Asymptotic non-null distributions of test statistics for redundancy in high-dimensional canonical correlation analysis

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Abstract

In this paper, we derive asymptotic non-null distributions of three test statistics, i.e., the likelihood ratio criterion, the Lawley-Hotelling criterion and the Bartlett-Nanda-Pillai criterion, for redundancy in high-dimensional canonical correlation analysis. Since our setting is that the dimension of one of two observation vectors may be large but does not exceed the sample size, we use the high-dimensional asymptotic framework such that the sample size and the dimension divided by the sample size tend to ∞ and some positive constant being included in $[0, 1)$, respectively, for evaluating asymptotic distributions. Additionally, we derive asymptotic null distributions of the three test statistics under the high-dimensional asymptotic framework by using the same transformation as when we derive asymptotic non-null distribution. We verified that new approximations based on derived asymptotic distributions are more accurate than those based on asymptotic distributions evaluated from the classical asymptotic framework, i.e., only the sample size tends to ∞.

1 Introduction

Canonical correlation analysis (CCA) is a statistical method employed to investigate the relationships between a pair of q- and p-dimensional random vectors, $\boldsymbol{x} = (x_1, \ldots, x_q)'$ and $\boldsymbol{y} = (y_1, \ldots, y_p)'$. Introductions to CCA are provided in many textbooks for applied statistical analysis (see, e.g., Srivastava, 2002, chap. 14.7; Timm, 2002, chap. 8.7), and it has widespread applications in many fields (e.g., Chung *et al.*, 2017; Prera *et al.*, 2014; Sakar *et al.*, 2016; Singh *et al.*, 2014; Zheng *et al.*, 2014). In an actual data analysis, it is important to clarify whether or not some variables are irrelevant for analysis. In CCA, one of methods to solve the problem is hypothesis testing for redundancy. Since a distribution of a test statistic plays an important rule in hypothesis testing, it has been studied by many authors. It is known that the null distribution of the likelihood ratio (LR) criterion for redundancy is distributed according to the Wilks' Lambda distribution (e.g., Fujikoshi, 1982). Since the Wilks' Lambda distribution is

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user-unfriendly, it is important to derive an asymptotic approximation of the distribution of the LR criterion. Fujikoshi (1982) derived an asymptotic expansion of the null distribution of the LR criterion under the large-sample (LS) asymptotic framework such that only the sample size approaches ∞. However, an approximation based on the asymptotic expansion worsen as the dimension of either of observation vectors increases. To avoid the deterioration of approximation accuracy, Sakurai (2009) and Wakaki and Fujikoshi (2012) used the asymptotic expansion of the null distribution of the LR criterion under the asymptotic framework such that the sample size and the dimension of either of observation vectors tend to ∞. On the other hand, as for a non-null distribution of the LR criterion, under the LS asymptotic framework, Suzukawa and Sato (1996) obtained an asymptotic distribution of the LR criterion under a local alternative hypothesis. These results have been derived from the fact that the LR criterion is distributed according to the Wilks' Lambda distribution under the null hypothesis. However, the asymptotic distribution of the LR criterion has not been derived under the fixed alternative hypothesis because the criterion is not distributed according to the Wilks' Lambda distribution under the hypothesis. In CCA, the Lawley-Hotelling (LH) criterion and the Bartlett-Nanda-Pillai (BNP) criterion are commonly used in the test for redundancy just like the LR criterion. However, since the two test statistics are not distributed according to the Wilks' Lambda distribution even under the null hypothesis, the asymptotic null distributions have not been derived under except the LS asymptotic framework.

Let q_2 be the number of elements of $\boldsymbol{x}$ assumed to be irrelevant variables and $q_1 = q - q_2$. In this paper, since our setting is that the dimension of one of two observation vectors may be large but does not exceed the sample size, we use the following high-dimensional (HD) asymptotic framework for evaluating asymptotic distributions:

$$n \to \infty, \ \frac{p + q_1}{n} \to c \in [0, 1), \ q_2 : \text{fixed}.$$

It should be emphasized that the LS asymptotic framework is included in the HD asymptotic framework as a special case. An aim of this paper is deriving asymptotic distributions of the three test statistics under a fixed alternative hypothesis by using the HD asymptotic framework. To derive the asymptotic distributions, we transform the test statistics by using the result in Yanagihara *et al*. (2017) because the test statistics are not distributed according to the Wilks' Lambda distribution under the fixed alternative hypothesis. In Yanagihara *et al*. (2017), the asymptotic null distribution of the LR criterion by using the HD asymptotic framework, but the asymptotic non-null distribution have not been derived. Additionally, we derive the asymptotic null distributions of the three test statistics under the HD asymptotic framework because those of the LH criterion and the BNP criterion have not been derived.

This paper is organized as follows: In section 2, we introduce three test statistics for redundancy in CCA. In section 3, we present our key lemmas and main results. In Section 4, we verify the accuracies of approximations under the HD asymptotic framework by conducting numerical experiments. Technical details are provided in the Appendix.

2 Test for redundancy hypothesis

In this section, we introduce three test statistics for redundancy in CCA. Let $\boldsymbol{z} = (\boldsymbol{x}', \boldsymbol{y}')'$ be a $(q+p)$-dimensional random vector distributed according to $(q+p)$-variate normal distribution with

$$E[\boldsymbol{z}] = \boldsymbol{\mu} = (\boldsymbol{\mu}_x', \boldsymbol{\mu}_y')' \ , \ Cov[\boldsymbol{z}] = \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{xx} & \boldsymbol{\Sigma}_{xy} \\ \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_{yy} \end{pmatrix},$$

where $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$ are q- and p-dimension mean vectors of $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{\Sigma}_{xx}$ and $\boldsymbol{\Sigma}_{yy}$ are $q \times q$ and $p \times p$ covariance matrices of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{\Sigma}_{xy}$ is the $q \times p$ covariance matrix of $\boldsymbol{x}$ and $\boldsymbol{y}$.

We partition $\boldsymbol{x}$ as $\boldsymbol{x} = (\boldsymbol{x}_1', \boldsymbol{x}_2')'$, $\boldsymbol{x}_1 : q_1 \times 1$, $\boldsymbol{x}_2 : q_2 \times 1$ and express $\boldsymbol{\mu}_x$, $\boldsymbol{\Sigma}_{xx}$ and $\boldsymbol{\Sigma}_{xy}$ corresponding to the division of $\boldsymbol{x}$ as follows:

$$\boldsymbol{\mu}_x = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')' \ , \ \boldsymbol{\Sigma}_{xx} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}, \ \boldsymbol{\Sigma}_{xy} = \begin{pmatrix} \boldsymbol{\Sigma}_{1y} \\ \boldsymbol{\Sigma}_{2y} \end{pmatrix}.$$

We are interested in whether or not the elements of $\boldsymbol{x}_2$ are irrelevant variables in CCA. Fujikoshi (1985) defined that $\boldsymbol{x}_2$ is irrelevant if the following null hypothesis is true:

$$H_0 : \mathrm{tr}(\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}) = \mathrm{tr}(\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{1y}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{y1}). \tag{1}$$

In addition, we note that the fixed alternative hypothesis is as follows:

$$H_1 : \mathrm{tr}(\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xy}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{yx}) \neq \mathrm{tr}(\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{1y}\boldsymbol{\Sigma}_{yy}^{-1}\boldsymbol{\Sigma}_{y1}). \tag{2}$$

The left-hand side and the right-hand side of the equation (1) or (2) are the sum of squares of the canonical correlation coefficients between $\boldsymbol{x}$ and $\boldsymbol{y}$, and the sum of squares of the canonical correlation coefficients between $\boldsymbol{x}_1$ and $\boldsymbol{y}$, respectively. Let $\boldsymbol{O}_{q_2,p}$ be a $q_2 \times p$ matrix of zeros. In particular, we note that (1) is equivalent to

$$H_0 : \boldsymbol{\Sigma}_{2y\cdot 1} = \boldsymbol{O}_{q_2,p}, \tag{3}$$

where $\boldsymbol{\Sigma}_{ab\cdot c} = \boldsymbol{\Sigma}_{ab} - \boldsymbol{\Sigma}_{ac}\boldsymbol{\Sigma}_{cc}^{-1}\boldsymbol{\Sigma}_{cb}$.

Let $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$ be n independent random vectors from $\boldsymbol{z}$, and let $\bar{\boldsymbol{z}}$ be the sample mean of $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$ given by $\bar{\boldsymbol{z}} = n^{-1}\sum_{i=1}^n \boldsymbol{z}_i$ and $\boldsymbol{S}$ be the usual unbiased estimator of $\boldsymbol{\Sigma}$ given by $\boldsymbol{S} = (n-1)^{-1}\sum_{i=1}^n (\boldsymbol{z}_i - \bar{\boldsymbol{z}})(\boldsymbol{z}_i - \bar{\boldsymbol{z}})'$, divided in the same way as we divided $\boldsymbol{\Sigma}$, as follows:

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{xx} & \boldsymbol{S}_{xy} \\ \boldsymbol{S}_{yx} & \boldsymbol{S}_{yy} \end{pmatrix} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \boldsymbol{S}_{1y} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \boldsymbol{S}_{2y} \\ \boldsymbol{S}_{y1} & \boldsymbol{S}_{y2} & \boldsymbol{S}_{yy} \end{pmatrix}.$$

Suppose that $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n \sim N_{q+p}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then, the three test statistics, i.e., the LR, LH and BNP criteria are expressed as follows:

$$T_{\mathrm{G}} = \begin{cases} \log \dfrac{|\boldsymbol{S}_{yy\cdot 1}|}{|\boldsymbol{S}_{yy\cdot x}|} & (\mathrm{G} = \mathrm{LR}) \\ \mathrm{tr}\left\{(\boldsymbol{S}_{yy\cdot 1} - \boldsymbol{S}_{yy\cdot x})\boldsymbol{S}_{yy\cdot x}^{-1}\right\} & (\mathrm{G} = \mathrm{LH}) \\ \mathrm{tr}\left\{(\boldsymbol{S}_{yy\cdot 1} - \boldsymbol{S}_{yy\cdot x})\boldsymbol{S}_{yy\cdot 1}^{-1}\right\} & (\mathrm{G} = \mathrm{BNP}) \end{cases},$$

where $\boldsymbol{S}_{ab\cdot c} = \boldsymbol{S}_{ab} - \boldsymbol{S}_{ac}\boldsymbol{S}_{cc}^{-1}\boldsymbol{S}_{cb}$.

3 Main results

In this section, we derive asymptotic non-null distributions of the three test statistics within the HD asymptotic framework. First of all, we present key lemmas which are required for deriving our main results. Under the HD asymptotic framework, it is a serious problem that increasing the dimension of $\boldsymbol{S}_{yy\cdot 1}$ and $\boldsymbol{S}_{yy\cdot x}$ with an increase in the dimension p. However, we can avoid the problem by using the following lemma.

Lemma 3.1 *Suppose that $\boldsymbol{V}$ is a $q_2 \times p$ matrix and $\boldsymbol{W}$ is a $p \times p$ non-singular matrix. Then,*

$$\frac{|\boldsymbol{W} + \boldsymbol{V}'\boldsymbol{V}|}{|\boldsymbol{W}|} = |\boldsymbol{I}_{q_2} + \boldsymbol{V}\boldsymbol{W}^{-1}\boldsymbol{V}'|, \tag{4}$$

$$\mathrm{tr}(\boldsymbol{V}'\boldsymbol{V}\boldsymbol{W}^{-1}) = \mathrm{tr}(\boldsymbol{V}\boldsymbol{W}^{-1}\boldsymbol{V}'), \tag{5}$$

$$\mathrm{tr}\{\boldsymbol{W}(\boldsymbol{W} + \boldsymbol{V}'\boldsymbol{V})^{-1}\} = p - \mathrm{tr}\{\boldsymbol{V}\boldsymbol{W}^{-1}\boldsymbol{V}'(\boldsymbol{I}_{q_2} + \boldsymbol{V}\boldsymbol{W}^{-1}\boldsymbol{V}')^{-1}\}. \tag{6}$$

The proof of the lemma is omitted because it is easy to obtain from elementary linear algebra. By applying this lemma to our problem, we can evaluate the asymptotic behaviors of the three test statistics from two random matrices of which dimensions do not increase with an increase in the dimension p. Yanagihara *et al*. (2017) derived the asymptotic null distribution of the LR criterion under the HD asymptotic framework by using the following lemma (as for the proof, see Yanagihara *et al*. 2017).

Lemma 3.2 *Let $\boldsymbol{\mathcal{E}}$, $\boldsymbol{A}$, and $\boldsymbol{B}_1$ be mutually independent random matrices, which are defined by*

$$\boldsymbol{\mathcal{E}} \sim N_{(n-1)\times p}(\boldsymbol{O}_{n-1,p}, \boldsymbol{I}_p \otimes \boldsymbol{I}_{n-1}),\ \boldsymbol{A} \sim N_{(n-1)\times(q-q_1)}(\boldsymbol{O}_{n-1,q-q_1}, \boldsymbol{I}_{q-q_1} \otimes \boldsymbol{I}_{n-1}),$$

and

$$\boldsymbol{B} \sim N_{(n-1)\times q}(\boldsymbol{O}_{n-1,q}, \boldsymbol{\Sigma}_{xx} \otimes \boldsymbol{I}_{n-1}),$$

where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{B}$ are independent, and let $\boldsymbol{B}_1$ denote the $(n-1) \times q_1$ matrix defined by $\boldsymbol{B} = (\boldsymbol{B}_1, \boldsymbol{B}_2)$. Then,

$$(n-1)\boldsymbol{S}_{yy\cdot x} = \boldsymbol{\Sigma}_{yy\cdot x}^{1/2}\boldsymbol{\mathcal{E}}'(\boldsymbol{I}_{n-1} - \boldsymbol{P})\boldsymbol{\mathcal{E}}\boldsymbol{\Sigma}_{yy\cdot x}^{1/2},$$

$$(n-1)\boldsymbol{S}_{yy\cdot 1} = \boldsymbol{\Sigma}_{yy\cdot x}^{1/2}(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}})\boldsymbol{\Sigma}_{yy\cdot x}^{1/2},$$

where $\boldsymbol{P} = \boldsymbol{B}(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'$, $\boldsymbol{P}_1 = \boldsymbol{B}_1(\boldsymbol{B}_1'\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1'$ and $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_{yy\cdot x}^{-1/2}\boldsymbol{\Sigma}_{2y\cdot 1}'\boldsymbol{\Sigma}_{22\cdot 1}^{-1/2}$.

It is easy to see from (3) that $\boldsymbol{\Gamma} = \boldsymbol{O}_{p,q_2}$ under H_0. Therefore, $\boldsymbol{\Gamma}$ is called non-centrality parameter matrix, and it is known that $\mathrm{rank}(\boldsymbol{\Gamma}) \neq 0$ under H_1. However, since it is clearly that $\boldsymbol{\Gamma} \neq \boldsymbol{O}_{p,q_2}$ under H_1, we can not apply the equations in Lemma 3.1 to the three test statistics. Therefore, we have to give another expressions of $\boldsymbol{S}_{yy\cdot 1}$ and $\boldsymbol{S}_{yy\cdot x}$ in order to apply the equations in Lemma 3.1 to the three test statistics. The following lemma is what the result in Lemma 3.2 is extended to (the proof is given in Appendix A):

Lemma 3.3 *Let $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ and $\boldsymbol{T}$ be mutually independent random matrices, which are defined by*

$$\boldsymbol{U}_1 \sim N_{(n-q-1)\times p}(\boldsymbol{O}_{n-q-1,p}, \boldsymbol{I}_p \otimes \boldsymbol{I}_{n-q-1}),\ \boldsymbol{U}_2 \sim N_{q_2\times p}(\boldsymbol{O}_{q_2,p}, \boldsymbol{I}_p \otimes \boldsymbol{I}_{q_2}),$$

and $\boldsymbol{T} = \{t_{ij}\}_{i,j=1}^{q_2}$ *is a* $q_2 \times q_2$ *lower triangular matrix of which elements are mutually independent random variables, which are defined by* $t_{ii}^2 \sim \chi^2(n - q_1 - i)$ *and* $t_{ij} \sim N(0,1)$ $(i > j)$*. Also, let* $\boldsymbol{J\Delta K}$ *be a singular value decomposition of* $\boldsymbol{\Gamma}$ *defined in Lemma 3.2, where* $\boldsymbol{J}$ *is a* p*-th orthogonal matrix,* $\boldsymbol{K}$ *is a* q_2*-th orthogonal matrix. Then,*

$$(n-1)\boldsymbol{S}_{yy\cdot x} = \boldsymbol{\Sigma}_{yy\cdot x}^{1/2}\boldsymbol{U}_1'\boldsymbol{U}_1\boldsymbol{\Sigma}_{yy\cdot x}^{1/2}, \tag{7}$$

$$(n-1)\boldsymbol{S}_{yy\cdot 1} = \boldsymbol{\Sigma}_{yy\cdot x}^{1/2}\{\boldsymbol{U}_1'\boldsymbol{U}_1 + \boldsymbol{J}(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)'(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)\boldsymbol{J}'\}\boldsymbol{\Sigma}_{yy\cdot x}^{1/2}. \tag{8}$$

By using Lemma 3.1 and Lemma 3.3, we can derive the following another expressions of T_{G} (the proof is given in Appendix B).

Lemma 3.4 *Suppose that* $p > q_2$*. Let* $\boldsymbol{D} = \boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2$*, where* $\boldsymbol{T}$*,* $\boldsymbol{\Delta}$ *and* $\boldsymbol{U}_2$ *are defined in Lemma 3.3, and* $\tilde{\boldsymbol{U}}$ *is a* $q_2 \times q_2$ *random matrix distributed according to* $W_{q_2}(n - p - q_1 - 1, \boldsymbol{I}_{q_2})$*, where* $\boldsymbol{D}$ *and* $\tilde{\boldsymbol{U}}$ *are mutually independent. Then,*

$$T_{\mathrm{G}} = \begin{cases} \log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{DD}'| & (\mathrm{G} = \mathrm{LR}) \\ \mathrm{tr}(\tilde{\boldsymbol{U}}^{-1}\boldsymbol{DD}') & (\mathrm{G} = \mathrm{LH}) \\ \mathrm{tr}\{\boldsymbol{DD}'(\boldsymbol{DD}' + \tilde{\boldsymbol{U}})^{-1}\} & (\mathrm{G} = \mathrm{BNP}) \end{cases}. \tag{9}$$

Next, we evaluate the asymptotic behaviors of T_{G} by using Lemma 3.4. The result is the following theorem (the proof is given in Appendix C):

Theorem 3.1 *Suppose that* $p > q_2$*. When* H_1 *is true, the asymptotic non-null distributions of* T_{G} *as* $n \to \infty$*,* $(p + q_1)/n \to c \in [0, 1)$ *is given by*

$$\sqrt{\tau_{\mathrm{G}}^2}\,(T_{\mathrm{G}} - \eta_{\mathrm{G}}) \xrightarrow{d} N(0,1),$$

where η_{G} *and* τ_{G}^2 *are standardizing constants given by*

$$\eta_{\mathrm{G}} = \begin{cases} q_2 \log\dfrac{n - q_1}{n - p - q_1} + \log|\boldsymbol{I}_p + \boldsymbol{\Gamma\Gamma}'| & (\mathrm{G} = \mathrm{LR}) \\ \dfrac{n - q_1}{n - p - q_1}\mathrm{tr}(\boldsymbol{\Gamma\Gamma}') + \dfrac{pq_2}{n - p - q_1} & (\mathrm{G} = \mathrm{LH}) \\ \dfrac{pq_2}{n - q_1} + \left(1 - \dfrac{p}{n - q_1}\right)\mathrm{tr}\left\{\boldsymbol{\Gamma\Gamma}'(\boldsymbol{I}_p + \boldsymbol{\Gamma\Gamma}')^{-1}\right\} & (\mathrm{G} = \mathrm{BNP}) \end{cases},$$

$$\tau_{\mathrm{G}}^2 = \begin{cases} \dfrac{(n - p - q_1)(n - q_1)^2}{2\xi_{\mathrm{LR}}} & (\mathrm{G} = \mathrm{LR}) \\ \dfrac{(n - p - q_1)^3}{2\xi_{\mathrm{LH}}} & (\mathrm{G} = \mathrm{LH}) \\ \dfrac{(n - q_1)^4}{2(n - p - q_1)\xi_{\mathrm{BNP}}} & (\mathrm{G} = \mathrm{BNP}) \end{cases}.$$

Here, ξ_{G} *are constants given by*

$$\begin{aligned} \xi_{\mathrm{LR}} &= (n - q_1)(2n - p - 2q_1)\mathrm{tr}\left\{\boldsymbol{\Gamma\Gamma}'(\boldsymbol{I}_p + \boldsymbol{\Gamma\Gamma}')^{-1}\right\} + p(n - q_1)\mathrm{tr}\left\{(\boldsymbol{I}_p + \boldsymbol{\Gamma\Gamma}')^{-1}\right\} \\ &\quad - p(n - q_1)(p - q_2), \\ \xi_{\mathrm{LH}} &= (n - q_1)(2n - p - 2q_1)\mathrm{tr}\left\{(\boldsymbol{\Gamma\Gamma}')^2\right\} + 2(n - q_1)^2\mathrm{tr}\,(\boldsymbol{\Gamma\Gamma}') + p(n - q_1)q_2, \\ \xi_{\mathrm{BNP}} &= (n - q_1)(2n - p - 2q_1)\mathrm{tr}\left\{(\boldsymbol{\Gamma\Gamma}')^2(\boldsymbol{I}_p + \boldsymbol{\Gamma\Gamma}')^{-4}\right\} + 2(n - q_1)^2\mathrm{tr}\left\{\boldsymbol{\Gamma\Gamma}'(\boldsymbol{I}_p + \boldsymbol{\Gamma\Gamma}')^{-4}\right\} \\ &\quad + p(n - q_1)\mathrm{tr}\left\{(\boldsymbol{I}_p + \boldsymbol{\Gamma\Gamma}')^{-4}\right\} - p(n - q_1)(p - q_2). \end{aligned}$$

We can also derive asymptotic non-null distributions of the three test statistics under the LS asymptotic framework. Although the non-null distributions can be derived by a different way from the method for proving Theorem 3.1, the result corresponds with the one in Theorem 3.1 when $p/n \to 0$ and $q_1/n \to 0$. Therefore, the proof is omitted.

Corollary 3.1 *When H_1 is true, the asymptotic non-null distributions of $T_{\rm G}$ as $n \to \infty$ is given by*

$$\sqrt{\nu_{\rm G}^2}\,(T_{\rm G} - \theta_{\rm G}) \xrightarrow{d} N(0,1),$$

where $\theta_{\rm G}$ and $\nu_{\rm G}^2$ are standardizing constants given by

$$\theta_{\rm G} = \begin{cases} \log|\boldsymbol{I}_p + \boldsymbol{\Gamma}\boldsymbol{\Gamma}'| & ({\rm G = LR}) \\ {\rm tr}(\boldsymbol{\Gamma}\boldsymbol{\Gamma}') & ({\rm G = LH}) \\ p - {\rm tr}\left\{(\boldsymbol{I}_p + \boldsymbol{\Gamma}\boldsymbol{\Gamma}')^{-1}\right\} & ({\rm G = BNP}) \end{cases},$$

$$\nu_{\rm G}^2 = \begin{cases} \dfrac{n}{4{\rm tr}\,\{\boldsymbol{\Gamma}\boldsymbol{\Gamma}'(\boldsymbol{I}_p + \boldsymbol{\Gamma}\boldsymbol{\Gamma}')^{-1}\}} & ({\rm G = LR}) \\ \dfrac{n}{4{\rm tr}\,\{(\boldsymbol{\Gamma}\boldsymbol{\Gamma}')^2\} + 4{\rm tr}\,(\boldsymbol{\Gamma}\boldsymbol{\Gamma}')} & ({\rm G = LH}) \\ \dfrac{n}{4{\rm tr}\,\{(\boldsymbol{\Gamma}\boldsymbol{\Gamma}')^2(\boldsymbol{I}_p + \boldsymbol{\Gamma}\boldsymbol{\Gamma}')^{-4}\} + 4{\rm tr}\,\{\boldsymbol{\Gamma}\boldsymbol{\Gamma}'(\boldsymbol{I}_p + \boldsymbol{\Gamma}\boldsymbol{\Gamma}')^{-4}\}} & ({\rm G = BNP}) \end{cases}.$$

Asymptotic null distributions of the three test statistics under the HD asymptotic framework can be obtained as the following theorem in the same way as the proof of Theorem 3.1 (the proof is given in Appendix D).

Theorem 3.2 *When H_0 is true, the asymptotic null distributions of $T_{\rm G}$ as $n \to \infty$, $(p+q_1)/n \to c \in [0,1)$ is given by*

$$\sqrt{\tau_{{\rm G},0}^2}\,(T_{\rm G} - \eta_{{\rm G},0}) \xrightarrow{d} \begin{cases} \dfrac{1}{\sqrt{2pq_2}}(\chi^2_{pq_2} - pq_2) & (p : \text{fixed}) \\ N(0,1) & (p \to \infty) \end{cases},$$

where $\eta_{{\rm G},0}$ and $\tau_{{\rm G},0}^2$ are standardizing constants given by

$$\eta_{{\rm G},0} = \begin{cases} q_2 \log \dfrac{n - q_1}{n - p - q_1} & ({\rm G = LR}) \\ \dfrac{pq_2}{n - p - q_1} & ({\rm G = LH}) \\ \dfrac{pq_2}{n - q_1} & ({\rm G = BNP}) \end{cases},$$

$$\tau_{{\rm G},0}^2 = \begin{cases} \dfrac{(n - p - q_1)(n - q_1)}{2pq_2} & ({\rm G = LR}) \\ \dfrac{(n - p - q_1)^3}{2(n - q_1)pq_2} & ({\rm G = LH}) \\ \dfrac{(n - q_1)^3}{2(n - p - q_1)pq_2} & ({\rm G = BNP}) \end{cases}.$$

It is easy to see that $(2pq_2)^{-1/2}\left(\chi^2_{pq_2} - pq_2\right) \xrightarrow{d} N(0,1)$ as $p \to \infty$. Hence, we suggest that $(2pq_2)^{-1/2}\left(\chi^2_{pq_2,\alpha} - pq_2\right)$ is used as the threshold value for the null distributions of $T_{\rm G}$ with significance level α, where $\chi^2_{pq_2,\alpha}$ is the upper 100α % point of the χ^2-distribution with pq_2 degrees of freedom. By using this approximation, we can test regardless of the size of p.

4 Numerical Studies

In this section, we compare the accuracies of approximations based on derived asymptotic distributions under the HD asymptotic framework with those based on asymptotic distributions under the LS asymptotic framework. The accuracies are evaluated from normal or chi-square approximation. First, we compared approximations under H_0. Let $\alpha_{\mathrm{G},1}$ and $\alpha_{\mathrm{G},2}$ be upper probabilities when asymptotic distributions under the HD and the LS asymptotic frameworks are used respectively, which are defined by

$$\alpha_{\mathrm{G},1} = P\left(nT_{\mathrm{G}} > \frac{n}{\sqrt{2pq_2\tau_{\mathrm{G},0}^2}}(\chi^2_{pq_2,\alpha} - pq_2) + n\eta_{\mathrm{G},0} \right), \ \alpha_{\mathrm{G},2} = P\left(nT_{\mathrm{G}} > \chi^2_{pq_2,\alpha}\right).$$

We compared $\alpha_{\mathrm{G},1}$ with $\alpha_{\mathrm{G},2}$ when $\alpha = 0.05$. Tables 1-5 show the $\alpha_{\mathrm{G},1}$ and $\alpha_{\mathrm{G},2}$ which were evaluated by Monte Carlo simulations with 10,000 iterations. Note that $c_1 = p/n$ and $c_2 = q_1/n$ in Tables. From Tables 1 and 2, the approximations under the LS asymptotic framework performed a little well but lose to those under the HD asymptotic framework even when at least one of p and q_1 is small. From Tables 3-5, the approximations under the HD asymptotic framework performed well than those under the LS asymptotic framework.

Table 1: $q_1 = 7, q_2 = 3$

n	p	$\alpha_{\mathrm{LR},1}$	$\alpha_{\mathrm{LR},2}$	$\alpha_{\mathrm{LH},1}$	$\alpha_{\mathrm{LH},2}$	$\alpha_{\mathrm{BNP},1}$	$\alpha_{\mathrm{BNP},2}$
100	10	0.08	0.17	0.10	0.29	0.05	0.08
200	10	0.07	0.10	0.08	0.14	0.05	0.07
300	10	0.06	0.07	0.07	0.09	0.04	0.05
500	10	0.06	0.06	0.07	0.07	0.05	0.05
100	20	0.09	0.37	0.12	0.66	0.05	0.09
200	20	0.07	0.16	0.09	0.28	0.05	0.07
300	20	0.07	0.12	0.08	0.18	0.05	0.06
500	20	0.07	0.08	0.07	0.11	0.05	0.06
100	30	0.10	0.67	0.15	0.94	0.05	0.10
200	30	0.07	0.24	0.08	0.49	0.05	0.07
300	30	0.07	0.15	0.08	0.29	0.05	0.06
500	30	0.06	0.10	0.06	0.16	0.05	0.06
100	50	0.14	0.99	0.23	1.00	0.06	0.10
200	50	0.08	0.56	0.10	0.91	0.05	0.07
300	50	0.07	0.31	0.08	0.64	0.05	0.06
500	50	0.06	0.16	0.07	0.33	0.05	0.06

Table 2: $p = 7, q_2 = 3$

n	q_1	$\alpha_{\text{LR},1}$	$\alpha_{\text{LR},2}$	$\alpha_{\text{LH},1}$	$\alpha_{\text{LH},2}$	$\alpha_{\text{BNP},1}$	$\alpha_{\text{BNP},2}$
100	10	0.09	0.16	0.10	0.24	0.05	0.10
200	10	0.07	0.09	0.08	0.12	0.05	0.07
300	10	0.07	0.07	0.07	0.09	0.05	0.06
500	10	0.07	0.07	0.07	0.07	0.05	0.06
100	20	0.09	0.30	0.11	0.39	0.05	0.21
200	20	0.07	0.13	0.08	0.16	0.05	0.10
300	20	0.07	0.10	0.08	0.12	0.05	0.08
500	20	0.07	0.08	0.07	0.08	0.05	0.07
100	30	0.09	0.48	0.11	0.57	0.05	0.37
200	30	0.07	0.18	0.08	0.22	0.05	0.14
300	30	0.07	0.12	0.07	0.14	0.05	0.10
500	30	0.06	0.08	0.07	0.09	0.05	0.08
100	50	0.10	0.86	0.14	0.91	0.05	0.81
200	50	0.07	0.33	0.09	0.37	0.05	0.28
300	50	0.07	0.19	0.08	0.22	0.06	0.17
500	50	0.07	0.11	0.07	0.12	0.05	0.10

Table 3: $q_1 = 7, q_2 = 3$

c_1	n	p	$\alpha_{\text{LR},1}$	$\alpha_{\text{LR},2}$	$\alpha_{\text{LH},1}$	$\alpha_{\text{LH},2}$	$\alpha_{\text{BNP},1}$	$\alpha_{\text{BNP},2}$
0.1	100	10	0.08	0.17	0.10	0.28	0.05	0.08
	200	20	0.07	0.16	0.09	0.28	0.05	0.07
	300	30	0.06	0.15	0.07	0.29	0.05	0.06
	500	50	0.06	0.17	0.07	0.33	0.05	0.06
0.3	100	30	0.10	0.67	0.15	0.94	0.05	0.10
	200	60	0.08	0.74	0.11	0.98	0.05	0.06
	300	90	0.07	0.81	0.09	0.99	0.05	0.05
	500	150	0.07	0.91	0.08	1.00	0.05	0.05
0.5	100	50	0.12	0.99	0.22	1.00	0.05	0.09
	200	100	0.09	1.00	0.14	1.00	0.05	0.04
	300	150	0.09	1.00	0.12	1.00	0.05	0.04
	500	250	0.08	1.00	0.10	1.00	0.05	0.03
0.8	100	80	0.30	1.00	0.58	1.00	0.06	0.01
	200	160	0.16	1.00	0.32	1.00	0.06	0.00
	300	240	0.13	1.00	0.24	1.00	0.06	0.00
	500	400	0.11	1.00	0.17	1.00	0.06	0.00

Table 4: $p = 7, q_2 = 3$

c_2	n	q_1	$\alpha_{\mathrm{LR},1}$	$\alpha_{\mathrm{LR},2}$	$\alpha_{\mathrm{LH},1}$	$\alpha_{\mathrm{LH},2}$	$\alpha_{\mathrm{BNP},1}$	$\alpha_{\mathrm{BNP},2}$
0.1	100	10	0.08	0.17	0.10	0.23	0.05	0.10
	200	20	0.07	0.13	0.08	0.16	0.05	0.10
	300	30	0.07	0.12	0.07	0.14	0.05	0.10
	500	50	0.06	0.11	0.07	0.12	0.05	0.10
0.3	100	30	0.08	0.48	0.11	0.57	0.05	0.38
	200	60	0.07	0.41	0.08	0.45	0.05	0.35
	300	90	0.07	0.39	0.08	0.42	0.05	0.36
	500	150	0.06	0.38	0.07	0.39	0.05	0.36
0.5	100	50	0.10	0.86	0.15	0.90	0.05	0.80
	200	100	0.08	0.80	0.10	0.83	0.05	0.77
	300	150	0.07	0.79	0.08	0.81	0.05	0.77
	500	250	0.06	0.77	0.07	0.79	0.04	0.76
0.8	100	80	0.22	1.00	0.39	1.00	0.05	1.00
	200	160	0.12	1.00	0.18	1.00	0.05	1.00
	300	240	0.10	1.00	0.13	1.00	0.05	1.00
	500	400	0.08	1.00	0.10	1.00	0.05	1.00

Table 5: $q_2 = 3$

c_1, c_2	n	p, q_1	$\alpha_{\mathrm{LR},1}$	$\alpha_{\mathrm{LR},2}$	$\alpha_{\mathrm{LH},1}$	$\alpha_{\mathrm{LH},2}$	$\alpha_{\mathrm{BNP},1}$	$\alpha_{\mathrm{BNP},2}$
0.1	100	10	0.08	0.21	0.11	0.33	0.05	0.11
	200	20	0.07	0.28	0.09	0.43	0.05	0.15
	300	30	0.07	0.34	0.08	0.51	0.05	0.18
	500	50	0.06	0.46	0.07	0.66	0.05	0.24
0.2	100	20	0.10	0.74	0.14	0.92	0.05	0.38
	200	40	0.08	0.91	0.11	0.99	0.05	0.60
	300	60	0.07	0.97	0.09	1.00	0.05	0.74
	500	100	0.06	1.00	0.08	1.00	0.05	0.91
0.3	100	30	0.13	1.00	0.22	1.00	0.05	0.89
	200	60	0.10	1.00	0.15	1.00	0.05	0.99
	300	90	0.08	1.00	0.12	1.00	0.05	1.00
	500	150	0.08	1.00	0.10	1.00	0.05	1.00
0.4	100	40	0.20	1.00	0.40	1.00	0.06	1.00
	200	80	0.13	1.00	0.24	1.00	0.05	1.00
	300	120	0.12	1.00	0.20	1.00	0.05	1.00
	500	200	0.10	1.00	0.16	1.00	0.06	1.00

Next, we compared approximations under H_1. It should be kept in mind that sometimes some elements of $\mathbf{\Gamma}$ defined in Lemma 3.2 become infinite and other times those are finite as at least one of p and q tends to ∞. Therefore, we treat the following two non-centrality matrices.

・Structure 1 (at least of elements of $\mathbf{\Gamma}$ tend to ∞)

$$\mathbf{\Gamma} = \begin{cases} (\mathbf{\Lambda}_1, \boldsymbol{O}_{q_2, p-q_2})' & (q_2 \le p) \\ (\mathbf{\Lambda}_1, \boldsymbol{O}_{p, q_2-p}) & (p < q_2) \end{cases}, \ \mathbf{\Lambda}_1 = \mathrm{diag}(\sqrt{pq}, 0, \ldots, 0).$$

・Structure 2 (all elements of $\mathbf{\Gamma}$ are bounded)

$$\mathbf{\Gamma} = \begin{cases} (\mathbf{\Lambda}_2, \boldsymbol{O}_{q_2, p-q_2})' & (q_2 \le p) \\ (\mathbf{\Lambda}_2, \boldsymbol{O}_{p, q_2-p}) & (p < q_2) \end{cases}, \ \mathbf{\Lambda}_2 = \mathrm{diag}\left(\sqrt{\frac{(p+1)(q+1)}{1+pq}}, 0, \ldots, 0\right).$$

It is clearly that the non-zero element of $\mathbf{\Gamma}$ tends to ∞ in Structure 1, and all elements of $\mathbf{\Gamma}$ are bounded in Structure 2 as at least one of p and q tends to ∞.

Let $\alpha_{\mathrm{G},3}$ and $\alpha_{\mathrm{G},4}$ be upper probabilities when asymptotic distributions under the HD and the LS asymptotic frameworks are used respectively, which are defined by

$$\alpha_{\mathrm{G},3} = P\left(T_{\mathrm{G}} > \frac{1}{\sqrt{\tau_{\mathrm{G}}^2}} z_\alpha + \eta_{\mathrm{G}}\right), \ \alpha_{\mathrm{G},4} = P\left(T_{\mathrm{G}} > \frac{1}{\sqrt{\nu_{\mathrm{G}}^2}} z_\alpha + \theta_{\mathrm{G}}\right),$$

where z_α is the upper 100α % point of the standard normal distribution. We compared $\alpha_{\mathrm{G},3}$ with $\alpha_{\mathrm{G},4}$ when $\alpha = 0.05$. Tables 6-15 show the $\alpha_{\mathrm{G},3}$ and $\alpha_{\mathrm{G},4}$ which were evaluated by Monte Carlo simulations with 10,000 iterations. Note that $c_1 = p/n$ and $c_2 = q_1/n$ in Tables. From Tables 6 and 7, the approximations under the LS asymptotic framework performed a little well but lose to those under the HD asymptotic framework even when at least one of p and q_1 is small. The reason is that the HD asymptotic framework includes in the LS asymptotic framework. From Tables 8-15, the approximations under the HD asymptotic framework performed well, but the approximations under the LS asymptotic framework are poor.

Table 6: Structure 1 : $q_1 = 7$, $q_2 = 3$

n	p	$\alpha_{\mathrm{LR},3}$	$\alpha_{\mathrm{LR},4}$	$\alpha_{\mathrm{LH},3}$	$\alpha_{\mathrm{LH},4}$	$\alpha_{\mathrm{BNP},3}$	$\alpha_{\mathrm{BNP},4}$
100	10	0.06	0.55	0.10	0.26	0.03	1.00
200	10	0.05	0.32	0.08	0.16	0.03	1.00
300	10	0.05	0.25	0.07	0.14	0.03	1.00
500	10	0.05	0.17	0.07	0.11	0.03	1.00
100	20	0.07	0.96	0.11	0.49	0.05	1.00
200	20	0.06	0.74	0.08	0.28	0.04	1.00
300	20	0.06	0.58	0.08	0.22	0.03	1.00
500	20	0.05	0.42	0.07	0.16	0.04	1.00
100	30	0.08	1.00	0.12	0.74	0.04	1.00
200	30	0.06	0.96	0.09	0.44	0.04	1.00
300	30	0.05	0.86	0.08	0.32	0.04	1.00
500	30	0.05	0.67	0.07	0.22	0.04	1.00
100	50	0.11	1.00	0.15	0.98	0.04	1.00
200	50	0.06	1.00	0.09	0.75	0.04	1.00
300	50	0.06	1.00	0.08	0.58	0.04	1.00
500	50	0.05	0.97	0.07	0.39	0.04	1.00

Table 7: Structure 1 : $p = 7$, $q_2 = 3$

n	q_1	$\alpha_{\text{LR},3}$	$\alpha_{\text{LR},4}$	$\alpha_{\text{LH},3}$	$\alpha_{\text{LH},4}$	$\alpha_{\text{BNP},3}$	$\alpha_{\text{BNP},4}$
100	10	0.06	0.37	0.11	0.21	0.03	1.00
200	10	0.05	0.21	0.08	0.14	0.03	1.00
300	10	0.05	0.17	0.07	0.12	0.03	1.00
500	10	0.05	0.13	0.07	0.09	0.03	1.00
100	20	0.06	0.43	0.11	0.25	0.03	1.00
200	20	0.05	0.23	0.08	0.15	0.03	1.00
300	20	0.05	0.18	0.08	0.13	0.03	1.00
500	20	0.05	0.13	0.07	0.10	0.03	1.00
100	30	0.06	0.50	0.11	0.28	0.03	1.00
200	30	0.05	0.25	0.08	0.16	0.03	1.00
300	30	0.05	0.19	0.08	0.13	0.03	1.00
500	30	0.05	0.14	0.07	0.10	0.03	1.00
100	50	0.07	0.69	0.13	0.40	0.03	1.00
200	50	0.05	0.31	0.09	0.19	0.03	1.00
300	50	0.05	0.22	0.08	0.15	0.03	1.00
500	50	0.05	0.15	0.07	0.11	0.03	1.00

Table 8: Structure 1 : $q_1 = 7, q_2 = 3$

c_1	n	p	$\alpha_{\text{LR},3}$	$\alpha_{\text{LR},4}$	$\alpha_{\text{LH},3}$	$\alpha_{\text{LH},4}$	$\alpha_{\text{BNP},3}$	$\alpha_{\text{BNP},4}$
0.1	100	10	0.06	0.54	0.10	0.25	0.04	1.00
	200	20	0.06	0.74	0.09	0.28	0.04	1.00
	300	30	0.05	0.86	0.07	0.31	0.04	1.00
	500	50	0.05	0.96	0.07	0.40	0.04	1.00
0.3	100	30	0.08	1.00	0.12	0.73	0.04	1.00
	200	60	0.07	1.00	0.09	0.88	0.05	1.00
	300	90	0.06	1.00	0.09	0.95	0.05	1.00
	500	150	0.06	1.00	0.08	0.99	0.05	1.00
0.5	100	50	0.11	1.00	0.16	0.99	0.05	1.00
	200	100	0.09	1.00	0.11	1.00	0.05	1.00
	300	150	0.08	1.00	0.09	1.00	0.05	1.00
	500	250	0.07	1.00	0.08	1.00	0.05	1.00
0.8	100	80	0.28	1.00	0.33	1.00	0.05	1.00
	200	160	0.15	1.00	0.19	1.00	0.06	1.00
	300	240	0.12	1.00	0.15	1.00	0.05	1.00
	500	400	0.10	1.00	0.12	1.00	0.05	1.00

Table 9: Structure 1 : $p = 7, q_2 = 3$

c_2	n	q_1	$\alpha_{\mathrm{LR},3}$	$\alpha_{\mathrm{LR},4}$	$\alpha_{\mathrm{LH},3}$	$\alpha_{\mathrm{LH},4}$	$\alpha_{\mathrm{BNP},3}$	$\alpha_{\mathrm{BNP},4}$
0.1	100	10	0.06	0.36	0.11	0.21	0.03	1.00
	200	20	0.05	0.23	0.08	0.15	0.03	1.00
	300	30	0.05	0.19	0.08	0.12	0.03	1.00
	500	50	0.05	0.15	0.07	0.11	0.03	1.00
0.3	100	30	0.06	0.50	0.11	0.29	0.03	1.00
	200	60	0.05	0.34	0.09	0.21	0.03	1.00
	300	90	0.06	0.28	0.08	0.18	0.03	1.00
	500	150	0.05	0.22	0.08	0.16	0.03	1.00
0.5	100	50	0.07	0.69	0.13	0.40	0.04	1.00
	200	100	0.06	0.48	0.10	0.29	0.03	1.00
	300	150	0.05	0.40	0.09	0.25	0.03	1.00
	500	250	0.05	0.33	0.08	0.22	0.03	1.00
0.8	100	80	0.16	0.98	0.26	0.76	0.04	1.00
	200	160	0.07	0.85	0.15	0.57	0.04	1.00
	300	240	0.06	0.76	0.12	0.50	0.03	1.00
	500	400	0.06	0.64	0.10	0.42	0.03	1.00

Table 10: Structure 1 : $q_2 = 3$

$c_1,\ c_2$	n	$p,\ q_1$	$\alpha_{\mathrm{LR},3}$	$\alpha_{\mathrm{LR},4}$	$\alpha_{\mathrm{LH},3}$	$\alpha_{\mathrm{LH},4}$	$\alpha_{\mathrm{BNP},3}$	$\alpha_{\mathrm{BNP},4}$
0.1	100	10	0.06	0.56	0.10	0.26	0.04	1.00
	200	20	0.05	0.78	0.08	0.31	0.04	1.00
	300	30	0.05	0.90	0.08	0.35	0.04	1.00
	500	50	0.06	0.98	0.07	0.45	0.04	1.00
0.2	100	20	0.07	0.98	0.12	0.57	0.04	1.00
	200	40	0.06	1.00	0.10	0.72	0.05	1.00
	300	60	0.06	1.00	0.09	0.82	0.04	1.00
	500	100	0.06	1.00	0.08	0.93	0.05	1.00
0.3	100	30	0.10	1.00	0.15	0.89	0.04	1.00
	200	60	0.08	1.00	0.12	0.97	0.05	1.00
	300	90	0.07	1.00	0.09	0.99	0.05	1.00
	500	150	0.07	1.00	0.09	1.00	0.05	1.00
0.4	100	40	0.19	1.00	0.24	1.00	0.06	1.00
	200	80	0.13	1.00	0.15	1.00	0.05	1.00
	300	120	0.11	1.00	0.14	1.00	0.05	1.00
	500	200	0.09	1.00	0.11	1.00	0.05	1.00

Table 11: Structure 2 : $q_1 = 7$, $q_2 = 3$

n	p	$\alpha_{\text{LR},3}$	$\alpha_{\text{LR},4}$	$\alpha_{\text{LH},3}$	$\alpha_{\text{LH},4}$	$\alpha_{\text{BNP},3}$	$\alpha_{\text{BNP},4}$
100	10	0.07	0.72	0.11	0.51	0.04	0.94
200	10	0.06	0.46	0.08	0.31	0.03	0.78
300	10	0.05	0.35	0.07	0.23	0.04	0.64
500	10	0.06	0.24	0.07	0.18	0.04	0.48
100	20	0.07	1.00	0.12	0.95	0.04	1.00
200	20	0.06	0.91	0.08	0.69	0.04	1.00
300	20	0.06	0.79	0.08	0.54	0.04	0.98
500	20	0.05	0.59	0.06	0.37	0.04	0.91
100	30	0.08	1.00	0.12	1.00	0.05	1.00
200	30	0.06	1.00	0.09	0.94	0.05	1.00
300	30	0.06	0.98	0.07	0.82	0.04	1.00
500	30	0.05	0.88	0.07	0.61	0.04	1.00
100	50	0.12	1.00	0.20	1.00	0.05	1.00
200	50	0.07	1.00	0.10	1.00	0.05	1.00
300	50	0.06	1.00	0.08	1.00	0.05	1.00
500	50	0.06	1.00	0.07	0.94	0.04	1.00

Table 12: Structure 2 : $p = 7$, $q_2 = 3$

n	q_1	$\alpha_{\text{LR},3}$	$\alpha_{\text{LR},4}$	$\alpha_{\text{LH},3}$	$\alpha_{\text{LH},4}$	$\alpha_{\text{BNP},3}$	$\alpha_{\text{BNP},4}$
100	10	0.06	0.50	0.10	0.37	0.03	0.78
200	10	0.05	0.29	0.07	0.22	0.03	0.53
300	10	0.06	0.22	0.08	0.17	0.03	0.41
500	10	0.05	0.17	0.07	0.14	0.04	0.30
100	20	0.07	0.58	0.11	0.43	0.03	0.84
200	20	0.06	0.33	0.08	0.25	0.03	0.56
300	20	0.06	0.24	0.08	0.19	0.03	0.44
500	20	0.05	0.17	0.07	0.14	0.03	0.30
100	30	0.07	0.66	0.11	0.49	0.03	0.88
200	30	0.06	0.37	0.09	0.27	0.04	0.61
300	30	0.05	0.26	0.07	0.20	0.03	0.45
500	30	0.05	0.18	0.07	0.15	0.04	0.32
100	50	0.08	0.82	0.13	0.67	0.04	0.95
200	50	0.06	0.41	0.09	0.31	0.03	0.67
300	50	0.05	0.28	0.07	0.21	0.03	0.50
500	50	0.05	0.19	0.06	0.15	0.04	0.33

Table 13: Structure 2 : $q_1 = 7, q_2 = 3$

c_1	n	p	$\alpha_{\text{LR},3}$	$\alpha_{\text{LR},4}$	$\alpha_{\text{LH},3}$	$\alpha_{\text{LH},4}$	$\alpha_{\text{BNP},3}$	$\alpha_{\text{BNP},4}$
0.1	100	10	0.06	0.72	0.10	0.51	0.04	0.94
	200	20	0.06	0.92	0.08	0.69	0.04	1.00
	300	30	0.06	0.98	0.08	0.82	0.04	1.00
	500	50	0.05	1.00	0.07	0.94	0.04	1.00
0.3	100	30	0.08	1.00	0.13	1.00	0.04	1.00
	200	60	0.07	1.00	0.10	1.00	0.05	1.00
	300	90	0.07	1.00	0.09	1.00	0.05	1.00
	500	150	0.06	1.00	0.08	1.00	0.05	1.00
0.5	100	50	0.12	1.00	0.19	1.00	0.05	1.00
	200	100	0.09	1.00	0.13	1.00	0.05	1.00
	300	150	0.08	1.00	0.11	1.00	0.05	1.00
	500	250	0.07	1.00	0.10	1.00	0.06	1.00
0.8	100	80	0.30	1.00	0.53	1.00	0.07	1.00
	200	160	0.16	1.00	0.29	1.00	0.06	1.00
	300	240	0.13	1.00	0.21	1.00	0.06	1.00
	500	400	0.10	1.00	0.16	1.00	0.06	1.00

Table 14: Structure 2 : $p = 7, q_2 = 3$

c_2	n	q_1	$\alpha_{\text{LR},3}$	$\alpha_{\text{LR},4}$	$\alpha_{\text{LH},3}$	$\alpha_{\text{LH},4}$	$\alpha_{\text{BNP},3}$	$\alpha_{\text{BNP},4}$
0.1	100	10	0.07	0.51	0.11	0.37	0.03	0.79
	200	20	0.06	0.33	0.08	0.24	0.03	0.57
	300	30	0.05	0.25	0.07	0.19	0.03	0.45
	500	50	0.05	0.20	0.07	0.16	0.04	0.33
0.3	100	30	0.07	0.66	0.11	0.50	0.03	0.88
	200	60	0.06	0.45	0.09	0.34	0.03	0.70
	300	90	0.05	0.36	0.08	0.28	0.03	0.59
	500	150	0.05	0.27	0.07	0.21	0.03	0.45
0.5	100	50	0.07	0.82	0.13	0.67	0.03	0.95
	200	100	0.06	0.62	0.10	0.48	0.03	0.84
	300	150	0.06	0.51	0.09	0.39	0.03	0.75
	500	250	0.06	0.41	0.08	0.32	0.03	0.62
0.8	100	80	0.18	1.00	0.34	0.99	0.04	1.00
	200	160	0.09	0.94	0.16	0.86	0.04	0.99
	300	240	0.07	0.87	0.12	0.75	0.04	0.97
	500	400	0.06	0.76	0.09	0.62	0.03	0.91

Table 15: Structure 2 : $q_2 = 3$

c_1, c_2	n	p, q_1	$\alpha_{\mathrm{LR},3}$	$\alpha_{\mathrm{LR},4}$	$\alpha_{\mathrm{LH},3}$	$\alpha_{\mathrm{LH},4}$	$\alpha_{\mathrm{BNP},3}$	$\alpha_{\mathrm{BNP},4}$
0.1	100	10	0.07	0.75	0.10	0.54	0.04	0.95
	200	20	0.06	0.94	0.09	0.75	0.04	1.00
	300	30	0.06	0.99	0.08	0.88	0.04	1.00
	500	50	0.05	1.00	0.07	0.98	0.04	1.00
0.2	100	20	0.08	1.00	0.13	0.98	0.05	1.00
	200	40	0.07	1.00	0.10	1.00	0.05	1.00
	300	60	0.07	1.00	0.09	1.00	0.05	1.00
	500	100	0.06	1.00	0.08	1.00	0.05	1.00
0.3	100	30	0.11	1.00	0.18	1.00	0.05	1.00
	200	60	0.08	1.00	0.13	1.00	0.04	1.00
	300	90	0.08	1.00	0.11	1.00	0.05	1.00
	500	150	0.07	1.00	0.09	1.00	0.05	1.00
0.4	100	40	0.19	1.00	0.34	1.00	0.06	1.00
	200	80	0.13	1.00	0.22	1.00	0.06	1.00
	300	120	0.11	1.00	0.18	1.00	0.05	1.00
	500	200	0.09	1.00	0.14	1.00	0.05	1.00

Appendix

A Proof of Lemma 3.3

At first, we show the (7) and (8). Considering the conditional distribution of $\boldsymbol{B}_2$ given $\boldsymbol{B}_1$ in Lemma 3.2, $\boldsymbol{B}_2$ is given by $\boldsymbol{B}_1$ and $\boldsymbol{A}$ defined in Lemma 3.2, as follows:

$$\boldsymbol{B}_2 = \boldsymbol{B}_1\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} + \boldsymbol{A}\boldsymbol{\Sigma}_{22\cdot 1}^{1/2}.$$

This implies that

$$\boldsymbol{A} = (\boldsymbol{B}_2 - \boldsymbol{B}_1\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12})\boldsymbol{\Sigma}_{22\cdot 1}^{-1/2}.$$

Notice that $\boldsymbol{P}\boldsymbol{B}_1 = \boldsymbol{B}_1$, $\boldsymbol{P}\boldsymbol{B}_2 = \boldsymbol{B}_2$ and $(\boldsymbol{I}_{n-1} - \boldsymbol{P})\boldsymbol{A} = \boldsymbol{O}_{n-1,q_2}$, where $\boldsymbol{P} = \boldsymbol{B}(\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'$ and $\boldsymbol{B} = (\boldsymbol{B}_1, \boldsymbol{B}_2)$. Hence, we have

$$(\boldsymbol{P} - \boldsymbol{P}_1)\boldsymbol{A} = (\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A} - (\boldsymbol{I}_{n-1} - \boldsymbol{P})\boldsymbol{A} = (\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A}, \tag{A.1}$$

where $\boldsymbol{P}_1 = \boldsymbol{B}_1(\boldsymbol{B}_1'\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1'$. From (A.1), by using $\boldsymbol{\Gamma}$ and $\boldsymbol{\mathcal{E}}$ defined in Lemma 3.2, we can calculate as follows:

$$\begin{aligned}(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}}) &= (\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}})'\Big\{(\boldsymbol{I}_{n-1} - \boldsymbol{P}) + (\boldsymbol{P} - \boldsymbol{P}_1)\Big\}(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}}) \\ &= (\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1} - \boldsymbol{P})(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}}) \\ &\quad + (\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}})'(\boldsymbol{P} - \boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}}) \\ &= \boldsymbol{\mathcal{E}}'(\boldsymbol{I}_{n-1} - \boldsymbol{P})\boldsymbol{\mathcal{E}} + (\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}})'(\boldsymbol{P} - \boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}' + \boldsymbol{\mathcal{E}}).\end{aligned}$$

Since $\boldsymbol{P}-\boldsymbol{P}_1$ is an idempotent matrix, the following equation is derived:

$$
\begin{aligned}
(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})'(\boldsymbol{P}-\boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}}) &= (\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})'(\boldsymbol{P}-\boldsymbol{P}_1)^2(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}}) \\
&= \Big\{(\boldsymbol{P}-\boldsymbol{P}_1)^2\boldsymbol{A}\boldsymbol{\Gamma}'+(\boldsymbol{P}-\boldsymbol{P}_1)\boldsymbol{\mathcal{E}}\Big\}'\Big\{(\boldsymbol{P}-\boldsymbol{P}_1)^2\boldsymbol{A}\boldsymbol{\Gamma}'+(\boldsymbol{P}-\boldsymbol{P}_1)\boldsymbol{\mathcal{E}}\Big\} \\
&= \Big\{(\boldsymbol{P}-\boldsymbol{P}_1)(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Gamma}'+(\boldsymbol{P}-\boldsymbol{P}_1)\boldsymbol{\mathcal{E}}\Big\}' \\
&\quad\times\Big\{(\boldsymbol{P}-\boldsymbol{P}_1)(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Gamma}'+(\boldsymbol{P}-\boldsymbol{P}_1)\boldsymbol{\mathcal{E}}\Big\} \\
&= \Big\{(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}}\Big\}'(\boldsymbol{P}-\boldsymbol{P}_1)\Big\{(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}}\Big\}.
\end{aligned}
$$

Let $\boldsymbol{Q}_1\boldsymbol{Q}_1'$ and $\boldsymbol{Q}\boldsymbol{Q}'$ be spectral decompositions of $\boldsymbol{P}-\boldsymbol{P}_1$ and $\boldsymbol{I}_{n-1}-\boldsymbol{P}$, respectively, where $\boldsymbol{Q}_1$ is an $(n-1)\times q_2$ matrix satisfying $\boldsymbol{Q}_1'\boldsymbol{Q}_1=\boldsymbol{I}_{q_2}$, and $\boldsymbol{Q}$ is an $(n-1)\times(n-q-1)$ matrix satisfying $\boldsymbol{Q}'\boldsymbol{Q}=\boldsymbol{I}_{n-q-1}$. Then, $(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})$ is expressed as follows:

$$
\begin{aligned}
&(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}}) \\
&= \boldsymbol{\mathcal{E}}'\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{\mathcal{E}}+\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\Big\}'\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\Big\}.
\end{aligned}
$$

Let $\boldsymbol{J}\boldsymbol{\Delta}\boldsymbol{K}'$ be a singular decomposition of $\boldsymbol{\Gamma}$, where $\boldsymbol{J}$ is a p-th orthogonal matrix and $\boldsymbol{K}$ is a q_2-th orthogonal matrix. Then, the another expression of $(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})$ is given as

$$
\begin{aligned}
&(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}}) \\
&= \boldsymbol{\mathcal{E}}'\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{\mathcal{E}}+\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{K}\boldsymbol{\Delta}'\boldsymbol{J}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\Big\}'\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{K}\boldsymbol{\Delta}'\boldsymbol{J}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\Big\} \\
&= \boldsymbol{\mathcal{E}}'\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{\mathcal{E}}+\boldsymbol{J}\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{K}\boldsymbol{\Delta}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}\Big\}'\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{K}\boldsymbol{\Delta}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}\Big\}\boldsymbol{J}'.
\end{aligned} \tag{A.2}
$$

Here, without loss of generality, we can replace $\boldsymbol{A}\boldsymbol{K}$ with $\boldsymbol{A}$. Then, (A.2) is expressed as follows:

$$
\begin{aligned}
&(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}})'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)(\boldsymbol{A}\boldsymbol{\Gamma}'+\boldsymbol{\mathcal{E}}) \\
&= \boldsymbol{\mathcal{E}}'\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{\mathcal{E}}+\boldsymbol{J}\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Delta}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}\Big\}'\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Delta}'+\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}\Big\}\boldsymbol{J}'.
\end{aligned}
$$

It is easy to see that $\boldsymbol{A}'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}$ is distributed according to $W_{q_2}(n-q_1-1,\boldsymbol{I}_{q_2})$ because $\boldsymbol{A}$ and $\boldsymbol{P}_1$ are mutually independent. Hence, we apply the Bartlett decomposition (see, e.g., Fujikoshi *et al.*, 2010, chap. 2.3) to $\boldsymbol{A}'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}$ as follows:

$$
\boldsymbol{A}'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}=\boldsymbol{T}\boldsymbol{T}', \tag{A.3}
$$

where $\boldsymbol{T}=\{t_{ij}\}_{i,j=1}^{q_2}$ is a $q_2\times q_2$ lower triangular matrix of which elements are mutually independent random variables, which are defined by $t_{ii}^2\sim\chi^2(n-q_1-i)$ and $t_{ij}\sim N(0,1)$ $(i>j)$, and $\boldsymbol{T}$, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{B}_1$ are mutually independent since $\boldsymbol{A}$, $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{B}_1$ are mutually independent. Let $\boldsymbol{H}\boldsymbol{L}\boldsymbol{G}'$ be a singular value decomposition of $\boldsymbol{T}$, where $\boldsymbol{L}$ is a diagonal matrix of which diagonal elements are the singular values of $\boldsymbol{T}$, and $\boldsymbol{H}$ and $\boldsymbol{G}$ are q_2-th orthogonal matrices. The following singular value decomposition of $\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}$ is also used:

$$
\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1}-\boldsymbol{P}_1)\boldsymbol{A}=\boldsymbol{C}\boldsymbol{L}\boldsymbol{H}'=\boldsymbol{R}\boldsymbol{T}',
$$

where $\boldsymbol{C}$ is a q_2-th orthogonal matrix and $\boldsymbol{R} = \boldsymbol{C}\boldsymbol{G}'$ because of

$$\begin{aligned}\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A}\}' \boldsymbol{Q}_1'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A} &= \boldsymbol{A}'(\boldsymbol{P} - \boldsymbol{P}_1)\boldsymbol{A} \\ &= \boldsymbol{A}'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A} - \boldsymbol{A}'(\boldsymbol{I}_{n-1} - \boldsymbol{P})\boldsymbol{A} \\ &= \boldsymbol{A}'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A}.\end{aligned}$$

By using those equations, the following equation can be derived:

$$\begin{aligned}&\boldsymbol{J}\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Delta}' + \boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}\Big\}'\Big\{\boldsymbol{Q}_1'(\boldsymbol{I}_{n-1} - \boldsymbol{P}_1)\boldsymbol{A}\boldsymbol{\Delta}' + \boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}\Big\}\boldsymbol{J}' \\ &= \boldsymbol{J}(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J})'(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J})\boldsymbol{J}'.\end{aligned}$$

Hence, by replacing $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ with $\boldsymbol{Q}'\boldsymbol{\mathcal{E}}$ and $\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}$, respectively, (7) and (8) can be derived.

Next, we show that $\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}$, $\boldsymbol{Q}'\boldsymbol{\mathcal{E}}$ and $\boldsymbol{T}$ are mutually independent. Let $\boldsymbol{Q}_2\boldsymbol{Q}_2'$ be the spectral decomposition of $\boldsymbol{I}_{n-1}-\boldsymbol{P}_1$, where $\boldsymbol{Q}_2$ is an $(n-1)\times(n-q_1-1)$ matrix satisfying $\boldsymbol{Q}_2'\boldsymbol{Q}_2 = \boldsymbol{I}_{n-q_1-1}$. Then, we can see that $\boldsymbol{T}$ is a function of $\boldsymbol{Q}_2'\boldsymbol{A}$ from (A.3). It is easy to see that $\boldsymbol{Q}_2'\boldsymbol{A}$ is distributed according to $N_{(n-q_1-1)\times q_2}(\boldsymbol{O}_{n-q_1-1,q_2}, \boldsymbol{I}_{q_2}\otimes\boldsymbol{I}_{n-q_1-1})$ because $\boldsymbol{Q}_2$ and $\boldsymbol{A}$ are mutually independent. In the same way as $\boldsymbol{Q}_2'\boldsymbol{A}$, it is easy to see that $\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}$ is distributed according to $N_{q_2\times p}(\boldsymbol{O}_{q_2,p}, \boldsymbol{I}_p\otimes\boldsymbol{I}_{q_2})$ because $\boldsymbol{T}$, $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{B}_1$ are mutually independent. We can also see that $\boldsymbol{Q}'\boldsymbol{\mathcal{E}}$ is distributed according to $N_{(n-q-1)\times p}(\boldsymbol{O}_{n-q-1,p}, \boldsymbol{I}_p \otimes \boldsymbol{I}_{n-q-1})$ because $\boldsymbol{Q}$ and $\boldsymbol{\mathcal{E}}$ are mutually independent. Since $\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}$ and $\boldsymbol{B}$ are independent and it is obtained that $\boldsymbol{Q}'\boldsymbol{Q}_1 = \boldsymbol{O}_{n-q-1,q_2}$ from the fact that $(\boldsymbol{I}_{n-1}-\boldsymbol{P})(\boldsymbol{P}-\boldsymbol{P}_1) = \boldsymbol{O}_{n-1,n-1}$, the following expectations can be calculated:

$$\begin{aligned}E\Big[\mathrm{vec}(\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J})\mathrm{vec}(\boldsymbol{Q}'\boldsymbol{\mathcal{E}})'\Big] &= E\Big[E\Big[\mathrm{vec}(\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J})\mathrm{vec}(\boldsymbol{Q}'\boldsymbol{\mathcal{E}})'|\boldsymbol{B}\Big]\Big] \\ &= E\Big[(\boldsymbol{J}' \otimes \boldsymbol{R}'\boldsymbol{Q}_1')\mathrm{vec}(\boldsymbol{\mathcal{E}})\mathrm{vec}(\boldsymbol{\mathcal{E}})'(\boldsymbol{I}_p \otimes \boldsymbol{Q})\Big] \\ &= \boldsymbol{O}_{q_2p,(n-q-1)p},\end{aligned}$$

$$E\Big[\mathrm{vec}(\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J})\mathrm{vec}(\boldsymbol{Q}_2'\boldsymbol{A})'\Big] = \boldsymbol{O}_{q_2p,(n-q_1-1)q_2},$$

$$E\Big[\mathrm{vec}(\boldsymbol{Q}'\boldsymbol{\mathcal{E}})\mathrm{vec}(\boldsymbol{Q}_2'\boldsymbol{A})'\Big] = \boldsymbol{O}_{(n-q-1)p,(n-q_1-1)q_2}.$$

Thus, from the property of the multivariate normal distribution, $\boldsymbol{R}'\boldsymbol{Q}_1'\boldsymbol{\mathcal{E}}\boldsymbol{J}$, $\boldsymbol{Q}'\boldsymbol{\mathcal{E}}$ and $\boldsymbol{T}$ are mutually independent. □

B Proof of Lemma 3.4

From Lemma 3.3, it is clear that $\boldsymbol{U}_1'\boldsymbol{U}_1 \sim W_p(n - q - 1, \boldsymbol{I}_p)$, and $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ and $\boldsymbol{T}$ are mutually independent. Let $\tilde{\boldsymbol{U}} = (\boldsymbol{D}\boldsymbol{D}')^{1/2}\{\boldsymbol{D}(\boldsymbol{U}_1'\boldsymbol{U}_1)^{-1}\boldsymbol{D}'\}^{-1}(\boldsymbol{D}\boldsymbol{D}')^{1/2}$. Then, by using the properties of Wishart distribution, we can see that $\tilde{\boldsymbol{U}} \sim W_{q_2}(n-p-q_1-1, \boldsymbol{I}_{q_2})$ and $\tilde{\boldsymbol{U}}$ and $\boldsymbol{D}$ are independent. Thus, from the equations (4), (5) and (6) in Lemma 3.1, we can derive the following expressions of $T_{\rm G}$:

$$\begin{aligned} T_{\rm LR} &= \log \frac{|\boldsymbol{U}_1'\boldsymbol{U}_1 + \boldsymbol{J}(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)'(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)\boldsymbol{J}'|}{|\boldsymbol{U}_1'\boldsymbol{U}_1|} \\ &= \log \frac{|\boldsymbol{U}_1'\boldsymbol{U}_1 + (\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)'(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)|}{|\boldsymbol{U}_1'\boldsymbol{U}_1|} \\ &= \log |\boldsymbol{I}_p + (\boldsymbol{U}_1'\boldsymbol{U}_1)^{-1}(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)'(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)| \\ &= \log |\boldsymbol{I}_{q_2} + (\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)(\boldsymbol{U}_1'\boldsymbol{U}_1)^{-1}(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)'| \\ &= \log |\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{D}\boldsymbol{D}'|, \end{aligned}$$

$$\begin{aligned} T_{\rm LH} &= {\rm tr}\left[\{\boldsymbol{U}_1'\boldsymbol{U}_1 + (\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)'(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2) - \boldsymbol{U}_1'\boldsymbol{U}_1\}(\boldsymbol{U}_1'\boldsymbol{U}_1)^{-1}\right] \\ &= {\rm tr}\left\{\boldsymbol{D}(\boldsymbol{U}_1'\boldsymbol{U}_1)^{-1}\boldsymbol{D}'\right\} \\ &= {\rm tr}(\tilde{\boldsymbol{U}}^{-1}\boldsymbol{D}\boldsymbol{D}'), \end{aligned}$$

$$\begin{aligned} T_{\rm BNP} &= p - {\rm tr}\left[\boldsymbol{U}_1'\boldsymbol{U}_1\{\boldsymbol{U}_1'\boldsymbol{U}_1 + (\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)'(\boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2)\}^{-1}\right] \\ &= p - {\rm tr}\{\boldsymbol{U}_1'\boldsymbol{U}_1(\boldsymbol{U}_1'\boldsymbol{U}_1 + \boldsymbol{D}'\boldsymbol{D})^{-1}\} \\ &= {\rm tr}\left[\boldsymbol{D}(\boldsymbol{U}_1'\boldsymbol{U}_1)^{-1}\boldsymbol{D}'\{\boldsymbol{I}_{q_2} + \boldsymbol{D}(\boldsymbol{U}_1'\boldsymbol{U}_1)^{-1}\boldsymbol{D}'\}^{-1}\right] \\ &= {\rm tr}\{\boldsymbol{D}\boldsymbol{D}'(\boldsymbol{D}\boldsymbol{D}' + \tilde{\boldsymbol{U}})^{-1}\}. \end{aligned}$$

Hence, the result in Lemma 3.4 can be derived. □

C Proof of Theorem 3.1

From the assumption $p > q_2$, we present $\boldsymbol{\Delta} = (\boldsymbol{\Lambda}, \boldsymbol{O}_{q_2,p-q_2})'$, where $\boldsymbol{\Lambda} = {\rm diag}(\lambda_1, \ldots, \lambda_{q_2})$ $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{q_2} \geq 0)$ of which the diagonal elements of $\boldsymbol{\Lambda}$ are the singular values of $\boldsymbol{\Gamma}$.

C.1 Case of the LR criterion

At first, we prove the case of the LR criterion. From the equation (9) in Lemma 3.4, $T_{\rm LR}$ can be expressed as follows:

$$T_{\rm LR} = \log |\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{D}\boldsymbol{D}'|,$$

where $\tilde{\boldsymbol{U}}$ and $\boldsymbol{D} = \boldsymbol{T}'\boldsymbol{\Delta}' + \boldsymbol{U}_2$ are random matrices defined in Lemma 3.4 and Lemma 3.3. Let $\boldsymbol{U}_2 = (\boldsymbol{U}_{21}, \boldsymbol{U}_{22})$, where $\boldsymbol{U}_{21}$ is a $q_2 \times q_2$ random matrix, and $\boldsymbol{U}_{22}$ is a $q_2 \times (p - q_2)$ random matrix. Then, $\boldsymbol{D}\boldsymbol{D}'$ is calculated

$$\boldsymbol{D}\boldsymbol{D}' = (\boldsymbol{T}'\boldsymbol{\Lambda}' + \boldsymbol{U}_{21}, \boldsymbol{U}_{22})(\boldsymbol{T}'\boldsymbol{\Lambda}' + \boldsymbol{U}_{21}, \boldsymbol{U}_{22})' = \boldsymbol{F}\boldsymbol{F}' + \boldsymbol{U}_{22}\boldsymbol{U}_{22}', \tag{C.1}$$

where $\boldsymbol{F} = \boldsymbol{T}'\boldsymbol{\Lambda} + \boldsymbol{U}_{21}$, and $\boldsymbol{T}$, $\boldsymbol{U}_{21}$, $\boldsymbol{U}_{22}$, and $\tilde{\boldsymbol{U}}$ are mutually independent. Here, it should be emphasized that we do not assume the specific Landau asymptotic order to the diagonal elements of $\boldsymbol{\Lambda}$. However, by using $\boldsymbol{\Phi}^2 = \boldsymbol{I}_{q_2} + \boldsymbol{\Lambda}^2$, we can see that $\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} = O(1)$ because of

$$||\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}||^2 = {\rm tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}) = \sum_{i=1}^{q_2} \frac{\lambda_i^2}{1 + \lambda_i^2} \leq q_2 < \infty.$$

From these expression, we can derive the following matrix form so that we can expand matrices under the HD asymptotic framework:

$$\begin{aligned}
T_{\mathrm{LR}} &= \log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{D}\boldsymbol{D}'| \\
&= \log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{F}\boldsymbol{F}' + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'| \\
&= \log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'| + \log|\boldsymbol{I}_{q_2} + (\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1}\boldsymbol{F}\boldsymbol{F}'| \\
&= \log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'| + \log|\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'(\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}| + \log|\boldsymbol{\Phi}^2|. \qquad \text{(C.2)}
\end{aligned}$$

Since $v_{ii} = (n - q_1 - i)^{-1/2}\{t_{ii}^2 - (n - q_1 - i)\} \xrightarrow{d} N(0, 2)$ $(n - p - q_1 \to \infty)$, we have

$$\begin{aligned}
t_{ii} &= \sqrt{n - q_1 - i}\,\Big(1 + \frac{1}{\sqrt{n - q_1 - i}}v_{ii}\Big)^{1/2} \\
&= \sqrt{n - q_1}\Big\{1 + \frac{1}{2\sqrt{n - q_1}}v_{ii} + O_p(n^{-1})\Big\}.
\end{aligned}$$

Thus, the following equation is derived:

$$\boldsymbol{T} = \sqrt{n - q_1}\boldsymbol{I}_{q_2} + \frac{1}{2}\boldsymbol{V}_1 + \boldsymbol{V}_2 + O_p(n^{-1/2}), \qquad \text{(C.3)}$$

where $\boldsymbol{V}_1 = \mathrm{diag}(v_{11}, \ldots, v_{q_2q_2})$ and $\boldsymbol{V}_2 = \boldsymbol{T} - \mathrm{diag}(t_{11}, \ldots, t_{q_2q_2})$. By using (C.3) and the definition of $\boldsymbol{F}$, $\boldsymbol{F}\boldsymbol{\Phi}^{-1}$ can be expanded as

$$\begin{aligned}
\boldsymbol{F}\boldsymbol{\Phi}^{-1} &= \boldsymbol{T}'\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} + \boldsymbol{U}_{21}\boldsymbol{\Phi}^{-1} \\
&= \sqrt{n - q_1}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} + \frac{1}{2}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\boldsymbol{V}_1 + \boldsymbol{V}_2'\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} + \boldsymbol{U}_{21}\boldsymbol{\Phi}^{-1} + O_p(n^{-1/2}). \qquad \text{(C.4)}
\end{aligned}$$

Suppose that

$$\boldsymbol{V}_3 = \frac{1}{\sqrt{n - p - q_1 - 1}}\{\tilde{\boldsymbol{U}} - (n - p - q_1 - 1)\boldsymbol{I}_{q_2}\},\ \boldsymbol{V}_4 = \frac{1}{\sqrt{p - q_2}}\{\boldsymbol{U}_{22} - (p - q_2)\boldsymbol{I}_{q_2}\}.$$

By using $\boldsymbol{V}_3$ and $\boldsymbol{V}_4$, we can expand $(\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1}$ as

$$(\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1} = \frac{1}{n - q_1}\Big\{\boldsymbol{I}_{q_2} - \frac{\sqrt{n - p - q_1}}{n - q_1}\boldsymbol{V}_3 - \frac{\sqrt{p - q_2}}{n - q_1}\boldsymbol{V}_4 + O_p(n^{-1})\Big\}. \qquad \text{(C.5)}$$

From (C.4), (C.5), $\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'(\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}$ can be expanded as

$$\begin{aligned}
\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'(\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1} = \boldsymbol{I}_{q_2} + \frac{1}{n - q_1}\{&\sqrt{n - q_1}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_1 + \sqrt{n - q_1}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\boldsymbol{V}_2\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} \\
&+ \sqrt{n - q_1}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\boldsymbol{V}_2'\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} + \sqrt{n - q_1}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\boldsymbol{U}_{21}\boldsymbol{\Phi}^{-1} \\
&+ \sqrt{n - q_1}\boldsymbol{\Phi}^{-1}\boldsymbol{U}_{21}'\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} - \sqrt{n - p - q_1}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\boldsymbol{V}_3\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1} \\
&- \sqrt{p - q_2}\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\boldsymbol{V}_4\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\} + O_p(n^{-1}).
\end{aligned}$$

By using the above equation, we can expand $\log|\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'(\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}|$ as

$$\begin{aligned}
\log|\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'(\tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}')^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}| =& \frac{1}{\sqrt{n - q_1}}\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_1) - \frac{\sqrt{n - p - q_1}}{n - q_1}\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_3) \\
&- \frac{\sqrt{p - q_2}}{n - q_1}\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_4) + \frac{2}{\sqrt{n - q_1}}\mathrm{tr}(\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-2}\boldsymbol{U}_{21}) \\
&+ O_p(n^{-1}). \qquad \text{(C.6)}
\end{aligned}$$

Also, by using $\boldsymbol{V}_3$, $\tilde{\boldsymbol{U}}^{-1}$ can be expanded as

$$\tilde{\boldsymbol{U}}^{-1} = \frac{1}{n-p-q_1}\Big\{\boldsymbol{I}_{q_2} - \frac{1}{\sqrt{n-p-q_1}}\boldsymbol{V}_3 + O_p(n^{-1})\Big\}. \tag{C.7}$$

Therefore, we can expand $\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'$ as

$$\begin{aligned}\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}' = &\frac{n-q_1}{n-p-q_1}\boldsymbol{I}_{q_2} + \frac{1}{n-p-q_1}\Big(-\frac{p}{\sqrt{n-p-q_1}}\boldsymbol{V}_3 + \sqrt{p-q_2}\boldsymbol{V}_4\Big)\\ &+ O_p(n^{-1}).\end{aligned}$$

By using the above equation, $\log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'|$ can be expanded as

$$\begin{aligned}\log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'| = &\, q_2 \log\frac{n-q_1}{n-p-q_1} - \frac{p}{(n-q_1)\sqrt{n-p-q_1}}\mathrm{tr}(\boldsymbol{V}_3)\\ &+ \frac{\sqrt{p-q_2}}{n-q_1}\mathrm{tr}(\boldsymbol{V}_4) + O_p(n^{-1}).\end{aligned} \tag{C.8}$$

From (C.2), (C.6), and (C.8), T_{LR} can be expanded as

$$\begin{aligned}T_{\mathrm{LR}} = &\, q_2\log\frac{n-q_1}{n-p-q_1} + \log|\boldsymbol{\Phi}^2| + \frac{1}{\sqrt{n-q_1}}\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_1)\\ &- \frac{\sqrt{n-p-q_1}}{n-q_1}\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_3) - \frac{p}{(n-q_1)\sqrt{n-p-q_1}}\mathrm{tr}(\boldsymbol{V}_3) - \frac{\sqrt{p-q_2}}{n-q_1}\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_4)\\ &+ \frac{\sqrt{p-q_2}}{n-q_1}\mathrm{tr}(\boldsymbol{V}_4) + \frac{2}{\sqrt{n-q_1}}\mathrm{tr}(\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-2}\boldsymbol{U}_{21}) + O_p(n^{-1}).\end{aligned} \tag{C.9}$$

Then, we can derive the following equation from (C.9):

$$\begin{aligned}&\sqrt{n-q_1}\left(T_{\mathrm{LR}} - q_2\log\frac{n-q_1}{n-p-q_1} - \log|\boldsymbol{\Phi}^2|\right)\\ &= \mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_1) - \mathrm{tr}\Big\{\Big(\sqrt{\frac{n-p-q_1}{n-q_1}}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2} + \frac{p}{\sqrt{n-q_1}\sqrt{n-p-q_1}}\boldsymbol{I}_{q_2}\Big)\boldsymbol{V}_3\Big\}\\ &\quad + \sqrt{\frac{p}{n-q_1}}\mathrm{tr}\Big\{\Big(\boldsymbol{I}_{q_2} - \boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\Big)\boldsymbol{V}_4\Big\} + 2\mathrm{tr}(\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-2}\boldsymbol{U}_{21}) + O_p(n^{-1/2}).\end{aligned} \tag{C.10}$$

It is easy to see that

$$\frac{\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\boldsymbol{V}_1)}{\sqrt{\mathrm{tr}(\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4})}} \xrightarrow{d} N(0,2),$$

$$\frac{\mathrm{tr}\Big\{\Big(\sqrt{\frac{n-p-q_1}{n-q_1}}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2} + \frac{p}{\sqrt{n-q_1}\sqrt{n-p-q_1}}\boldsymbol{I}_{q_2}\Big)\boldsymbol{V}_3\Big\}}{\sqrt{\mathrm{tr}\Big[\Big\{\left(1-\frac{q_1}{n}\right)^2\boldsymbol{\Lambda}^4 + 2\frac{p}{n}\left(1-\frac{q_1}{n}\right)\boldsymbol{\Lambda}^2 + \left(\frac{p}{n}\right)^2\boldsymbol{I}_{q_2}\Big\}\boldsymbol{\Phi}^{-4}\Big]}} \xrightarrow{d} N\Big(0, \frac{2}{(1-c)(1-c_2)}\Big),$$

$$\sqrt{\frac{p}{n-q_1}}\cdot\frac{\mathrm{tr}\Big\{\Big(\boldsymbol{I}_{q_2} - \boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}\Big)\boldsymbol{V}_4\Big\}}{\sqrt{\mathrm{tr}(\boldsymbol{\Phi}^{-4})}} \xrightarrow{d} \begin{cases} U(0) & (c_1 = 0)\\ N\left(0, \dfrac{2c_1}{1-c_2}\right) & (c_1 \neq 0)\end{cases},$$

$$\frac{\mathrm{tr}(\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-2}\boldsymbol{U}_{21})}{\sqrt{\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-4})}} \sim N(0,1),$$

and

$$\log|\mathbf{\Phi}^2| = \log|\boldsymbol{I}_p + \mathbf{\Gamma}\mathbf{\Gamma}'|,\ \mathrm{tr}(\mathbf{\Phi}^{-2}) = \mathrm{tr}\left\{(\boldsymbol{I}_p + \mathbf{\Gamma}\mathbf{\Gamma}')^{-1}\right\} - (p - q_2),$$

where $U(\cdot)$ is the degenerate distribution, and c_1 and c_2 are the limit of p/n and q_1/n, respectively. Hence, we can derive the non-null asymptotic distribution of the LR criterion by normalizing the asymptotic variance. □

C.2 Case of the LH criterion

Next, we prove the case of the LH criterion. From the equation (9) in Lemma 3.4, T_{LH} can be expressed as follows:

$$T_{\mathrm{LH}} = \mathrm{tr}(\tilde{\boldsymbol{U}}^{-1}\boldsymbol{D}\boldsymbol{D}').$$

By using (C.1), we can derive the following equation:

$$T_{\mathrm{LH}} = \mathrm{tr}(\boldsymbol{F}'\tilde{\boldsymbol{U}}^{-1}\boldsymbol{F}) + \mathrm{tr}(\tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'). \tag{C.11}$$

It is easy to see that $(1+\lambda_1)^{-1}\mathbf{\Lambda} = O(1)$ because of

$$||(1+\lambda_1)^{-1}\mathbf{\Lambda}||^2 = \sum_{i=1}^{q_2}\left(\frac{\lambda_i}{1+\lambda_1}\right)^2 \leq q_2 < \infty.$$

From (C.3), $(1+\lambda_1)^{-1}\boldsymbol{F}$ can be expanded as

$$\begin{aligned}(1+\lambda_1)^{-1}\boldsymbol{F} = \sqrt{n-q_1}(1+\lambda_1)^{-1}\mathbf{\Lambda} + \frac{1}{2}(1+\lambda_1)^{-1}\mathbf{\Lambda}\boldsymbol{V}_1 + \boldsymbol{V}_2'(1+\lambda_1)^{-1}\mathbf{\Lambda}\\ + (1+\lambda_1)^{-1}\boldsymbol{U}_{21} + O_p(n^{-1/2}).\end{aligned} \tag{C.12}$$

From (C.7) and (C.12), we can expand $(1+\lambda_1)^{-1}\boldsymbol{F}'\tilde{\boldsymbol{U}}^{-1}\boldsymbol{F}(1+\lambda_1)^{-1}$ as

$$\begin{aligned}(1+\lambda_1)^{-1}\boldsymbol{F}'\tilde{\boldsymbol{U}}^{-1}\boldsymbol{F}(1+\lambda_1)^{-1} = \frac{1}{n-p-q_1}\Big\{&(n-q_1)(1+\lambda_1)^{-2}\mathbf{\Lambda}^2 + \sqrt{n-q_1}(1+\lambda_1)^{-2}\mathbf{\Lambda}^2\boldsymbol{V}_1\\ &+ \sqrt{n-q_1}(1+\lambda_1)^{-1}\mathbf{\Lambda}\boldsymbol{V}_2(1+\lambda_1)^{-1}\mathbf{\Lambda}\\ &+ \sqrt{n-q_1}(1+\lambda_1)^{-1}\mathbf{\Lambda}\boldsymbol{V}_2'(1+\lambda_1)^{-1}\mathbf{\Lambda}\\ &- \frac{n-q_1}{\sqrt{n-p-q_1}}(1+\lambda_1)^{-1}\mathbf{\Lambda}\boldsymbol{V}_3(1+\lambda_1)^{-1}\mathbf{\Lambda}\\ &+ \sqrt{n-q_1}(1+\lambda_1)^{-1}\mathbf{\Lambda}(1+\lambda_1)^{-1}\boldsymbol{U}_{21}\\ &+ \sqrt{n-q_1}(1+\lambda_1)^{-1}\boldsymbol{U}_{21}'(1+\lambda_1)^{-1}\mathbf{\Lambda}\Big\} + O_p(n^{-1}).\end{aligned} \tag{C.13}$$

From (C.13), $(1+\lambda_1)^{-2}\mathrm{tr}(\boldsymbol{F}'\tilde{\boldsymbol{U}}^{-1}\boldsymbol{F})$ can be expanded as follows:

$$\begin{aligned}(1+\lambda_1)^{-2}\mathrm{tr}(\boldsymbol{F}'\tilde{\boldsymbol{U}}^{-1}\boldsymbol{F}) = &\frac{n-q_1}{n-p-q_1}\mathrm{tr}\{(1+\lambda_1)^{-2}\mathbf{\Lambda}^2\} + \frac{\sqrt{n-q_1}}{n-p-q_1}\mathrm{tr}\{(1+\lambda_1)^{-2}\mathbf{\Lambda}^2\boldsymbol{V}_1\}\\ &- \frac{n-q_1}{(n-p-q_1)^{3/2}}\mathrm{tr}\{(1+\lambda_1)^{-2}\mathbf{\Lambda}^2\boldsymbol{V}_3\}\\ &+ \frac{2\sqrt{n-q_1}}{n-p-q_1}\mathrm{tr}\{(1+\lambda_1)^{-2}\mathbf{\Lambda}\boldsymbol{U}_{21}\} + O_p(n^{-1}).\end{aligned} \tag{C.14}$$

By using $\boldsymbol{V}_3$ and $\boldsymbol{V}_4$, $\tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'$ and $(1+\lambda_1)^{-2}\mathrm{tr}(\tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}')$ can be expanded as

$$\begin{aligned}\tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}' = {} & \frac{1}{n-p-q_1}\Big\{\sqrt{p-q_2}\boldsymbol{V}_4 + (p-q_2)\boldsymbol{I}_{q_2} - \sqrt{\frac{p-q_2}{n-p-q_1}}\boldsymbol{V}_3\boldsymbol{V}_4 - \frac{p-q_2}{\sqrt{n-p-q_1}}\boldsymbol{V}_3\Big\} \\ & + O_p(pn^{-2}),\end{aligned}$$

and

$$\begin{aligned}(1+\lambda_1)^{-2}\mathrm{tr}(\tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_{22}\boldsymbol{U}_{22}') = {} & \frac{p-q_2}{n-p-q_1}(1+\lambda_1)^{-2}q_2 - \frac{p-q_2}{(n-p-q_1)^{3/2}}\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{V}_3\} \\ & + \frac{\sqrt{p-q_2}}{n-p-q_1}\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{V}_4\} \\ & - \frac{\sqrt{p-q_2}}{(n-p-q_1)^{3/2}}\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{V}_3\boldsymbol{V}_4\} + O_p(pn^{-2}). \qquad \text{(C.15)}\end{aligned}$$

From (C.11), (C.14) and (C.15), we can derive the following equation:

$$\begin{aligned}& (1+\lambda_1)^{-2}\frac{n-p-q_1}{\sqrt{n-q_1}}\left\{T_{\mathrm{LH}} - \frac{n-q_1}{n-p-q_1}\mathrm{tr}(\boldsymbol{\Lambda}^2) - \frac{pq_2}{n-p-q_1}\right\} \\ & = \mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{\Lambda}^2\boldsymbol{V}_1\} \\ & \quad - \mathrm{tr}\Big\{\Big(\sqrt{\frac{n-q_1}{n-p-q_1}}(1+\lambda_1)^{-2}\boldsymbol{\Lambda}^2 + \frac{p}{\sqrt{n-p-q_1}\sqrt{n-q_1}}(1+\lambda_1)^{-2}\boldsymbol{I}_{q_2}\Big)\boldsymbol{V}_3\Big\} \\ & \quad + (1+\lambda_1)^{-2}\sqrt{\frac{p}{n-q_1}}\mathrm{tr}(\boldsymbol{V}_4) + 2\mathrm{tr}\{(1+\lambda_1)^{-1}\boldsymbol{\Lambda}\boldsymbol{U}_{21}\} + O_p(n^{-1/2}). \qquad \text{(C.16)}\end{aligned}$$

It is easy to see that

$$\frac{\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{V}_1)}{\sqrt{\mathrm{tr}(\boldsymbol{\Lambda}^4)}} \xrightarrow{d} N(0,2),$$

$$\frac{\mathrm{tr}\Big\{\Big(\sqrt{\frac{n-q_1}{n-p-q_1}}\boldsymbol{\Lambda}^2 + \frac{p}{\sqrt{n-p-q_1}\sqrt{n-q_1}}\boldsymbol{I}_{q_2}\Big)\boldsymbol{V}_3\Big\}}{\sqrt{\mathrm{tr}\Big\{\left(1-\frac{q_1}{n}\right)^2\boldsymbol{\Lambda}^4 + 2\frac{p}{n}\left(1-\frac{q_1}{n}\right)\boldsymbol{\Lambda}^2 + \left(\frac{p}{n}\right)^2\boldsymbol{I}_{q_2}\Big\}}} \xrightarrow{d} N\Big(0, \frac{2}{(1-c)(1-c_2)}\Big),$$

$$\sqrt{\frac{p}{n-q_1}}\mathrm{tr}(\boldsymbol{V}_4) \xrightarrow{d} \begin{cases} U(0) & (c_1 = 0) \\ N\left(0, \dfrac{2c_1q_2}{1-c_2}\right) & (c_1 \neq 0), \end{cases}$$

$$\frac{\mathrm{tr}(\boldsymbol{\Lambda}\boldsymbol{U}_{21})}{\sqrt{\mathrm{tr}(\boldsymbol{\Lambda}^2)}} \sim N(0,1),$$

and

$$\mathrm{tr}(\boldsymbol{\Lambda}^2) = \mathrm{tr}(\boldsymbol{\Gamma}\boldsymbol{\Gamma}'),\ \mathrm{tr}(\boldsymbol{\Lambda}^4) = \mathrm{tr}\left\{(\boldsymbol{\Gamma}\boldsymbol{\Gamma}')^2\right\}.$$

From the above equations, we can derive the result of the case of the LH criterion by normalizing the asymptotic variance. □

C.3 Case of the BNP criterion

Finally, we prove the case of the BNP criterion. From the equation (9) in Lemma 3.4, T_{BNP} can be expressed as follows:

$$T_{\mathrm{BNP}} = \mathrm{tr}\{\boldsymbol{D}\boldsymbol{D}'(\boldsymbol{D}\boldsymbol{D}' + \tilde{\boldsymbol{U}})^{-1}\}.$$

By applying the formula of inverse of the sum of matrices to $(\boldsymbol{D}\boldsymbol{D}' + \tilde{\boldsymbol{U}})^{-1}$, the above equation can be expressed as follows:

$$\begin{aligned}
T_{\mathrm{BNP}} &= \mathrm{tr}(\boldsymbol{D}\boldsymbol{D}'\boldsymbol{U}_3^{-1}) \\
&\quad - \mathrm{tr}\Big\{\boldsymbol{D}\boldsymbol{D}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}(\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1})^{-1}\boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\Big\} \\
&= \mathrm{tr}(\boldsymbol{F}\boldsymbol{F}'\boldsymbol{U}_3^{-1}) + \mathrm{tr}(\boldsymbol{U}_{22}\boldsymbol{U}_{22}'\boldsymbol{U}_3^{-1}) \\
&\quad - \mathrm{tr}\Big\{\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}(\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1})^{-1}\boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\Big\} \\
&\quad - \mathrm{tr}\Big\{\boldsymbol{U}_{22}\boldsymbol{U}_{22}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}(\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1})^{-1}\boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\Big\}, \qquad \text{(C.17)}
\end{aligned}$$

where $\boldsymbol{U}_3 = \tilde{\boldsymbol{U}} + \boldsymbol{U}_{22}\boldsymbol{U}_{22}'$. By using $\boldsymbol{V}_3$ and $\boldsymbol{V}_4$, $\boldsymbol{U}_3^{-1}$ can be expanded as

$$\begin{aligned}
\boldsymbol{U}_3^{-1} &= \frac{1}{n-q-1}\left(\boldsymbol{I}_{q_2} + \frac{n-p-q_1-1}{n-q-1}\boldsymbol{V}_3 + \frac{\sqrt{p-q_2}}{n-q-1}\boldsymbol{V}_4\right)^{-1} \\
&= \frac{1}{n-q_1}\left\{\boldsymbol{I}_{q_2} - \frac{n-p-q_1}{n-q_1}\boldsymbol{V}_3 - \frac{\sqrt{p}}{n-q_1}\boldsymbol{V}_4 + O_p(n^{-1})\right\}. \qquad \text{(C.18)}
\end{aligned}$$

From (C.18), we can expand the following equations as

$$\begin{aligned}
\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{F}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\} &= \mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{\Lambda}^2\} + \frac{1}{\sqrt{n-q_1}}\mathrm{tr}\{(1+\lambda_1)^{-1}\boldsymbol{\Lambda}^2\boldsymbol{V}_1\} \\
&\quad - \frac{\sqrt{n-p-q_1}}{n-q_1}\mathrm{tr}\{(1+\lambda_1)^{-1}\boldsymbol{\Lambda}^2\boldsymbol{V}_3\} - \frac{\sqrt{p}}{n-q_1}\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{\Lambda}^2\boldsymbol{V}_4\} \\
&\quad + \frac{2}{\sqrt{n-q_1}}\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{\Lambda}\boldsymbol{U}_{21}\} + O_p(n^{-1}), \qquad \text{(C.19)} \\
\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{U}_{22}\boldsymbol{U}_{22}'\boldsymbol{U}_3^{-1}\} &= \frac{p}{n-q_1}(1+\lambda_1)^{-2}q_2 - \frac{p\sqrt{n-p-q_1}}{(n-q_1)^2}\mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{V}_3\} \\
&\quad + \frac{p}{n-q_1}(1+\lambda_1)^{-2}\mathrm{tr}\left\{\left(\frac{1}{\sqrt{p-q_2}} - \frac{\sqrt{p}}{n-q_1}\right)\boldsymbol{V}_4\right\} + O_p(pn^{-2}), \\
&\qquad \text{(C.20)}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{tr}\Big\{(1+\lambda_1)^{-2}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}(\boldsymbol{\Phi}^{-2} + \boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1})^{-1}\boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\Big\} \\
&= \mathrm{tr}\{(1+\lambda_1)^{-2}\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-2}\} + \frac{1}{\sqrt{n-q_1}}\mathrm{tr}\left\{(1+\lambda_1)^{-2}(2\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-2} - \boldsymbol{\Lambda}^6\boldsymbol{\Phi}^{-4})\boldsymbol{V}_1\right\} \\
&\quad - \frac{\sqrt{n-p-q_1}}{n-q_1}\mathrm{tr}\left\{(1+\lambda_1)^{-2}(2\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-2} - \boldsymbol{\Lambda}^6\boldsymbol{\Phi}^{-4})\boldsymbol{V}_3\right\} \\
&\quad - \frac{\sqrt{p}}{n-q_1}\mathrm{tr}\left\{(1+\lambda_1)^{-2}(2\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-2} - \boldsymbol{\Lambda}^6\boldsymbol{\Phi}^{-4})\boldsymbol{V}_4\right\} \\
&\quad + \frac{2}{\sqrt{n-q_1}}\mathrm{tr}\left\{(1+\lambda_1)^{-2}(2\boldsymbol{\Lambda}^3\boldsymbol{\Phi}^{-2} - \boldsymbol{\Lambda}^5\boldsymbol{\Phi}^{-4})\boldsymbol{U}_{21}\right\} + O_p(n^{-1}), \qquad \text{(C.21)}
\end{aligned}$$

and

$$
\begin{aligned}
&\operatorname{tr}\Big\{\boldsymbol{U}_{22}\boldsymbol{U}_{22}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1}(\boldsymbol{\Phi}^{-2}+\boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\boldsymbol{F}\boldsymbol{\Phi}^{-1})^{-1}\boldsymbol{\Phi}^{-1}\boldsymbol{F}'\boldsymbol{U}_3^{-1}\Big\} \\
&= \frac{p}{n-q_1}\operatorname{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}) + \frac{p}{(n-q_1)^{3/2}}\operatorname{tr}\left\{(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}-\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4})\boldsymbol{V}_1\right\} \\
&\quad - \frac{p\sqrt{n-p-q_1}}{(n-q_1)^2}\operatorname{tr}\left\{(2\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}-\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4})\boldsymbol{V}_3\right\} \\
&\quad + \frac{\sqrt{p}}{(n-q_1)^2}\operatorname{tr}\left[\left\{(n-2p-q_1)\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}+p\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4}\right\}\boldsymbol{V}_4\right] \\
&\quad + \frac{2p}{(n-q_1)^{3/2}}\operatorname{tr}\left\{(\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-2}-\boldsymbol{\Lambda}^3\boldsymbol{\Phi}^{-4})\boldsymbol{U}_{21}\right\} + O_p(n^{-1}). \qquad \text{(C.22)}
\end{aligned}
$$

From (C.17), (C.19), (C.20), (C.21) and (C.22), we can derive the following equation:

$$
\begin{aligned}
&(1+\lambda_1)^{-2}\frac{(n-q_1)^{3/2}}{n-p-q_1}\left\{T_{\text{BNP}}-\frac{pq_2}{n-q_1}-\left(1-\frac{p}{n-q_1}\right)\operatorname{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2})\right\} \\
&= (1+\lambda_1)^{-2}\operatorname{tr}\left\{\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}(\boldsymbol{I}_{q_2}-\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2})\boldsymbol{V}_1\right\} \\
&\quad -(1+\lambda_1)^{-2}\sqrt{\frac{n-q_1}{n-p-q_1}}\operatorname{tr}\left\{\left(\frac{n-2p-q_1}{n-q_1}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}-\frac{n-p-q_1}{n-q_1}\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4}+\frac{p}{n-q_1}\boldsymbol{I}_{q_2}\right)\boldsymbol{V}_3\right\} \\
&\quad -(1+\lambda_1)^{-2}\frac{\sqrt{p}\sqrt{n-q_1}}{n-p-q_1}\operatorname{tr}\left\{\left(\frac{2(n-p-q_1)}{n-q_1}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}-\frac{n-p-q_1}{n-q_1}\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4}-\frac{n-p-q_1}{n-q_1}\boldsymbol{I}_{q_2}\right)\boldsymbol{V}_4\right\} \\
&\quad +2(1+\lambda_1)^{-2}\operatorname{tr}\left\{\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-2}(\boldsymbol{I}_{q_2}-\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2})\boldsymbol{U}_{21}\right\}+O_p(n^{-1/2}).
\end{aligned}
$$

It is easy to see that

$$
\frac{\operatorname{tr}\left\{\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}(\boldsymbol{I}_{q_2}-\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2})\boldsymbol{V}_1\right\}}{\sqrt{\operatorname{tr}(\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-8})}} \xrightarrow{d} N(0,2),
$$

$$
\begin{aligned}
&\sqrt{\frac{n-q_1}{n-p-q_1}}\cdot\frac{\operatorname{tr}\left\{\left(\frac{n-2p-q_1}{n-q_1}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}-\frac{n-p-q_1}{n-q_1}\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4}+\frac{p}{n-q_1}\boldsymbol{I}_{q_2}\right)\boldsymbol{V}_3\right\}}{\sqrt{\left(1-\frac{q_1}{n}\right)^2\operatorname{tr}(\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-8})+2\frac{p}{n}\left(1-\frac{q_1}{n}\right)\operatorname{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-8})+\left(\frac{p}{n}\right)^2\operatorname{tr}(\boldsymbol{\Phi}^{-8})}} \\
&\xrightarrow{d} N\left(0,\frac{2}{(1-c)(1-c_2)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\sqrt{p}\sqrt{n-q_1}}{n-p-q_1}\cdot\frac{\operatorname{tr}\left\{\left(\frac{2(n-p-q_1)}{n-q_1}\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2}-\frac{n-p-q_1}{n-q_1}\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-4}-\frac{n-p-q_1}{n-q_1}\boldsymbol{I}_{q_2}\right)\boldsymbol{V}_4\right\}}{\sqrt{\operatorname{tr}(\boldsymbol{\Phi}^{-8})}} \\
&\xrightarrow{d}\begin{cases} U(0) & (c_1=0) \\ N\left(0,\dfrac{2c_1}{1-c_2}\right) & (c_1\neq 0)\end{cases},
\end{aligned}
$$

$$
\frac{\operatorname{tr}\left\{\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-2}(\boldsymbol{I}_{q_2}-\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-2})\boldsymbol{U}_{21}\right\}}{\sqrt{\operatorname{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-8})}} \sim N(0,1),
$$

and

$$
\operatorname{tr}(\boldsymbol{\Lambda}^4\boldsymbol{\Phi}^{-8}) = \operatorname{tr}\left\{(\boldsymbol{\Gamma}\boldsymbol{\Gamma}')^2(\boldsymbol{I}_p+\boldsymbol{\Gamma}\boldsymbol{\Gamma}')^{-4}\right\},
$$

$$\mathrm{tr}(\boldsymbol{\Lambda}^2\boldsymbol{\Phi}^{-8}) = \mathrm{tr}\left\{\boldsymbol{\Gamma}\boldsymbol{\Gamma}'(\boldsymbol{I}_p + \boldsymbol{\Gamma}\boldsymbol{\Gamma}')^{-4}\right\},$$

$$\mathrm{tr}(\boldsymbol{\Phi}^{-8}) = \mathrm{tr}\left\{(\boldsymbol{I}_p + \boldsymbol{\Gamma}\boldsymbol{\Gamma}')^{-4}\right\} - (p - q_2).$$

From the above equations, we can derive the result of the case of the BNP criterion by normalizing the asymptotic variance. □

D Proof of Theorem 3.2

D.1 Case of the LR criterion

From the equation (9) in Lemma 3.4, T_{LR} can be expressed as follows:

$$T_{\mathrm{LR}} = \log|\boldsymbol{I}_{q_2} + \tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_2\boldsymbol{U}_2'|. \tag{D.1}$$

By using $\boldsymbol{V}_4$, $\boldsymbol{U}_2\boldsymbol{U}_2'$ can be expressed as follows:

$$\begin{aligned}\boldsymbol{U}_2\boldsymbol{U}_2' &= \boldsymbol{U}_{21}\boldsymbol{U}_{21}' + \boldsymbol{U}_{22}\boldsymbol{U}_{22}' \\ &= \boldsymbol{U}_{21}\boldsymbol{U}_{21}' + (p - q_2)\boldsymbol{I}_{q_2} + \sqrt{p - q_2}\boldsymbol{V}_4,\end{aligned}$$

where $\boldsymbol{U}_2 = (\boldsymbol{U}_{21}, \boldsymbol{U}_{22})$ and $\boldsymbol{U}_{22}\boldsymbol{U}_{22}' = (p - q_2)\left\{\boldsymbol{I}_{q_2} + (p - q_2)^{-1/2}\boldsymbol{V}_4\right\}$. Then, by using (C.7), $\tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_2\boldsymbol{U}_2'$ can be expanded as

$$\begin{aligned}&\tilde{\boldsymbol{U}}^{-1}\boldsymbol{U}_2\boldsymbol{U}_2' \\ &= \frac{1}{n - p - q_1}\left\{(p - q_2)\boldsymbol{I}_{q_2} + \boldsymbol{U}_{21}\boldsymbol{U}_{21}' - \frac{p}{\sqrt{n - p - q_1}}\boldsymbol{V}_3 + \sqrt{p - q_2}\boldsymbol{V}_4 + O_p(p^{1/2}n^{-1/2})\right\}.\end{aligned} \tag{D.2}$$

Therefore, from (D.1) and (D.2), we can derive the following equation:

$$\begin{aligned}\frac{n - q_1}{\sqrt{p}}\left(T_{\mathrm{LR}} - q_2\log\frac{n - q_1}{n - p - q_1}\right) &= -\frac{q_2^2}{\sqrt{p}} - \sqrt{\frac{p}{n - p - q_1}}\mathrm{tr}(\boldsymbol{V}_3) \\ &\quad + \sqrt{\frac{p - q_2}{p}}\mathrm{tr}(\boldsymbol{V}_4) + \frac{1}{\sqrt{p}}\mathrm{tr}(\boldsymbol{U}_{21}\boldsymbol{U}_{21}') + O_p(n^{-1/2}).\end{aligned}$$

It is easy to see that

$$\sqrt{\frac{p - q_2}{p}}\mathrm{tr}(\boldsymbol{V}_4) \sim \sqrt{\frac{p - q_2}{p}}\frac{1}{\sqrt{p - q_2}}\left\{\chi^2_{q_2(p - q_2)} - q_2(p - q_2)\right\} \xrightarrow{d} N(0, 2q_2)\ (p \to \infty),$$

and

$$\mathrm{tr}(\boldsymbol{U}_{21}\boldsymbol{U}_{21}') \sim \chi^2_{q_2^2}.$$

Hence, we can derive the result of the case of the LR criterion by normalizing the asymptotic variance. □

D.2 Case of the LH criterion

From the equation (9) in Lemma 3.4, T_{LH} can be expressed as follows:

$$T_{\mathrm{LH}} = \mathrm{tr}(\boldsymbol{U}_2\boldsymbol{U}_2'\tilde{\boldsymbol{U}}^{-1}).$$

From (D.2), we can derive the following equation:

$$\begin{aligned}\frac{n-p-q_1}{\sqrt{p}}\left(T_{\mathrm{LH}} - \frac{pq_2}{n-p-q_1}\right) &= -\frac{q_2^2}{\sqrt{p}} - \sqrt{\frac{p}{n-p-q_1}}\mathrm{tr}(\boldsymbol{V}_3) \\ &\quad + \sqrt{\frac{p-q_2}{p}}\mathrm{tr}(\boldsymbol{V}_4) + \frac{1}{\sqrt{p}}\mathrm{tr}(\boldsymbol{U}_{21}\boldsymbol{U}_{21}') + O_p(n^{-1/2}).\end{aligned}$$

Hence, we can derive the result of the case of the LH by normalizing the asymptotic variance.□

D.3 Case of the BNP criterion

From the equation (9) in Lemma 3.4, T_{BNP} can be expressed as follows:

$$T_{\mathrm{BNP}} = \mathrm{tr}\{\boldsymbol{U}_2\boldsymbol{U}_2'(\boldsymbol{U}_2\boldsymbol{U}_2' + \tilde{\boldsymbol{U}})^{-1}\}.$$

By using $\boldsymbol{V}_3$ and $\boldsymbol{V}_4$, $\boldsymbol{U}_2\boldsymbol{U}_2' + \tilde{\boldsymbol{U}}$ can be expressed as

$$\boldsymbol{U}_2\boldsymbol{U}_2' + \tilde{\boldsymbol{U}} = (n-q-1)\boldsymbol{I}_{q_2} + \sqrt{n-p-q_1-1}\boldsymbol{V}_3 + \sqrt{p-q_2}\boldsymbol{V}_4 + \boldsymbol{U}_{21}\boldsymbol{U}_{21}',$$

Then, we can expand $(\boldsymbol{U}_2\boldsymbol{U}_2' + \tilde{\boldsymbol{U}})^{-1}$ as

$$(\boldsymbol{U}_2\boldsymbol{U}_2' + \tilde{\boldsymbol{U}})^{-1} = \frac{1}{n-q_1}\left\{\boldsymbol{I}_{q_2} - \frac{\sqrt{n-p-q_1}}{n-q_1}\boldsymbol{V}_3 - \frac{\sqrt{p-q_2}}{n-q_1}\boldsymbol{V}_4 - \frac{1}{n-q_1}\boldsymbol{U}_{21}\boldsymbol{U}_{21}' + O_p(n^{-1})\right\}.$$

Therefore, the following equation can be derived.

$$\begin{aligned}&\boldsymbol{U}_2\boldsymbol{U}_2'(\boldsymbol{U}_2\boldsymbol{U}_2' + \tilde{\boldsymbol{U}})^{-1} \\ &= \frac{p-q_2}{n-q_1}\left\{\boldsymbol{I}_{q_2} - \frac{\sqrt{n-p-q_1}}{n-q_1}\boldsymbol{V}_3 - \frac{\sqrt{p-q_2}}{n-q_1}\boldsymbol{V}_4 + \frac{1}{\sqrt{p-q_2}}\boldsymbol{V}_4 + \frac{1}{p-q_2}\boldsymbol{U}_{21}\boldsymbol{U}_{21}' + O_p(p^{-1/2}n^{-1/2})\right\}.\end{aligned}$$

Using the above equation, we can drive the following equation.

$$\begin{aligned}\frac{(n-q_1)^2}{\sqrt{p}(n-p-q_1)}\left(T_{\mathrm{BNP}} - \frac{pq_2}{n-q_1}\right) &= -\frac{(n-q_1)q_2^2}{\sqrt{p}(n-p-q_1)} - \sqrt{\frac{p}{n-p-q_1}}\mathrm{tr}(\boldsymbol{V}_3) \\ &\quad + \sqrt{\frac{p-q_2}{p}}\mathrm{tr}(\boldsymbol{V}_4) + \frac{n-q_1}{\sqrt{p}(n-p-q_1)}\mathrm{tr}(\boldsymbol{U}_{21}\boldsymbol{U}_{21}') + O_p(n^{-1/2}).\end{aligned}$$

Hence, we can derive the result of the case of the BNP criterion by normalizing the asymptotic variance. □

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