Estimation of multivariate 3rd moment for high-dimensional data and its application for testing multivariate normality

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Abstract

This paper is concerned with the multivariate 3rd moment and its estimation. Mardia (1970) and Srivastava (1984) proposed the multivariate skewness and its estimator, independently. However, these estimators can not defined for the case in which the dimension p is larger than the sample size N. In this paper, we treat the multivariate 3rd moment γ which is defined by using Hadamard product of observation vectors, and propose an estimate of γ which is well defined when $p > N$. Based on the estimator, we propose new test for multivariate normality. Simulation results revealed that our proposed test has good accuracy.

Keywords: Multivariate 3rd moment, Hadamard product, testing multivariate normality, (n, p)-asymptotic
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1 Introduction

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be p-variate random vectors which are independently and identically distributed as a p-dimensional distribution $F_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Mardia (1970) defined multivariate skewness and kurtosis as

$$\beta_{\mathrm{M1},p} = E[\{(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\}^3],$$
$$\beta_{\mathrm{M2},p} = E[\{(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\}^2],$$

respectively. When $p = 1$, $\beta_{1,p}$ and $\beta_{2,p}$ are reduced to the ordinal univariate squared skewness and kurtosis, respectively. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ be a random sample drown from a population with distribution $F_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Mardia [10] proposed sample counterparts of $\beta_{\mathrm{M1},p}$ and $\beta_{\mathrm{M2},p}$ as

$$b_{\mathrm{M1},p} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{(\boldsymbol{x}_i-\bar{\boldsymbol{x}})'\boldsymbol{S}^{-1}(\boldsymbol{x}_j-\bar{\boldsymbol{x}})\}^3,$$
$$b_{\mathrm{M2},p} = \frac{1}{N}\sum_{i=1}^{N}\{(\boldsymbol{x}_i-\bar{\boldsymbol{x}})'\boldsymbol{S}^{-1}(\boldsymbol{x}_i-\bar{\boldsymbol{x}})\}^2,$$

respectively, where

$$\bar{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i, \quad \boldsymbol{S} = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i-\bar{\boldsymbol{x}})(\boldsymbol{x}_i-\bar{\boldsymbol{x}})'.$$

Srivastava [8] (as cited in a more recent book by this author [9]) defined other multivariate skewness $\beta_{\mathrm{S1},p}$ and kurtosis $\beta_{\mathrm{S2},p}$, which are described as follows: Let $\boldsymbol{\Gamma} = (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p)$ be an orthogonal matrix such that $\boldsymbol{\Gamma} = (\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p)$, where $\boldsymbol{D}_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$. Let $y_\ell = \boldsymbol{\gamma}_\ell'\boldsymbol{x}$ and $\theta_\ell = \boldsymbol{\gamma}_\ell'\boldsymbol{\mu}$ for $\ell = 1, \ldots, p$.

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Then $\beta_{\mathrm{S}1,p}$ and $\beta_{\mathrm{S}2,p}$ are given by

$$\beta_{\mathrm{S}1,p} = \frac{1}{p}\sum_{\ell=1}^{p}\left\{\frac{E[(y_\ell-\theta_\ell)^3]}{\lambda_\ell^{3/2}}^2\right\},$$
$$\beta_{\mathrm{S}2,p} = \frac{1}{p}\sum_{\ell=1}^{p}\frac{E[(y_\ell-\theta_\ell)^4]}{\lambda_\ell^2},$$

respectively. Srivastava [8] proposed sample counterparts of $\beta_{\mathrm{S}1,p}$ and $\beta_{\mathrm{S}2,p}$ as $b_{\mathrm{S}1,p}$ and $b_{\mathrm{S}2,p}$, respectively. To describe them, let $\{N/(N-1)\}\boldsymbol{S} = \boldsymbol{H}\boldsymbol{D}_u\boldsymbol{H}'$, where $\boldsymbol{H} = (\boldsymbol{h}_1,\dots,\boldsymbol{h}_p)$ is an orthogonal matrix and $\boldsymbol{D}_u = \mathrm{diag}(u_1,\dots,u_p)$. With being

$$y_{\ell i} = \boldsymbol{h}_\ell'\boldsymbol{x}_i \qquad (\ell = 1,\dots,p, i = 1,\dots,N),$$

$b_{\mathrm{S}1,p}$ and $b_{\mathrm{S}2,p}$ are given by

$$b_{\mathrm{S}1,p} = \frac{1}{p}\sum_{\ell=1}^{p}\left\{\frac{u_\ell^{-3/2}}{N}\sum_{i=1}^{N}(y_{\ell i}-\bar{y}_\ell)^3\right\}^2,$$
$$b_{\mathrm{S}2,p} = \frac{1}{Np}\sum_{\ell=1}^{p}u_\ell^{-2}\sum_{i=1}^{N}(y_{\ell i}-\bar{y}_\ell)^4,$$

respectively, where $\bar{y}_\ell = N^{-1}\sum_{i=1}^{N}y_{\ell i}$.

These sample counterparts of multivariate skewness and kurtosis are applied for testing multivariate normality, i.e., testing null hypothesis that population distribution is multivariate normal. Under multivariate normality, Mardia [10] showed that $(N/6)b_{\mathrm{M}1,p}$ converges in distribution to $\chi^2(f_{\mathrm{M}1})$, chi-squared distribution with $f_{\mathrm{M}1}$ degrees of freedom, and $\sqrt{N}\{b_{\mathrm{M}2,p} - p(p+2)\}/\{8p(p+2)\}^{1/2}$ converges in distribution to $N(0,1)$. Here, $f_{\mathrm{M}1} = p(p+1)(p+2)/6$. Thus, if

$$\frac{N}{6}b_{\mathrm{M}1,p} \geq \chi^2_{f_{\mathrm{M}1}}(1-\alpha) \tag{1}$$

non-normality of the data is expected, where $\chi^2_{f_{\mathrm{M}1}}(1-\alpha)$ is the $100(1-\alpha)\%$ point of $\chi^2(f_{\mathrm{M}1})$. Similarly, the normality is rejected if

$$\left|\sqrt{\frac{N}{8p(p+2)}}b_{\mathrm{M}2,p} - p(p+2)\right| \geq z_{1-\alpha/2},$$

where $z_{1-\alpha/2}$ is the $100(1-\alpha/2)\%$ point of the standard normal distribution. Srivastava [8] showed that $\{Np/6\}b_{\mathrm{S}1,p}$ converges in distribution to $\chi^2(p)$ under multivariate normality, and showed that $(Np/24)^{1/2}(b_{\mathrm{S}2,p}-3)$ converges in distribution to $N(0,1)$. Based on these convergences, Srivastava [8] proposed testing criterion which rejects the normality of the data if

$$\frac{Np}{6}b_{\mathrm{S}1,p} \geq \chi^2_p(1-\alpha), \tag{2}$$

and proposed the testing criterion that the normality is rejected if

$$\left|\sqrt{\frac{Np}{24}}(b_{\mathrm{S}2,p}-3)\right| \geq z_{1-\alpha/2}.$$

Some other methods to assess the multivariate normality have been studied. For a review of the results, see, e.g., Henze [3] and Mecklin and Mundfrom [11].

In last years, we encounter more and more problems in applications when p is comparable with N or even exceeds it. For example, financial data, consumer data, network data and medical data have this feature. When $p > N$, it is impossible to define $b_{\mathrm{M}1,p}$, $b_{\mathrm{M}2,p}$, $b_{\mathrm{S}1,p}$ and $b_{\mathrm{S}2,p}$, because the sample covariance matrix $\boldsymbol{S}$ becomes singular.

Instead of dealing with $\beta_{\mathrm{M}2,p}$ or $\beta_{\mathrm{S}2,p}$, Himeno and Yamada [4] has treated

$$\kappa = E[\{(\boldsymbol{x}-\boldsymbol{\mu})'(\boldsymbol{x}-\boldsymbol{\mu})\}] - 2\operatorname{tr}\boldsymbol{\Sigma}^2 - (\operatorname{tr}\boldsymbol{\Sigma})^2.$$

When $\boldsymbol{\Sigma}$ equals to identity matrix $\boldsymbol{I}_p$, κ is the same as the another definition of the Mardia [10]'s multivariate kurtosis $\beta_{2\mathrm{M},1} - p(p+2)$. Himeno and Yamada [4] gave the unbiased estimator of κ as the sample counterpart of κ, which is as follows:

$$\hat{\kappa} = -\frac{1}{(N-2)(N-3)}\{2N^2\operatorname{tr}\boldsymbol{S}^2 + N^2(\operatorname{tr}\boldsymbol{S})^2 - N(N+1)Q\},$$

where

$$Q = \frac{1}{N-1}\sum_{i=1}^{N}\{(\boldsymbol{x}-\bar{\boldsymbol{x}})'(\boldsymbol{x}-\bar{\boldsymbol{x}})\}^2.$$

It is noted that $\hat{\kappa}$ is well defined for the case in which $p > N$. Himeno and Yamada [4] showed that $(N/8)^{1/2}p^{-1}(\hat{\kappa}/\widehat{a_2})$ converges in distribution to $N(0,1)$ as N and p go to infinity together under the following assumptions: (1) $(N-1)/p$ converges to a positive constant as N, p go to infinity; (2) $a_i = \operatorname{tr}\boldsymbol{\Sigma}^i/p$ converges to a positive constant as $p \to \infty$, $i = 1, \ldots, 4$. Here, $\widehat{a_2}$ is an unbiased estimator of a_2, which is as follows:

$$\widehat{a_2} = \frac{1}{(N-1)(N-2)(N-3)}\{N(N-1)(N-2)\operatorname{tr}\boldsymbol{S}^2 + N(\operatorname{tr}\boldsymbol{S})^2 - (N-1)^2Q\}.$$

They proposed testing criterion which rejects the normality of the data if

$$\left|\sqrt{\frac{N}{8}}\frac{1}{p}\frac{\hat{\kappa}}{\widehat{a_2}}\right| \geq z_{1-\alpha/2}.$$

It is reported in Himeno and Yamada [4] that the power of this test gets large as N and p become large for local alternative hypothesis that population distribution is multivariate t.

In this paper, we treat other multivariate 3rd moment γ instead of dealing with $\beta_{\mathrm{M}1,p}$ or $\beta_{1\mathrm{S}1,p}$. An estimator of γ is proposed that is well defined for the case in which $p > N$. Based on $\hat{\gamma}$, we propose a new test for assessing multivariate normality.

The present paper is organized as follows. In Section 2, we propose γ, and gave the estimator $\hat{\gamma}$. Asymptotic normality of $\hat{\gamma}$ is shown under the asymptotic framework that $N, p \to \infty$ for the case in which population distribution is multivariate normal. We give a testing criterion for multivariate normality based on $\hat{\gamma}$. Some results of small-scale simulation including to see attained significance level and empirical power are reported in Section 3. We give concluding remarks in Section 4. All technical proofs are relegated to the Appendix.

2 Multivariate 3rd moment and its estimator

Assume that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are independently and identically distributed (i.i.d.), and assume the following multivariate linear model:

$$\boldsymbol{x}_i = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{\varepsilon}_i, \tag{3}$$

where $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_N$ are i.i.d. as a p-dimensional distribution $F_p(\boldsymbol{0}, \boldsymbol{I}_p)$. For random vector $\boldsymbol{x} = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\boldsymbol{\varepsilon}$ with $\boldsymbol{\varepsilon} \sim F_p(\boldsymbol{0}, \boldsymbol{I}_p)$, let

$$\gamma = \frac{E[\{\odot^3(\boldsymbol{x}-\boldsymbol{\mu})\}'\boldsymbol{1}_p]}{\sqrt{\boldsymbol{1}_p'(\odot^3\boldsymbol{\Sigma})\boldsymbol{1}_p}},$$

where the notation "$\odot^i\boldsymbol{A}$" stands for the Hadamard product of i matrices $\boldsymbol{A}$, i.e.,

$$\odot^i\boldsymbol{A} = \boldsymbol{A}\odot\boldsymbol{A}\odot\cdots\odot\boldsymbol{A} \quad (i \text{ times}).$$

We note that $\odot^i\boldsymbol{A}$ is positive definite, which is guaranteed by Schur's product theorem (cf. Schott [7]). When $p = 1$, γ is reduced to the univariate skewness. It is hard to obtain analytic form for the unbiased estimator of γ. Instead of getting it, we construct the estimator by taking the ratio of unbiased estimator for the numerator of γ divided by for the denominator, which an estimator we proposed is as follows.

$$\hat{\gamma} = \frac{E[\{\widehat{\odot^3(\boldsymbol{x}-\boldsymbol{\mu})\}'\mathbf{1}_p}]}{\sqrt{\widehat{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}}} = \frac{T}{\sqrt{S}},$$

where

$$T = \frac{4}{3{}_N\mathrm{P}_3}\sum_{i,j,k}^{N}{}^{*}\left[\odot^3\left\{\boldsymbol{x}_i - \frac{1}{2}(\boldsymbol{x}_j+\boldsymbol{x}_k)\right\}\right]'\mathbf{1}_p,$$

$$S = \frac{1}{8{}_N\mathrm{P}_6}\sum_{\substack{k,\ell,\alpha,\beta,\\ \gamma,\delta}}^{N}{}^{*}\ \mathbf{1}_p'\{(\boldsymbol{x}_k-\boldsymbol{x}_\ell)(\boldsymbol{x}_k-\boldsymbol{x}_\ell)'\odot(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)'\odot(\boldsymbol{x}_\gamma-\boldsymbol{x}_\delta)(\boldsymbol{x}_\gamma-\boldsymbol{x}_\delta)'\}\mathbf{1}_p.$$

Here, the notation $\sum^*$ stands for the sum of all pairs of indices which are not equal. For example, $\sum_{i,j,k}{}^* = \sum_{i=1}\sum_{j=1,j\neq i}\sum_{k=1,k\neq i,k\neq j}$.

2.1 Asymptotic distribution of $\hat{\gamma}$ and a test for γ

Firstly, we show the asymptotic normality of $\hat{\gamma}$ as $N,p\to\infty$ under the assumption that $F_p(\mathbf{0},\boldsymbol{I}_p) = N_p(\mathbf{0},\boldsymbol{I}_p)$. It is noted that T is invariant under the location shift. Hence, without loss of generality, we may assume $\boldsymbol{\mu}=\mathbf{0}$. It can be described that

$$T = \frac{4}{3{}_N\mathrm{P}_3}\sum_{i,j,k}^{N}{}^{*}\odot^3\left[\boldsymbol{\Sigma}^{1/2}\left\{\boldsymbol{z}_i - \frac{1}{2}(\boldsymbol{z}_j+\boldsymbol{z}_k)\right\}\right]'\mathbf{1}_p.$$

For deriving the anaclitic form of $\mathrm{Var}(T)$, we decompose T as

$$\begin{aligned}
T &= \frac{1}{N}\sum_{\ell=1}^{p}\sum_{i=1}^{N}(\boldsymbol{a}_\ell'\boldsymbol{z}_i)^3 - \frac{3}{N(N-1)}\sum_{\ell=1}^{p}\sum_{i,j}^{N}{}^{*}(\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2\boldsymbol{a}_\ell'\boldsymbol{z}_j \\
&\quad + \frac{2}{N(N-1)(N-2)}\sum_{\ell=1}^{p}\sum_{i,j,k}^{N}{}^{*}\boldsymbol{a}_\ell'\boldsymbol{z}_i\boldsymbol{a}_\ell'\boldsymbol{z}_j\boldsymbol{a}_\ell'\boldsymbol{z}_k \\
&= \frac{1}{N}\sum_{\ell=1}^{p}\sum_{i=1}^{N}\left\{(\boldsymbol{a}_\ell'\boldsymbol{z}_i)^3 - 3\boldsymbol{a}_\ell'\boldsymbol{a}_\ell\boldsymbol{a}_\ell'\boldsymbol{z}_i\right\} - \frac{3}{N(N-1)}\sum_{\ell=1}^{p}\left\{\sum_{i,j}^{N}{}^{*}(\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2\boldsymbol{a}_\ell'\boldsymbol{z}_j - (N-1)\boldsymbol{a}_\ell'\boldsymbol{a}_\ell\sum_{i=1}^{N}\boldsymbol{a}_\ell'\boldsymbol{z}_i\right\} \\
&\quad + \frac{2}{N(N-1)(N-2)}\sum_{\ell=1}^{p}\sum_{i,j,k}^{N}{}^{*}\boldsymbol{a}_\ell'\boldsymbol{z}_i\boldsymbol{a}_\ell'\boldsymbol{z}_j\boldsymbol{a}_\ell'\boldsymbol{z}_k \\
&= T_1 + T_2 + T_3,
\end{aligned}$$

where $\boldsymbol{\Sigma}^{1/2} = \boldsymbol{A} = (\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)'$. After much algebraic calculations, it is shown that $\mathrm{Cov}(T_1,T_2)=0$, $\mathrm{Cov}(T_2,T_3)=0$ and $\mathrm{Cov}(T_1,T_3)=0$, and so

$$\sigma_T^2 = \mathrm{Var}(T) = \mathrm{Var}(T_1)+\mathrm{Var}(T_2)+\mathrm{Var}(T_3).$$

We obtain analytic forms of $\mathrm{Var}(T_1)$, $\mathrm{Var}(T_2)$ and $\mathrm{Var}(T_3)$, respectively, which are as follows.

$$\mathrm{Var}(T_1) = \frac{6}{N}\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p, \tag{4}$$

$$\mathrm{Var}(T_2) = \frac{18}{N(N-1)}\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p, \tag{5}$$

$$\mathrm{Var}(T_3) = \frac{24}{N(N-1)(N-2)}\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p. \tag{6}$$

These proofs are given in Appendix. From (4), (5) and (6), we have

$$\mathrm{Var}(T) = \frac{6N}{(N-1)(N-2)} \mathbf{1}_p'(\odot^3 \boldsymbol{\Sigma}) \mathbf{1}_p.$$

It is found that

$$\frac{\mathrm{Var}(T_2)}{\sigma_T^2} = \frac{3(N-2)}{N^2} \to 0 \quad (N, p \to \infty), \qquad \frac{\mathrm{Var}(T_3)}{\sigma_T^2} = \frac{4}{N^2} \to 0 \quad (N, p \to \infty).$$

From Chebyshev inequality we have

$$\frac{1}{\sigma_T} T_2 \xrightarrow{p} 0 \quad (N, p \to \infty), \qquad \frac{1}{\sigma_T} T_3 \xrightarrow{p} 0 \quad (N, p \to \infty).$$

These probability convergences imply that

$$\frac{1}{\sigma_T}(T - T_1) \xrightarrow{p} 0 \quad (N, p \to \infty).$$

And we can express that

$$\frac{1}{\sigma_T} T_1 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[\frac{1}{\sigma_T \sqrt{N}} \sum_{\ell=1}^{p} \left\{ (\boldsymbol{a}_\ell' \boldsymbol{z}_i)^3 - 3 \boldsymbol{a}_\ell' \boldsymbol{a}_\ell \boldsymbol{a}_\ell' \boldsymbol{z}_i \right\} \right] = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\eta_i}{\sigma_T}.$$

It is shown that $E[\eta_i] = 0$, and is shown in Appendix that

$$\mathrm{Var}\left(\frac{\eta_i}{\sigma_T} \right) = \frac{6}{N \sigma_T^2} \mathbf{1}_p'(\odot^3 \boldsymbol{\Sigma}) \mathbf{1}_p, \tag{7}$$

which converges to 1 as $N, p \to \infty$. By virtue of the ordinal central limit theorem, we find that the asymptotic distribution of $\sigma_T^{-1} T_1$ is $N(0, 1)$ as $N, p \to \infty$. Using Slutsly theorem, the asymptotic normality of T can be proved, which is given as the following theorem.

Theorem 1. *As $N, p \to \infty$,*

$$\frac{1}{\sigma_T} T \xrightarrow{\mathcal{D}} N(0, 1).$$

In Appendix, we prove the rate consistency for $S / \mathbf{1}_p'(\odot^3 \boldsymbol{\Sigma}) \mathbf{1}_p$ under the condition C;

$$\mathrm{C} : \limsup_{p} \frac{\mathbf{1}_p'(\odot^3 \boldsymbol{\Sigma}_+) \mathbf{1}_p}{\mathbf{1}_p'(\odot^3 \boldsymbol{\Sigma}) \mathbf{1}_p} < \infty,$$

where $\boldsymbol{\Sigma}_+ = (|\sigma_{ij}|)$ for $\boldsymbol{\Sigma} = (\sigma_{ij})$. The result is given as the following theorem.

Theorem 2. *Suppose that the condition* C *holds. Then, as $N, p \to \infty$,*

$$\frac{S}{\mathbf{1}_p'(\odot^3 \boldsymbol{\Sigma}) \mathbf{1}_p} \xrightarrow{p} 1.$$

From these theorems (Theorem 1 and Theorem 2) and Slutsky's theorem, we find that

$$\sqrt{\frac{N}{6}} \hat{\gamma} \xrightarrow{\mathcal{D}} N(0, 1) \tag{8}$$

as $N, p \to \infty$ under the condition that C holds.

As an application, assessing the assumption of multivariate normality is considered by testing $\gamma = 0$. The null hypothesis that $\gamma = 0$ is rejected with significance level α for the case in which $|\sqrt{N/6} \hat{\gamma}| > z_{1-\alpha/2}$, where z_α is the 100 α% point of the standard normal distribution.

3 Numerical results

3.1 Simulation

In order to see the accuracy of the asymptotic normality (8), we tried a numerical simulation. It can be expressed that

$$S = \frac{1}{{}_N\mathrm{P}_3}A - \frac{3}{{}_N\mathrm{P}_4}B + \frac{3}{{}_N\mathrm{P}_5}C - \frac{1}{{}_N\mathrm{P}_6}D, \tag{9}$$

where

$$A = \sum_{k,\ell,\alpha}^{N}{}^{*}\mathbf{1}_p'(\boldsymbol{x}_k\boldsymbol{x}_k' \odot \boldsymbol{x}_\ell\boldsymbol{x}_\ell' \odot \boldsymbol{x}_\alpha\boldsymbol{x}_\alpha')\mathbf{1}_p, \qquad B = \sum_{k,\ell,\alpha,\beta}^{N}{}^{*}\mathbf{1}_p'(\boldsymbol{x}_k\boldsymbol{x}_k' \odot \boldsymbol{x}_\ell\boldsymbol{x}_\ell' \odot \boldsymbol{x}_\alpha\boldsymbol{x}_\beta')\mathbf{1}_p,$$

$$C = \sum_{k,\ell,\alpha,\beta,\gamma}^{N}{}^{*}\mathbf{1}_p'(\boldsymbol{x}_k\boldsymbol{x}_k' \odot \boldsymbol{x}_\ell\boldsymbol{x}_\alpha' \odot \boldsymbol{x}_\beta\boldsymbol{x}_\gamma')\mathbf{1}_p, \qquad D = \sum_{k,\ell,\alpha,\beta,\gamma,\delta}^{N}{}^{*}\mathbf{1}_p'(\boldsymbol{x}_k\boldsymbol{x}_\ell' \odot \boldsymbol{x}_\alpha\boldsymbol{x}_\beta' \odot \boldsymbol{x}_\gamma\boldsymbol{x}_\delta')\mathbf{1}_p.$$

To compute A, it is needed to calculate triple sum, which is performed by triple-loop calculations. We see that B, C, D and T are needed to perform multi-loop calculations. Generally, it takes a lot time to carry out multi-loop calculations. To save computing time, we shall provide other expressions for A, B, C, D and T.

Put

$$\begin{aligned}
\boldsymbol{X}_{ij} &= \sum_{k=1}^{N}(\odot^i\boldsymbol{x}_k)(\odot^j\boldsymbol{x}_k)' && (i,j=1,\ldots,3),\\
\boldsymbol{X}_{0i} &= \boldsymbol{X}_{i0}' = \left\{\sum_{k=1}^{N}(\odot^i\boldsymbol{x}_k)\mathbf{1}_p'\right\}' && (i=1,\ldots,3),\\
\boldsymbol{s}_k &= \sum_{i=1}^{N}(\odot^k\boldsymbol{x}_i) && (k=1,\ldots,3).
\end{aligned}$$

Then, it holds that

$$\begin{aligned}
T &= \frac{N}{(N-1)(N-2)}\boldsymbol{s}_3'\mathbf{1}_p - \frac{3}{(N-1)(N-2)}(\boldsymbol{s}_2\odot\boldsymbol{s}_1)'\mathbf{1}_p + \frac{2}{N(N-1)(N-2)}(\odot^3\boldsymbol{s}_1)'\mathbf{1}_p,\\
A &= \mathbf{1}_p'\left[(\odot^3\boldsymbol{X}_{11}) - 3\boldsymbol{X}_{11}\odot\boldsymbol{X}_{22} + 2\boldsymbol{X}_{33}\right]\mathbf{1}_p,\\
B &= \mathbf{1}_p'\left[-(\odot^3\boldsymbol{X}_{11}) + 5\boldsymbol{X}_{11}\odot\boldsymbol{X}_{22} - 6\boldsymbol{X}_{33} - 4\boldsymbol{X}_{10}\odot\boldsymbol{X}_{11}\odot\boldsymbol{X}_{12} + 4\boldsymbol{X}_{10}\odot\boldsymbol{X}_{23}\right.\\
&\qquad \left.+2\boldsymbol{X}_{12}\odot\boldsymbol{X}_{21} + \boldsymbol{X}_{01}\odot\boldsymbol{X}_{10}\odot(\odot^2\boldsymbol{X}_{11}) - \boldsymbol{X}_{01}\odot\boldsymbol{X}_{10}\odot\boldsymbol{X}_{22}\right]\mathbf{1}_p,\\
C &= \mathbf{1}_p'\left[2(\odot^3\boldsymbol{X}_{11}) - 14\boldsymbol{X}_{11}\odot\boldsymbol{X}_{22} + 24\boldsymbol{X}_{33} + 16\boldsymbol{X}_{10}\odot\boldsymbol{X}_{11}\odot\boldsymbol{X}_{12} - 24\boldsymbol{X}_{10}\odot\boldsymbol{X}_{23}\right.\\
&\qquad - 8\boldsymbol{X}_{12}\odot\boldsymbol{X}_{21} - 4\boldsymbol{X}_{01}\odot\boldsymbol{X}_{10}\odot(\odot^2\boldsymbol{X}_{11}) + 8\boldsymbol{X}_{01}\odot\boldsymbol{X}_{10}\odot\boldsymbol{X}_{22} - 4\boldsymbol{X}_{01}\odot(\odot^2\boldsymbol{X}_{10})\odot\boldsymbol{X}_{12}\\
&\qquad + 4(\odot^2\boldsymbol{X}_{10})\odot\boldsymbol{X}_{13} + 4\boldsymbol{X}_{01}\odot\boldsymbol{X}_{12}\odot\boldsymbol{X}_{20} - 4\boldsymbol{X}_{13}\odot\boldsymbol{X}_{20} - 2\boldsymbol{X}_{02}\odot(\odot^2\boldsymbol{X}_{10})\odot\boldsymbol{X}_{11}\\
&\qquad \left.+\boldsymbol{X}_{02}\odot\boldsymbol{X}_{11}\odot\boldsymbol{X}_{20} + (\odot^2\boldsymbol{X}_{01})\odot(\odot^2\boldsymbol{X}_{10})\odot\boldsymbol{X}_{11}\right]\mathbf{1}_p,\\
D &= \{(\odot^3\boldsymbol{s}_1)'\mathbf{1}_p - 3(\boldsymbol{s}_2\odot\boldsymbol{s}_1)'\mathbf{1}_p + 2\boldsymbol{s}_3'\mathbf{1}_p\}^2 - 6A - 18B - 9C.
\end{aligned}$$

One can see that these expressions have no multi-sum expression.

Generate the data based on the model (3). Firstly, we checked the asymptotic normality of $\hat{\gamma}$ given in (8). Make 10^4 samples of the size N, each of which is constructed by p-dimensional vectors which are i.i.d. as $N_p(\mathbf{0},\boldsymbol{\Sigma})$. In this study, the following two cases for $\boldsymbol{\Sigma}$ are treated: (1) $\boldsymbol{\Sigma} = \boldsymbol{I}_p$; (2) $\boldsymbol{\Sigma}$ has Toeplitz structure such that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\mathrm{T}} = \boldsymbol{D}^{1/2}\boldsymbol{P}\boldsymbol{D}^{1/2}$, where $\boldsymbol{D} = \mathrm{diag}(d_1,\ldots,d_p)$ with $d_i = 5+(-1)^{i-1}(p-i+1)/p$, and $\boldsymbol{P} = (0.1^{|i-j|})$. We note that these two cases satisfy the condition C. Figure 1 shows Q-Q plot of $\sqrt{N/6}\hat{\gamma}$ when $\boldsymbol{\Sigma} = \boldsymbol{I}_p$ for the case in which $(N,p) = (60,60)$, and Figure 2 shows Q-Q plot when $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\mathrm{T}}$. We can see the goodness of fit of the standard normal distribution to $\sqrt{N/6}\hat{\gamma}$ from these figures. This numerical simulations were carried out for several other dimensions

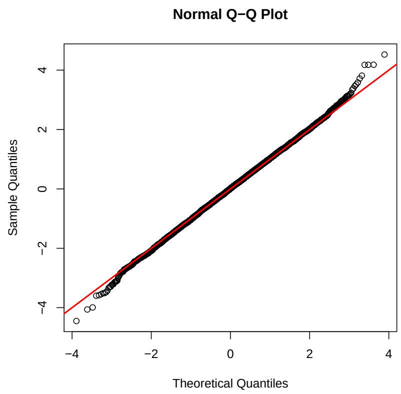

Figure 1: Q-Q plot of $\sqrt{10}\hat{\gamma}$ when $\boldsymbol{\Sigma} = \boldsymbol{I}_{10}$

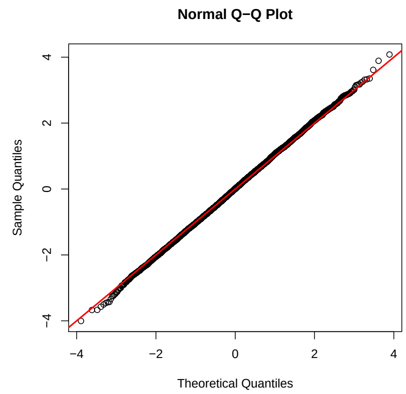

Figure 2: Q-Q plot of $\sqrt{10}\hat{\gamma}$ when $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\mathrm{T}}$

as well, $p = 120$, 240, 480, and also for other sample sizes, $N = 120$, 240, 480, but due to similarity of the graphs, only a selection is reported here.

Next, we checked the attained significance level(ASL) for testing the null hypothesis that $H_0 : F_p(\mathbf{0}, \boldsymbol{I}_p) = N_p(\mathbf{0}, \boldsymbol{I}_p)$ under the assumption of model (3) by using the following criterion:

$$|\sqrt{N/6}\hat{\gamma}| > z_{1-\alpha/2} \implies \text{reject the null hypothesis that } H_0 : F_p(\mathbf{0}, \boldsymbol{I}_p) = N_p(\mathbf{0}, \boldsymbol{I}_p). \tag{10}$$

To compute ASL, Monte Carlo simulation with 10^4 replication was done for the nominal level $\alpha = 0.05$. We computed for the case in which $N = 60$,120, 240, 480, and $p = 60$, 120, 240, 480, and wrote these values in Table 1. The settings of $\boldsymbol{\Sigma}$ treated in this paper are as follows.

Case 1. $\boldsymbol{\Sigma} = \boldsymbol{I}_p$.
Case 2. $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_{\mathrm{T}}$.
Case 3. Generate a sparse positive definite covariance matrix $\boldsymbol{\Sigma}$ by Algorithm 1. Set sparsity level $s = 0.7$.

Algorithm 1 The algorithm for generating sparse covariance matrix $\boldsymbol{\Sigma}$

1: Construct $p \times p$ matrix $\boldsymbol{R} = (r_{ij})$ as follows. For each $i < j$,

$$r_{ij} = \begin{cases} \mathrm{Unif}(0,1) & \text{with probability } (1-s) \times 0.75, \\ \mathrm{Unif}(-1,0) & \text{with probability } (1-s) \times 0.75, \\ 0 & \text{with probability } s, \end{cases}$$

where s is the level of sparsity in $\boldsymbol{R}$. Then, we set $r_{ji} = r_{ij}$ to obtain symmetry. Furthermore, set the diagonal elements of $\boldsymbol{R}$ to equal 1.

2: Prepare a $p \times p$ diagonal matrix $\boldsymbol{D} = (d_{ij})$ such that $d_{11}, \dots, d_{pp}$ are i.i.d. chi-squared distribution with 1 degree of freedom.

3: Create $\boldsymbol{A} = \boldsymbol{DRD}$.

4: In order to obtain positive definiteness of $\boldsymbol{\Sigma}$, calculate the minimum eigenvalue $e_{\min}$ of $\boldsymbol{A}$. Set

$$\boldsymbol{\Sigma} = \begin{cases} \boldsymbol{A} + (-e_{\min} + 0.1)\boldsymbol{I}_p & \text{if } e_{\min} \le 0, \\ \boldsymbol{A} & \text{otherwise.} \end{cases}$$

Case 4. Use the same generating method as Case 3 with sparsity level $s = 0.4$.
Case 5. Use the same generating method as Case 3 with sparsity level $s = 0.1$.

For Case 3-5, we repeat the Monte Carlo simulation 10^2 times, and wrote the average of these 10^2 values in the table. The value in parenthesis is the standard error.

Table 1: Attained significance level of the proposed test

N	p	Case 1	Case 2	Case 3(S.E)	Case 4(S.E)	Case 5(S.E)
60	60	0.058	0.058	0.056(0.0022)	0.056(0.0026)	0.056(0.0024)
	120	0.055	0.055	0.056(0.0024)	0.056(0.0021)	0.056(0.0020)
	240	0.055	0.052	0.056(0.0025)	0.056(0.0021)	0.056(0.0023)
	480	0.056	0.050	0.056(0.0025)	0.056(0.0021)	0.056(0.0022)
120	60	0.053	0.055	0.053(0.0022)	0.053(0.0024)	0.053(0.0021)
	120	0.053	0.055	0.053(0.0022)	0.053(0.0021)	0.053(0.0019)
	240	0.055	0.052	0.053(0.0023)	0.053(0.0023)	0.053(0.0024)
	480	0.054	0.050	0.052(0.0021)	0.053(0.0023)	0.053(0.0023)
240	60	0.054	0.052	0.052(0.0025)	0.052(0.0023)	0.052(0.0022)
	120	0.052	0.053	0.051(0.0025)	0.052(0.0020)	0.051(0.0020)
	240	0.050	0.049	0.051(0.0021)	0.051(0.0018)	0.051(0.0020)
	480	0.049	0.050	0.051(0.0020)	0.052(0.0022)	0.052(0.0020)
480	60	0.051	0.051	0.051(0.0022)	0.051(0.0023)	0.051(0.0023)
	120	0.051	0.054	0.051(0.0022)	0.051(0.0021)	0.051(0.0022)
	240	0.050	0.049	0.051(0.0020)	0.051(0.0021)	0.051(0.0022)
	480	0.049	0.049	0.050(0.0023)	0.051(0.0024)	0.051(0.0021)

We can see from Table 1 that ASL of our statistic is little affeected by the sparsity of the covariance matrix. It is observed that the accuracy of the approximation gets better as N and p become large.

Thirdly, we checked low-dimensional performance of the proposed test. The setting of dimensionality is $p = 10$. We compare ASL of our proposed test with the one of Mardia [10]'s test (1) and the one of Srivastava [8]'s test (2) for the case in which $N = 40, 50, 60, 70, 80, 90, 100$ and $\boldsymbol{\Sigma} = \boldsymbol{I}_{10}$. Computing methodology is the same as the one in Table 1. The symbol "P" denotes the value of ASL for proposed test (10), "M" denotes for Mardia [10]'s test (1), and "S" denotes for Srivastava [8]'s test (2). We can see that ASL of our proposed test overestimate, whereas ASLs of Mardia [10]'s test and Srivastava [8]'s test underestimate. Our approximation seems to be good for the case in which $N \geq 70$, whereas other two approximations are not good. The precisions of all three tests gets better as N becomes large.

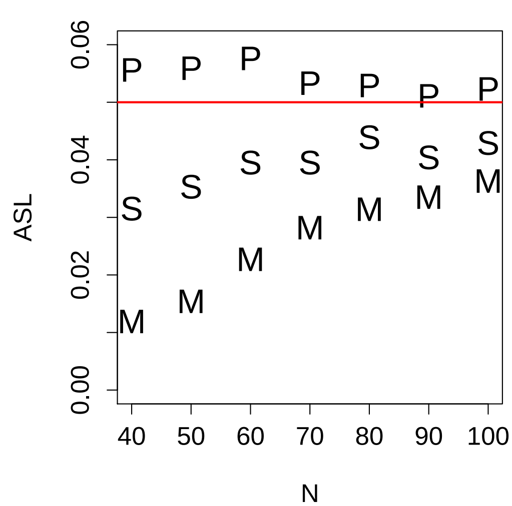

Figure 3: Comparison of ASL when $p = 10$

Finally, we checked the empirical power(EMP) of the proposed test by Monte Carlo simulation.

The setting of the local alternative hypothesis treated in this paper is as follows: For $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_p)' \sim F_p(\mathbf{0}, \boldsymbol{I}_p)$,

$$\varepsilon_i = \sqrt{\frac{\nu}{2}}\left(\frac{c_i}{\nu} - 1\right) \quad (i = 1, \ldots, p),$$

where $c_1, \ldots, c_p$ are i.i.d. as the chi-squared distribution with ν degrees of freedom. We computed the EMP based on 10^3 repetition for the case in which $\alpha = 0.05$ and $\nu = 10000$. The settings of N and p are the same as the ones in Table 1. We carried out for all models of $\boldsymbol{\Sigma}$ written in Table 1, but due to similarity of the tendencies, Case 2-5 are omitted here. We can see from Table 2 that EMP are monotone increasing for p and N.

Table 2: Empirical power of the proposed test for Case 1

$N \setminus p$	60	120	240	480
60	0.11	0.17	0.29	0.50
120	0.16	0.29	0.52	0.78
240	0.32	0.51	0.78	0.98
480	0.49	0.78	0.98	1.00

3.2 Real data analysis

We applied our test to data-set which is used in Kubokawa and Srivastava [6]. The first applied data is colon data which is constructed by 2000 (p) genes expression levels on 22 normal colon tissues and 40 tumor colon tissues. We preprocessed the data by applying 10 logarithmic transformation. The second applied data-set is leukemia data which is constructed by 3571(p) gene expressions on 47 patients suffering from acute lymphoblastic leukemia (ALL, 47 cases) and 25 patients suffering from acute myeloid leukemia (AML, 25 cases). The data-set are preprocessed by following protocol written in Dudoit et al. [1].

For the colon data, the p-value of our test is 0.000 for normal colon tissues and 0.000 for tumor colon tissues. For the leukemia data, we observed that both the p-value of our test for ALL and for AML are 0.000 for the leukemia data. These results indicate that the multivariate normality assumption on both sets of data cannot be rejected at the usual significance level 5%.

4 Concluding remarks

This paper treats multivariate 3rd moment γ which is defined by using Hadamard product of observation vectors. When the dimension equals to 1, γ becomes univariate skewness. An estimator of γ, denoted as $\hat{\gamma}$, which is well defined for $p > N$ is proposed. Based on $\hat{\gamma}$, testing criterion for assessing multivariate normality is given.

We prepare the expression of the testing statistic without having multi-sum. Calculating time becomes shorten. The performance of the testing criterion, in terms of the attained significance level and the empirical power, is shown through simulations for several setting of sample sizes and dimensions. We apply our proposed test to microarray data sets, some of the most popular are in high-dimensional data.

To assess the normality of high-dimensional data, we recommend to use not only our proposed test but also Himeno and Yamada [4]'s test based on $\hat{\kappa}$ together. Omnibus test using $\hat{\gamma}$ and $\hat{\kappa}$, which imitates Jarque and Bera [5]'s test, is a future work.

A Moment of statistic

In this section, analytic forms of $\mathrm{Var}(T_1)$, $\mathrm{Var}(T_2)$ and $\mathrm{Var}(T_3)$ are proposed. We derive them by using these results (Lemma 1), proofs of which are tedious but simple, therefore skipped.

Lemma 1. *Let $\boldsymbol{z}$ be a random vector distributed as $N_p(\boldsymbol{0}, \boldsymbol{I}_p)$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ be constant vectors. Then,*

$$
\begin{aligned}
&E[(\boldsymbol{a}'\boldsymbol{z})^2] = \boldsymbol{a}'\boldsymbol{a},\\
&E[(\boldsymbol{a}'\boldsymbol{z})^4] = 3(\boldsymbol{a}'\boldsymbol{a})^2,\\
&E[(\boldsymbol{a}'\boldsymbol{z})^2(\boldsymbol{b}'\boldsymbol{z})^2] = 2(\boldsymbol{a}'\boldsymbol{b})^2 + \boldsymbol{a}'\boldsymbol{a}\boldsymbol{b}'\boldsymbol{b},\\
&E[(\boldsymbol{a}'\boldsymbol{z})^3\boldsymbol{b}'\boldsymbol{z}] = 3\boldsymbol{a}'\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b},\\
&E[(\boldsymbol{a}'\boldsymbol{z})^6] = 15(\boldsymbol{a}'\boldsymbol{a})^3,\\
&E[(\boldsymbol{a}'\boldsymbol{z})^3(\boldsymbol{b}'\boldsymbol{z})^3] = 6(\boldsymbol{a}'\boldsymbol{b})^3 + 9\boldsymbol{a}'\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}\boldsymbol{b}'\boldsymbol{b}.
\end{aligned}
$$

Here, we write analytic forms of $\mathrm{Var}(T_1)$, $\mathrm{Var}(T_2)$ and $\mathrm{Var}(T_3)$ in the following lemma.

Lemma 2. *For T_1, T_2 and T_3, as defined in Section 2.1,*

$$
\begin{aligned}
\mathrm{Var}(T_1) &= \frac{6}{N}\boldsymbol{1}_p'(\odot^3\boldsymbol{\Sigma})\boldsymbol{1}_p,\\
\mathrm{Var}(T_2) &= \frac{18}{N(N-1)}\boldsymbol{1}_p'(\odot^3\boldsymbol{\Sigma})\boldsymbol{1}_p,\\
\mathrm{Var}(T_3) &= \frac{24}{N(N-1)(N-2)}\boldsymbol{1}_p'(\odot^3\boldsymbol{\Sigma})\boldsymbol{1}_p.
\end{aligned}
$$

Proof. In Section 3, we expressed that

$$
T_1 = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\eta_i,
$$

where

$$
\eta_i = \frac{1}{\sqrt{N}}\sum_{\ell=1}^{p}\left\{(\boldsymbol{a}_\ell'\boldsymbol{z}_i)^3 - 3\boldsymbol{a}_\ell'\boldsymbol{a}_\ell\boldsymbol{z}_\ell'\boldsymbol{z}_i\right\}.
$$

Since $\eta_1, \ldots, \eta_N$ are i.i.d., it holds that

$$
\mathrm{Var}(T_1) = \mathrm{Var}(\eta) = \mathrm{Var}\left(\frac{1}{\sqrt{N}}\sum_{\ell=1}^{p}\{(\boldsymbol{a}_\ell'\boldsymbol{z})^3 - 3\boldsymbol{a}_\ell'\boldsymbol{a}_\ell\boldsymbol{a}_\ell'\boldsymbol{z}\}\right),
$$

where $\boldsymbol{z} \sim N_p(\boldsymbol{0}, \boldsymbol{I}_p)$. We find that $E[\zeta] = 0$, and so

$$
\begin{aligned}
\mathrm{Var}(\zeta) &= E[\zeta^2]\\
&= E\left[\frac{1}{N}\sum_{\ell=1}^{p}\{(\boldsymbol{a}_\ell'\boldsymbol{z})^3 - 3\boldsymbol{a}_\ell'\boldsymbol{a}_\ell\boldsymbol{a}_\ell'\boldsymbol{z}\}^2\right.\\
&\qquad\left. + \frac{1}{N}\sum_{\ell,\alpha}^{p}{}^{*}\{(\boldsymbol{a}_\ell'\boldsymbol{z})^3 - 3\boldsymbol{a}_\ell'\boldsymbol{a}_\ell\boldsymbol{a}_\ell'\boldsymbol{z}\}\{(\boldsymbol{a}_\alpha'\boldsymbol{z})^3 - 3\boldsymbol{a}_\alpha'\boldsymbol{a}_\alpha\boldsymbol{a}_\alpha'\boldsymbol{z}\}\right]\\
&= \left[\frac{1}{N}\sum_{\ell=1}^{p}\{(15-18+9)(\boldsymbol{a}_\ell'\boldsymbol{a}_\ell)^3\}\right.\\
&\qquad\left. + \frac{1}{N}\sum_{\ell,\alpha}^{p}{}^{*}\{6(\boldsymbol{a}_\ell'\boldsymbol{a}_\ell)^3 + (9-9-9+9)\boldsymbol{a}_\ell'\boldsymbol{a}_\ell\boldsymbol{a}_\ell'\boldsymbol{a}_\alpha\boldsymbol{a}_\alpha'\boldsymbol{a}_\alpha\}\right]\\
&= \frac{6}{N}\left\{\sum_{\ell=1}^{p}(\boldsymbol{a}_\ell'\boldsymbol{a}_\ell)^3 + \sum_{\ell,\alpha}^{p}{}^{*}(\boldsymbol{a}_\ell'\boldsymbol{a}_\alpha)^3\right\}\\
&= \frac{6}{N}\boldsymbol{1}_p'(\odot^3\boldsymbol{A}^2)\boldsymbol{1}_p\\
&= \frac{6}{N}\boldsymbol{1}_p'(\odot^3\boldsymbol{\Sigma})\boldsymbol{1}_p,
\end{aligned}
$$

where the third equality follows from Lemma 1.

Since $E[T_2]=0$, we have

$$
\begin{aligned}
\mathrm{Var}(T_2) &= E[T_2^2] \\
&= \frac{9}{N^2(N-1)^2} E\left[\sum_{\ell=1}^{p} \left\{ \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^4 (\boldsymbol{a}_\ell'\boldsymbol{z}_j)^2 - (N-1)\boldsymbol{a}_\ell'\boldsymbol{a}_\ell \sum_{i=1}^{N} \boldsymbol{a}_\ell'\boldsymbol{z}_i \right\}^2 \right. \\
&\quad + \sum_{\ell,\alpha}^{p}{}^{*} \left\{ \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^4 (\boldsymbol{a}_\ell'\boldsymbol{z}_j)^2 - (N-1)\boldsymbol{a}_\ell'\boldsymbol{a}_\ell \sum_{i=1}^{N} \boldsymbol{a}_\ell'\boldsymbol{z}_i \right\} \\
&\quad \left. \cdot \left\{ \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\alpha'\boldsymbol{z}_i)^4 (\boldsymbol{a}_\alpha'\boldsymbol{z}_j)^2 - (N-1)\boldsymbol{a}_\alpha'\boldsymbol{a}_\alpha \sum_{i=1}^{N} \boldsymbol{a}_\alpha'\boldsymbol{z}_i \right\} \right] \\
&= \frac{9}{N^2(N-1)^2} E\left[\sum_{\ell=1}^{p} \left\{ \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^4 (\boldsymbol{a}_\ell'\boldsymbol{z}_j)^2 + \sum_{i,j,k}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2 (\boldsymbol{a}_\ell'\boldsymbol{z}_j)^2 (\boldsymbol{a}_\ell'\boldsymbol{z}_k)^2 \right. \right. \\
&\quad \left. + (N-1)^2 (\boldsymbol{a}_\ell'\boldsymbol{a}_\ell)^2 \sum_{i=1}^{N} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2 - 2(N-1)\boldsymbol{a}_\ell'\boldsymbol{a}_\ell \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2 (\boldsymbol{a}_\ell'\boldsymbol{z}_j)^2 \right\} \\
&\quad + \sum_{\ell,\alpha}^{p}{}^{*} \left\{ \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2 (\boldsymbol{a}_\alpha'\boldsymbol{z}_i)^2 \boldsymbol{a}_\ell'\boldsymbol{z}_j \boldsymbol{a}_\alpha'\boldsymbol{z}_j + \sum_{i,j,k}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2 \boldsymbol{a}_\ell'\boldsymbol{z}_j \boldsymbol{a}_\alpha'\boldsymbol{z}_j (\boldsymbol{a}_\alpha'\boldsymbol{z}_k)^2 \right. \\
&\quad - (N-1)\boldsymbol{a}_\alpha'\boldsymbol{a}_\alpha \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2 \boldsymbol{a}_\ell'\boldsymbol{z}_j \boldsymbol{a}_\alpha'\boldsymbol{z}_j - (N-1)\boldsymbol{a}_\ell'\boldsymbol{a}_\ell \sum_{i,j}^{N}{}^{*} (\boldsymbol{a}_\alpha'\boldsymbol{z}_i)^2 \boldsymbol{a}_\alpha'\boldsymbol{z}_j \boldsymbol{a}_\ell'\boldsymbol{z}_j \\
&\quad \left. \left. + (N-1)^2 \boldsymbol{a}_\ell'\boldsymbol{a}_\ell \boldsymbol{a}_\alpha'\boldsymbol{a}_\alpha \sum_{i=1}^{N} \boldsymbol{a}_\ell'\boldsymbol{z}_i \boldsymbol{a}_\alpha'\boldsymbol{z}_i \right\} \right] \\
&= \frac{9}{N^2(N-1)^2} \left[\sum_{\ell=1}^{p} \left\{ 3N(N-1) + N(N-1)(N-2) + N(N-1)^2 \right. \right. \\
&\quad \left. -2N(N-1)^2 \right\} (\boldsymbol{a}_\ell'\boldsymbol{a}_\ell)^3 + \sum_{\ell,\alpha}^{p}{}^{*} \left\{ N(N-1)\{2(\boldsymbol{a}_\ell'\boldsymbol{a}_\alpha)^2 + \boldsymbol{a}_\ell'\boldsymbol{a}_\ell \boldsymbol{a}_\alpha'\boldsymbol{a}_\alpha\} \boldsymbol{a}_\ell'\boldsymbol{a}_\alpha \right. \\
&\quad \left. \left. + \{N(N-1)(N-2) - N(N-1)^2 - N(N-1)^2 + N(N-1)^2\} \boldsymbol{a}_\ell'\boldsymbol{a}_\ell \boldsymbol{a}_\ell'\boldsymbol{a}_\alpha \boldsymbol{a}_\alpha'\boldsymbol{a}_\alpha \right\} \right] \\
&= \frac{9}{N^2(N-1)^2} \left[2N(N-1) \sum_{\ell=1}^{p} (\boldsymbol{a}_\ell'\boldsymbol{a}_\ell)^3 + 2N(N-1) \sum_{\ell,\alpha}^{p}{}^{*} (\boldsymbol{a}_\ell'\boldsymbol{a}_\alpha)^3 \right] \\
&= \frac{18}{N(N-1)} \mathbf{1}_p' (\odot^3 \boldsymbol{A}^2) \mathbf{1}_p \\
&= \frac{18}{N(N-1)} \mathbf{1}_p' (\odot^3 \boldsymbol{\Sigma}) \mathbf{1}_p,
\end{aligned}
$$

where the fourth equality follows from Lemma 1.

Since $E[T_3]=0$, we have

$$\begin{aligned}
\mathrm{Var}(T_3) &= E[T_3^2] \\
&= \frac{4}{N^2(N-1)^2(N-2)^2} E\left[\sum_{\ell=1}^{p}\left(\sum_{i,j,k}^{N}{}^{*} \boldsymbol{a}_\ell'\boldsymbol{z}_i\boldsymbol{a}_\ell'\boldsymbol{z}_j\boldsymbol{a}_\ell'\boldsymbol{z}_k\right)^2\right. \\
&\quad \left. + \sum_{\ell,\alpha}^{p}{}^{*}\left(\sum_{\ell=1}^{p}\sum_{i,j,k}^{N}{}^{*}\boldsymbol{a}_\ell'\boldsymbol{z}_i\boldsymbol{a}_\ell'\boldsymbol{z}_j\boldsymbol{a}_\ell'\boldsymbol{z}_k\right)\left(\sum_{i,j,k}^{N}{}^{*}\boldsymbol{a}_\alpha'\boldsymbol{z}_i\boldsymbol{a}_\alpha'\boldsymbol{z}_j\boldsymbol{a}_\alpha'\boldsymbol{z}_k\right)\right] \\
&= \frac{4}{N^2(N-1)^2(N-2)^2} E\left[6\sum_{\ell=1}^{p}\sum_{i,j,k}^{N}{}^{*}(\boldsymbol{a}_\ell'\boldsymbol{z}_i)^2(\boldsymbol{a}_\ell'\boldsymbol{z}_j)^2(\boldsymbol{a}_\ell'\boldsymbol{z}_k)^2\right. \\
&\quad \left. + 6\sum_{\ell,\alpha}^{p}{}^{*}\sum_{i,j,k}^{N}{}^{*}\boldsymbol{a}_\ell'\boldsymbol{z}_i\boldsymbol{z}_i'\boldsymbol{a}_\alpha\boldsymbol{a}_\ell'\boldsymbol{z}_j\boldsymbol{z}_j'\boldsymbol{a}_\alpha\boldsymbol{a}_\ell'\boldsymbol{z}_k\boldsymbol{z}_k'\boldsymbol{a}_\alpha\right] \\
&= \frac{24}{N(N-1)(N-2)}\left[\sum_{\ell=1}^{p}(\boldsymbol{a}_\ell'\boldsymbol{a}_\ell)^3 + \sum_{\ell,\alpha}^{p}{}^{*}(\boldsymbol{a}_\ell'\boldsymbol{a}_\alpha)^3\right] \\
&= \frac{24}{N(N-1)(N-2)}\boldsymbol{1}_p'(\odot^3\boldsymbol{A}^2)\boldsymbol{1}_p \\
&= \frac{24}{N(N-1)(N-2)}\boldsymbol{1}_p'(\odot^3\boldsymbol{\Sigma})\boldsymbol{1}_p,
\end{aligned}$$

where the fourth equality follows from Lemma 1. □

B Proof of Theorem 2

In this section, we give a proof of Theorem 2. Before proving it, we prepare three lemmas. The first two lemmas (Lemma 3 and Lemma 4) treat formula about multi-sum, and the last lemma treats the order of the expectations which is used to prove Theorem 2. Since proofs of Lemma 3 and Lemma 4 are tedious but simple, we skipped here.

Lemma 3. *The following equations hold:*

$$\begin{aligned}
\sum_{i,j,s,t}^{p}{}^{*}\sigma_{ij}^2\sigma_{st}^2\sigma_{is}\sigma_{jt} &= \mathrm{tr}\{(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\}^2 - \mathrm{tr}\,\boldsymbol{D}\boldsymbol{\Sigma}(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\boldsymbol{D} - \mathrm{tr}\,\boldsymbol{D}(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}(\odot^2\boldsymbol{\Sigma}) - \boldsymbol{1}_p'(\odot^3\boldsymbol{\Sigma})^2\boldsymbol{1}_p \\
&\quad + 2\boldsymbol{1}_p'\{\boldsymbol{D}^3(\odot^3\boldsymbol{\Sigma})\}\boldsymbol{1}_p - \sum_{i,j,s}^{p}{}^{*}\sigma_{ij}^3\sigma_{is}^3 - \sum_{i,j,s}^{p}{}^{*}\sigma_{ij}^2\sigma_{is}\sigma_{jj}\sigma_{js}^2 - \sum_{i,j,s}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^2\sigma_{is}^2\sigma_{js}, \\
\sum_{i,j,s,t}^{p}{}^{*}\sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js} &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js} - \mathrm{tr}\,\boldsymbol{D}(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}(\odot^2\boldsymbol{\Sigma}) \\
&\quad - 2\,\mathrm{tr}\,\boldsymbol{D}(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}(\odot^2\boldsymbol{\Sigma}) + 2\boldsymbol{1}_p'\{\boldsymbol{D}^3(\odot^3\boldsymbol{\Sigma})\}\boldsymbol{1}_p - 3\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^2\sigma_{is}^2\sigma_{js}.
\end{aligned}$$

Lemma 4. *The following equations hold:*

$$\sum_{i,j,s}^{p}{}^{*}\sigma_{ij}^3\sigma_{is}^3 = \mathbf{1}_p'(\odot^3\mathbf{\Sigma})^2\mathbf{1}_p - \mathbf{1}_p'\{\boldsymbol{D}^3(\odot^3\mathbf{\Sigma})\}\mathbf{1}_p - \sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{ij}^3 - \sum_{i,j}^{p}{}^{*}\sigma_{ij}^6,$$

$$\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}\sigma_{is}\sigma_{js}^2 = \mathrm{tr}\,\boldsymbol{D}\mathbf{\Sigma}(\odot^2\mathbf{\Sigma})\mathbf{\Sigma}\boldsymbol{D} - \mathbf{1}_p'\{\boldsymbol{D}^3(\odot^3\mathbf{\Sigma})\}\mathbf{1}_p - \sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{ij}^3 - \sum_{i,j}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}^2\sigma_{jj}^2,$$

$$\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^2\sigma_{is}^2\sigma_{js} = \mathrm{tr}\,\boldsymbol{D}(\odot^2\mathbf{\Sigma})\mathbf{\Sigma}(\odot^2\mathbf{\Sigma}) - \mathbf{1}_p'\{\boldsymbol{D}^3(\odot^3\mathbf{\Sigma})\}\mathbf{1}_p - \sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{ij}^3 - \sum_{i,j}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^4\sigma_{jj}.$$

Lemma 5. *Let*

$$Q_{k\ell\alpha\beta\gamma\delta} = \sum_{i=1}^{p}\sum_{j=1}^{p}\boldsymbol{a}_i'\boldsymbol{z}_k\boldsymbol{z}_\ell\boldsymbol{a}_j\boldsymbol{a}_i\boldsymbol{z}_\alpha\boldsymbol{z}_\beta\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{z}_\gamma\boldsymbol{z}_\delta'\boldsymbol{a}_j,$$

where $\mathbf{\Sigma}^{1/2} = \boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)'$, *and* $\boldsymbol{z}_k$, $\boldsymbol{z}_\ell$, $\boldsymbol{z}_\alpha$, $\boldsymbol{z}_\beta$, $\boldsymbol{z}_\gamma$ *and* $\boldsymbol{z}_\delta$ *are i.i.d. as* $N_p(\mathbf{0}, \boldsymbol{I}_p)$. *Then,*

$$E[Q_{112233}^2] = O\left(\max\left\{\{\mathbf{1}_p'(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2, \left|\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js}\right|\right\}\right), \tag{11}$$

$$E[Q_{112234}^2] = O\left(\max\left\{\{\mathbf{1}_p'(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2, \left|\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js}\right|\right\}\right), \tag{12}$$

$$E[Q_{112345}^2] = O\left(\{\mathbf{1}_p'(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2\right), \tag{13}$$

$$E[Q_{123456}^2] = O\left(\{\mathbf{1}_p'(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2\right). \tag{14}$$

In addition, under the assumption C,

$$E[Q_{112233}^2] = O\left(\{\mathbf{1}_p'(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2\right), \quad E[Q_{112234}^2] = O\left(\{\mathbf{1}_p'(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2\right).$$

Proof. Since (12), (13) and (14) can be proved by using the same way to show (11), we only show this, and omit other three here. It can be expressed that

$$\begin{aligned}
E[Q_{112233}^2] = E\Bigg[&\sum_{i=1}^{p}(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_i)^2\\
&+ \sum_{i,j}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_i)(\boldsymbol{a}_j'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_j\boldsymbol{a}_j'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_j\boldsymbol{a}_j'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_j)\\
&+ 2\sum_{i,j}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_j)^2\\
&+ 4\sum_{i,j,s}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_j)(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_s\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_s\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_s)\\
&+ \sum_{i,j,s,t}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_j)(\boldsymbol{a}_s'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_t\boldsymbol{a}_s'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_t\boldsymbol{a}_s'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_t)\\
&+ 4\sum_{i,j}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_i)(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_s\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_s\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_s)\\
&+ 2\sum_{i,j}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_i)(\boldsymbol{a}_j'\boldsymbol{z}_1\boldsymbol{z}_1\boldsymbol{a}_s\boldsymbol{a}_j'\boldsymbol{z}_2\boldsymbol{z}_2\boldsymbol{a}_s\boldsymbol{a}_j'\boldsymbol{z}_3\boldsymbol{z}_3\boldsymbol{a}_s)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{p}\{3(\boldsymbol{a}_i'\boldsymbol{a}_i)^2\}^3 + \sum_{i,j}^{p}{}^{*}\{\boldsymbol{a}_i'\boldsymbol{a}_i\boldsymbol{a}_j'\boldsymbol{a}_j + 2(\boldsymbol{a}_i'\boldsymbol{a}_j)^2\}^3 + 2\sum_{i,j}^{p}{}^{*}\{\boldsymbol{a}_i'\boldsymbol{a}_i\boldsymbol{a}_j'\boldsymbol{a}_j + 2(\boldsymbol{a}_i'\boldsymbol{a}_j)^2\}^3 \\
&\quad + 4\sum_{i,j,s}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{a}_i\boldsymbol{a}_j'\boldsymbol{a}_s + 2\boldsymbol{a}_i'\boldsymbol{a}_j\boldsymbol{a}_i'\boldsymbol{a}_s)^3 + \sum_{i,j,s,t}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{a}_j\boldsymbol{a}_s'\boldsymbol{a}_t + \boldsymbol{a}_i'\boldsymbol{a}_s\boldsymbol{a}_j'\boldsymbol{a}_t + \boldsymbol{a}_i'\boldsymbol{a}_t\boldsymbol{a}_j'\boldsymbol{a}_s)^3 \\
&\quad + 4\sum_{i,j}^{p}{}^{*}(3\boldsymbol{a}_i'\boldsymbol{a}_i\boldsymbol{a}_i'\boldsymbol{a}_j)^3 + 2\sum_{i,j,s}^{p}{}^{*}(\boldsymbol{a}_i'\boldsymbol{a}_i\boldsymbol{a}_j'\boldsymbol{a}_s + 2\boldsymbol{a}_i'\boldsymbol{a}_j\boldsymbol{a}_j'\boldsymbol{a}_s)^3 \\
&= 27\sum_{i=1}^{p}\sigma_{ii}^6 + 3\sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{jj}^3 + 18\sum_{i,j}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}^2\sigma_{jj}^2 + 36\sum_{i,j}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^4\sigma_{jj} + 24\sum_{i,j}^{p}{}^{*}\sigma_{ij}^6 \\
&\quad + 108\sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{ij}^3 + 48\sum_{i,j,s}^{p}{}^{*}\sigma_{ij}^3\sigma_{is}^3 + 6\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}^3\sigma_{js}^3 + 36\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}\sigma_{is}\sigma_{js}^2 \\
&\quad + 72\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^2\sigma_{is}^2\sigma_{js} + 3\sum_{i,j,s,t}^{p}{}^{*}\sigma_{ij}^3\sigma_{st}^3 + 18\sum_{i,j,s,t}^{p}{}^{*}\sigma_{ij}^2\sigma_{st}^2\sigma_{is}\sigma_{jt} + 6\sum_{i,j,s,t}^{p}{}^{*}\sigma_{ij}\sigma_{is}\sigma_{it}\sigma_{js}\sigma_{jt}\sigma_{st} \\
&= 3\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2 + 24\sum_{i=1}^{p}\sigma_{ii}^6 + 18\sum_{i,j}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}^2\sigma_{jj}^2 + 36\sum_{i,j}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^4\sigma_{jj} + 18\sum_{i,j}^{p}{}^{*}\sigma_{ij}^6 \\
&\quad + 96\sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{ij}^3 + 36\sum_{i,j,s}^{p}{}^{*}\sigma_{ij}^3\sigma_{is}^3 + 36\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}\sigma_{is}\sigma_{js}^2 + 72\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}\sigma_{js}\sigma_{ij}^2\sigma_{is}^2 \\
&\quad + 18\sum_{i,j,s,t}^{p}{}^{*}\sigma_{ij}^2\sigma_{st}^2\sigma_{is}\sigma_{jt} + 6\sum_{i,j,s,t}^{p}{}^{*}\sigma_{ij}\sigma_{is}\sigma_{it}\sigma_{js}\sigma_{jt}\sigma_{st} \\
&= 3\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2 + 18\operatorname{tr}\{(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\}^2 - 18\operatorname{tr}\boldsymbol{D}\boldsymbol{\Sigma}(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\boldsymbol{D} - 36\operatorname{tr}\boldsymbol{D}(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}(\odot^2\boldsymbol{\Sigma}) \\
&\quad - 18\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})^2\mathbf{1}_p + 48\mathbf{1}_p'\{\boldsymbol{D}^3(\odot^3\boldsymbol{\Sigma})\}\mathbf{1}_p + 24\sum_{i=1}^{p}\sigma_{ii}^6 + 18\sum_{i,j}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}^2\sigma_{jj}^2 + 36\sum_{i,j}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^4\sigma_{jj} \\
&\quad + 18\sum_{i,j}^{p}{}^{*}\sigma_{ij}^6 + 96\sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{ij}^3 + 18\sum_{i,j,s}^{p}{}^{*}\sigma_{ij}^3\sigma_{is}^3 + 18\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}^2\sigma_{ij}\sigma_{is}\sigma_{js}^2 + 36\sum_{i,j,s}^{p}{}^{*}\sigma_{ii}\sigma_{ij}^2\sigma_{is}^2\sigma_{js} \\
&\quad + 6\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sigma_{ij}\sigma_{is}\sigma_{it}\sigma_{js}\sigma_{jt}\sigma_{st} \\
&= 3\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2 + 18\operatorname{tr}\{(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\}^2 - 24\mathbf{1}_p'\{\boldsymbol{D}^3(\odot^3\boldsymbol{\Sigma})\}\mathbf{1}_p + 24\sum_{i=1}^{p}\sigma_{ii}^6 + 24\sum_{i,j}^{p}{}^{*}\sigma_{ii}^3\sigma_{ij}^3 \\
&\quad + 6\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sigma_{ij}\sigma_{is}\sigma_{it}\sigma_{js}\sigma_{jt}\sigma_{st} \\
&= 3\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2 + 18\operatorname{tr}\{(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\}^2 + 6\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sigma_{ij}\sigma_{is}\sigma_{it}\sigma_{js}\sigma_{jt}\sigma_{st},
\end{aligned}
$$

where the second equality follows from Lemma 1, the third equality from bottom follows from Lemma 3 and the second equality from bottom follows from Lemma 4. Since $\odot^2\boldsymbol{\Sigma}$ is positive definite, we have

$$
\operatorname{tr}\{(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\}^2 \le \{\operatorname{tr}(\odot^2\boldsymbol{\Sigma})\boldsymbol{\Sigma}\}^2,
$$

where the right-hand side of the inequality can be written as $\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2$. From this result, it is found that

$$
E[Q_{112233}^2] = O\left(\max\left\{\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2, \left|\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{s=1}^{p}\sum_{t=1}^{p}\sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js}\right|\right\}\right).
$$

Note that

$$\begin{aligned}\sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js} &\le \sigma_{ij}\sigma_{st}(\sqrt{|\sigma_{is}|})^2(\sqrt{|\sigma_{jt}|})^2\sigma_{js}\sigma_{it}\\ &\le \frac{\sigma_{ij}^2\sigma_{st}^2|\sigma_{is}||\sigma_{jt}| + \sigma_{sj}^2\sigma_{ti}^2|\sigma_{is}||\sigma_{jt}|}{2},\end{aligned}$$

From this inequality, we have

$$\begin{aligned}\left|\sum_{i=1}^p\sum_{j=1}^p\sum_{s=1}^p\sum_{t=1}^p \sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js}\right| &\le \sum_{i=1}^p\sum_{j=1}^p\sum_{s=1}^p\sum_{t=1}^p |\sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js}|\\ &\le \frac{1}{2}\left[\mathrm{tr}\{(\odot^2\boldsymbol{\Sigma}_+)\boldsymbol{\Sigma}_+\}^2 + \mathrm{tr}\{(\odot^2\boldsymbol{\Sigma}_+)\boldsymbol{\Sigma}_+\}^2\right]\\ &\le \left[\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma}_+)\mathbf{1}_p\right]^2.\end{aligned}$$

Hence under the assumption C,

$$\sum_{i=1}^p\sum_{j=1}^p\sum_{s=1}^p\sum_{t=1}^p \sigma_{ij}\sigma_{st}\sigma_{is}\sigma_{jt}\sigma_{it}\sigma_{js} = O\left(\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2\right),$$

and so

$$E[Q_{112233}^2] = O\left(\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2\right).$$

□

Proof of Theorem 2. Since S is unbiased estimator of $\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p$, it is sufficient to show that

$$\mathrm{Var}(S/\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p) \to 0 \tag{15}$$

as $N, p \to \infty$. Since S is invariant under location shift, without loss of generality, we may assume that $\boldsymbol{\mu} = \mathbf{0}$. Each of A, B, C and D in (9) can be written as

$$\begin{aligned}A &= \sum_{k,\ell,\alpha}^{N}{}^{*}\mathbf{1}_p'\{(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_k\boldsymbol{z}_k'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\ell\boldsymbol{z}_\ell'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\alpha\boldsymbol{z}_\alpha'\boldsymbol{\Sigma}^{1/2})\}\mathbf{1}_p,\\ B &= \sum_{k,\ell,\alpha,\beta}^{N}{}^{*}\mathbf{1}_p'\{(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_k\boldsymbol{z}_k'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\ell\boldsymbol{z}_\ell'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\alpha\boldsymbol{z}_\beta'\boldsymbol{\Sigma}^{1/2})\}\mathbf{1}_p,\\ C &= \sum_{k,\ell,\alpha,\beta,\gamma}^{N}{}^{*}\mathbf{1}_p'\{(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_k\boldsymbol{z}_k'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\ell\boldsymbol{z}_\alpha'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\beta\boldsymbol{z}_\gamma'\boldsymbol{\Sigma}^{1/2})\}\mathbf{1}_p,\\ D &= \sum_{k,\ell,\alpha,\beta,\gamma,\delta}^{N}{}^{*}\mathbf{1}_p'\{(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_k\boldsymbol{z}_\ell'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\alpha\boldsymbol{z}_\beta'\boldsymbol{\Sigma}^{1/2})\odot(\boldsymbol{\Sigma}^{1/2}\boldsymbol{z}_\gamma\boldsymbol{z}_\delta'\boldsymbol{\Sigma}^{1/2})\}\mathbf{1}_p.\end{aligned}$$

From (9), we can express that

$$\begin{aligned}&\left(S/\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p - 1\right)^2\\ &= \left[\left\{\frac{1}{{}_N\mathrm{P}_3}\frac{A}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p} - 1\right\} - \frac{3}{{}_N\mathrm{P}_4}\frac{B}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p} + \frac{3}{{}_N\mathrm{P}_5}\frac{C}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p} - \frac{1}{{}_N\mathrm{P}_6}\frac{D}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right]^2\\ &\le 4\left[\left\{\frac{1}{{}_N\mathrm{P}_3}\frac{A}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p} - 1\right\}^2 + \left(\frac{3}{{}_N\mathrm{P}_4}\right)^2\left(\frac{B}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right)^2 + \left(\frac{3}{{}_N\mathrm{P}_5}\right)^2\left(\frac{C}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right)^2\right.\\ &\quad\left. + \left(\frac{1}{{}_N\mathrm{P}_6}\right)^2\left(\frac{D}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right)^2\right],\end{aligned}$$

where the inequality follows from Jensen inequality. Thus, it is described that

$$
\begin{aligned}
\mathrm{Var}\left(\frac{S}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right) \le 4\Bigg[& E\left[\left\{\frac{1}{{}_N\mathrm{P}_3}\frac{A}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}-1\right\}^2\right]+\left(\frac{3}{{}_N\mathrm{P}_4}\right)^2 E\left[\left(\frac{B}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right)^2\right] \\
& +\left(\frac{3}{{}_N\mathrm{P}_5}\right)^2 E\left[\left(\frac{C}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right)^2\right]+\left(\frac{1}{{}_N\mathrm{P}_6}\right)^2 E\left[\left(\frac{D}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right)^2\right]\Bigg]. \qquad (16)
\end{aligned}
$$

For the first term in the right-hand side of the inequality (16), expanding $\{.\}^2$, and excluding the term whose expectation becomes 0, we have

$$
\begin{aligned}
E\left[\left\{\frac{1}{{}_N\mathrm{P}_3}\frac{A}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}-1\right\}^2\right] &= \left(\frac{1}{{}_N\mathrm{P}_3\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right)^2 E\Bigg[6\sum_{k,\ell,\alpha}^{N}{}^{*}Q_{kk\ell\ell\alpha\alpha}^2+18\sum_{k,\ell,\alpha,\beta}^{N}{}^{*}Q_{kk\ell\ell\alpha\alpha}Q_{kk\ell\ell\beta\beta} \\
&\quad +9\sum_{k,\ell,\alpha,\beta,\gamma}^{N}{}^{*}Q_{kk\ell\ell\alpha\alpha}Q_{kk\beta\beta\gamma\gamma}+\sum_{k,\ell,\alpha,\beta,\gamma,\delta}^{N}{}^{*}Q_{kk\ell\ell\alpha\alpha}Q_{\beta\beta\gamma\gamma\delta\delta}\Bigg] \\
&= \left(\frac{1}{{}_N\mathrm{P}_3}\right)^2[6{}_N\mathrm{P}_3E[A_1]+18{}_N\mathrm{P}_4E[A_2]+9{}_N\mathrm{P}_5E[A_3]] \\
&\quad +\left\{\frac{{}_N\mathrm{P}_6}{({}_N\mathrm{P}_3)^2}-1\right\} \\
&= \left(\frac{1}{{}_N\mathrm{P}_3}\right)^2[6{}_N\mathrm{P}_3E[A_1]+18{}_N\mathrm{P}_4E[A_2]+9{}_N\mathrm{P}_5E[A_3]] \\
&\quad -\frac{3(3N^2-15N+20)}{{}_N\mathrm{P}_3}, \qquad (17)
\end{aligned}
$$

where

$$
E[A_1]=\frac{E[Q_{112233}^2]}{\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2},\quad E[A_2]=\frac{E[Q_{112233}Q_{112244}]}{\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2},\quad E[A_3]=\frac{E[Q_{112233}Q_{114455}]}{\{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p\}^2}.
$$

Using the same calculation method, the second-fourth terms in the right-hand side of the inequality (16) can be written as

$$
\begin{aligned}
E\left[\left\{\frac{1}{{}_N\mathrm{P}_4}\frac{B}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right\}^2\right] &= \left(\frac{1}{{}_N\mathrm{P}_4}\right)^2[2{}_N\mathrm{P}_4E[B_1]+2{}_N\mathrm{P}_4E[B_2]+4{}_N\mathrm{P}_5E[B_3]+4{}_N\mathrm{P}_5E[B_4] \\
&\quad +2{}_N\mathrm{P}_6E[B_5]+2{}_N\mathrm{P}_6E[B_6]], \qquad (18) \\
E\left[\left\{\frac{1}{{}_N\mathrm{P}_5}\frac{C}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right\}^2\right] &= \left(\frac{1}{{}_N\mathrm{P}_5}\right)^2[4{}_N\mathrm{P}_5E[C_1]+8{}_N\mathrm{P}_5E[C_2]+8{}_N\mathrm{P}_5E[C_3]+4{}_N\mathrm{P}_5E[C_4] \\
&\quad +4{}_N\mathrm{P}_6E[C_5]+8{}_N\mathrm{P}_6E[C_6]+8{}_N\mathrm{P}_6E[C_7]+4{}_N\mathrm{P}_6E[C_8]], \qquad (19) \\
E\left[\left\{\frac{1}{{}_N\mathrm{P}_6}\frac{D}{\mathbf{1}_p'(\odot^3\boldsymbol{\Sigma})\mathbf{1}_p}\right\}^2\right] &= \left(\frac{1}{{}_N\mathrm{P}_6}\right)^2[36{}_N\mathrm{P}_6E[D_1]+324{}_N\mathrm{P}_6E[D_2]+324{}_N\mathrm{P}_6E[D_3] \\
&\quad +36{}_N\mathrm{P}_6E[D_4]], \qquad (20)
\end{aligned}
$$

where

$$E[B_1] = \frac{E[Q^2_{112234}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[B_2] = \frac{E[Q_{112234}Q_{112243}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[B_3] = \frac{E[Q_{112234}Q_{115534}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2},$$
$$E[B_4] = \frac{E[Q_{112234}Q_{115543}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[B_5] = \frac{E[Q_{112234}Q_{556634}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[B_6] = \frac{E[Q_{112234}Q_{556643}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2},$$
$$E[C_1] = \frac{E[Q^2_{112345}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[C_2] = \frac{E[Q_{112345}Q_{112354}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[C_3] = \frac{E[Q_{112345}Q_{112435}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2},$$
$$E[C_4] = \frac{E[Q_{112345}Q_{113254}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[C_5] = \frac{E[Q_{112345}Q_{662345}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[C_6] = \frac{E[Q_{112345}Q_{662354}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2},$$
$$E[C_7] = \frac{E[Q_{112345}Q_{662435}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[C_8] = \frac{E[Q_{112345}Q_{663254}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2},$$
$$E[D_1] = \frac{E[Q^2_{123456}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[D_2] = \frac{E[Q_{123456}Q_{123465}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}, \quad E[D_3] = \frac{E[Q_{123456}Q_{124365}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2},$$
$$E[D_4] = \frac{E[Q_{123456}Q_{214365}]}{\{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p\}^2}.$$

From Lemma 5, we have

$$E[A_1] = O(1), \quad E[B_1] = O(1), \quad E[C_1] = O(1), \quad E[D_1] = O(1).$$

It follows from Cauchy-Schwarz inequality that

$$\begin{aligned} |E[A_j]| &\le E[A_1] \qquad (j = 2, 3, 4), \\ |E[B_j]| &\le E[B_1] \qquad (j = 2, \dots, 6), \\ |E[C_j]| &\le E[C_1] \qquad (j = 2, \dots, 8), \\ |E[D_j]| &\le E[D_1] \qquad (j = 2, 3, 4). \end{aligned}$$

From them, we find that

$$\begin{aligned} E[A_j] &= O(1) \qquad (j = 2, 3, 4), \\ E[B_j] &= O(1) \qquad (j = 2, \dots, 6), \\ E[C_j] &= O(1) \qquad (j = 2, \dots, 8), \\ E[D_j] &= O(1) \qquad (j = 2, 3, 4). \end{aligned}$$

Combining these results with (17), (18), (19) and (20), it is found that

$$\begin{aligned} E\left[\left\{\frac{1}{{}_N\mathrm{P}_3}\frac{A}{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p} - 1\right\}^2\right] &= O(N^{-1}), \\ E\left[\left\{\frac{1}{{}_N\mathrm{P}_4}\frac{B}{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p}\right\}^2\right] &= O(N^{-2}), \\ E\left[\left\{\frac{1}{{}_N\mathrm{P}_5}\frac{C}{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p}\right\}^2\right] &= O(N^{-4}), \\ E\left[\left\{\frac{1}{{}_N\mathrm{P}_6}\frac{C}{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p}\right\}^2\right] &= O(N^{-6}), \end{aligned}$$

and so

$$\mathrm{Var}\left(\frac{S}{\mathbf{1}'_p(\odot^3\mathbf{\Sigma})\mathbf{1}_p}\right) = O(N^{-1}),$$

which converges to 0 as $N, p \to \infty$. $\square$

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