1}] = \log\mathrm{E}[\Lambda_1^{-2it}] \\
&= \log\Gamma\left(\frac{N-(p-1)-k}{2}-Nit\right) - \log\Gamma\left(\frac{N-(p-1)-k}{2}\right) \\
&\quad + \log\Gamma\left(\frac{N-k}{2}\right) - \log\Gamma\left(\frac{N-k}{2}-Nit\right) \\
&\quad + \sum_{l=1}^{p-2}\left\{\log\Gamma\left(\frac{N-l-1}{2}-Nit\right) - \log\Gamma\left(\frac{N-l-1}{2}\right)\right. \\
&\qquad \left. + \log\Gamma\left(\frac{N-1}{2}\right) - \log\Gamma\left(\frac{N-1}{2}-Nit\right)\right\} - N(p-1)it\log(p-1) \\
&\quad + (p-1)\left\{\log\Gamma\left(\frac{N-1}{2}-Nit\right) - \log\Gamma\left(\frac{N-1}{2}\right)\right\} \\
&\quad + \log\Gamma\left(\frac{(N-1)(p-1)}{2}\right) - \log\Gamma\left(\frac{(N-1)(p-1)}{2}-N(p-1)it\right).
\end{aligned}$$

Here, the expansion formula for the log-gamma function is given by the following lemma (see, e.g., Barnes, 1899).

Lemma 2.2. Let z be a complex number, and let α be a constant. Then, it holds that

$$\log\Gamma(z+\alpha) = \left(z+\alpha-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \frac{1}{2}\beta_2(\alpha)z^{-1} + \varepsilon,$$

where $|\arg z| \le \pi$, $\beta_2(\alpha) = \alpha^2 - \alpha + 1/6$, $|\varepsilon| \le C|z|^{-2}$ and C is a positive constant.

From Lemma 2.2, the log-characteristic function $\log\varphi(t)$ can be expanded as

$$\log\varphi(t) = -\frac{f}{2}\log(1-2it) + \frac{M_{p,k}}{N}\{(1-2it)^{-1}-1\} + \varepsilon_1,$$

where $f = (p^2+p-4)/2$, $|\varepsilon_1| \le C_1 N^{-2}$ and

$$M_{p,k} = \frac{(p-1)^2 + 2(p-1)(k+1)}{4} + \frac{(p-2)(p-1)(2p+9)}{24} + \frac{(p-2)(4p-3)}{6(p-1)}. \tag{2.9}$$

Note that C_1 does not depend on t. Hence, the characteristic function of $-2\log\Lambda_1$ can be written as

$$\varphi(t) = e^{\log\varphi(t)} = (1-2it)^{-\frac{f}{2}} + \frac{M_{p,k}}{N}\left\{(1-2it)^{-\frac{f+2}{2}} - (1-2it)^{-\frac{f}{2}}\right\} + \varepsilon_2. \quad (2.10)$$

Here, it is easily checked that there exists a positive constant C_2 such that C_2 does not depend on t and $|\varepsilon_2| \le |t|^{-f/2}C_2N^{-2}$. Inverting (2.10), the distribution function of $-2\log\Lambda_1$ is given by

$$\mathrm{P}(-2\log\Lambda_1 \le c) = G_f(c) + \frac{M_{p,k}}{N}\{G_{f+2}(c) - G_f(c)\} + \varepsilon_3, \quad (2.11)$$

where $G_s(\cdot)$ is the distribution function of the chi-squared distribution with s degrees of freedom, and

$$|\varepsilon_3| \le \frac{C_2}{\pi N^2}\int_{-1}^{1}\left|\frac{\sin(ct)}{t}\right|dt + \frac{C_2}{\pi N^2}\int_{|t|>1}\frac{1}{|t|^{1+f/2}}dt = \frac{cC_3}{N^2} + \frac{C_4}{N^2}, \quad (C_3, C_4 > 0).$$

Therefore, from (2.7) and (2.11) we obtain the asymptotic expansion of the null distribution of $-2\log\Lambda$ as

$$\mathrm{P}(-2\log\Lambda \le c) = G_f(c) + \frac{M_{p,k}}{N}\{G_{f+2}(c) - G_f(c)\} + O(N^{-2}). \quad (2.12)$$

3. Asymptotic expansion of the null distribution when $\lambda^2 = 0$

Suppose that the hypothesis H_0 is true and $\lambda^2 = 0$. Let U be a random variable distributed as $Beta((N-k)/2, (N-1)(p-1)/2)$. Here, since s_1 and s_2 are independent, and $\lambda^2 = 0$, i.e., $\tau^2 = \sigma^2$, from (2.3) it holds that

$$\frac{s_1}{s_1+s_2} = \frac{\tau^2(s_1/\tau^2)}{\tau^2(s_1/\tau^2) + \sigma^2(s_2/\sigma^2)} = \frac{s_1/\tau^2}{s_1/\tau^2 + s_2/\sigma^2} \sim Beta\left(\frac{N-k}{2}, \frac{(N-1)(p-1)}{2}\right).$$

Hence, $s_1/(s_1+s_2)$ and U have the same distribution. Thus, from the definition of Λ, the distribution of $-2\log\Lambda$ is written by using U as

$$\mathrm{P}(-2\log\Lambda \le c) = \mathrm{P}(-2\log\Lambda_1 \le c)\mathrm{P}(U \ge 1/p) + \mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p), \quad (3.1)$$

where $W = -N\log\{p^p(p-1)^{-(p-1)}U(1-U)^{p-1}\}$. Next, we expand $\mathrm{P}(U \ge 1/p)$ and $\mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p)$.

3.1. Expansion of $\mathrm{P}(U \geq 1/p)$

Since U is distributed as $Beta((N-k)/2, (N-1)(p-1)/2)$, it holds that

$$\begin{aligned}\mathrm{P}\left(U < \frac{1}{p}\right) &= \frac{1}{B\left(\frac{N-k}{2}, \frac{(N-1)(p-1)}{2}\right)} \int_0^{\frac{1}{p}} u^{\frac{N-k}{2}-1}(1-u)^{\frac{(N-1)(p-1)}{2}-1} du \\ &= \varepsilon + \frac{1}{B\left(\frac{N-k}{2}, \frac{(N-1)(p-1)}{2}\right)} \int_{\frac{1}{p^{2p}}}^{\frac{1}{p}} u^{\frac{N-k}{2}-1}(1-u)^{\frac{(N-1)(p-1)}{2}-1} du, \\ \varepsilon &= \frac{1}{B\left(\frac{N-k}{2}, \frac{(N-1)(p-1)}{2}\right)} \int_0^{\frac{1}{p^{2p}}} u^{\frac{N-k}{2}-1}(1-u)^{\frac{(N-1)(p-1)}{2}-1} du,\end{aligned} \tag{3.2}$$

where $B(\cdot,\cdot)$ is the beta function. From Stirling's formula, the inverse of the beta function is expanded as

$$\begin{aligned}&\frac{1}{B\left(\frac{N-k}{2}, \frac{(N-1)(p-1)}{2}\right)} \\ &= \frac{\sqrt{N}}{2\sqrt{\pi}} \left\{\frac{p^p}{(p-1)^{p-1}}\right\}^{\frac{N}{2}} \frac{p^{-\frac{p+k}{2}}}{(p-1)^{-\frac{p}{2}}} \left\{e^{-\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)} + O(N^{-2})\right\}.\end{aligned} \tag{3.3}$$

From (3.3), for large N, it holds that

$$|\varepsilon| \leq C\sqrt{N}\left\{\frac{p^p}{(p-1)^{p-1}}\right\}^{\frac{N}{2}} \frac{p^{-pN}}{N} \leq p^{-\frac{N}{2}} = O(a_1^{-N}), \tag{3.4}$$

where a_1 and C are positive constants, and $a_1 > 1$.

On the other hand, the integral of (3.2) can be expressed as

$$\int_{\frac{1}{p^{2p}}}^{\frac{1}{p}} u^{\frac{N-k}{2}-1}(1-u)^{\frac{(N-1)(p-1)}{2}-1} du = \int_{\frac{1}{p^{2p}}}^{\frac{1}{p}} e^{\frac{N}{2}\log u(1-u)^{p-1}} u^{-\frac{k}{2}-1}(1-u)^{-\frac{p-1}{2}-1} du. \tag{3.5}$$

Let $f(u) = \log u(1-u)^{p-1}$ and $g(u) = u^{-k/2-1}(1-u)^{-(p-1)/2-1}$. Note that $f'(1/p) = 0$. Then, for any u satisfying $1/p^{2p} \leq u \leq 1/p$, using Taylor expansion at $u = 1/p$, $f(u)$ and $g(u)$ can be expanded as

$$\begin{aligned}f(u) &= s_0 + s_2\left(u - \frac{1}{p}\right)^2 + s_3\left(u - \frac{1}{p}\right)^3 + s_4\left(u - \frac{1}{p}\right)^4 + \varepsilon\left(u - \frac{1}{p}\right)^5, \\ g(u) &= g_0 + g_1\left(u - \frac{1}{p}\right) + g_2\left(u - \frac{1}{p}\right)^2 + \delta\left(u - \frac{1}{p}\right)^3,\end{aligned} \tag{3.6}$$

where

$$s_0 = f\left(\frac{1}{p}\right),\ s_2 = \frac{1}{2}f''\left(\frac{1}{p}\right),\ s_3 = \frac{1}{6}f'''\left(\frac{1}{p}\right),\ s_4 = \frac{1}{24}f^{(4)}\left(\frac{1}{p}\right),\ \varepsilon = \frac{1}{120}f^{(5)}(p^\star),$$

$$g_0 = g\left(\frac{1}{p}\right),\ g_1 = g'\left(\frac{1}{p}\right),\ g_2 = \frac{1}{2}g''\left(\frac{1}{p}\right),\ \delta = \frac{1}{6}f'''(\tilde{p}),$$

and $p^\star, \tilde{p} \in [1/p^{2p}, 1/p]$. Note that ε and δ are bounded. From (3.6), the right hand side of (3.5) is given by

$$e^{\frac{N}{2}s_0}\int_{\frac{1}{p^{2p}}}^{\frac{1}{p}}\Bigg[e^{\frac{N}{2}s_2\left(u-\frac{1}{p}\right)^2}e^{\frac{N}{2}\left\{s_3\left(u-\frac{1}{p}\right)^3+s_4\left(u-\frac{1}{p}\right)^4+\varepsilon\left(u-\frac{1}{p}\right)^5\right\}} \times\left\{g_0+g_1\left(u-\frac{1}{p}\right)+g_2\left(u-\frac{1}{p}\right)^2+\delta\left(u-\frac{1}{p}\right)^3\right\}\Bigg]du. \tag{3.7}$$

Let $\sqrt{-s_2}=t$, and let $\sqrt{N}t(u-1/p)=y$. Then, (3.7) can be written as

$$e^{\frac{N}{2}s_0}\frac{1}{\sqrt{N}t}\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}e^{\frac{s_3y^3}{2\sqrt{N}t^3}+\frac{s_4y^4}{2Nt^4}+\frac{\tilde{\varepsilon}y^5}{N\sqrt{N}}}\times\left(g_0+\frac{g_1y}{\sqrt{N}t}+\frac{g_2y^2}{Nt^2}+\frac{\tilde{\delta}y^3}{N\sqrt{N}}\right)dy, \tag{3.8}$$

where $\tilde{\varepsilon}$ and $\tilde{\delta}$ are bounded, and $T=t(p^{-1}-p^{-2p})$, $\tilde{\varepsilon}=2^{-1}t^{-5}\varepsilon$ and $\tilde{\delta}=t^{-3}\delta$. Let

$$w_1=\frac{s_3}{2t^3},\ w_2=\frac{s_4}{2t^4},\ z_1=\frac{g_1}{t},\ z_2=\frac{g_2}{t^2}.$$

Then, using $e^x=1+x+2^{-1}x^2+6^{-1}e^{x^*}x^3$, $x^*\in[0,x]$, (3.8) is expanded as

$$\begin{aligned}&Kg_0\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}dy+\frac{K}{\sqrt{N}}\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}(z_1y+g_0w_1y^3)dy\\&+\frac{K}{N}\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}\left\{z_2y^2+(g_0w_2+z_1w_1)y^4+\frac{g_0w_1^2}{2}y^6\right\}dy+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4,\end{aligned} \tag{3.9}$$

where

$$K=e^{\frac{N}{2}s_0}\frac{1}{\sqrt{N}t},\ \varepsilon_1=\frac{K}{N\sqrt{N}}\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}e^{\frac{N}{2}\alpha}\tilde{\delta}y^3dy,\ \varepsilon_2=K\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}g_0\omega dy,$$

$$\varepsilon_3=K\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}\frac{z_1y}{\sqrt{N}}\left(\omega+\frac{w_2y^4}{N}+\frac{w_1^2y^6}{2N}\right)dy,$$

$$\varepsilon_4=K\int_{-\sqrt{N}T}^{0}e^{-\frac{y^2}{2}}\frac{z_2y^2}{N}\left(\omega+\frac{w_2y^4}{N}+\frac{w_1^2y^6}{2N}+\frac{w_1y^3}{\sqrt{N}}\right)dy,$$

and

$$\frac{N}{2}\alpha=\frac{s_3y^3}{2\sqrt{N}t^3}+\frac{s_4y^4}{2Nt^4}+\frac{\tilde{\varepsilon}y^5}{N\sqrt{N}}=\frac{N}{2}\frac{1}{6}f'''(u^*)\left(u-\frac{1}{p}\right)^3,$$

$$\omega=\frac{\tilde{\varepsilon}y^5+w_1w_2y^7}{N\sqrt{N}}+\frac{w_2^2y^8+2w_1\tilde{\varepsilon}y^8}{2N^2}+\frac{(\tilde{\varepsilon})^2y^{10}}{2N^3}+\frac{w_2\tilde{\varepsilon}y^9}{N^2\sqrt{N}}+\frac{e^{\alpha^*}}{6}\left(\frac{w_1y^3}{\sqrt{N}}+\frac{w_2y^4}{N}+\frac{\tilde{\varepsilon}y^5}{N\sqrt{N}}\right)^3.$$

Note that $\alpha^*\le 0$, because, for any u satisfying $1/p^{2p}\le u\le 1/p$, $f'''(u)$ is non-negative, i.e.,

$$\frac{s_3y^3}{2\sqrt{N}t^3}+\frac{s_4y^4}{2Nt^4}+\frac{\tilde{\varepsilon}y^5}{N\sqrt{N}}=\frac{N}{2}\frac{1}{6}f'''(u^*)\left(u-\frac{1}{p}\right)^3\le 0.$$

Next, we evaluate each term in (3.9). It is easily checked that

$$\int_{-\sqrt{N}T}^{0} e^{-\frac{y^2}{2}} y^v dy = 2^{\frac{v-1}{2}} \Gamma\left(\frac{v+1}{2}\right)(-1)^v + O(c^{-N}), \quad (c > 1),$$

for any non-negative integer v. Thus, using this result, the first three terms in (3.9) are given by

$$\begin{aligned} &Kg_0 \int_{-\sqrt{N}T}^{0} e^{-\frac{y^2}{2}} dy = Kg_0 \left\{ \frac{\sqrt{2\pi}}{2} + O(c_1^{-N}) \right\}, \\ &\frac{K}{\sqrt{N}} \int_{-\sqrt{N}T}^{0} e^{-\frac{y^2}{2}} (z_1 y + g_0 w_1 y^3) dy = -\frac{K}{\sqrt{N}} \{ z_1 + 2g_0 w_1 + O(c_2^{-N}) \}, \\ &\frac{K}{N} \int_{-\sqrt{N}T}^{0} e^{-\frac{y^2}{2}} \left\{ z_2 y^2 + (g_0 w_2 + z_1 w_1) y^4 + \frac{g_0 w_1^2}{2} y^6 \right\} dy \\ &\quad = \frac{K}{N} \left\{ \frac{\sqrt{2\pi}}{2} z_2 + (g_0 w_2 + z_1 w_1) \frac{3\sqrt{2\pi}}{2} + \frac{15 g_0 w_1^2}{4} \sqrt{2\pi} + O(c_3^{-N}) \right\}, \end{aligned} \tag{3.10}$$

respectively, where $c_1, c_2, c_3 > 1$. Similarly, using $e^{N\alpha/2} \leq 1$, $e^{\alpha^*} \leq 1$ and $|y|^{m_1} \leq |y|^{m_2} + 1$, $(m_1 \leq m_2)$, we have

$$\begin{aligned} &|\varepsilon_1| \leq \frac{KC}{N\sqrt{N}} \int_0^{\infty} e^{-\frac{y^2}{2}} y^3 dy = K \times O(N^{-\frac{3}{2}}), \\ &|\varepsilon_i| \leq \frac{K}{N\sqrt{N}} \int_0^{\infty} e^{-\frac{y^2}{2}} (C_1 y^{13+i} + C_2) dy = K \times O(N^{-\frac{3}{2}}), \quad (i = 2, 3, 4), \end{aligned} \tag{3.11}$$

where $C, C_1, C_2 > 0$. Here, g_0, w_1, w_2, z_1 , z_2 and K are written by using p, k and N as

$$\begin{aligned} &K = \frac{1}{\sqrt{N}} \left\{ \frac{(p-1)^{p-1}}{p^p} \right\}^{\frac{N}{2}} \frac{\sqrt{2(p-1)}}{p\sqrt{p}}, \; g_0 = \frac{p^{\frac{k+p}{2}}}{(p-1)^{\frac{p}{2}}} \frac{p\sqrt{p}}{\sqrt{p-1}}, \\ &w_1 = \frac{\sqrt{2}(p-2)}{3\sqrt{p(p-1)}}, \; w_2 = -\frac{p^2 - 3p + 3}{2p(p-1)}, \; z_1 = \frac{p}{\sqrt{2}(p-1)} \frac{p^{\frac{k+p}{2}}}{(p-1)^{\frac{p}{2}}} \{2 + (1-p)(1+k)\}, \\ &z_2 = \frac{\sqrt{p(p-1)}}{4} \frac{p^{\frac{k+p}{2}}}{(p-1)^{\frac{p}{2}}} \left\{ (k+2)(k+4) - \frac{2(k+2)(p+1)}{p-1} + \frac{(p+1)(p+3)}{(p-1)^2} \right\}, \end{aligned} \tag{3.12}$$

respectively. Hence, from (3.9), (3.10), (3.11) and (3.12) we obtain

$$
\begin{aligned}
&\int_{\frac{1}{p^{2p}}}^{\frac{1}{p}} u^{\frac{N-k}{2}-1}(1-u)^{\frac{(N-1)(p-1)}{2}-1}du = \int_{\frac{1}{p^{2p}}}^{\frac{1}{p}} e^{\frac{N}{2}\log u(1-u)^{p-1}} u^{-\frac{k}{2}-1}(1-u)^{-\frac{p-1}{2}-1}du \\
&= \frac{1}{\sqrt{N}}\left\{\frac{(p-1)^{(p-1)}}{p^p}\right\}^{\frac{N}{2}} \frac{p^{\frac{p+k}{2}}}{(p-1)^{\frac{p}{2}}} \\
&\quad \times \left\{\sqrt{\pi} + \frac{1}{\sqrt{N}}\frac{3k(p-1)-p-1}{3\sqrt{p(p-1)}} + \frac{\sqrt{\pi}}{N}\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)} + O(N^{-\frac{3}{2}})\right\} \\
&= \frac{1}{\sqrt{N}}\left\{\frac{(p-1)^{(p-1)}}{p^p}\right\}^{\frac{N}{2}} \frac{p^{\frac{p+k}{2}}}{(p-1)^{\frac{p}{2}}} \\
&\quad \times \left\{\sqrt{\pi}e^{\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)} + \frac{1}{\sqrt{N}}\frac{3k(p-1)-p-1}{3\sqrt{p(p-1)}} + O(N^{-\frac{3}{2}})\right\}.
\end{aligned}
\tag{3.13}
$$

Finally, from (3.2), (3.3), (3.4) and (3.13), the probability $\mathrm{P}(U < 1/p)$ is expanded as

$$
\begin{aligned}
\mathrm{P}(U < 1/p) &= \frac{1}{2\sqrt{\pi}}\left\{\sqrt{\pi} + \frac{1}{\sqrt{N}}\frac{3k(p-1)-p-1}{3\sqrt{p(p-1)}} + O(N^{-\frac{3}{2}})\right\} \\
&= \frac{1}{2} + \frac{1}{\sqrt{N}}\frac{3k(p-1)-p-1}{6\sqrt{\pi p(p-1)}} + O(N^{-\frac{3}{2}}).
\end{aligned}
$$

This implies

$$
\mathrm{P}(U \geq 1/p) = 1 - \mathrm{P}(U < 1/p) = \frac{1}{2} - \frac{1}{\sqrt{N}}\frac{3k(p-1)-p-1}{6\sqrt{\pi p(p-1)}} + O(N^{-\frac{3}{2}}). \tag{3.14}
$$

3.2. Expansion of $\mathrm{P}(-2\log\Lambda_1 + W \leq c, U < 1/p)$

From the property of the conditional probability, $\mathrm{P}(-2\log\Lambda_1 + W \leq c, U < 1/p)$ can be written as

$$
\mathrm{P}(-2\log\Lambda_1 + W \leq c, U < 1/p) = \mathrm{P}(U < 1/p) \times \mathrm{P}(-2\log\Lambda_1 + W \leq c | U < 1/p). \tag{3.15}
$$

Let $Y = W|(U < 1/p)$, and let $\psi(y)$ be a probability density function of Y. Since Λ_1 and (W, U) are independent, Λ_1 and Y are also independent. Hence, it holds that

$$
\begin{aligned}
\mathrm{P}(-2\log\Lambda_1 + W \leq c | U < 1/p) &= \mathrm{P}(-2\log\Lambda_1 + Y \leq c) = \mathrm{P}(-2\log\Lambda_1 \leq c - Y) \\
&= \int \mathrm{P}(-2\log\Lambda_1 \leq c - y)\psi(y)dy.
\end{aligned}
\tag{3.16}
$$

From (2.11), the integral of (3.16) can be expressed as

$$\begin{aligned}\int \mathrm{P}(-2\log\Lambda_1 \le c-y)\psi(y)dy &= \int_0^c \mathrm{P}(-2\log\Lambda_1 \le c-y)\psi(y)dy \\ &= \int_0^c G_f(c-y)\psi(y)dy \\ &\quad + \frac{M_{p,k}}{N}\int_0^c \{G_{f+2}(c-y)-G_f(c-y)\}\psi(y)dy \\ &\quad + \int_0^c \varepsilon\psi(y)dy,\end{aligned} \tag{3.17}$$

because both $-2\log\Lambda_1$ and Y are non-negative. Here, the last integral of (3.17) satisfies that

$$\begin{aligned}\left|\int_0^c \varepsilon\psi(y)dy\right| &\le \frac{1}{N^2}\int_0^c \{(c-y)C_3+C_4\}\psi(y)dy \\ &\le \frac{1}{N^2}\int_0^c \{cC_3+C_4\}\psi(y)dy \\ &\le \frac{\{cC_3+C_4\}}{N^2}\int \psi(y)dy = \frac{\{cC_3+C_4\}}{N^2} = O(N^{-2}).\end{aligned} \tag{3.18}$$

Next, we expand $\psi(y)$.

We consider the following relation,

$$y = -N\log\frac{p^p}{(p-1)^{p-1}}u(1-u)^{p-1} = -N\log\frac{p^p}{(p-1)^{p-1}} - N\log u(1-u)^{p-1}, \tag{3.19}$$

where y is defined on $(0,c]$ and c is a positive constant. Similarly, u is defined on $[u^*, 1/p)$, and u^* is a positive number satisfying

$$-N\log\frac{p^p}{(p-1)^{p-1}}u^*(1-u^*)^{p-1} = c. \tag{3.20}$$

Let $f(u) = \log u(1-u)^{p-1}$. Then, the following expansion holds (see, Appendix).

$$u - \frac{1}{p} = -\frac{b_1}{\sqrt{N}}\sqrt{y} + \frac{b_2}{N}y + \frac{b_3}{N\sqrt{N}}y\sqrt{y} + \frac{\varepsilon}{N^2}y^2, \tag{3.21}$$

where

$$\begin{aligned}b_1 &= \sqrt{-\frac{2}{f''(1/p)}},\ b_2 = \frac{b_1^4}{12}f'''(1/p), \\ b_3 &= \frac{\{f'''(1/p)\}^2 b_1^7}{288} - \frac{f'''(1/p)b_1^3 b_2}{4} - \frac{f^{(4)}(1/p)b_1^5}{48},\ \varepsilon = O(1).\end{aligned}$$

On the other hand, given the event $U < 1/p$, a conditional probability density function $\phi(u)$ of U can be written as

$$\phi(u) = \begin{cases} \frac{1}{\mathrm{P}(U<1/p)B\left(\frac{N-k}{2}, \frac{(N-1)(p-1)}{2}\right)} u^{\frac{N-k}{2}-1}(1-u)^{\frac{(N-1)(p-1)}{2}-1} & (0<u<1/p) \\ 0 & (\text{otherwise}) \end{cases}.$$

Under $u < 1/p$, form (3.19) it holds that

$$\left|\frac{du}{dy}\right| = \frac{1}{\left|\frac{dy}{du}\right|} = -\frac{u(1-u)}{Np(u-1/p)}. \tag{3.22}$$

Hence, $\psi(y)$ is given by using the transformation formula and $\phi(u)$ as

$$\begin{aligned}\psi(y) &= \frac{1}{\mathrm{P}(U<1/p)B\left(\frac{N-k}{2}, \frac{(N-1)(p-1)}{2}\right)} u^{\frac{N-k}{2}-1}(1-u)^{\frac{(N-1)(p-1)}{2}-1}\left|\frac{du}{dy}\right| \\ &= \frac{1}{\mathrm{P}(U<1/p)}\beta^{-1}u^{N/2}(1-u)^{N(p-1)/2}u^{-k/2}(1-u)^{-(p-1)/2}\frac{1}{-Np(u-1/p)},\end{aligned}$$

where $\beta = B((N-k)/2, (N-1)(p-1)/2)$. Noting that

$$u^{\frac{N}{2}}(1-u)^{\frac{N(p-1)}{2}} = \left\{\frac{(p-1)^{p-1}}{p^p}\right\}^{\frac{N}{2}} e^{-\frac{y}{2}},$$

using (3.3), $\psi(y)$ can be expressed as

$$\begin{aligned}\psi(y) = &\frac{1}{\mathrm{P}(U<1/p)}\frac{1}{2\sqrt{\pi}}\frac{p^{-\frac{p+k}{2}-1}}{(p-1)^{-\frac{p}{2}}}\left\{e^{-\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)} + O(N^{-2})\right\} \\ &\times e^{-\frac{y}{2}}g(u)\frac{1}{-\sqrt{N}(u-1/p)},\end{aligned} \tag{3.23}$$

where $g(u) = u^{-k/2}(1-u)^{-(p-1)/2}$. Here, from (3.20), there exists a positive constant u^{**} such that $u^{**} < u^*$ for any N. Thus, using Taylor expansion at $u = 1/p$, $g(u)$ can be expanded as

$$\begin{aligned}g(u) &= g_0 + g_1(u-1/p) + \frac{1}{2}g_2(u-1/p)^2 + \delta_1(u-1/p)^3, \\ g_0 &= g(1/p),\ g_1 = g'(1/p),\ g_2 = g''(1/p),\ \delta_1 = O(1).\end{aligned} \tag{3.24}$$

Substituting (3.21) into (3.24), we obtain

$$\begin{aligned}g(u) &= g_0 + \frac{q_1}{\sqrt{N}}\sqrt{y} + \frac{q_2}{N}y + \frac{\varepsilon_1}{N\sqrt{N}}y\sqrt{y}, \\ q_1 &= -g_1b_1,\ q_2 = g_1b_2 + \frac{1}{2}g_2b_1^2,\ \varepsilon_1 = O(1).\end{aligned} \tag{3.25}$$

Similarly, substituting (3.21) into (3.22), we can write

$$\begin{aligned}\frac{1}{-\sqrt{N}(u-1/p)} &= \frac{1}{b_1\sqrt{y}\,(1-\delta^*)}, \\ \delta^* = \frac{c_1}{\sqrt{N}}\sqrt{y} + \frac{c_2}{N}y + \frac{\delta_2}{N\sqrt{N}}y\sqrt{y} &= o(1),\ c_1 = \frac{b_2}{b_1},\ c_2 = \frac{b_3}{b_1},\ \delta_2 = O(1).\end{aligned} \tag{3.26}$$

Since $(1-x)^{-1} = 1 + x + x^2 + \delta_3x^3,\ \delta_3 = O(1)$ when $x = o(1)$, we have

$$\frac{1}{1-\delta^*} = 1 + \frac{r_1}{\sqrt{N}}\sqrt{y} + \frac{r_2}{N}y + \frac{\varepsilon_2}{N\sqrt{N}}y\sqrt{y}, \tag{3.27}$$

where $r_1 = c_1$, $r_2 = c_2 + c_1^2$ and $\varepsilon_2 = O(1)$. Therefore, substituting (3.25), (3.26) and (3.27) into (3.23) yields

$$\begin{aligned}\psi(y) &= K\frac{1}{\sqrt{y}}e^{-\frac{y}{2}}\left(g_0 + \frac{q_1\sqrt{y}}{\sqrt{N}} + \frac{q_2 y}{N} + \frac{\varepsilon_1 y\sqrt{y}}{N\sqrt{N}}\right)\left(1 + \frac{r_1\sqrt{y}}{\sqrt{N}} + \frac{r_2 y}{N} + \frac{\varepsilon_2 y\sqrt{y}}{N\sqrt{N}}\right)\\ &= K\frac{1}{\sqrt{y}}e^{-\frac{y}{2}}\left(g_0 + \frac{v_1}{\sqrt{N}}\sqrt{y} + \frac{v_2}{N}y + \frac{\varepsilon_3}{N\sqrt{N}}y\sqrt{y}\right),\end{aligned} \tag{3.28}$$

where K, v_1, v_2 and ε_3 are given by

$$\begin{aligned}K &= \frac{1}{b_1}\,\frac{1}{\mathrm{P}(U < 1/p)}\,\frac{1}{2\sqrt{\pi}}\,\frac{p^{-\frac{p+k}{2}-1}}{(p-1)^{-\frac{p}{2}}}\left\{e^{-\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)} + O(N^{-2})\right\},\\ v_1 &= g_0 r_1 + q_1,\ v_2 = g_0 r_2 + q_1 r_1 + q_2,\ \varepsilon_3 = O(1),\end{aligned}$$

respectively. Let

$$J_0 = Kg_0\Gamma\left(\frac{1}{2}\right)2^{\frac{1}{2}},\ J_1 = Kv_1\Gamma\left(\frac{2}{2}\right)2^{\frac{2}{2}},\ J_2 = Kv_2\Gamma\left(\frac{3}{2}\right)2^{\frac{3}{2}}.$$

Furthermore, let $h_s(y)$ be a probability density function of the chi-squared distribution with s degrees of freedom. Then, from (3.28), the conditional probability density function $\psi(y)$ can be expressed as

$$\psi(y) = J_0 h_1(y) + \frac{J_1}{\sqrt{N}}h_2(y) + \frac{J_2}{N}h_3(y) + \frac{K\varepsilon_3 e^{-\frac{y}{2}}}{N\sqrt{N}}y. \tag{3.29}$$

Using the convolution formula

$$G_{f+s}(c) = \int_0^c G_f(c-y)h_s(y)dy,$$

substituting (3.29) into (3.17) yields

$$\begin{aligned}\mathrm{P}(-2\log\Lambda_1 + Y \le c) &= J_0 G_{f+1}(c) + \frac{J_1}{\sqrt{N}}G_{f+2}(c) + \frac{J_2}{N}G_{f+3}(c)\\ &\quad + \frac{J_0 M_{p,k}}{N}\{G_{f+3}(c) - G_{f+1}(c)\} + \varepsilon,\end{aligned} \tag{3.30}$$

where ε is given by

$$\begin{aligned}\varepsilon = &\frac{J_1 M_{p,k}}{N\sqrt{N}}\{G_{f+4}(c) - G_{f+2}(c)\} + \frac{J_2 M_{p,k}}{N^2}\{G_{f+5}(c) - G_{f+3}(c)\}\\ &+ \frac{K\varepsilon_3}{N\sqrt{N}}\int_0^c e^{-\frac{y}{2}}yG_f(c-y)dy\\ &+ \frac{K\varepsilon_3 M_{p,k}}{N^2\sqrt{N}}\int_0^c e^{-\frac{y}{2}}y\{G_{f+2}(c-y) - G_f(c-y)\}dy + O(N^{-2}).\end{aligned}$$

Here, K, J_0, J_1 and J_2 are given by

$$\begin{aligned}
K &= \frac{1}{\mathrm{P}(U<1/p)}\frac{1}{2\sqrt{2\pi}}\frac{p^{-\frac{p+k}{2}+\frac{1}{2}}}{(p-1)^{-\frac{p}{2}+\frac{1}{2}}}\left\{e^{-\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)}+O(N^{-2})\right\},\\
J_0 &= \frac{1}{2}\frac{1}{\mathrm{P}(U<1/p)}\left\{e^{-\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)}+O(N^{-2})\right\},\\
J_1 &= \frac{3k(p-1)-p-1}{6\sqrt{\pi p(p-1)}}\frac{1}{\mathrm{P}(U<1/p)}\left\{e^{-\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)}+O(N^{-2})\right\},\\
J_2 &= \frac{1}{2}\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\frac{1}{\mathrm{P}(U<1/p)}\left\{e^{-\frac{1}{N}\left(\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\right)}+O(N^{-2})\right\},
\end{aligned} \tag{3.31}$$

respectively. In addition, it is easily checked that $\varepsilon = O(N^{-3/2})$ because J_1, J_2, $M_{p,k}$, $G_s(c)$, $G_{s+2}(c) - G_s(c)$, K, ε_3 and $ye^{-y/2}$ are bounded under $0 < y \le c$. Therefore, from (3.15), (3.18), (3.30) and (3.31) we obtain

$$\begin{aligned}
&\mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p) = \mathrm{P}(-2\log\Lambda_1 + Y \le c)\times \mathrm{P}(U<1/p)\\
&\quad = \frac{1}{2}G_{f+1}(c) + \frac{1}{\sqrt{N}}\frac{3k(p-1)-p-1}{6\sqrt{\pi p(p-1)}}G_{f+2}(c)\\
&\qquad + \frac{1}{N}\frac{3k^2(p-1)^2+5p^2-2p-1}{12p(p-1)}\left\{\frac{1}{2}G_{f+3}(c) - \frac{1}{2}G_{f+1}(c)\right\}\\
&\qquad + \frac{M_{p,k}}{N}\left\{\frac{1}{2}G_{f+3}(c) - \frac{1}{2}G_{f+1}(c)\right\} + O(N^{-3/2}).
\end{aligned} \tag{3.32}$$

3.3. Final result

Substituting (2.11), (3.14) and (3.32) into (3.1), we derive the asymptotic expansion of the null distribution of $-2\log\Lambda$ as

$$\begin{aligned}
\mathrm{P}(-2\log\Lambda \le c) &= \frac{1}{2}G_f(c) + \frac{1}{2}G_{f+1}(c) + \frac{A_1}{\sqrt{N}}\{G_{f+2}(c) - G_f(c)\}\\
&\quad + \frac{A_2}{N}\{G_{f+2}(c) - G_f(c)\} + \frac{A_3}{N}\{G_{f+3}(c) - G_{f+1}(c)\} + O(N^{-3/2}),
\end{aligned} \tag{3.33}$$

where

$$A_1 = \frac{3k(p-1)-p-1}{6\sqrt{\pi p(p-1)}},\ A_2 = \frac{M_{p,k}}{2},\ A_3 = \frac{3k^2(p-1)^2+5p^2-2p-1}{24p(p-1)} + A_2. \tag{3.34}$$

4. Conservativeness and power

In Section 2 and 3, we derived the asymptotic expansions (2.12) and (3.33) when $\lambda^2 > 0$ and $\lambda^2 = 0$, respectively. For testing the hypothesis (1.3), we should use the asymptotic expansion (2.12) if $\lambda^2 > 0$ is true, on the other hand, we should use (3.33) if $\lambda^2 = 0$ is true. However, we do not know which is true because λ^2 is the unknown parameter. Hence, which expansion should we use, that is the problem. Nevertheless, one of the answers to this problem is given by the following theorem.

Theorem 4.1. Let Λ be the LR criterion for testing $\mathrm{H}_0 : \boldsymbol{\Sigma} = \lambda^2 \mathbf{1}_p \mathbf{1}_p' + \sigma^2 \boldsymbol{I}_p,\ \lambda^2 \geq 0,\ \sigma^2 > 0$ v.s. H_1 : not H_0 under the parallel profile model (1.1). Then, under H_0 it holds that

$$\mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_0) \leq \mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_0, \lambda^2 = 0).$$

Proof. Suppose that H_0 is true and $\lambda^2 > 0$. Under H_0, from Section 2, $-2\log\Lambda$ can be written as

$$-2\log\Lambda = \begin{cases} -2\log\Lambda_1, & \text{if } s_1/f_1 \geq s_2/f_2, \\ -2\log\Lambda_1 + Q, & \text{if } s_1/f_1 < s_2/f_2, \end{cases}$$

where, $f_1,\ f_2,\ s_1,\ s_2$ and Λ_1 are defined in Section 2, and Q is given by

$$Q = -N\log\left\{\frac{p^p}{(p-1)^{p-1}}\left(\frac{s_1}{s_1+s_2}\right)\left(1-\frac{s_1}{s_1+s_2}\right)^{p-1}\right\}. \tag{4.1}$$

Note that $\Lambda_1,\ s_1$ and s_2 are mutually independent, and

$$s_1 \sim \tau^2\chi^2_{N-k}, \quad s_2 \sim \sigma^2\chi^2_{(N-1)(p-1)}, \quad \tau^2 = p\lambda^2 + \sigma^2.$$

From the definition of Λ we have

$$\begin{aligned}\mathrm{P}(-2\log\Lambda \leq c \mid \mathrm{H}_0, \lambda^2 > 0) &= \mathrm{P}(-2\log\Lambda_1 \leq c, s_1/s_2 \geq (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0) \\ &\quad + \mathrm{P}(-2\log\Lambda_1 + Q \leq c, s_1/s_2 < (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0).\end{aligned} \tag{4.2}$$

Since $\sigma^2/\tau^2 < 1$, if $s_1/s_2 < (p-1)^{-1}$ then

$$\frac{1}{p} > \frac{s_1}{s_1+s_2} = \frac{s_1/\tau^2}{s_1/\tau^2 + (\sigma^2/\tau^2)\times s_2/\sigma^2} > \frac{s_1/\tau^2}{s_1/\tau^2 + s_2/\sigma^2} \equiv U \text{ (say) } > 0. \tag{4.3}$$

Note that $U \sim Beta((N-k)/2, (N-1)(p-1)/2)$. From (4.3) it holds that

$$Q \leq -N\log\frac{p^p}{(p-1)^{p-1}}U(1-U)^{p-1} \equiv W \text{ (say)},$$

because the function

$$-N\log\frac{p^p}{(p-1)^{p-1}}x(1-x)^{p-1}$$

is a decreasing function on $(0, 1/p)$. Thus, the following inequality holds.

$$\begin{aligned}&\mathrm{P}(-2\log\Lambda_1 + W \leq c, s_1/s_2 < (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0) \\ &\leq \mathrm{P}(-2\log\Lambda_1 + Q \leq c, s_1/s_2 < (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0).\end{aligned} \tag{4.4}$$

Hence, using (4.2) and (4.4) it holds that

$$\begin{aligned}\mathrm{P}(-2\log\Lambda \le c \mid \mathrm{H}_0, \lambda^2 > 0) \ge{}& \mathrm{P}(-2\log\Lambda_1 \le c, s_1/s_2 \ge (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0) \\ &+ \mathrm{P}(-2\log\Lambda_1 + W \le c, s_1/s_2 < (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0).\end{aligned} \tag{4.5}$$

Furthermore, the right hand side of (4.5) satisfies that

$$\begin{aligned}&\mathrm{P}(-2\log\Lambda_1 \le c, s_1/s_2 \ge (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0) \\ &\quad + \mathrm{P}(-2\log\Lambda_1 + W \le c, s_1/s_2 < (p-1)^{-1} \mid \mathrm{H}_0, \lambda^2 > 0) \\ &= \mathrm{P}\left(-2\log\Lambda_1 \le c, \frac{s_1/\tau^2}{s_2/\sigma^2} \ge \frac{1}{p-1} \,\middle|\, \mathrm{H}_0, \lambda^2 > 0\right) \\ &\quad + \mathrm{P}\left(-2\log\Lambda_1 \le c, \frac{1}{p-1} > \frac{s_1/\tau^2}{s_2/\sigma^2} \ge \frac{\sigma^2}{\tau^2}\frac{1}{p-1} \,\middle|\, \mathrm{H}_0, \lambda^2 > 0\right) \\ &\quad + \mathrm{P}\left(-2\log\Lambda_1 + W \le c, \frac{s_1/\tau^2}{s_2/\sigma^2} < \frac{1}{p-1} \,\middle|\, \mathrm{H}_0, \lambda^2 > 0\right) \\ &\quad - \mathrm{P}\left(-2\log\Lambda_1 + W \le c, \frac{1}{p-1} > \frac{s_1/\tau^2}{s_2/\sigma^2} \ge \frac{\sigma^2}{\tau^2}\frac{1}{p-1} \,\middle|\, \mathrm{H}_0, \lambda^2 > 0\right) \\ &= \mathrm{P}(-2\log\Lambda_1 \le c, U \ge 1/p \mid \mathrm{H}_0, \lambda^2 > 0) + \mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p \mid \mathrm{H}_0, \lambda^2 > 0) \\ &\quad + \mathrm{P}\left(-2\log\Lambda_1 \le c, \frac{1}{p-1} > \frac{s_1/\tau^2}{s_2/\sigma^2} \ge \frac{\sigma^2}{\tau^2}\frac{1}{p-1} \,\middle|\, \mathrm{H}_0, \lambda^2 > 0\right) \\ &\quad - \mathrm{P}\left(-2\log\Lambda_1 + W \le c, \frac{1}{p-1} > \frac{s_1/\tau^2}{s_2/\sigma^2} \ge \frac{\sigma^2}{\tau^2}\frac{1}{p-1} \,\middle|\, \mathrm{H}_0, \lambda^2 > 0\right) \\ &\ge \mathrm{P}(-2\log\Lambda_1 \le c, U \ge 1/p \mid \mathrm{H}_0, \lambda^2 > 0) + \mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p \mid \mathrm{H}_0, \lambda^2 > 0).\end{aligned} \tag{4.6}$$

Here, the last inequality is derived by $W \ge 0$. Noting that Λ_1, U and W do not depend on λ^2, substituting (4.6) into (4.5) yields

$$\begin{aligned}&\mathrm{P}(-2\log\Lambda \le c \mid \mathrm{H}_0, \lambda^2 > 0) \\ &\ge \mathrm{P}(-2\log\Lambda_1 \le c, U \ge 1/p \mid \mathrm{H}_0, \lambda^2 > 0) + \mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p \mid \mathrm{H}_0, \lambda^2 > 0) \\ &= \mathrm{P}(-2\log\Lambda_1 \le c \mid \mathrm{H}_0, \lambda^2 > 0)\mathrm{P}(U \ge 1/p \mid \mathrm{H}_0, \lambda^2 > 0) \\ &\quad + \mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p \mid \mathrm{H}_0, \lambda^2 > 0) \\ &= \mathrm{P}(-2\log\Lambda_1 \le c \mid \mathrm{H}_0, \lambda^2 = 0)\mathrm{P}(U \ge 1/p \mid \mathrm{H}_0, \lambda^2 = 0) \\ &\quad + \mathrm{P}(-2\log\Lambda_1 + W \le c, U < 1/p \mid \mathrm{H}_0, \lambda^2 = 0) \\ &= (3.1) = \mathrm{P}(-2\log\Lambda \le c \mid \mathrm{H}_0, \lambda^2 = 0).\end{aligned} \tag{4.7}$$

This implies that $\mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_0) \le \mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_0, \lambda^2 = 0)$ □

From Theorem 4.1, the actual test size of the test using the null distribution under $\lambda^2 = 0$, is always smaller than the nominal test size α even if $\lambda^2 > 0$ is true. The nominal test size α can be chosen freely by the analyst, and α should not be overestimated. In this sense, this testing method is conservative (the safest). For this reason, we suggest using the asymptotic expansion (3.33) for testing the hypothesis (1.3).

On the other hand, in the certain alternative hypothesis, the power of the test using the null distribution under $\lambda^2 = 0$ is given by the following corollary.

Corollary 4.1. Let Λ be the LR criterion defined as in Theorem 4.1, and let

$$\mathrm{H}_{11} : \boldsymbol{\Sigma} = \rho \mathbf{1}_p \mathbf{1}_p' + \sigma^2 \boldsymbol{I}_p, \ 0 > \rho > -\frac{\sigma^2}{p}, \ \sigma^2 > 0.$$

Then, under H_{11} it holds that

$$\mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_0, \lambda^2 = 0) \le \mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_{11}).$$

Proof. Suppose that H_{11} is true. Under H_{11}, since Σ has the exchangeable covariance structure, $-2\log\Lambda$ can be written as

$$-2\log\Lambda = \begin{cases} -2\log\Lambda_1, & \text{if } s_1/f_1 \ge s_2/f_2, \\ -2\log\Lambda_1 + Q, & \text{if } s_1/f_1 < s_2/f_2, \end{cases}$$

where, $f_1, \ f_2, \ s_1, \ s_2$ and Λ_1 are defined in Section 2, and Q is given by (4.1). Note that $\Lambda_1, \ s_1$ and s_2 are mutually independent, and

$$s_1 \sim \upsilon^2 \chi^2_{N-k}, \quad s_2 \sim \sigma^2 \chi^2_{(N-1)(p-1)}, \quad \upsilon^2 = p\rho + \sigma^2.$$

Noting that $\sigma^2/\upsilon^2 > 1$ and Y is non-negative, we have

$$\begin{aligned}
&\mathrm{P}(-2\log\Lambda \le c \mid \mathrm{H}_{11}) \\
&= \mathrm{P}(-2\log\Lambda_1 \le c, s_1/s_2 \ge (p-1)^{-1} \mid \mathrm{H}_{11}) \\
&\quad + \mathrm{P}(-2\log\Lambda_1 + Q \le c, s_1/s_2 < (p-1)^{-1} \mid \mathrm{H}_{11}) \\
&= \mathrm{P}\left(-2\log\Lambda_1 \le c, \frac{s_1/\upsilon^2}{s_2/\sigma^2} \ge \frac{1}{p-1} \,\middle|\, \mathrm{H}_{11}\right) \\
&\quad - \mathrm{P}\left(-2\log\Lambda_1 \le c, \frac{1}{p-1} \le \frac{s_1/\upsilon^2}{s_2/\sigma^2} < \frac{\sigma^2}{\upsilon^2}\frac{1}{p-1} \,\middle|\, \mathrm{H}_{11}\right) \\
&\quad + \mathrm{P}\left(-2\log\Lambda_1 + Q \le c, \frac{s_1/\upsilon^2}{s_2/\sigma^2} < \frac{1}{p-1} \,\middle|\, \mathrm{H}_{11}\right) \\
&\quad + \mathrm{P}\left(-2\log\Lambda_1 + Q \le c, \frac{1}{p-1} \le \frac{s_1/\upsilon^2}{s_2/\sigma^2} < \frac{\sigma^2}{\upsilon^2}\frac{1}{p-1} \,\middle|\, \mathrm{H}_{11}\right) \\
&\le \mathrm{P}\left(-2\log\Lambda_1 \le c, \frac{s_1/\upsilon^2}{s_2/\sigma^2} \ge \frac{1}{p-1} \,\middle|\, \mathrm{H}_{11}\right) \\
&\quad + \mathrm{P}\left(-2\log\Lambda_1 + Q \le c, \frac{s_1/\upsilon^2}{s_2/\sigma^2} < \frac{1}{p-1} \,\middle|\, \mathrm{H}_{11}\right).
\end{aligned}$$

Since $\sigma^2/\upsilon^2 > 1$, if $(s_1/\upsilon^2)/(s_2/\sigma^2) < (p-1)^{-1}$ then

$$0 < \frac{s_1}{s_1 + s_2} = \frac{s_1/\upsilon^2}{s_1/\upsilon^2 + (\sigma^2/\upsilon^2) \times s_2/\sigma^2} < \frac{s_1/\upsilon^2}{s_1/\upsilon^2 + s_2/\sigma^2} \equiv U^* \text{ (say) } < \frac{1}{p-1}. \quad (4.8)$$

Note that $U^* \sim Beta((N-k)/2, (N-1)(p-1)/2)$. From (4.8) it holds that

$$Q \geq -N \log \frac{p^p}{(p-1)^{p-1}} U^*(1-U^*)^{p-1} \equiv W^* \text{ (say)}.$$

Therefore, similarly as to evaluations of (4.4) and (4.7) in the proof of Theorem 4.1, we obtain

$$\mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_0, \lambda^2 = 0) \leq \mathrm{P}(-2\log\Lambda > c \mid \mathrm{H}_{11}).$$

□

Corollary 4.1 implies that, for the fixed nominal test size α, the actual test size (or power) of the test using the null distribution under $\lambda^2 = 0$, is larger than α under H_{11}. Therefore, this testing method is also better from the viewpoint of the power.

Remark 4.1. The null hypothesis H_{0^*} considered by Srivastava and Singull (2012) includes H_{11}. Hence, for the fixed nominal test size α, the actual test size (or power) of the test using the M-LRT proposed by Srivastava and Singull (2012), is equal to α under H_{11}. Therefore, under the certain alternative hypothesis H_{11}, the test using the null distribution of R-LRT proposed by Yokoyama (1995) under $\lambda^2 = 0$, is better than the test using the M-LRT from the viewpoint of the power.

5. Numerical experiments

To compare the accuracy of the approximations based on the limiting distribution and the asymptotic expansion, we compute the actual test sizes (ATSs) of $-2\log\Lambda$. Let

$$\begin{aligned}
F^{(\lambda^2>0)}(c) &= G_f(c) + \frac{M_{p,k}}{N}\{G_{f+2}(c) - G_f(c)\}, \\
F^{(\lambda^2=0)}(c) &= \frac{1}{2}G_f(c) + \frac{1}{2}G_{f+1}(c) + \frac{A_1}{\sqrt{N}}\{G_{f+2}(c) - G_f(c)\} \\
&\quad + \frac{A_2}{N}\{G_{f+2}(c) - G_f(c)\} + \frac{A_3}{N}\{G_{f+3}(c) - G_{f+1}(c)\},
\end{aligned}$$

where $M_{p,k}$ is given by (2.9) and $A_1,\ A_2,\ A_3$ are given by (3.34). Let α be the nominal test size. Then, from 100,000 monte carlo simulation runs, in the case of $\lambda^2 > 0$ the ATSs of $-2\log\Lambda$ based on the limiting distribution and the asymptotic expansion are computed as

$$\hat{\alpha}_{\lambda^2>0} = \frac{\#\{t\ |G_f(t) > 1-\alpha\}}{100000},\ \tilde{\alpha}_{\lambda^2>0} = \frac{\#\{t\ |F^{(\lambda^2>0)}(t) > 1-\alpha\}}{100000},$$

respectively. Similarly, in the case of $\lambda^2 = 0$ these are computed as

$$\hat{\alpha}_{\lambda^2=0} = \frac{\#\{t\ |0.5 \times (G_f(t) + G_{f+1}(t)) > 1-\alpha\}}{100000},\ \tilde{\alpha}_{\lambda^2=0} = \frac{\#\{t\ |F^{(\lambda^2=0)}(t) > 1-\alpha\}}{100000},$$

respectively. Here, t is the values of $-2\log\Lambda$ calculated from the simulation data. Under the null hypothesis we considered the following six cases.

$$
\begin{array}{llll}
(1) & p=2,\ k=2,\ \lambda^2=1,\ \sigma^2=1, & (2) & p=4,\ k=2,\ \lambda^2=1,\ \sigma^2=1, \\
(3) & p=2,\ k=5,\ \lambda^2=1,\ \sigma^2=1, & (4) & p=2,\ k=2,\ \lambda^2=0,\ \sigma^2=1, \\
(5) & p=4,\ k=2,\ \lambda^2=0,\ \sigma^2=1, & (6) & p=2,\ k=5,\ \lambda^2=0,\ \sigma^2=1.
\end{array}
$$

The ATSs in the cases (1)–(3) and (4)–(6) are given in Table 1 and Table 2, respectively, for $\alpha = 0.05$.

Table 1. The ATSs in the cases (1)–(3)

	Sample size N	Nominal 5% test $\hat{\alpha}_{\lambda^2>0}$	$\tilde{\alpha}_{\lambda^2>0}$
Case (1)	10	0.11776	0.07249
	20	0.07384	0.05382
	30	0.06532	0.05226
	40	0.06047	0.05005
	50	0.05953	0.05133
	80	0.05671	0.05180
	100	0.05430	0.05001
Case (2)	10	0.24649	0.13524
	20	0.11172	0.06703
	30	0.08576	0.05714
	40	0.07563	0.05454
	50	0.06871	0.05267
	80	0.06177	0.05079
	100	0.05813	0.04986
Case (3)	10	0.28712	0.18017
	20	0.10971	0.06863
	30	0.08318	0.05713
	40	0.07303	0.05395
	50	0.06786	0.05352
	80	0.06188	0.05272
	100	0.05654	0.04930

Table 2. The ATSs in the cases (4)–(6)

	Sample size N	Nominal 5% test $\hat{\alpha}_{\lambda^2=0}$	$\tilde{\alpha}_{\lambda^2=0}$
Case (4)	10	0.12881	0.07344
	20	0.08451	0.05691
	30	0.07110	0.05210
	40	0.06607	0.05208
	50	0.06320	0.05196
	80	0.05849	0.05034
	100	0.05646	0.04952
Case (5)	10	0.26490	0.14340
	20	0.11929	0.06738
	30	0.09053	0.05782
	40	0.07823	0.05406
	50	0.07120	0.05237
	80	0.06294	0.05108
	100	0.06075	0.05071
Case (6)	10	0.38579	0.22048
	20	0.16566	0.08472
	30	0.11993	0.06411
	40	0.10114	0.05883
	50	0.09308	0.05777
	80	0.07656	0.05188
	100	0.07278	0.05203

From Table 1 and 2 we can see that the accuracy of the approximations of $\tilde{\alpha}_{\lambda^2>0}$ and $\tilde{\alpha}_{\lambda^2=0}$ are better than that of $\hat{\alpha}_{\lambda^2>0}$ and $\hat{\alpha}_{\lambda^2=0}$, respectively. However, for the small sample size, the approximation of the asymptotic expansion is still not good when p and k are not very small. Nevertheless, for any natural number s, applying the same techniques used in Section 2 and 3 we can derive the asymptotic expansion up to the order of N^{-s}. Therefore, further improvement in the accuracy of the approximation is possible.

6. Conclusion

We derived the asymptotic expansions of the null distribution of the L-LRT proposed by Yokoyama (1995). Numerical experiments showed that the accuracy of the approximation of the asymptotic expansion is better than that of the limiting distribution. Furthermore, Applying the same techniques used in Section 2 and 3, further improvement in the accuracy of the approximation is possible. Therefore, "the accuracy of the approximation" of the test using the L-LRT was improved.

On the other hand, from Theorem 4.1, we showed that the test Assuming $\lambda^2 = 0$ is the safest. In addition, from Corollary 4.1, we also showed that the power of the test assuming $\lambda^2 = 0$ is larger than the nominal test size α in the certain alternative hypothesis H_{11}. Recall that the test using the M-LRT proposed by Srivastava and Singull (2012) is not detected the hypothesis H_{11}. Hence, under H_{11}, the test using the L-LRT is better than the test using the M-LRT.

Appendix : derivation of (3.21)

Consider the following relation

$$y = -N \log \frac{p^p}{(p-1)^{p-1}} u(1-u)^{p-1} = -N \log \frac{p^p}{(p-1)^{p-1}} - N \log u(1-u)^{p-1},$$

where y is defined on $(0, c]$ and c is a positive constant. Similarly, u is defined on $[u^*, 1/p)$ and u^* is a positive number satisfying

$$-N \log \frac{p^p}{(p-1)^{p-1}} u^*(1-u^*)^{p-1} = c.$$

Here, for simplicity, we denote by $\mathrm{N}(x; \varepsilon)$ the ε-neighborhood of x. Let

$$f(u) = \log u(1-u)^{p-1}.$$

Using Taylor expansion at $u = 1/p$, $f(u)$ can be expanded as

$$f(u) = f(1/p) + f'(1/p)(u - 1/p) + \frac{1}{2} f''(u_1)(u - 1/p)^2, \ u^* \le u_1 < 1/p.$$

Nothing that $-N\{f(u) - f(1/p)\} = y$, $f'(1/p) = 0$ and $u < 1/p$, we have

$$u - \frac{1}{p} = -\frac{\sqrt{y}}{\sqrt{N}} \sqrt{-\frac{2}{f''(u_1)}} \equiv -\frac{\sqrt{y}}{\sqrt{N}} c_1, \tag{A.1}$$

where $f''(u_1)$ and c_1 are bounded. Again, using Taylor expansion at $u = 1/p$, $f(u)$ can be expressed as

$$f(u) = f(1/p) + \frac{1}{2}f''(1/p)(u-1/p)^2 + \frac{1}{6}f'''(u_2)(u-1/p)^3,\ u^* \leq u_2 < 1/p.$$

From this expansion and (A.1), we obtain

$$u - \frac{1}{p} = -\frac{\sqrt{y}}{\sqrt{N}}\sqrt{-\frac{2}{f''(1/p)}}\sqrt{1 + \frac{1}{6}f'''(u_2)\frac{\sqrt{y}}{\sqrt{N}}c_1^3} \equiv -\frac{\sqrt{y}}{\sqrt{N}}\sqrt{-\frac{2}{f''(1/p)}}\sqrt{1 + \frac{\varepsilon_1\sqrt{y}}{\sqrt{N}}}, \tag{A.2}$$

where $\varepsilon_1\sqrt{y}N^{-1/2} = o(1)$ because $f'''(u_2)$, c_1 and y are bounded. Let $g(x) = (1-x)^{1/2}$. Then, using Taylor expansion at $x = 0$, $g(x)$ is expressed as

$$\sqrt{1-x} = 1 + g'(x^*)x,\ x^* \in \mathrm{N}(0;|x|).$$

Thus, we have

$$\sqrt{1 + \frac{\varepsilon_1\sqrt{y}}{\sqrt{N}}} = 1 - g'(\delta_1)\frac{\varepsilon_1\sqrt{y}}{\sqrt{N}},\ \delta_1 \in \mathrm{N}(0;|\varepsilon_1|\sqrt{y}N^{-1/2}). \tag{A.3}$$

Note that $g'(\delta_1)$ is bounded because $\delta_1 = o(1)$. Therefore, from (A.2) and (A.3) we obtain

$$u - \frac{1}{p} = -\frac{\sqrt{y}}{\sqrt{N}}\sqrt{-\frac{2}{f''(1/p)}} + \frac{y}{N}g'(\delta_1)\varepsilon_1\sqrt{-\frac{2}{f''(1/p)}} \equiv -\frac{\sqrt{y}}{\sqrt{N}}b_1 + \frac{y}{N}c_2, \tag{A.4}$$

where c_2 is bounded and b_1 is given by

$$b_1 = \sqrt{-\frac{2}{f''(1/p)}}.$$

Similarly, from (A.1) and (A.4), using Taylor expansion at $u = 1/p$, $f(u)$ can be expanded as

$$\begin{aligned}
f(u) &= f(1/p) + \frac{1}{2}f''(1/p)(u-1/p)^2 + \frac{1}{6}f'''(1/p)(u-1/p)^3 + \frac{1}{24}f^{(4)}(u_3)(u-1/p)^4, \\
&= f(1/p) + \frac{1}{2}f''(1/p)(u-1/p)^2 + \frac{1}{6}f'''(1/p)\left(-\frac{b_1^3}{N\sqrt{N}}y\sqrt{y}\right) \\
&\quad + \frac{1}{6}f'''(1/p)\left(\frac{3b_1^2c_2}{N^2}y^2 - \frac{3b_1c_2^2y^2\sqrt{y}}{N^2\sqrt{N}} + \frac{c_2^3}{N^3}y^3\right) + \frac{1}{24}f^{(4)}(u_3)\frac{c_1^4}{N^2}y^2 \\
&= f(1/p) + \frac{1}{2}f''(1/p)(u-1/p)^2 + \frac{1}{6}f'''(1/p)\left(-\frac{b_1^3}{N\sqrt{N}}y\sqrt{y}\right) + \frac{y^2}{N^2}\varepsilon_2,
\end{aligned}$$

where u_3 is defined on $[u^*, 1/p)$ and ε_2 is bounded. Hence, from this expansion, we have

$$u - \frac{1}{p} = -\frac{\sqrt{y}}{\sqrt{N}}b_1\sqrt{1 - \left\{\frac{f'''(1/p)}{6\sqrt{N}}b_1^3\sqrt{y} - \frac{\varepsilon_2}{N}y\right\}} \equiv -\frac{\sqrt{y}}{\sqrt{N}}b_1\sqrt{1-\varepsilon_3}, \tag{A.5}$$

where $\varepsilon_3 = o(1)$. Using

$$\sqrt{1-x} = 1 - \frac{1}{2}x + \frac{1}{2}g''(x^*)x^2, \ x^* \in \mathrm{N}(0; |x|),$$

$(1-\varepsilon_3)^{1/2}$ can be expressed as

$$\begin{aligned}\sqrt{1-\varepsilon_3} &= 1 - \frac{1}{2}\varepsilon_3 + \frac{1}{2}g''(\delta_2)\varepsilon_3^2 \\ &= 1 - \frac{f'''(1/p)}{12\sqrt{N}}b_1^3\sqrt{y} + \frac{\varepsilon_2 y}{2N} + \frac{g''(\delta_2)}{2}\frac{y}{N}\left\{\frac{f'''(1/p)}{6}b_1^3 - \frac{\varepsilon_2}{\sqrt{N}}\sqrt{y}\right\}^2 \\ &= 1 - \frac{f'''(1/p)}{12\sqrt{N}}b_1^3\sqrt{y} + \frac{y}{N}\varepsilon_4, \ \delta_2 \in \mathrm{N}(0; |\varepsilon_3|), \qquad \text{(A.6)}\end{aligned}$$

where $g''(\delta_2)$ and ε_4 are bounded because $\delta_2 = o(1)$. Therefore, substituting (A.6) into (A.5) yields

$$u - \frac{1}{p} = -\frac{\sqrt{y}}{\sqrt{N}}b_1 + \frac{y}{N}\frac{b_1^4}{12}f'''(1/p) - \frac{y\sqrt{y}}{N\sqrt{N}}b_1\varepsilon_4 = -\frac{\sqrt{y}}{\sqrt{N}}b_1 + \frac{y}{N}b_2 + \frac{y\sqrt{y}}{N\sqrt{N}}\varepsilon_5, \quad \text{(A.7)}$$

$$b_2 = \frac{b_1^4}{12}f'''(1/p),$$

where ε_5 is bounded.

Finally, using Taylor expansion at $u = 1/p$ for $f(u)$ up to the fifth order, the expansion for $g(x)$ up to the second order, (A.1) and (A.7), we obtain

$$u - \frac{1}{p} = -\frac{b_1}{\sqrt{N}}\sqrt{y} + \frac{b_2}{N}y + \frac{b_3}{N\sqrt{N}}y\sqrt{y} + \frac{\varepsilon}{N^2}y^2, \ \varepsilon = O(1),$$

$$b_3 = \frac{\{f'''(1/p)\}^2 b_1^7}{288} - \frac{f'''(1/p)b_1^3 b_2}{4} - \frac{f^{(4)}(1/p)b_1^5}{48}.$$

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