Simultaneous Testing of Mean Vectors and Covariance Matrices with Monotone Missing Data

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Abstract

In this paper, we consider the multi-sample problem of simultaneous testing of the mean vectors and the covariance matrices when the data sets have a same monotone pattern of missing observations. We give the likelihood ratio test (LRT) statistic and we propose the modified LRT statistics by using the decomposition of the LR. Finally, we investigate the asymptotic behavior of the upper percentiles of these test statistics by Monte Carlo simulation.

Key Words and Phrases: Asymptotic expansion, Modified likelihood ratio test statistic, k-step monotone missing data.

1 Introduction

Testing the problem with missing data is an important problem in statistical data analyses. A variety of statistical procedures to deal with missing data have been developed by many authors. In this paper, we consider the problem of simultaneous testing of the mean vectors and the covariance matrices under monotone missing data. For non-missing and multivariate normality in the case of one-sample problem, the modified LRT statistics were discussed by Davis (1971), Muirhead (1982), Siotani Hayakawa and Fujikoshi (1985), and Srivastava (2002) and so on. Hao and Krishnamoorthy (2001) and Hosoya and Seo (2015) gave the LRT statistic and modified LRT statistics using a decomposition of the LRT statistic when the data have a two-step monotone pattern that is missing observations. As an extension of two-step monotone missing case, in this paper, the LRT and modified LRT statistics are given under multivariate normality with a general k-step monotone missing data pattern. As with one-sample case, for two or a general m-sample problem, the LRT and the modified LRT have been discussed by Muirhead (1982) and so on. Recently, Hosoya and Seo (2016) gave some modified LRT statistic under two-step monotone missing data. However, there is not LRT and modified LRT for a general m-sample case under a general k-step monotone missing data.

The maximum likelihood estimators (MLEs) of the mean vectors and the covariance matrices with monotone missing data are key estimators to derive the LRT. For the one-sample problem case, Jinadasa and Tracy (1992) gave the MLEs as closed form in the case of the general k-step monotone missing data. Yagi and Seo (2015) obtained the MLEs for the m-sample as an extension of the results in Jinadasa and Tracy (1992). Using the results, we give the LRT and the modified LRTs. The remainder of this paper is organized as follows. In Section 2, we review the LRT and modified LRT statistics for the equality of several normal populations in the case of non-missing data. Further, we describe the LRT and modified LRT statistics for testing the hypothesis that a mean vector and a covariance matrix are equal to a given vector and matrix. In addition, we review the LRT and modified LRT statistics for testing the equality of several covariance matrices and testing the hypothesis that a covariance matrix is equal to a given matrix in the case of non-missing data. In Section 3, we present assumptions and notations to be used throughout this paper. In Section 4, we deal with LRT and modified LRT statistics for the multi-sample and one-sample problems under a general k-step monotone missing data. In Section 5, some simulation results for the m-sample ($m = 2, 3$) and one-sample cases are presented to investigate the accuracy of the approximation to the upper percentiles of the null distributions of modified LRT statistics.

2 Complete data case

In this section, we review testing the equality of several normal populations and testing the equality of several covariance matrices for non-missing and normality.

2.1 Testing equality of several normal populations

Let $\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{n^{(\ell)}}^{(\ell)}$ be independently distributed as $N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)})$, $\ell = 1, 2, \ldots, m$. Then the ℓ-th sample mean vector and the matrix of sums of squares and products are denoted as

$$\overline{\boldsymbol{x}}^{(\ell)} = \frac{1}{n^{(\ell)}} \sum_{j=1}^{n^{(\ell)}} \boldsymbol{x}_j^{(\ell)},$$

$$\boldsymbol{E}^{(\ell)} = \sum_{j=1}^{n^{(\ell)}} (\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})(\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})',$$

respectively. We consider the following hypothesis,

$$H_{01} : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \cdots = \boldsymbol{\mu}^{(m)},\ \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)} \text{ vs. } H_{11} : \text{not } H_{01}.$$

Then the LRT is based on the statistic

$$\lambda_{\mathrm{S}(m)} = \frac{\prod_{\ell=1}^{m} \left| \frac{1}{n^{(\ell)}} \boldsymbol{E}^{(\ell)} \right|^{\frac{n^{(\ell)}}{2}}}{\left| \frac{1}{n} \boldsymbol{W} \right|^{\frac{n}{2}}},$$

where $n = \sum_{\ell=1}^{m} n^{(\ell)}$,

$$\boldsymbol{W} = \sum_{\ell=1}^{m} \sum_{j=1}^{n^{(\ell)}} (\boldsymbol{x}_j^{(\ell)} - \overline{\overline{\boldsymbol{x}}})(\boldsymbol{x}_j^{(\ell)} - \overline{\overline{\boldsymbol{x}}})',$$

$$\overline{\overline{\boldsymbol{x}}} = \frac{1}{n} \sum_{\ell=1}^{m} n^{(\ell)} \overline{\boldsymbol{x}}^{(\ell)}.$$

We note that

$$\boldsymbol{W} = \sum_{\ell=1}^{m} \boldsymbol{E}^{(\ell)} + \sum_{\ell=1}^{m} n^{(\ell)} (\overline{\boldsymbol{x}}^{(\ell)} - \overline{\overline{\boldsymbol{x}}})(\overline{\boldsymbol{x}}^{(\ell)} - \overline{\overline{\boldsymbol{x}}})'.$$

Further, when the null hypothesis $H_{01} : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \cdots = \boldsymbol{\mu}^{(m)},\ \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)}$ is true, an asymptotic expansion for the distribution of $-2\rho_1 \log \lambda_{\mathrm{S}(m)}$ is derived by Muirhead (1982, p.308), where

$$\rho_1 = 1 - \frac{2p^2 + 9p + 11}{6(m-1)(p+3)n} \left(\sum_{\ell=1}^{m} \frac{n}{n^{(\ell)}} - 1 \right).$$

We note that an approximate test of size α of H_{01} is to reject H_{01} if $-2\rho_1 \log \lambda_{\mathrm{S}(m)} > \chi^2_{f_1,\alpha}$, where $\chi^2_{f_1,\alpha}$ denotes the upper 100α percentile of the χ^2 distribution with $f_1(= p(p+3)(m-1)/2)$ degrees of freedom, and the value of ρ_1 makes the term of order n^{-1} vanish in the asymptotic expansion of the distribution of $-2\rho_1 \log \lambda_{\mathrm{S}(m)}$. That is, this means that the error in this approximation is of order n^{-2}. For one-sample case, on the other hand, the hypothesis can be written as

$$H_{02} : \boldsymbol{\mu} = \mathbf{0},\ \boldsymbol{\Sigma} = \boldsymbol{I} \text{ vs. } H_{12} : \text{not } H_{02}.$$

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be independently distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then the LRT is based on the statistic

$$\lambda_{\mathrm{S}(1)} = e^{\frac{np}{2}} \left| \frac{1}{n} \boldsymbol{E} \right|^{\frac{n}{2}} \mathrm{etr}\left(-\frac{1}{2} \boldsymbol{E} \right) \exp\left(-\frac{1}{2} n \overline{\boldsymbol{x}}' \overline{\boldsymbol{x}} \right),$$

where

$$\boldsymbol{E} = \sum_{j=1}^{n} (\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})', \quad \overline{\boldsymbol{x}} = \frac{1}{n} \sum_{j=1}^{n} \boldsymbol{x}_j.$$

Similarly, the modified LRT statistic $-2\rho_2 \log \lambda_{\mathrm{S}(1)}$ is given by Muirhead (1982, p.370), where

$$\rho_2 = 1 - \frac{2p^2 + 9p + 11}{6n(p+3)}.$$

Therefore, we note that the cumulative distribution function of $-2\rho_2 \log \lambda_{\mathrm{S}(1)}$ can be expressed as

$$\Pr\{-2\rho_2 \log \lambda_{\mathrm{S}(1)} \leq x\} = G_{f_2}(x) + O(n^{-2}),$$

where $G_{f_2}(x)$ is the cumulative distribution function of the χ^2 distribution with $f_2 = p(p+3)/2$ degrees of freedom.

2.2 Testing equality of several covariance matrices

In this subsection, since testing the equality of several covariance matrices is used for testing equality of several normal populations with monotone missing data, we review the LRT and MLRT statistics for testing the equality of several covariance matrices with non-missing data. For the derivation of these results, see Muirhead (1982).

We first consider the hypothesis test,

$$H_{03} : \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)} \text{ vs. } H_{13} : \text{not } H_{03}.$$

With the same assumption of m-sample problem in Subsection 2.1, the LR is expressed as

$$\lambda_{\mathrm{V}(m)} = \frac{\displaystyle\prod_{\ell=1}^{m} \left| \frac{1}{n^{(\ell)}-1} \boldsymbol{E}^{(\ell)} \right|^{\frac{n^{(\ell)}-1}{2}}}{\displaystyle\left| \frac{1}{n-m} \sum_{\ell=1}^{m} \boldsymbol{E}^{(\ell)} \right|^{\frac{n-m}{2}}},$$

and the modified LRT statistic can be expressed as $-2\rho_3 \log \lambda_{\mathrm{V}(m)}$, where

$$\rho_3 = 1 - \frac{2p^2+3p-1}{6(p+1)(m-1)(n-m)} \left(\sum_{\ell=1}^{m} \frac{n-m}{n^{(\ell)}-1} - 1 \right).$$

We note that $\lambda_{\mathrm{V}(m)}$ is the slightly modified statistic in order to obtain the unbiased test. As with Subsection 2.1, for one-sample case, the LR for the hypothesis $H_{04} : \boldsymbol{\Sigma} = \boldsymbol{I}$ vs. H_{14} : not H_{04}, is expressed as

$$\lambda_{\mathrm{V}(1)} = e^{\frac{(n-1)p}{2}} \left| \frac{1}{n-1} \boldsymbol{E} \right|^{\frac{n-1}{2}} \mathrm{etr} \left(-\frac{1}{2} \boldsymbol{E} \right),$$

and the modified LRT statistic can be expressed as $-2\rho_4 \log \lambda_{\mathrm{V}(1)}$, where

$$\rho_4 = 1 - \frac{2p^2+3p-1}{6(n-1)(p+1)}.$$

Also, the cumulative distribution functions of $-2\rho_3 \log \lambda_{\mathrm{V}(m)}$ and $-2\rho_4 \log \lambda_{\mathrm{V}(1)}$ are given by

$$\Pr\{-2\rho_3 \log \lambda_{\mathrm{V}(m)} \le x\} = G_{f_3}(x) + O(n^{-2}),$$
$$\Pr\{-2\rho_4 \log \lambda_{\mathrm{V}(1)} \le x\} = G_{f_4}(x) + O(n^{-2}),$$

respectively, where $G_{f_i}(x)$, $i = 3, 4$ is the cumulative distribution function of the χ^2 distribution with f_i degrees of freedom, and $f_3 = p(p+1)(m-1)/2$, $f_4 = p(p+1)/2$.

3 Assumptions and notations

In this section, we give the assumptions and notations as preliminaries. Let $\boldsymbol{x}_i^{(\ell)}$ be a $p_i \times 1$ normal random vector with mean vector $\boldsymbol{\mu}_i^{(\ell)}$ and the covariance matrix $\boldsymbol{\Sigma}_i^{(\ell)}$, where $\boldsymbol{\mu}_i^{(\ell)} = (\boldsymbol{\mu}^{(\ell)})_i = (\mu_1^{(\ell)}, \mu_2^{(\ell)}, \ldots, \mu_{p_i}^{(\ell)})'$ and $\boldsymbol{\Sigma}_i^{(\ell)}$ is the $p_i \times p_i$ principal submatrix of $\boldsymbol{\Sigma}^{(\ell)} (= \boldsymbol{\Sigma}_1^{(\ell)})$, with $p = p_1 > p_2 > \cdots > p_k > 0$, $i = 1, 2, \ldots, k$, $\ell = 1, 2, \ldots, m$. Suppose that $\boldsymbol{x}_{i1}^{(\ell)}, \boldsymbol{x}_{i2}^{(\ell)}, \ldots, \boldsymbol{x}_{in_i^{(\ell)}}^{(\ell)}$ are independent and identically distributed samples from $\boldsymbol{x}_i$,

$i = 1, 2, \ldots, k$, $n_1^{(\ell)} > p$. Further, let $\boldsymbol{x}_i^{(\ell)}$, $i = 1, 2, \ldots, k$ be independent mutually. Note that k denotes the number of steps. As for the partitions of $\boldsymbol{\Sigma}^{(\ell)}$, for $1 \leq i < j \leq k$, let $(\boldsymbol{\Sigma}_i^{(\ell)})_j$ be the principal submatrix of $\boldsymbol{\Sigma}_i^{(\ell)}$ of order $p_j \times p_j$; we define, for $i = 2, 3, \ldots, k$, $\ell = 1, 2, \ldots, m$,

$$\boldsymbol{\Sigma}_i^{(\ell)} = (\boldsymbol{\Sigma}_1^{(\ell)})_i, \quad \boldsymbol{\Sigma}_1^{(\ell)} = \boldsymbol{\Sigma}^{(\ell)} = \begin{pmatrix} \boldsymbol{\Sigma}_i^{(\ell)} & \boldsymbol{\Sigma}_{i2}^{(\ell)} \\ \boldsymbol{\Sigma}_{i2}^{(\ell)\prime} & \boldsymbol{\Sigma}_{i3}^{(\ell)} \end{pmatrix}, \quad \boldsymbol{\Sigma}_{i-1}^{(\ell)} = \begin{pmatrix} \boldsymbol{\Sigma}_i^{(\ell)} & \boldsymbol{\Sigma}_{(i-1,2)}^{(\ell)} \\ \boldsymbol{\Sigma}_{(i-1,2)}^{(\ell)\prime} & \boldsymbol{\Sigma}_{(i-1,3)}^{(\ell)} \end{pmatrix}.$$

We note that in one-sample problem, the condition is $n_i > p_i$ instead of $n_1^{(\ell)} > p$.

4 Monotone missing data case

In this section, we consider the LRT and the modified LRT statistics for simultaneous testing of the mean vectors and the covariance matrices for multi-sample and one-sample cases under a general k-step monotone missing data.

4.1 Multi-sample problem

In this subsection, we will consider the LRT statistic for testing the equality of m normal populations with k-step monotone missing data. With the assumptions and notations as in Section 3, the LR for $H_{01} : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \cdots = \boldsymbol{\mu}^{(m)}$, $\boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)}$ vs. H_{11} : not H_{01}, can be obtained as

$$\tau_{(m)} = \frac{\displaystyle\prod_{\ell=1}^{m}\prod_{i=1}^{k}\left|\widehat{\boldsymbol{\Sigma}}_i^{(\ell)}\right|^{\frac{n_i^{(\ell)}}{2}}}{\displaystyle\prod_{i=1}^{k}\left|\widetilde{\boldsymbol{\Sigma}}_i\right|^{\frac{\nu_i}{2}}},$$

where $\nu_i = \sum_{\ell=1}^{m} n_i^{(\ell)}$, $i = 1, 2, \ldots, k$, $\widehat{\boldsymbol{\Sigma}}_i^{(\ell)}$ is the MLE of $\boldsymbol{\Sigma}_i$ from the ℓ-th population under H_{11}, and $\widetilde{\boldsymbol{\Sigma}}_i$ is the MLE of $\boldsymbol{\Sigma}_i$ under H_{01}. We will express $\widehat{\boldsymbol{\Sigma}}^{(\ell)}$ and $\widetilde{\boldsymbol{\Sigma}}$ concretely.

Let

$$\overline{\boldsymbol{x}}_i^{(\ell)} = \frac{1}{n_i^{(\ell)}} \sum_{j=1}^{n_i^{(\ell)}} \boldsymbol{x}_{ij}^{(\ell)}, \quad \boldsymbol{E}_i^{(\ell)} = \sum_{j=1}^{n_i^{(\ell)}} (\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i^{(\ell)})(\boldsymbol{x}_{ij}^{(\ell)} - \overline{\boldsymbol{x}}_i^{(\ell)})', \quad i = 1, 2, \ldots, k$$

$$\boldsymbol{d}_1^{(\ell)} = \overline{\boldsymbol{x}}_1^{(\ell)}, \quad \boldsymbol{d}_i^{(\ell)} = \frac{n_i^{(\ell)}}{N_{i+1}^{(\ell)}} \left[\overline{\boldsymbol{x}}_i^{(\ell)} - \frac{1}{N_i^{(\ell)}} \sum_{j=1}^{i-1} n_j^{(\ell)} (\overline{\boldsymbol{x}}_j^{(\ell)})_i \right], \quad i = 2, 3, \ldots, k$$

$$N_1^{(\ell)} = 0, \ N_{i+1}^{(\ell)} = N_i^{(\ell)} + n_i^{(\ell)} \ (= \sum_{j=1}^{i} n_j^{(\ell)}), \quad i = 1, 2, \ldots, k.$$

Then, we note that $\widehat{\boldsymbol{\Sigma}}^{(\ell)}$, $\ell = 1, 2, \ldots, m$ is given by

$$\widehat{\boldsymbol{\Sigma}}^{(\ell)} = \frac{1}{n_1^{(\ell)}} \boldsymbol{H}_1^{(\ell)} + \sum_{i=2}^{k} \frac{1}{N_{i+1}^{(\ell)}} \boldsymbol{F}_i^{(\ell)} \left[\boldsymbol{H}_i^{(\ell)} - \frac{n_i^{(\ell)}}{N_i^{(\ell)}} \boldsymbol{L}_{i-1,1}^{(\ell)} \right] \boldsymbol{F}_i^{(\ell)\prime},$$

where

$$\begin{aligned}
&\boldsymbol{H}_1^{(\ell)} = \boldsymbol{E}_1^{(\ell)}, \ \boldsymbol{H}_i^{(\ell)} = \boldsymbol{E}_i^{(\ell)} + \frac{N_i^{(\ell)} N_{i+1}^{(\ell)}}{n_i^{(\ell)}} \boldsymbol{d}_i^{(\ell)} \boldsymbol{d}_i^{(\ell)\prime}, \ i = 2, 3, \ldots, k, \\
&\boldsymbol{L}_1^{(\ell)} = \boldsymbol{H}_1^{(\ell)}, \ \boldsymbol{L}_i^{(\ell)} = (\boldsymbol{L}_{i-1}^{(\ell)})_i + \boldsymbol{H}_i^{(\ell)}, \ i = 2, 3, \ldots, k, \\
&\boldsymbol{L}_{i1}^{(\ell)} = (\boldsymbol{L}_i^{(\ell)})_{i+1}, \ \boldsymbol{L}_i^{(\ell)} = \begin{pmatrix} \boldsymbol{L}_{i1}^{(\ell)} & \boldsymbol{L}_{i2}^{(\ell)} \\ \boldsymbol{L}_{i2}^{(\ell)\prime} & \boldsymbol{L}_{i3}^{(\ell)} \end{pmatrix}, \ i = 1, 2, \ldots, k-1, \\
&\boldsymbol{G}_1^{(\ell)} = \boldsymbol{I}_{p_1}, \ \boldsymbol{G}_{i+1}^{(\ell)} = \begin{pmatrix} \boldsymbol{I}_{p_{i+1}} \\ \boldsymbol{L}_{i2}^{(\ell)\prime} \boldsymbol{L}_{i1}^{(\ell)-1} \end{pmatrix}, \ i = 1, 2, \ldots, k-1, \\
&\boldsymbol{F}_1^{(\ell)} = \boldsymbol{G}_1^{(\ell)}, \ \boldsymbol{F}_i^{(\ell)} = \boldsymbol{F}_{i-1}^{(\ell)} \boldsymbol{G}_i^{(\ell)}, \ i = 2, 3, \ldots, k.
\end{aligned}$$

On the other hand, let

$$M_i = \sum_{\ell=1}^{m} N_i^{(\ell)}, \quad i = 1, 2, \ldots, k+1,$$

$$\overline{\overline{\boldsymbol{x}}}_i = \frac{1}{\nu_i} \sum_{\ell=1}^{m} n_i^{(\ell)} \overline{\boldsymbol{x}}_i^{(\ell)}, \quad \boldsymbol{W}_i = \sum_{\ell=1}^{m} \sum_{j=1}^{n_i^{(\ell)}} (\boldsymbol{x}_{ij}^{(\ell)} - \overline{\overline{\boldsymbol{x}}}_i)(\boldsymbol{x}_{ij}^{(\ell)} - \overline{\overline{\boldsymbol{x}}}_i)', \quad i = 1, 2, \ldots, k,$$

$$\boldsymbol{h}_1 = \overline{\overline{\boldsymbol{x}}}_1, \quad \boldsymbol{h}_i = \frac{\nu_i}{M_{i+1}} \left[\overline{\overline{\boldsymbol{x}}}_i - \frac{1}{M_i} \sum_{j=1}^{i-1} \nu_j (\overline{\overline{\boldsymbol{x}}}_j)_i \right], \quad i = 2, 3, \ldots, k.$$

Then, the MLE of $\boldsymbol{\Sigma}$ under H_{01} is given by

$$\widetilde{\boldsymbol{\Sigma}} = \frac{1}{\nu_1} \boldsymbol{V}_1 + \sum_{i=2}^{k} \frac{1}{M_{i+1}} \boldsymbol{P}_i \left[\boldsymbol{V}_i - \frac{\nu_i}{M_i} \boldsymbol{R}_{i-1,1} \right] \boldsymbol{P}_i',$$

where

$$\boldsymbol{V}_1 = \boldsymbol{W}_1, \quad \boldsymbol{V}_i = \boldsymbol{W}_i + \frac{M_i M_{i+1}}{\nu_i}\boldsymbol{h}_i\boldsymbol{h}_i', \quad i = 2, 3, \ldots, k,$$
$$\boldsymbol{R}_1 = \boldsymbol{V}_1, \ \boldsymbol{R}_i = (\boldsymbol{R}_{i-1})_i + \boldsymbol{V}_i, \ i = 2, 3, \ldots, k,$$
$$\boldsymbol{R}_{i1} = (\boldsymbol{R}_i)_{i+1}, \ \boldsymbol{R}_i = \begin{pmatrix} \boldsymbol{R}_{i1} & \boldsymbol{R}_{i2} \\ \boldsymbol{R}_{i2}' & \boldsymbol{R}_{i3} \end{pmatrix}, \ i = 1, 2, \ldots, k-1,$$
$$\boldsymbol{Q}_1 = \boldsymbol{I}_{p_1}, \ \boldsymbol{Q}_{i+1} = \begin{pmatrix} \boldsymbol{I}_{p_{i+1}} \\ \boldsymbol{R}_{i2}'\boldsymbol{R}_{i1}^{-1} \end{pmatrix}, \ i = 1, 2, \ldots, k-1,$$
$$\boldsymbol{P}_1 = \boldsymbol{Q}_1, \ \boldsymbol{P}_i = \boldsymbol{P}_{i-1}\boldsymbol{Q}_i, \ i = 2, 3, \ldots, k.$$

We note that the above results essentially coincide with one-sample case of Jinadasa and Tracy (1992). Thus, we can obtain the LRT statistic $-2\log\tau_{(m)}$ whose null distribution is asymptotically distributed as a χ^2 distribution with f_1 degrees of freedom when the sample sizes are large.

Since the 1-st step data of the monotone missing data is a complete data part, $\lambda_{\mathrm{V}(m)}$ and ρ_3 for the 1-st step data are given by

$$\lambda_{\mathrm{V}(m)} = \frac{\prod_{\ell=1}^{m} \left| \frac{1}{n_1^{(\ell)} - 1}\boldsymbol{E}_1^{(\ell)} \right|^{\frac{n_1^{(\ell)}-1}{2}}}{\left| \frac{1}{\nu_1 - m}\sum_{\ell=1}^{m}\boldsymbol{E}_1^{(\ell)} \right|^{\frac{\nu_1 - m}{2}}},$$

and

$$\rho_3 = 1 - \frac{2p^2 + 3p - 1}{6(p+1)(m-1)(\nu_1 - m)}\left(\sum_{\ell=1}^{m} \frac{\nu_1 - m}{n_1^{(\ell)} - 1} - 1 \right),$$

respectively. Therefore, we can decompose $\tau_{(m)}$ as $\tau_{(m)} = \lambda_{\mathrm{V}(m)}\gamma_{(m)}^{\ddagger}$, and we propose the modified LRT statistic as $-2\log\tau_{(m)}^{\ddagger}$, where $\tau_{(m)}^{\ddagger} = \lambda_{\mathrm{V}(m)}^{\rho_3}\gamma_{(m)}^{\ddagger}$. Similarly, for the 1-st step data, $\lambda_{\mathrm{S}(m)}$ and ρ_1 are given by

$$\lambda_{\mathrm{S}(m)} = \frac{\prod_{\ell=1}^{m} \left| \frac{1}{n_1^{(\ell)}}\boldsymbol{E}_1^{(\ell)} \right|^{\frac{n_1^{(\ell)}}{2}}}{\left| \frac{1}{\nu_1}\boldsymbol{W}_1 \right|^{\frac{\nu_1}{2}}},$$

and

$$\rho_1 = 1 - \frac{2p^2 + 9p + 11}{6(m-1)(p+3)\nu_1}\left(\sum_{\ell=1}^{m} \frac{\nu_1}{n_1^{(\ell)}} - 1 \right),$$

respectively. Using $\lambda_{\mathrm{S}(m)}$ in this case, we can write $\tau_{(m)}$ as $\tau_{(m)} = \lambda_{\mathrm{S}(m)}\gamma^*_{(m)}$. Therefore we propose the other modified LRT statistic as $-2\log\tau^*_{(m)}$, where $\tau^*_{(m)} = \lambda^{\rho_1}_{\mathrm{S}(m)}\gamma^*_{(m)}$, It is expected that null distribution of these modified statistics converge faster to the χ^2 distribution than that of the LRT statistic when the sample sizes become large.

4.2 One-sample problem

We consider the following hypothesis,

$$H_{02} : \boldsymbol{\mu} = \boldsymbol{\mu}_0,\ \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 \text{ vs. } H_{12} : \text{not } H_{02}.$$

Without loss of generality, we can assume that $\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$, and define

$$\begin{aligned}
&\boldsymbol{E}_i = \sum_{j=1}^{n_i}(\boldsymbol{x}_{ij} - \overline{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij} - \overline{\boldsymbol{x}}_i)',\ \ \overline{\boldsymbol{x}}_i = \frac{1}{n_i}\sum_{j=1}^{n_i}\boldsymbol{x}_{ij},\quad i = 1, 2, \ldots, k,\\
&\boldsymbol{d}_1 = \overline{\boldsymbol{x}}_1,\quad \boldsymbol{d}_i = \frac{n_i}{N_{i+1}}\left[\overline{\boldsymbol{x}}_i - \frac{1}{N_i}\sum_{j=1}^{i-1} n_j(\overline{\boldsymbol{x}}_j)_i\right],\quad i = 2, 3, \ldots, k,\\
&N_1 = 0,\quad N_{i+1} = N_i + n_i\ (= \sum_{j=1}^{i} n_j),\quad i = 1, 2, \ldots, k.
\end{aligned}$$

Then, the LR is given by

$$\tau_{(1)} = \prod_{i=1}^{k}\left[e^{\frac{n_i p_i}{2}}|\widehat{\boldsymbol{\Sigma}}_i|^{\frac{n_i}{2}}\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{E}_i\right)\exp\left(-\frac{1}{2}n_i\overline{\boldsymbol{x}}_i'\overline{\boldsymbol{x}}_i\right)\right],$$

where $\widehat{\boldsymbol{\Sigma}}_i$ is the MLE of $\boldsymbol{\Sigma}_i$ under H_{12}. That is, $\widehat{\boldsymbol{\Sigma}}$ is given by

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{n_1}\boldsymbol{H}_1 + \sum_{i=2}^{k}\frac{1}{N_{i+1}}\boldsymbol{F}_i\left[\boldsymbol{H}_i - \frac{n_i}{N_i}\boldsymbol{L}_{i-1,1}\right]\boldsymbol{F}_i',$$

where

$$\begin{aligned}
&\boldsymbol{H}_1 = \boldsymbol{E}_1,\ \boldsymbol{H}_i = \boldsymbol{E}_i + \frac{N_i N_{i+1}}{n_i}\boldsymbol{d}_i\boldsymbol{d}_i',\ i = 2, 3, \ldots, k,\\
&\boldsymbol{L}_1 = \boldsymbol{H}_1,\ \boldsymbol{L}_i = (\boldsymbol{L}_{i-1})_i + \boldsymbol{H}_i,\ i = 2, 3, \ldots, k,\\
&\boldsymbol{L}_{i1} = (\boldsymbol{L}_i)_{i+1},\ \boldsymbol{L}_i = \begin{pmatrix}\boldsymbol{L}_{i1} & \boldsymbol{L}_{i2}\\ \boldsymbol{L}_{i2}' & \boldsymbol{L}_{i3}\end{pmatrix},\ i = 1, 2, \ldots, k-1,\\
&\boldsymbol{G}_1 = \boldsymbol{I}_{p_1},\ \boldsymbol{G}_{i+1} = \begin{pmatrix}\boldsymbol{I}_{p_{i+1}}\\ \boldsymbol{L}_{i2}'\boldsymbol{L}_{i1}^{-1}\end{pmatrix},\ i = 1, 2, \ldots, k-1,\\
&\boldsymbol{F}_1 = \boldsymbol{G}_1,\ \boldsymbol{F}_i = \boldsymbol{F}_{i-1}\boldsymbol{G}_i,\ i = 2, 3, \ldots, k.
\end{aligned}$$

This result of MLE is obtained by Jinadasa and Tracy (1992, p.45).

By decomposition of $\tau_{(1)}$, as with Subsection 4.1, we propose two modified LRT statistics, $-2\log\tau^{\ddagger}_{(1)}$ and $-2\log\tau^{*}_{(1)}$, where $\tau^{\ddagger}_{(1)} = \lambda^{\rho_4}_{\mathrm{V}(1)}\gamma^{\ddagger}_{(1)}$, $\tau^{*}_{(1)} = \lambda^{\rho_2}_{\mathrm{S}(1)}\gamma^{*}_{(1)}$,

$$\lambda_{\mathrm{V}(1)} = e^{\frac{(n_1-1)p}{2}}\left|\frac{1}{n_1-1}\boldsymbol{E}_1\right|^{\frac{n_1-1}{2}}\mathrm{etr}\left(-\frac{1}{2}\boldsymbol{E}_1\right),$$
$$\rho_4 = 1 - \frac{2p^2+3p-1}{6(n_1-1)(p+1)},$$

and

$$\lambda_{\mathrm{S}(1)} = e^{\frac{n_1 p}{2}}\left|\frac{1}{n_1}\boldsymbol{E}_1\right|^{\frac{n_1}{2}}\mathrm{etr}\left(-\frac{1}{2}\boldsymbol{E}_1\right)\exp\left(-\frac{1}{2}n_1\overline{\boldsymbol{x}}_1'\,\overline{\boldsymbol{x}}_1\right),$$
$$\rho_2 = 1 - \frac{2p^2+9p+11}{6n_1(p+3)}.$$

5 Simulation studies

In this section, we will study the numerical accuracy and the asymptotic behaviors of the χ^2 approximations for the proposed statistics under the multi-sample and one-sample problems. In order to investigate the accuracy of the approximation for the multi-sample case (or one-sample case), we compute the upper 100α percentiles of $-2\log\tau_{(m)}$, $-2\log\tau^{\ddagger}_{(m)}$, and $-2\log\tau^{*}_{(m)}$ (or $-2\log\tau_{(1)}$, $-2\log\tau^{\ddagger}_{(1)}$, and $-2\log\tau^{*}_{(1)}$) by Monte Carlo simulation. For each parameter, the simulation involved 10^6 times based on the normal random vectors $\boldsymbol{x}^{(\ell)}_{ij}$ (or $\boldsymbol{x}_{ij}$) generated from $N_{p_i}(\boldsymbol{0}, \boldsymbol{I}_{p_i})$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n^{(\ell)}_i$, $\ell = 1, 2, \ldots, m$. Computations are carried out for the following cases (with $\alpha = 0.05, 0.01$):

(I) Multi-sample, Five-step Case:

$(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$,

$n^{(\ell)}_1 = 20, 30, 50, 100, 200, 300,\ n^{(\ell)}_2 = n^{(\ell)}_3 = \cdots = n^{(\ell)}_5 = 10, 15, 20, 50,$

$\ell = 1, 2, \ldots, m,\ m = 2, 3.$

(II) One-sample Case:

(II-I) Two-step Case:

$(p_1, p_2) = (4, 2),\ (8, 4),$

$n_1 = 10, 20, 30, 50, 100,\ n_2 = 10, 20, 50, 100,$

(II-II) Two-step Case:

$(p_1, p_2) = (15, 12),$

$n_1 = 20, 30, 50, 100, 200, 300,\ n_2 = 15, 20, 50, 100,$

(II-III) Three-step Case:

$(p_1, p_2, p_3) = (15, 12, 9),$

$n_1 = 20, 30, 50, 100, 200, 300,\ n_2 = n_3 = 15, 20, 50, 100,$

(II-IV) Five-step Case:

$(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3),$

$n_1 = 20, 30, 50, 100, 200, 300,\ n_2 = n_3 = \cdots = n_5 = 15, 20, 50, 100.$

Further, we compute the actual type I error rates for the upper percentiles of $-2\log\tau_{(m)}$, $-2\log\tau_{(1)}$, $-2\log\tau^{\ddagger}_{(m)}$, $-2\log\tau^{\ddagger}_{(1)}$, $-2\log\tau^{*}_{(m)}$, and $-2\log\tau^{*}_{(1)}$ given by

$$\begin{aligned}
\alpha_{\tau_{(m)}} &= \Pr\{-2\log\tau_{(m)} > \chi^2_{f_1,\alpha}\}, & \alpha_{\tau_{(1)}} &= \Pr\{-2\log\tau_{(1)} > \chi^2_{f_2,\alpha}\},\\
\alpha_{\tau^{\ddagger}_{(m)}} &= \Pr\{-2\log\tau^{\ddagger}_{(m)} > \chi^2_{f_1,\alpha}\}, & \alpha_{\tau^{\ddagger}_{(1)}} &= \Pr\{-2\log\tau^{\ddagger}_{(1)} > \chi^2_{f_2,\alpha}\},\\
\alpha_{\tau^{*}_{(m)}} &= \Pr\{-2\log\tau^{*}_{(m)} > \chi^2_{f_1,\alpha}\}, & \alpha_{\tau^{*}_{(1)}} &= \Pr\{-2\log\tau^{*}_{(1)} > \chi^2_{f_2,\alpha}\},
\end{aligned}$$

respectively, where $\chi^2_{f_i,\alpha}$ is the upper 100α percentile of the χ^2 distribution with f_i $(i = 1, 2)$ degrees of freedom.

In Tables 1 and 2, we provide the simulated results of Case (I). It may be noted from Tables 1 and 2 that the values of $\alpha_{\tau^{\ddagger}_{(m)}}$ and $\alpha_{\tau^{*}_{(m)}}$ are approximately 0.05 when $n_1^{(\ell)} \geq 300$. In Tables 3 – 5, we provide the simulated results of two-step cases (Cases (II-I), (II-II)). It may be noted from Tables 3 – 5 that the values of (i) $\alpha_{\tau^{\ddagger}_{(1)}}$ and (ii) $\alpha_{\tau^{*}_{(1)}}$ are approximately

0.05 in the following cases:

(i) $\alpha_{\tau^{\ddagger}_{(1)}}$:

(1) $(p_1, p_2) = (4, 2),\ n_1 \geq 100,$

(2) $(p_1, p_2) = (8, 4),\ n_1 \geq 100$ and $n_2 \geq 20,$

(3) $(p_1, p_2) = (15, 12),$ (a) $n_1 \geq 200,$ (b) $n_1 = n_2,\ 50 \leq n_1 \leq 100.$

(ii) $\alpha_{\tau^{*}_{(1)}}$:

(1) $(p_1, p_2) = (4, 2),\ n_1 \geq 20,$

(2) $(p_1, p_2) = (8, 4),\ n_1 \geq 30,$

(3) $(p_1, p_2) = (15, 12),$ (a) $n_1 \geq 50$ and $n_2 = 15,$ (b) $n_1 \geq 200.$

In Tables 6 and 7, we provide the simulated results of three-step and five-step cases (Cases (II-III), (II-IV)). It may be noted from Tables 6 and 7 that the value of $\alpha_{\tau^{\ddagger}_{(1)}}$ is approximately 0.05 when $n_1 \geq 100$ and $n_2 = n_3 \geq 20$, and the value of $\alpha_{\tau^{*}_{(1)}}$ is approximately 0.05 when $n_1 \geq 200$. Comparing among the three cases, $(p_1, p_2) = (15, 12)$, $(p_1, p_2, p_3) = (15, 12, 9)$, and $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$ in tables, it may be noted that both $\alpha_{\tau^{\ddagger}_{(1)}}$ and $\alpha_{\tau^{*}_{(1)}}$ are not depend on the number of steps.

In conclusion, we gave the LRT statistics of testing the equality of several multivariate normal populations when the data sets have general k-step monotone missing data pattern. In addition, we gave the result for one-sample problem. By the decomposition of the LR derived in this paper, we developed two modified LRT statistics for both of multi-sample and one-sample cases. The simulation studies showed that the LRT statistics are distributed as χ^2 distribution when the sample sizes are considerably large. However, the modified LRT statistics are approximately distributed as χ^2 distribution even when the sample sizes are not large.

Table 1: Simulated values and type I error rates: $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$, $m = 2$

Sample Size		Upper Percentile			Type I Error Rate		
$n_1^{(\ell)}$	$n_2^{(\ell)} = n_3^{(\ell)} = \cdots = n_5^{(\ell)}$	$-2\log\tau_{(m)}$	$-2\log\tau^{\ddagger}_{(m)}$	$-2\log\tau^{*}_{(m)}$	$\alpha_{\tau_{(m)}}$	$\alpha_{\tau^{\ddagger}_{(m)}}$	$\alpha_{\tau^{*}_{(m)}}$
		$\alpha = 0.05$					
20	10	275.57	171.66	155.35	.989	.098	.024
30	10	217.93	166.57	157.40	.768	.070	.027
50	10	192.19	165.81	160.77	.399	.066	.038
100	10	176.86	164.83	162.43	.172	.060	.046
200	10	169.83	164.06	162.86	.098	.055	.049
300	10	167.63	163.81	163.03	.080	.054	.050
20	15	269.82	166.62	150.55	.983	.067	.015
30	15	214.83	164.06	154.99	.730	.055	.021
50	15	190.68	164.61	159.62	.376	.058	.034
100	15	176.45	164.49	162.13	.167	.058	.045
200	15	169.84	164.06	162.90	.097	.055	.049
300	15	167.58	163.77	162.99	.079	.054	.049
20	20	266.13	163.50	147.47	.977	.052	.011
30	20	212.50	162.09	153.12	.701	.045	.017
50	20	189.55	163.67	158.73	.356	.053	.030
100	20	175.99	164.11	161.76	.162	.056	.043
200	20	169.66	163.92	162.75	.096	.055	.048
300	20	167.58	163.79	163.01	.079	.054	.049
20	50	257.24	155.57	139.89	.956	.026	.005
30	50	206.11	156.60	147.86	.608	.025	.009
50	50	185.88	160.59	155.79	.297	.038	.022
100	50	174.48	162.85	160.56	.145	.049	.038
200	50	169.01	163.36	162.22	.091	.051	.045
300	50	167.21	163.46	162.70	.077	.052	.048
		$\alpha = 0.01$					
20	10	300.25	188.85	171.88	.959	.034	.006
30	10	235.73	181.13	171.39	.548	.018	.005
50	10	207.47	179.32	173.91	.185	.015	.007
100	10	191.00	178.08	175.52	.054	.013	.009
200	10	183.41	177.16	175.92	.025	.011	.010
300	10	180.90	176.80	175.96	.019	.011	.010
20	15	294.28	183.98	167.17	.941	.021	.004
30	15	232.43	178.57	169.05	.501	.013	.004
50	15	206.04	178.25	172.98	.168	.013	.007
100	15	190.56	177.74	175.19	.052	.012	.009
200	15	183.41	177.21	175.96	.025	.012	.010
300	15	180.99	176.87	176.05	.019	.011	.010
20	20	290.38	180.76	164.12	.926	.016	.003
30	20	229.88	176.61	167.24	.465	.011	.003
50	20	204.85	177.34	172.13	.156	.012	.006
100	20	190.02	177.33	174.83	.049	.012	.008
200	20	183.13	176.96	175.72	.024	.011	.009
300	20	181.01	176.94	176.11	.019	.011	.010
20	50	281.11	172.89	156.64	.877	.007	.001
30	50	223.15	171.10	162.12	.367	.005	.002
50	50	200.72	174.16	169.15	.118	.008	.004
100	50	188.39	176.09	173.67	.042	.010	.007
200	50	182.59	176.58	175.37	.022	.011	.009
300	50	180.59	176.59	175.78	.018	.011	.010

Note. $\chi^2_{f_1,0.05} = 163.12$, $\chi^2_{f_1,0.01} = 176.14$, $f_1 = 135$, $\ell = 1, 2$.

Table 2: Simulated values and type I error rates: $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$, $m = 3$

Sample Size		Upper Percentile			Type I Error Rate		
$n_1^{(\ell)}$	$n_2^{(\ell)} = n_3^{(\ell)} = \cdots = n_5^{(\ell)}$	$-2\log\tau_{(m)}$	$-2\log\tau_{(m)}^{\ddagger}$	$-2\log\tau_{(m)}^{*}$	$\alpha_{\tau_{(m)}}$	$\alpha_{\tau_{(m)}^{\ddagger}}$	$\alpha_{\tau_{(m)}^{*}}$
			$\alpha = 0.05$				
20	10	486.74	322.22	296.42	.999	.106	.020
30	10	398.78	314.56	299.54	.898	.072	.023
50	10	357.46	313.55	305.16	.527	.069	.036
100	10	332.54	312.32	308.28	.218	.063	.046
200	10	320.77	311.02	309.05	.114	.057	.049
300	10	316.89	310.47	309.16	.088	.055	.049
20	15	477.31	313.85	288.31	.998	.066	.011
30	15	393.52	309.89	295.00	.868	.052	.015
50	15	355.12	311.46	303.16	.495	.059	.030
100	15	331.60	311.49	307.50	.209	.059	.043
200	15	320.46	310.73	308.77	.112	.056	.048
300	15	316.90	310.50	309.19	.088	.055	.049
20	20	471.15	308.27	282.91	.997	.047	.007
30	20	389.72	306.54	291.79	.844	.041	.011
50	20	353.33	309.91	301.65	.471	.052	.026
100	20	330.92	310.93	306.93	.203	.057	.041
200	20	320.45	310.74	308.76	.111	.056	.048
300	20	316.60	310.22	308.91	.087	.054	.048
20	50	456.21	294.61	269.52	.992	.018	.002
30	50	378.99	297.07	282.58	.756	.019	.005
50	50	346.93	304.32	296.24	.388	.033	.016
100	50	328.34	308.63	304.77	.177	.047	.034
200	50	319.52	309.94	308.02	.105	.053	.045
300	50	316.27	309.95	308.67	.085	.053	.047
			$\alpha = 0.01$				
20	10	517.33	345.03	318.46	.995	.037	.005
30	10	422.06	333.98	318.39	.743	.019	.004
50	10	377.95	331.98	323.22	.281	.016	.007
100	10	351.45	330.21	325.94	.075	.014	.009
200	10	339.01	328.77	326.67	.030	.012	.010
300	10	335.09	328.27	326.91	.022	.011	.010
20	15	508.06	336.87	310.66	.991	.021	.002
30	15	416.40	329.26	313.94	.691	.012	.003
50	15	375.42	329.85	321.25	.255	.013	.006
100	15	350.52	329.43	325.24	.070	.013	.008
200	15	338.68	328.46	326.39	.029	.012	.009
300	15	334.94	328.18	326.80	.021	.011	.010
20	20	501.08	331.01	304.87	.987	.014	.001
30	20	412.47	326.00	310.83	.652	.009	.002
50	20	373.65	328.32	319.76	.235	.011	.005
100	20	349.80	328.87	324.69	.067	.012	.008
200	20	338.84	328.65	326.60	.029	.012	.010
300	20	334.67	327.94	326.56	.021	.011	.010
20	50	485.96	317.35	291.84	.967	.005	.000
30	50	401.20	316.52	301.70	.527	.004	.001
50	50	367.02	322.83	314.53	.175	.007	.003
100	50	347.19	326.73	322.69	.055	.010	.006
200	50	337.88	327.86	325.86	.027	.011	.009
300	50	334.34	327.72	326.36	.020	.011	.009

Note. $\chi^2_{f_1,0.05} = 309.33$, $\chi^2_{f_1,0.01} = 326.98$, $f_1 = 270$, $\ell = 1, 2, 3$.

Table 3: Simulated values and type I error rates: $(p_1, p_2) = (4, 2)$

Sample Size		Upper Percentile			Type I Error Rate		
n_1	n_2	$-2\log\tau_{(1)}$	$-2\log\tau^{\ddagger}_{(1)}$	$-2\log\tau^{*}_{(1)}$	$\alpha_{\tau_{(1)}}$	$\alpha_{\tau^{\ddagger}_{(1)}}$	$\alpha_{\tau^{*}_{(1)}}$
			$\alpha = 0.05$				
10	10	29.38	26.23	24.13	.159	.091	.056
20	10	26.11	24.77	23.76	.090	.066	.051
30	10	25.24	24.38	23.71	.074	.060	.050
50	10	24.58	24.08	23.68	.063	.056	.050
100	10	24.16	23.91	23.71	.057	.053	.050
10	20	29.21	26.11	24.07	.154	.089	.055
20	20	26.00	24.69	23.71	.087	.065	.050
30	20	25.16	24.31	23.66	.073	.059	.050
50	20	24.55	24.06	23.66	.063	.055	.050
100	20	24.14	23.89	23.69	.057	.053	.050
10	50	29.10	26.01	24.03	.150	.086	.055
20	50	25.87	24.58	23.64	.085	.063	.049
30	50	25.12	24.30	23.67	.072	.059	.050
50	50	24.53	24.05	23.67	.062	.055	.050
100	50	24.10	23.86	23.66	.056	.052	.050
10	100	28.99	25.95	23.99	.149	.085	.054
20	100	25.84	24.57	23.65	.084	.063	.050
30	100	25.02	24.22	23.61	.071	.058	.049
50	100	24.47	24.00	23.64	.062	.055	.049
100	100	24.07	23.83	23.65	.055	.052	.049
			$\alpha = 0.01$				
10	10	36.38	32.45	29.99	.053	.024	.013
20	10	32.14	30.48	29.28	.023	.015	.010
30	10	31.05	29.99	29.19	.017	.013	.010
50	10	30.31	29.69	29.20	.014	.012	.010
100	10	29.74	29.43	29.19	.012	.011	.010
10	20	36.17	32.31	29.89	.051	.023	.012
20	20	32.02	30.40	29.24	.022	.015	.010
30	20	30.95	29.95	29.15	.017	.013	.010
50	20	30.19	29.59	29.12	.014	.011	.010
100	20	29.70	29.40	29.15	.012	.011	.010
10	50	36.05	32.25	29.92	.050	.023	.013
20	50	31.85	30.30	29.21	.021	.014	.010
30	50	30.91	29.92	29.18	.017	.013	.010
50	50	30.18	29.59	29.14	.014	.011	.010
100	50	29.68	29.39	29.15	.012	.011	.010
10	100	35.89	32.11	29.82	.048	.022	.012
20	100	31.80	30.27	29.20	.021	.014	.010
30	100	30.87	29.89	29.19	.016	.012	.010
50	100	30.12	29.55	29.10	.013	.011	.010
100	100	29.67	29.38	29.17	.012	.011	.010

Note. $\chi^2_{f_2,0.05} = 23.68$, $\chi^2_{f_2,0.01} = 29.14$, $f_2 = 14$.

Table 4: Simulated values and type I error rates: $(p_1, p_2) = (8, 4)$

Sample Size		Upper Percentile			Type I Error Rate		
n_1	n_2	$-2\log\tau_{(1)}$	$-2\log\tau^{\ddagger}_{(1)}$	$-2\log\tau^{*}_{(1)}$	$\alpha_{\tau_{(1)}}$	$\alpha_{\tau^{\ddagger}_{(1)}}$	$\alpha_{\tau^{*}_{(1)}}$
			$\alpha = 0.05$				
10	10	106.52	79.50	72.56	.766	.315	.187
20	10	72.82	64.28	61.40	.238	.091	.058
30	10	67.91	62.68	60.80	.146	.072	.053
50	10	64.63	61.65	60.54	.096	.061	.051
100	10	62.54	61.10	60.55	.070	.056	.051
10	20	105.86	78.90	72.07	.756	.301	.178
20	20	72.46	64.02	61.18	.230	.087	.056
30	20	67.73	62.54	60.70	.142	.070	.052
50	20	64.54	61.59	60.50	.095	.061	.050
100	20	62.42	61.00	60.45	.069	.055	.050
10	50	105.17	78.36	71.59	.746	.290	.171
20	50	72.05	63.71	60.95	.220	.083	.054
30	50	67.42	62.32	60.52	.137	.068	.050
50	50	64.38	61.48	60.42	.093	.059	.049
100	50	62.41	61.00	60.47	.069	.055	.050
10	100	105.02	78.24	71.48	.741	.285	.168
20	100	71.83	63.56	60.84	.216	.081	.053
30	100	67.22	62.18	60.41	.135	.066	.049
50	100	64.26	61.40	60.36	.091	.059	.049
100	100	62.34	60.96	60.43	.069	.054	.050
			$\alpha = 0.01$				
10	10	124.72	92.81	85.10	.581	.154	.078
20	10	82.91	73.24	70.02	.089	.023	.013
30	10	77.31	71.36	69.27	.044	.017	.011
50	10	73.42	70.05	68.81	.025	.013	.010
100	10	71.00	69.38	68.75	.016	.012	.010
10	20	124.21	92.46	84.82	.569	.146	.074
20	20	82.46	72.94	69.80	.085	.022	.012
30	20	77.00	71.15	69.09	.043	.016	.011
50	20	73.33	70.00	68.77	.024	.013	.010
100	20	70.87	69.27	68.65	.016	.011	.010
10	50	123.54	91.81	84.34	.556	.139	.070
20	50	82.06	72.68	69.66	.080	.021	.012
30	50	76.67	70.94	68.96	.041	.016	.011
50	50	73.12	69.89	68.70	.023	.013	.010
100	50	70.83	69.25	68.67	.015	.011	.010
10	100	123.02	91.50	84.06	.552	.137	.069
20	100	81.88	72.53	69.55	.078	.021	.012
30	100	76.39	70.74	68.83	.039	.015	.010
50	100	73.07	69.88	68.71	.023	.013	.010
100	100	70.83	69.27	68.68	.015	.011	.010

Note. $\chi^2_{f_2,0.05} = 60.48$, $\chi^2_{f_2,0.01} = 68.71$, $f_2 = 44$.

Table 5: Simulated values and type I error rates: $(p_1, p_2) = (15, 12)$

Sample Size		Upper Percentile			Type I Error Rate		
n_1	n_2	$-2\log\tau_{(1)}$	$-2\log\tau^{\ddagger}_{(1)}$	$-2\log\tau^{*}_{(1)}$	$\alpha_{\tau_{(1)}}$	$\alpha_{\tau^{\ddagger}_{(1)}}$	$\alpha_{\tau^{*}_{(1)}}$
			$\alpha = 0.05$				
20	15	230.95	173.82	165.16	.846	.120	.060
30	15	197.71	166.84	161.49	.483	.072	.042
50	15	182.02	165.35	162.21	.240	.063	.045
100	15	172.25	164.40	162.85	.121	.057	.049
200	15	167.62	163.80	163.03	.079	.054	.050
300	15	166.10	163.58	163.06	.069	.053	.050
20	20	228.37	171.63	163.02	.823	.101	.050
30	20	196.27	165.66	160.32	.459	.064	.037
50	20	181.30	164.74	161.64	.230	.060	.042
100	20	171.95	164.15	162.60	.118	.056	.047
200	20	167.61	163.80	163.03	.079	.054	.050
300	20	166.08	163.55	163.04	.068	.052	.050
20	50	221.47	165.59	157.19	.753	.062	.029
30	50	191.54	161.67	156.49	.382	.043	.024
50	50	178.85	162.70	159.68	.197	.048	.034
100	50	171.16	163.50	161.99	.109	.052	.044
200	50	167.29	163.53	162.76	.077	.052	.048
300	50	166.05	163.56	163.05	.068	.052	.050
20	100	218.25	162.82	154.53	.712	.049	.022
30	100	188.93	159.45	154.38	.338	.034	.019
50	100	176.99	161.12	158.16	.173	.040	.029
100	100	170.18	162.68	161.22	.101	.048	.040
200	100	166.91	163.21	162.46	.075	.051	.047
300	100	165.86	163.38	162.89	.067	.052	.049
			$\alpha = 0.01$				
20	15	251.64	190.33	181.22	.671	.041	.017
30	15	213.85	180.90	175.24	.251	.018	.009
50	15	196.69	178.87	175.52	.087	.014	.009
100	15	185.91	177.49	175.83	.033	.012	.010
200	15	180.95	176.83	176.00	.019	.011	.010
300	15	179.30	176.58	176.03	.015	.011	.010
20	20	248.89	187.98	178.98	.639	.033	.014
30	20	212.30	179.74	174.17	.232	.016	.008
50	20	195.94	178.25	174.94	.082	.013	.008
100	20	185.74	177.40	175.72	.032	.012	.009
200	20	180.98	176.89	176.06	.019	.011	.010
300	20	179.41	176.69	176.13	.015	.011	.010
20	50	241.62	182.05	173.33	.543	.019	.007
30	50	207.24	175.76	170.39	.175	.010	.005
50	50	193.24	176.18	173.03	.065	.010	.007
100	50	184.83	176.66	175.05	.029	.011	.009
200	50	180.70	176.68	175.86	.018	.011	.010
300	50	179.33	176.66	176.10	.015	.011	.010
20	100	238.50	179.25	170.66	.495	.014	.005
30	100	204.38	173.40	168.15	.146	.007	.003
50	100	191.24	174.62	171.53	.054	.008	.005
100	100	183.78	175.86	174.31	.026	.010	.008
200	100	180.21	176.29	175.50	.017	.010	.009
300	100	179.05	176.40	175.87	.015	.010	.010

Note. $\chi^2_{f_2,0.05} = 163.12$, $\chi^2_{f_2,0.01} = 176.14$, $f_2 = 135$.

Table 6: Simulated values and type I error rates: $(p_1, p_2, p_3) = (15, 12, 9)$

Sample Size		Upper Percentile			Type I Error Rate		
n_1	$n_2 = n_3$	$-2\log\tau_{(1)}$	$-2\log\tau^{\ddagger}_{(1)}$	$-2\log\tau^{*}_{(1)}$	$\alpha_{\tau_{(1)}}$	$\alpha_{\tau^{\ddagger}_{(1)}}$	$\alpha_{\tau^{*}_{(1)}}$
			$\alpha = 0.05$				
20	15	228.47	171.69	163.12	.824	.102	.050
30	15	196.15	165.64	160.35	.457	.064	.037
50	15	181.17	164.65	161.57	.228	.059	.042
100	15	172.09	164.28	162.75	.118	.057	.048
200	15	167.56	163.75	162.99	.079	.054	.049
300	15	166.03	163.51	163.00	.068	.052	.049
20	20	225.95	169.63	161.11	.801	.087	.042
30	20	194.71	164.39	159.15	.434	.057	.032
50	20	180.51	164.14	161.06	.219	.056	.040
100	20	171.71	163.96	162.44	.115	.055	.046
200	20	167.50	163.70	162.94	.079	.053	.049
300	20	166.08	163.57	163.06	.068	.053	.050
20	50	219.94	164.38	156.06	.735	.056	.026
30	50	190.38	160.71	155.62	.362	.039	.022
50	50	178.02	162.06	159.08	.185	.045	.032
100	50	170.67	163.12	161.63	.106	.050	.042
200	50	167.18	163.45	162.70	.076	.052	.048
300	50	165.87	163.39	162.88	.067	.051	.049
20	100	217.28	162.04	153.78	.700	.045	.020
30	100	188.11	158.81	153.77	.326	.031	.018
50	100	176.33	160.62	157.71	.165	.038	.027
100	100	169.85	162.42	160.98	.098	.046	.039
200	100	166.69	163.02	162.28	.073	.049	.046
300	100	165.68	163.23	162.73	.066	.051	.048
			$\alpha = 0.01$				
20	15	249.02	188.25	179.33	.639	.033	.014
30	15	212.18	179.73	174.16	.231	.016	.008
50	15	195.84	178.20	174.92	.081	.013	.009
100	15	185.91	177.57	175.94	.032	.012	.010
200	15	180.91	176.82	176.00	.018	.011	.010
300	15	179.26	176.57	176.01	.015	.011	.010
20	20	246.44	186.18	177.37	.607	.028	.011
30	20	210.55	178.52	173.06	.212	.013	.007
50	20	194.92	177.50	174.26	.076	.012	.008
100	20	185.40	177.12	175.51	.031	.011	.009
200	20	180.87	176.81	175.98	.018	.011	.010
300	20	179.40	176.70	176.16	.015	.011	.010
20	50	240.17	180.95	172.31	.521	.016	.007
30	50	206.15	174.94	169.64	.162	.009	.004
50	50	192.41	175.57	172.48	.060	.009	.006
100	50	184.30	176.28	174.71	.027	.010	.008
200	50	180.54	176.56	175.77	.018	.011	.010
300	50	179.11	176.46	175.91	.015	.010	.010
20	100	237.41	178.41	169.84	.481	.013	.005
30	100	203.71	172.94	167.71	.138	.007	.003
50	100	190.50	174.06	171.04	.051	.008	.005
100	100	183.35	175.56	174.05	.025	.009	.007
200	100	180.05	176.15	175.38	.016	.010	.009
300	100	178.98	176.37	175.84	.014	.010	.010

Note. $\chi^2_{f_2,0.05} = 163.12$, $\chi^2_{f_2,0.01} = 176.14$, $f_2 = 135$.

Table 7: Simulated values and type I error rates: $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$

Sample Size		Upper Percentile			Type I Error Rate		
n_1	$n_2 = n_3 = \cdots = n_5$	$-2\log\tau_{(1)}$	$-2\log\tau^{\ddagger}_{(1)}$	$-2\log\tau^{*}_{(1)}$	$\alpha_{\tau_{(1)}}$	$\alpha_{\tau^{\ddagger}_{(1)}}$	$\alpha_{\tau^{*}_{(1)}}$
			$\alpha = 0.05$				
20	15	227.98	171.39	162.86	.819	.099	.049
30	15	195.82	165.36	160.10	.451	.063	.036
50	15	181.02	164.58	161.50	.226	.058	.042
100	15	171.92	164.16	162.62	.117	.056	.047
200	15	167.57	163.77	163.00	.079	.054	.049
300	15	166.08	163.56	163.06	.068	.052	.050
20	20	225.59	169.32	160.82	.797	.084	.041
30	20	194.25	164.05	158.82	.426	.055	.032
50	20	180.24	163.92	160.88	.215	.054	.039
100	20	171.65	163.93	162.42	.115	.055	.046
200	20	167.47	163.69	162.92	.078	.053	.049
300	20	166.06	163.55	163.04	.068	.052	.050
20	50	219.78	164.25	155.94	.732	.055	.025
30	50	190.13	160.52	155.43	.358	.038	.021
50	50	177.82	161.89	158.95	.183	.044	.031
100	50	170.61	163.06	161.60	.105	.050	.042
200	50	167.03	163.31	162.58	.076	.051	.047
300	50	165.91	163.44	162.94	.067	.052	.049
20	100	217.13	161.93	153.69	.698	.045	.020
30	100	187.96	158.68	153.67	.322	.031	.018
50	100	176.22	160.56	157.64	.164	.038	.027
100	100	169.73	162.34	160.90	.097	.046	.039
200	100	166.72	163.07	162.35	.073	.050	.046
300	100	165.62	163.19	162.70	.065	.050	.048
			$\alpha = 0.01$				
20	15	248.50	187.95	178.99	.633	.032	.014
30	15	211.94	179.57	174.07	.226	.015	.008
50	15	195.69	178.13	174.89	.080	.013	.008
100	15	185.79	177.48	175.86	.031	.012	.010
200	15	180.84	176.77	175.95	.018	.011	.010
300	15	179.33	176.64	176.09	.015	.011	.010
20	20	246.12	185.83	176.99	.601	.027	.011
30	20	210.22	178.15	172.72	.207	.013	.006
50	20	194.70	177.32	174.14	.075	.012	.008
100	20	185.34	177.09	175.49	.031	.011	.009
200	20	180.74	176.68	175.85	.018	.011	.010
300	20	179.25	176.56	176.02	.015	.011	.010
20	50	240.03	180.79	172.17	.518	.016	.006
30	50	205.58	174.58	169.33	.159	.008	.004
50	50	192.11	175.30	172.23	.059	.009	.006
100	50	184.27	176.26	174.71	.027	.010	.008
200	50	180.44	176.45	175.67	.017	.010	.009
300	50	178.99	176.35	175.83	.015	.010	.010
20	100	237.11	178.29	169.83	.478	.013	.005
30	100	203.65	172.88	167.73	.137	.007	.003
50	100	190.45	174.09	171.10	.050	.008	.005
100	100	183.38	175.59	174.09	.024	.009	.008
200	100	180.05	176.18	175.40	.017	.010	.009
300	100	178.87	176.27	175.76	.014	.010	.010

Note. $\chi^2_{f_2,0.05} = 163.12$, $\chi^2_{f_2,0.01} = 176.14$, $f_2 = 135$.

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