Second Order Expansions for Distributions of Statistics and Its Quantiles Based on Random Size Samples

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Abstract:
In practice, we often encounter situations where a sample size is not defined in advance and can be a random value. In the present paper second order Chebyshev–Edgeworth and Cornish–Fisher expansions based of Student's t- and Laplace distributions and their quantiles are derived for distributions of statistics constructed for random size samples of a special kind. We use a general transfer theorem, which allows to construct the expansions for distributions of randomly normalized statistics from the corresponding expansions for distributions of the non-randomly normalized statistics and for a distribution of the random size of the underlying sample. Recently, interest in Cornish-Fisher expansions has increased because of study in risk management. Widespread risk measure Value at Risk (VaR) substantially depends on the quantiles of the loss function, which is connected with description of investment portfolio of financial instruments.

Key words: Chebyshev–Edgeworth Expansions; Cornish–Fisher Expansions; Samples with Random Sizes; Laplace distribution; Student's t-distribution

MSC classification: 60E05, 60G50, 60F05, 62E17

1 Introduction

In classical problems of mathematical statistics sample size is traditionally considered to be deterministic and plays role of a known parameter which is, as a rule, sufficiently large. However, we often encounter situations where a sample size is not defined in advance and can be a random variable. Usually it happens when data are accumulated for a time interval, which duration cannot be considered fixed for different reasons. In this case the problem is raised how to estimate different parameters of the given statistic, for example, the distribution function or quantiles based on random size samples. In Gnedenko (1989) it is demonstrated that asymptotic behaviour of statistics based on random size samples can radically differ from the the properties of non-random sample size statistics. Limit distributions for random indexed sequences and its application are considered e.g. in the monograph Gnedenko and Korolev (1996). In Bening, Galieva and Korolev (2012, 2013) general transfer theorems are proved to obtain

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rates of convergence and first order Edgeworth type expansions of asymptotically normal statistics based on random size samples. Their statements depend on corresponding results for the distributions of the considered non-randomly normalized statistic and of the random size of the underlying sample.

In the present paper results of Bening, Galieva and Korolev (2012, 2013) are refined and extended: second order Chebyshev-Edgeworth and Cornish-Fisher expansions based of Student's t- and Laplace distributions and their quantiles are derived for samples with random size of a special kind. The importance and different practical applications of these limit distributions are discussed e.g. in Bening and Korolev (2005, 2008) and in Schluter and Trede (2016). Classical Cornish-Fisher expansions based on quantiles of the standard normal distribution were introduced in Cornish and Fisher (1937), their generalizations were proposed in Hill and Davis (1968), see also Ulyanov (2011).

Recently, interest in Cornish-Fisher expansions has increased because of study in risk management. For example, widespread risk measure Value at Risk (VaR) substantially depends on the quantiles of the loss function, which is connected with description of investment portfolio of financial instruments, for example, see Jaschke (2002).

We use the following notations and symbols: $\mathbb{R}$ as real numbers, $\mathbb{N} := \{1, 2, ...\}$ as positive integers and $\mathbb{I}_A(x)$ as indicator function.

Let $X, X_1, X_2, \ldots \in \mathbb{R}$ and $N_1, N_2, \ldots \in \mathbb{N}$ be random variables on the same probability space $(\Omega, \mathbb{A}, \mathbb{P})$. In statistics the random variables $X_1, X_2, \ldots, X_m$ are observations with non-random size sample $m \in \mathbb{N}$ and let N_n be random size of the underlying sample, which depends of natural parameter $n \in \mathbb{N}$. We suppose that random variables $N_n \in \mathbb{N}$ with $n \in \mathbb{N}$ are independent of random variables $X_1, X_2, \ldots$.

Let $T_m := T_m(X_1, \ldots, X_m)$ be some statistic of a sample with non-random sample size $m \in \mathbb{N}$. Define the random variable T_{N_n} for every $n \in \mathbb{N}$, supposing that

$$T_{N_n}(\omega) := T_{N_n(\omega)}\left(X_1(\omega), \ldots, X_{N_n(\omega)}\right), \ \ \omega \in \Omega, \tag{1}$$

i.e. T_{N_n} is some statistic obtained from a random sample $X_1, X_2, \ldots, X_{N_n}$ on the basis of statistics T_m but with random sample size N_n.

In the present paper we consider a sequence of independent identically distributed (i.i.d.) random variables $X, X_1, X_2, \ldots$ with

$$\left.\begin{array}{c} \mathbb{E}\,|X|^5 < \infty, \ \ \mathbb{E}(X) = \mu, \ \ 0 < \mathrm{Var}(X) = \sigma^2, \\ \text{skewness}\ \lambda_3 = \sigma^{-3}\,\mathbb{E}\,(X-\mu)^3 \quad \text{and} \ \ \text{kurtosis}\ \lambda_4 = \sigma^{-4}\,\mathbb{E}\,(X-\mu)^4 \ - \ 3 \end{array}\right\} \tag{2}$$

and suppose that random variable X admits Cramér's condition:

$$\limsup_{|t|\to\infty} \left|\mathbb{E} e^{itX}\right| < 1. \tag{3}$$

As statistic T_m we choose the asymptotically normal sample mean

$$T_m = (X_1 + \cdots + X_m)/m\,, \quad m = 1, 2, ..., \tag{4}$$

Then we have, see Petrov (1995, Theorem 5.18 with $k = 5$),

$$\sup_x \left|\mathbb{P}(\sigma^{-1}\sqrt{m}(T_m - \mu) \le x) - \Phi_{2;m}(x)\right| \le C m^{-3/2}, \tag{5}$$

where C does not depend on m and with second order asymptotic expansion

$$\Phi_{2;m}(x) = \Phi(x) - \left(\frac{\lambda_3}{6\sqrt{m}} H_2(x) + \frac{1}{m}\left(\frac{\lambda_4}{24} H_3(x) + \frac{\lambda_3^2}{72} H_5(x)\right)\right)\varphi(x),$$

$\Phi(x)$ and $\varphi(x)$ being distribution function and density of standard normal random variable and Chebyshev-Hermite polynomials

$$H_2(x) = x^2 - 1\,, \quad H_3(x) = x^3 - 3x \quad \text{and} \quad H_5(x) = x^5 - 10x^3 + 15x\,. \tag{6}$$

Consider now the random mean T_{N_n}, based on statistic (4) with a random number N_n of summands $X_1, X_2, ..., X_{N_n}$:

$$T_{N_n} = (X_1 + X_2 + ... + X_{N_n})/N_n, \quad n \in \mathbb{N}. \tag{7}$$

Let g_n be a sequence of positive real numbers: $0 < g_n \uparrow \infty$. Suppose that $N_n \to \infty$ in probability as $n \to \infty$. The limit laws of $\mathbb{P}\Big(\sigma^{-1}\sqrt{g_n}(T_{N_n} - \mu) \le x\Big)$ are scale mixtures of normal distributions with zero mean, depending on the random sample size N_n. In Section 2 we give auxiliary statements to find Chebyshev-Edgeworth and Cornish-Fisher expansions for the normalized random mean T_{N_n}. In Section 3 as random sample size N_n we consider the negative binomial distribution, shifted by 1, with success probability $p = 1/n$. This is one of the leading distributions for count models. Here $g_n = \mathbb{E}N_n$ andN_n/g_n tends to the Gamma distribution and the normalized random mean T_{N_n} to Student's t-distribution. In Section 4 the random size N_n is the maximum of n i.i.d. discrete Pareto random variables with tail parameter 1, where N_n/g_n with $g_n = n$ tends to the reciprocal exponential distribution. In this case, the Laplace law is the limit distribution of the normalized random mean T_{N_n}. In both cases we prove first a second Edgeworth type expansion for N_n/g_n and then second order Chebyshev-Edgeworth and Cornish-Fisher expansions. The proofs are collected in Section 5.

2 Two Auxiliary Transfer Propositions

We suppose that the following condition on the statistic $T_m = T_m\,(X_1, \dots, X_m)$ is satisfied for a non-random sample size $m \in \mathbb{N}$:

Condition 1: *There exist differentiable bounded functions $f_1(x), f_2(x)$ and real numbers $a > 1$, $C_1 > 0$ such that for all integer $m \ge 1$*

$$\sup_x \left|\mathbb{P}\Big(\sigma^{-1}\sqrt{m}(T_m - \mu) \le x\Big) - \Phi(x) - m^{-1/2} f_1(x) - m^{-1} f_2(x)\right| \le C_1 m^{-a}. \tag{8}$$

Consider now the statistic $T_{N_n} = T_{N_n}\,(X_1, \dots, X_{N_n})$ with a random number $N_n = N_n \in \mathbb{N}$ of observations $X_1, X_2, ...X_{N_n}$.

Suppose that distribution functions of the normalized random sample size N_n satisfy the following condition.

Condition 2: *There exist a distribution function $H(y)$ with $H(0+) = 0$, a* *function of bounded variation $h_2(y)$, a sequence $0 < g_n \uparrow \infty$ and real numbers $b > 1$ and $C_2 > 0$ such that for all integer $n \geq 1$*

$$\sup_{y \geq 0} \left| \mathbb{P}\big(g_n^{-1} N_n \leq y\big) - H(y) - n^{-1} h_2(y) \right| \leq C_2 n^{-b}. \tag{9}$$

Proposition 1. *Let $X, X_1, X_2, \dots$ be i.i.d. random variables satisfying* (2) *and Cramér's condition* (3)*, i.e. due to* (5) *Condition 1 holds and* (8) *is satisfied with $f_1(x) = -\dfrac{\lambda_3}{6} H_2(x)\varphi(x)$, $f_2(x) = -\Big(\dfrac{\lambda_4}{24} H_3(x) + \dfrac{\lambda_3^2}{72} H_5(x)\Big)\varphi(x)$ and $a = 3/2$.*

Suppose for the random sample size N_n Condition 2 with (9) *is complied with the additional assumptions*

$$h_2(0) = 0, \;\; H(g_n^{-1}) \leq c_0\, n^{-\gamma} \;\; \textit{and} \;\; h_2(g_n^{-1}) \leq c_1\, n^{1-\gamma} \;\; \textit{for some} \;\; \gamma > 1\,, \tag{10}$$

then there exists a constant $C_3 = C_3(\lambda_3, \lambda_4, C_2) > 0$ such that $\forall n \in \mathbb{N}$

$$\sup_x \left| \mathbb{P}\big(\sigma^{-1}\sqrt{g_n}(T_{N_n} - \mu) \leq x\big) - G_{2,n}(x) \right| \leq C_1 \mathbb{E}(N_n^{-a}) + C_3 n^{-\min\{b,\gamma\}},$$

where

$$\begin{aligned} G_{2;n}(x) \;\; &= \;\; \int_0^\infty \Phi(x\sqrt{y})dH(y) + \frac{1}{\sqrt{g_n}} \int_0^\infty \frac{f_1(x\sqrt{y})}{\sqrt{y}} dH(y) \\ &\quad + \frac{1}{g_n} \int_0^\infty \frac{f_2(x\sqrt{y})}{y} dH(y) + \frac{1}{n} \int_0^\infty \Phi(x\sqrt{y}) dh_2(y)\,. \end{aligned} \tag{11}$$

Similar result is proved in Bening, Galieva and Korolev (2013, Theorem 3.1). However our Proposition 1 has more simple form of $G_{2;n}(x)$ which is more suitable for applications.

Let $F_n(x)$ be a sequence of distribution functions and each of it admit a Chebyshev-Edgeworth expansion in powers of $g_n^{-1/2}$ with $0 < g_n \uparrow \infty$ as $n \to \infty$:

$$F_n(x) = G(x) + g(x)\left(a_1(x) g_n^{-1/2} + a_2(x) g_n^{-1}\right) + R(g_n), \;\; R(g_n) = \mathcal{O}(g_n^{-1}) \tag{12}$$

if $n \to \infty$, where $g(x)$ is the density of the continuous limit distribution $G(x)$.

Proposition 2. *Let $F_n(x)$ be given by* (12) *and let $x(u)$ and u be quantiles of distributions F_n and G with the same order, i.e. $F_n(x(u)) = G(u)$. Then the following expansion occurs:*

$$x(u) = u + b_1(u) g_n^{-1/2} + b_2(u) g_n^{-1} + R^*(g_n), \;\; R^*(g_n) = \mathcal{O}(g_n^{-1}), \quad n \to \infty,$$

with

$$b_1(u) = -a_1(u) \quad \textit{and} \quad b_2(u) = \frac{g'(u)}{2\,g(u)} a_1^2(u) + a_1'(u) a_1(u) - a_2(u)\,.$$

Proposition 2 is a direct consequence of more general statements, see e.g. Fujikoshi, Ulyanov and Shimizu (2010, Chapter 5.6.1) or Ulyanov, Aoshima and Fujikoshi (2016).

3 Chebyshev-Edgeworth and Cornish-Fisher Expansions with Student's Limit Distribution

Student's t-distribution function $S_\nu(x)$ is an absolutely continuous probability distribution function given by the density

$$s_\nu(x) = \frac{\Gamma\big((\nu+1)/2\big)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}, \quad x \in \mathbb{R}, \tag{13}$$

where $\nu > 0$ is a real shape parameter. If the parameter ν is positive integer, then it is called the number of degrees of freedom. Student's t-distributions are the limit laws for the statistics T_{N_n} given in (1), if the statistic $T_m := T_m(X_1, \ldots, X_m)$ is asymptotic normal, the random variables $X_1, X_2, \ldots$ have finite variances and the sample sizes N_n of are drawn from a negative binomial distribution, independent of $X_1, X_2, \ldots$, and $N_n \to \infty$ in probability if $n \to \infty$.

The sample size $N_n(r)$ is negative binomial distributed (shifted by 1) with success probability $1/n$, having probability mass function

$$\mathbb{P}(N_n(r) = j) = \frac{\Gamma(j+r-1)}{(j-1)!\,\Gamma(r)}\left(\frac{1}{n}\right)^r\left(1-\frac{1}{n}\right)^{j-1}, j = 1, 2, \ldots\; r > 0. \tag{14}$$

Since $N_n(r) \in \{1, 2, \ldots\}$ the random mean $T_{N_n(r)}$ is well-defined. If the parameter r is positive integer, then $r \geq 1$ is the predefined number of successes and $N_n(r) - 1$ is the random number of failures until the experiment is stopped. Moreover, for fixed $k \in \mathbb{N}$, see formula (5.31) in Johnson, Kemp and Kotz (2005, p. 218) with renamed notations, the distribution function is

$$\mathbb{P}(N_n(r) \leq k) = \sum\nolimits_{j=1}^{k} \frac{\Gamma(j+r-1)}{(j-1)!\,\Gamma(r)}\left(\frac{1}{n}\right)^r\left(1-\frac{1}{n}\right)^{j-1} = \frac{B_{1/n}(r,k)}{B(r,k)} \tag{15}$$

with beta function $B(r,k) = \Gamma(k)\,\Gamma(r)/\Gamma(k+r)$ and incomplete beta function

$$B_{1/n}(r,k) = \int_0^{1/n} u^{r-1}\,(1-u)^{k-1} du \overset{u=t/(1+t)}{=} \int_0^{1/(n-1)} \frac{t^{r-1}}{(1+t)^{k+r}} dt. \tag{16}$$

Define $g_n := \mathbb{E}(N_n(r)) = r\,(n-1) + 1$ then

$$\sup\nolimits_{x>0} |\mathbb{P}(N_n(r)/\mathbb{E}(N_n(r)) \leq x) - G_{r,r}(x)| \to 0 \quad \text{as} \quad n \to \infty,$$

where $G_{\alpha,\beta}(x)$ is the gamma distribution function with the shape $\alpha > 0$ and rate $\beta > 0$, having density

$$g_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,\mathbb{I}_{(0\,,\,\infty)}(x), \quad x \in \mathbb{R}. \tag{17}$$

For an asymptotically normal random mean T_m the limit distribution of $\mathbb{P}\left(\sigma^{-1}\sqrt{r(n-1)+1}(T_{N_n(r)} - \mu) \leq x\right)$ is Student's t-distribution $S_{2r}(x)$ having density function (13) with shape parameter $\nu = 2r$, see Bening and Korolev (2005, Corollary 2.1), or Schluter and Trede (2016, Theorem 1). Bening, Galieva and Korolev (2012, 2013) proved under more moment conditions rates of convergence and first order Edgeworth type expansions of asymptotically normal statistics T_m based on samples with random size $N_n(r)$. In order to obtain second order Edgeworth-type expansion for the normalized $T_{N_n(r)}$ we have to prove second order Edgeworth-type expansion for $N_n(r)/\mathbb{E}(N_n(r))$, see (9) in Condition 2 above. Since the limit Gamma distribution $G_{r,r}(x)$ is a continuous function and $\mathbb{P}(N_n(r)/\mathbb{E}(N_n(r)) \leq x)$ is a step function, it is necessary to add a discontinuous function to overcome the jumps..

Theorem 1. *Let $r > 1$ and the discrete random variable $N_n(r)$ have probability mass function* (14) *and $\mathbb{E}N_n(r) = r(n-1)+1$. For $x > 0$, and all $n \in \mathbb{N}$ there exists a real number $C_2(r) > 0$ such that*

$$\sup_{x\geq 0}\left|\mathbb{P}\left(\frac{N_n(r)}{r(n-1)+1} \leq x\right) - G_{2;n}(x)\right| \leq C_2(r)\, n^{-\min\{r,2\}}, \qquad (18)$$

where

$$G_{2;n}(x) = G_{r,r}(x) + \frac{a_0 G_{r,r}(x) + a_1 G_{r+1,r}(x) + a_2 G_{r+2,r}(x)}{2\,x\,n}\, \mathbb{I}_{[2,\infty)}(n) \qquad (19)$$

$$= G_{r,r}(x) + \frac{g_{r,r}(x)\big((x-1)(2-r) + 2Q_1\big((r(n-1)+1)x\big)\big)}{2\,r\,n}\, \mathbb{I}_{[2,\infty)}(n), \qquad (20)$$

$$a_1 = -2(x+r) + 1 - 2Q_1\big((r(n-1)+1)x\big),\ a_2 = r+1,\ a_0 = -(a_1+a_2),$$

$Q_1(y) = 1/2-(y-[y])$ *and $[y]$ is the integer part of y with $y-1 < [y] \leq y$.* (21)

Remark 1. Formula (20) shows that (9) is satisfied with $H(x) = G_{r,r}(x)$, $h_2(x) = \big((x-1)\,(2-r) + 2Q_1\big(r(n-1)+1)x\big)\big)g_{r,r}(x)\mathbb{I}_{[2,\infty)}(n)/(2r)$ for $x > 0$, $b = \min\{r,2\}$ and $g_n = r(n-1)+1$.

Figure 1 shows the approximation of $\mathbb{P}\left(N_n(r) \leq (r(n-1)+1)x\right)$ by $G_{2,2}(x)$ and $G_{2,2}(x) + h_2(x)/n$.

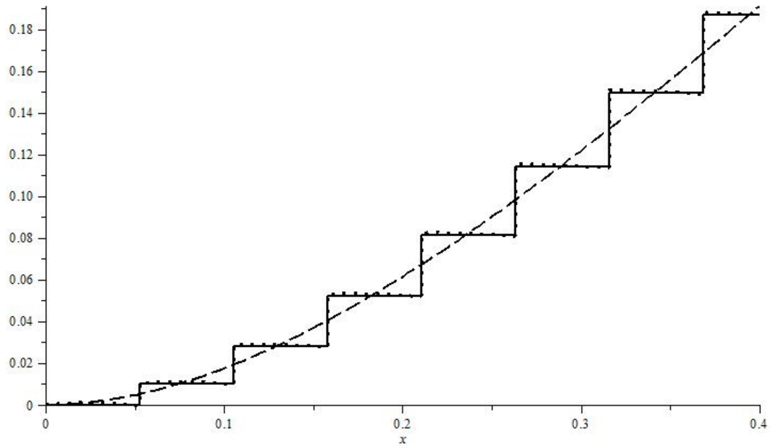

Figure 1: Distribution function $\mathbb{P}\left(N_n(r) \leq (r(n-1)+1)x\right)$ (solid line), the limit law $G_{2,2}(x)$ (dashed line) and the second approximation $G_{2,2}(x)+h_2(x)/n$ (doted line) with $n = 10$ and $r = 2$

Remark 2. Coefficients a_0, a_1, a_2 in (19) meet condition $a_0 + a_1 + a_2 = 0$. Ulyanov, Aoshima and Fujikoshi (2016) mention a wide class of statistics allowing similar representation like (19).
Remark 3. Integration by parts allows the passage from (19) to (20) since

$$G_{r+1,r}(x) = -(x/r)g_{r,r}(x) + G_{r,r}(x) \tag{22}$$

and

$$G_{r+2,r}(x) = -\big(x^2/(r+1) + x/r\big)g_{r,r}(x) + G_{r,r}(x).$$

In the following case the bound (18) is trivial:
Remark 4. If $n = 1$ then $\mathbb{P}\big(N_1(r) = 1\big) = 1$ and the left-hand side in (18) takes its maximum at $x = 1$ leading to $C_2(r) = \min\{G_{r,r}(1), 1 - G_{r,r}(1)\} < 1$.

In addition to the expansion of $N_n(r)$ a bound of $\mathbb{E}(N_n(r))^{-a}$ is required, where m^{-a} is rate of convergence of Edgeworth expansion for T_m, see (8).

Lemma 1. *Let $r > 1$. The random variable $N_n(r)$ is defined by* (14)*, then*

$$\mathbb{E}\big(N_n(r)\big)^{-3/2} \le C(r) \begin{cases} n^{-r}, & 1 < r < 3/2 \\ \ln(n)\, n^{-3/2}, & r = 3/2 \\ n^{-3/2}, & r > 3/2 \end{cases} \tag{23}$$

In case of $r = 3/2$ the convergence rate in (23) *cannot be improved.*

Now we present the second order Chebyshev-Edgeworth expansion for the standardized random mean $T_{N_n(r)}$.

Theorem 2. *Let $X, X_1, X_2, \ldots$ be i.i.d. random variables, where X satisfies* (2) *and Cramér's condition* (3)*. Let the discrete random variable $N_n = N_n(r)$ with parameter $r > 1$ have probability mass function* (14)*, being independent of $X_1, X_2, \ldots$. Consider the statistic $T_{N_n} = N_n^{-1}(X_1 + \ldots + X_{N_n})$. For the asymptotically normal statistics T_m the asymptotic expansion* (5) *and for the random size $N_n(r)$ with $r > 1$ the asymptotic expansion* (18) *hold with $g_n = \mathbb{E}\big(N_n(r)\big) = r(n-1)+1$. Then there exists a constant $C = C(r) > 0$ such that*

$$\sup_x \left|\mathbb{P}\left(\sigma^{-1}\sqrt{g_n}(T_{N_n} - \mu) \le x\right) - S_{2r;n}(x)\right| \le C \begin{cases} n^{-r}, & 1 < r < 3/2 \\ \ln(n)\, n^{-3/2}, & r = 3/2 \\ n^{-3/2}, & r > 3/2 \end{cases} \tag{24}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} S_{2r;n}(x) &= S_{2r}(x) - \frac{\lambda_3\,\big((r-1)x^2 - r\big)\, s_{2r}(x)}{3\,(2\,r-1)\sqrt{g_n}} \\ &\quad - \frac{x\, s_{2r}(x)}{36\,(2r-1)\,g_n}\Bigg\{\frac{2\,\lambda_3^2\,\big((r-2)(r-3)x^4 + 10\,r\,(2-r)x^2 + 15r^2\big)}{2r + x^2} \\ &\quad + 3\,\lambda_4\,\big((r-2)x^2 - 3r\big) + 9(r-2)(x^2+1)\Bigg\} \end{aligned}$$

In case of $r = 3/2$ the convergence rate in (24) *cannot be improved.*

Figure 2 shows the advantage of the Chebyshev-Edgeworth expansion versus the limit law in approximating the empirical distribution function.

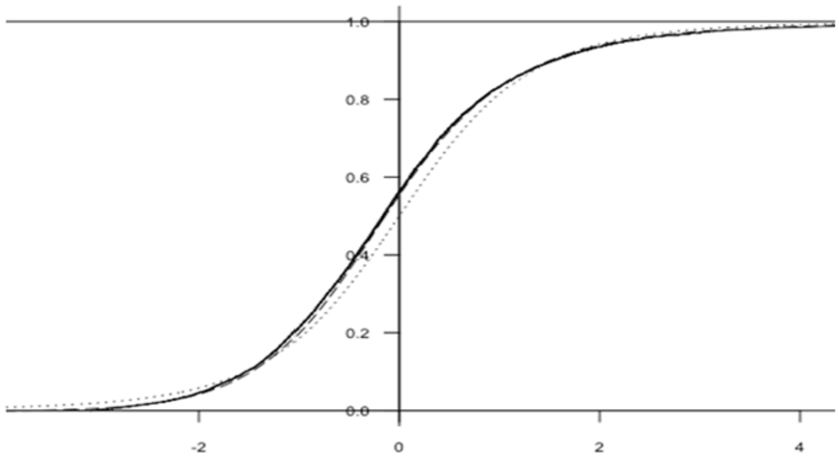

Figure 2: Empirical distribution function $\mathbb{P}\left(\sigma^{-1}\sqrt{g_n}(T_{N_n}-\mu)\le x\right)$ (solid line), limit Student law $S_{2r}(x)$ (doted line) and second approximation $S_{2r;n}(x)$ (dashed line) for random mean of $N_{10}(2)$ independent χ_1^2 random variables with $\lambda_3=\sqrt{8}$ and $\lambda_4=12$

Remark 5. Bening, Galieva and Korolev (2013) proved

$$\sup_x\left|\mathbb{P}\left(\sigma^{-1}\sqrt{g_n}(T_{N_n}-\mu)\le x\right)-S_{2r}(x)+\frac{\lambda_3\ f_r(x)}{6\sqrt{g_n}}\right|\le C(r)\,R(n)$$

with $R(n)=n^{-\min\{r,1\}}$ for $r>1/2, r\neq 1$ or $R(n)=n^{-1}\ln(n)$ for $r=1$ and

$$f_r(x)=\int_0^\infty\frac{(1-x^2y)}{\sqrt{y}}\varphi(x\sqrt{y})\,dS_{2r}(y)=\frac{2\left((r-1)x^2-r\right)\ s_{2r}(x)}{(2\,r-1)}.$$

The latter integral was calculated in Markov, Monakhov and Ulyanov (2016), where also a first order Cornish-Fisher expansion was presented (with some computational inaccuracies).

Using the second order Chebyshev-Edgeworth expansion in Theorem 2 and transfer Proposition 2 we obtain the following statement:

Theorem 3. *Let $x=x_\alpha$ and $u=u_\alpha$ be α-quantiles of standardized statistic $\sigma^{-1}\sqrt{r(n-1)+1}\left(T_{N_n(r)}-\mu\right)$ and of the limit Student's t-distribution $S_{2r}(x)$, respectively. Then for $u\neq 0$ with previous definitions the following asymptotic expansion holds*

$$x=u+\frac{\lambda_3\ \left((r-1)u^2-r\right)}{3\,(2\,r-1)\sqrt{g_n}}+\frac{1}{g_n}B_2(u)+\begin{cases}\mathcal{O}(n^{-r}), & 1<r<3/2\\ \mathcal{O}(\ln(n)\,n^{-3/2}), & r=3/2\\ \mathcal{O}(n^{-3/2}), & r>3/2\end{cases}$$

as $n \to \infty$, *where*

$$\begin{aligned} B_2(u) &= -\frac{(2r+1)u}{2(2r+u^2)}\,\frac{\lambda_3^2\big((r-1)u^2-r\big)^2}{9\,(2r-1)^2}+\frac{2\,\lambda_3^2(r-1)u\big((r-1)u^2-r\big)}{9\,(2r-1)^2} \\ &\quad+\frac{u}{36\,(2r-1)}\Bigg\{\frac{2\,\lambda_3^2\,\big((r-2)(r-3)u^4+10\,r\,(2-r)u^2+15r^2\big)}{2r+u^2} \\ &\quad+3\,\lambda_4\,\big((r-2)u^2-3r\big)+9(r-2)(u^2+1)\Bigg\} \end{aligned}$$

Figure 3 shows a Q–Q plot comparing the empirical quantiles of a randomly generated standardized random mean on the horizontal axis to the quantiles based on its Cornish-Fisher approximation on the vertical axis.

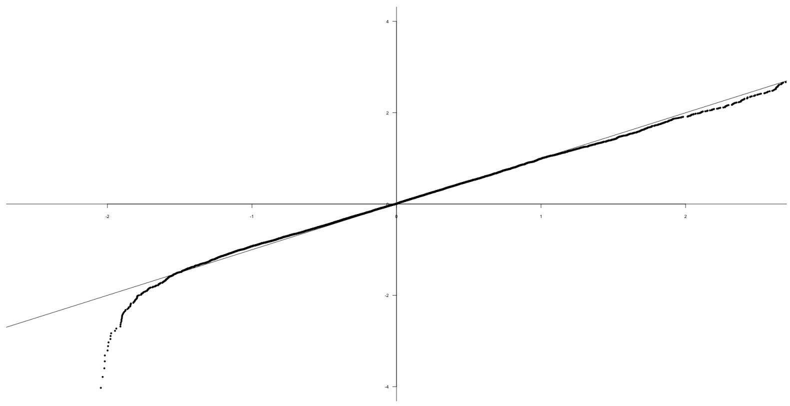

Figure 3: QQ-plot for quantiles of Cornish-Fisher approximation against the quantiles of the empirical quantiles of a standardized random mean of $N_{10}(2)$ independent χ^2_1 random variables with $\lambda_3 = \sqrt{8}$ and $\lambda_4 = 12$

Remark 6. Since $g_n = r(n-1)+1$ and $|g_n^{-\gamma} - (rn)^{-\gamma}| \le C(r)n^{-\gamma-1}$, the factors $g_n^{-1/2}$ and g_n^{-1} in both Chebyshev-Edgeworth and Cornish-Fisher expansions may be replaced by $(rn)^{-1/2}$ and $(rn)^{-1}$, respectively.

4 Chebyshev-Edgeworth and Cornish-Fisher Expansions with Laplace Limit Distribution

Let $Y(s)$ be discrete Pareto II distributed with parameter $s > 0$ and probability mass function

$$\mathbb{P}(Y(s) = k) = \frac{s}{s+k-1} - \frac{s}{s+k}, \quad k \in \mathbb{N} = \{1, 2, ...\}, \quad s > 0, \tag{25}$$

which is a particular class of a more general model of discrete Pareto distributions, obtained by discretization continuous Pareto II (Lomax) distributions on integers, considered e.g. in Buddana and Kozubowski (2014), where also main properties and characteristics of discrete Pareto distributions are discussed.

Let $s \geq 1$ be integer, $Z_1, Z_2, ...$ be independent and identically distributed random variables with some continuous distribution function and

$$Y(s) = \min\{k \geq 1 : \max\nolimits_{1 \leq j \leq s} Z_j < \max\nolimits_{s+1 \leq j \leq s+k} Z_j\},$$

then $Y(s)$ has probability mass function (25). The random variable $Y(s)$ plays an important part in the theory of records and describes the number of further trials required to obtain the next extreme observation, see e.g. Wilks (1959) or Bening and Korolev (2008).

The telescoping nature of the probabilities in (25) leads to a compact form of the distribution function:

$$\mathbb{P}\big(Y(s) \leq k\big) = \sum\nolimits_{m=1}^{k} \mathbb{P}\big(Y(s) = m\big) = 1 - \frac{s}{s+k} = \frac{k}{s+k}, \ k \in \mathbb{N}, \ s > 0. \quad (26)$$

Now, let $Y_1(s), Y_2(s), ..., s > 0$, be independent random variables with the same distribution (26). Define the random variable

$$N_n(s) = \max_{1 \leq j \leq n} Y_j(s) \quad \text{with} \quad \mathbb{P}(N_n(s) \leq k) = \left(\frac{k}{s+k}\right)^n, \ k \in \mathbb{N}, \ s > 0, \quad (27)$$

and using $\mathbb{P}(N_n(s) = k) = \mathbb{P}(N_n(s) \leq k) - \mathbb{P}(N_n(s) \leq k-1)$ we find

$$\mathbb{P}(N_n(s) = k) = \left(\frac{k}{s+k}\right)^n - \left(\frac{k-1}{s+k-1}\right)^n, \ k \in \mathbb{N}, \ s > 0. \quad (28)$$

Consider now a random mean $T_{N_n(s)}$ given in (7) with a random number $N_n(s)$ of observations $X_1, X_2, ...$, where $N_n(s)$ has probability mass function (28). Bening and Korolev (2008) proved for integer $s \geq 1$

$$\lim\nolimits_{n \to \infty} \mathbb{P}\left(N_n(s) \leq n\, x\right) = H(x) = e^{-s/x}\, \mathbb{I}_{(0\,,\,\infty)}(x) \quad (29)$$

and the limit distribution of $\mathbb{P}\left(\sigma^{-1}\sqrt{n}(T_{N_n(s)} - \mu) \leq x\right)$ is the Laplace distribution function $L_{1/\sqrt{s}}(x)$ having density function

$$l_{1/\sqrt{s}}(x) = \frac{\sqrt{2\,s}}{2}\, e^{-\sqrt{2\,s}|x|}, \quad x \in \mathbb{R}. \quad (30)$$

In Bening, Galieva and Korolev (2012, 2013) rates of convergence in (29) and a first order asymptotic expansion for $\mathbb{P}\Big(\sigma^{-1}\sqrt{n}(T_{N_n}(s) - \mu) \leq x\Big)$ are proved for integer $s \geq 1$.

Now we prove an Edgeworth-type expansion for $\mathbb{P}\big(N_n(s)/n \leq x\big)$ satisfying (9).

Theorem 4. *Let the discrete random variable $N_n = N_n(s)$ have probability mass function (28). For $x > 0$, fixed $s \geq s_0 > 0$ and all $n \in \mathbb{N}$ then there exists a real number $C_2(s) > 0$ such that*

$$\sup_{x>0} \left| \mathbb{P}\left(\frac{N_n(s)}{n} \leq x\right) - e^{-s/x} \left\{ 1 + \frac{s\,\big(s - 1 + 2Q_1(n\,x)\big)}{2\,x^2\,n} \right\} \right| \leq \frac{C_2(s)}{n^2} \quad (31)$$

where $Q_1(x)$ is defined in (21).

Remark 7. Formula (31) shows that Condition 2, see (9), is satisfied with $g_n = n$, $H(x) = e^{-s/x}$, $h_2(x) = e^{-s/x}\, s\left(s - 1 + 2Q_1(n\,x)\right) / (2\,x^2)$ and $\beta = 2$.

Figure 4 shows the approximation of $\mathbb{P}\left(N_n(s) \le n\,x\right)$ by $e^{-s/x}$ and $e^{-s/x} + h_2(x)/n$, $x > 0$.

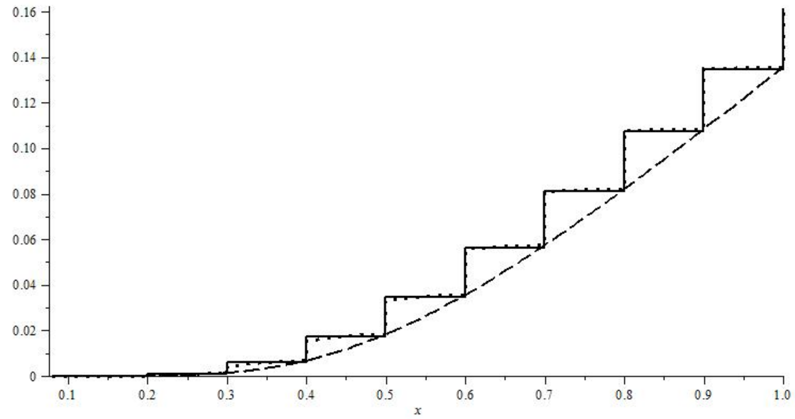

Figure 4: Distribution function $\mathbb{P}\left(N_n(s) \le n\,x\right)$ (solid line), limit law $e^{-s/x}$ (dashed line) and second approximations $e^{-s/x} + h_2(x)/n$ (doted line) with $n = 10$ and $s = 2$

Remark 8. Lyamin (2010) proved a first order bound in (31) for integer $s \ge 1$

$$\left|\mathbb{P}\left(\frac{N_n(s)}{n} \le x\right) - e^{-s/x}\right| \le \frac{C(s)}{n},\ C(s) = \begin{cases} 8e^{-2}/3 = 0.360..., & s = 1, n \ge 2 \\ 2e^{-2} = 0.270..., & s \ge 2, n \ge 1 \end{cases}.$$

In case $n = 1$ and $s = 1$ we have $\mathbb{P}\left(N_1(1) \le x\right) = 0$ for $0 < x < 1$ and

$$\sup_{0<x<1} \left|\mathbb{P}\left(N_1(1) \le x\right) - e^{-1/x}\right| = \sup_{0<x<1} e^{-1/x} = e^{-1} = 0.367\ldots.$$

Remark 9. The random variable $N_n(s)$ is a discrete one with integer values $k \ge 1$. Therefore the distribution function $\mathbb{P}\left(N_n(s) \le n\,x\right)$ is discontinuous with discontinuity points $x = k/n$, $k = 1, 2, ...$, whereas the limit distribution $H(x) = e^{-s/x}\,\mathbb{I}_{(0\,,\,\infty)}(x)$ is continuous. In the interval $(x, x + 1/n]$ with $x > 0$ the distribution function $\mathbb{P}\left(N_n(s) \le n\,x\right)$ has only one jump point $([nx] + 1)/n$. The increase of limit distribution $H(x)$ over the interval $(x, x + 1/n]$ is

$$H(x + 1/n) - H(x) = e^{-s/(x+1/n)} - e^{-s/x} = s\,e^{-s/x}/(n\,x^2) + \mathcal{O}(n^{-2}),\ n \to \infty$$

which is equivalent to the jump at $([nx] + 1)/n$ of the discontinuity correcting function in (31).

Remark 10. The continuous function $e^{-s/x}\mathbb{I}_{(0\,,\,\infty)}(x)$, $s > 0$, is the distribution function of the reciprocal random variable $W(s) = 1/V(s)$, where $V(s)$ is exponentially distributed with rate parameter $s > 0$.

In Theorems 2 and 3 the normalizing sequence is $g_n = \mathbb{E}\big(N_n(r)\big)$, if the random size is negative binomial distributed, see (15). In Theorem 4 and in next Theorem 5 the normalizing sequence is now $g_n = n$, since both $\mathbb{P}\big(N_n(s) \le n\,x\big)$ and $e^{-s/x}\mathbb{I}_{(0\,,\,\infty)}(x)$ for fixed $s > 0$ are heavy tailed with shape parameter 1.

Lemma 2. *For random size $N_n(s)$ with probabilities* (28) *and random variable $W(s)$ with distribution function $e^{-s/x}\mathbb{I}_{(0\,,\,\infty)}(x)$, $s > 0$ and $1 \le r < 2$ we have*

i) $\mathbb{E}\big(N_n(s)\big) = \infty$ *and* $\mathbb{E}\big(W(s)\big) = \infty$,

ii) first absolute pseudo moment $\nu_1 = \int_0^\infty x\big|d\big(\mathbb{P}\big(N_n(s) \le n\,x\big) - e^{-s/x}\big)\big| = \infty$,

iii) absolute difference moment $\chi_r = \int_0^\infty x^{r-1}\big|\mathbb{P}\big(N_n(s) \le n\,x\big) - e^{-s/x}\big|dx < \infty$.

On pseudo moments and some of their generalizations see e.g. Christoph and Wolf (1993, Chapter 2).

In addition to the expansion of $N_n(s)$ a bound of $\mathbb{E}(N_n(s))^{-3/2}$ is required, where $m^{-3/2}$ is rate of convergence of Edgeworth expansion for T_m, see (8).

Lemma 3. *For random size $N_n(s)$ with probabilities* (28) *with reals $s \ge s_0 > 0$ and arbitrary small $s_0 > 0$ and $n \ge 1$ we have*

$$\mathbb{E}\big(N_n(s)\big)^{-3/2} \le \left\{\begin{matrix} \zeta(3/2) = 2.612... & n = 1 \\ \dfrac{sn}{\Gamma(5/2)(n-3/2)^{5/2}}, & n \ge 2 \end{matrix}\right\} \le C(s)n^{-3/2}.$$

Now we present the second order Chebyshev-Edgeworth expansion for the standardized random mean $T_{N_n(s)}$.

Theorem 5. *Let $X, X_1, X_2, ...$ be i.i.d. random variables, where X satisfies* (2) *and Cramér's condition* (3). *Let the discrete random variable $N_n = N_n(s)$ with real parameter $s \ge s_0 > 0$ have probability mass function* (28), *being independent of $X_1, X_2, ...$ Consider the statistic $T_{N_n} = N_n^{-1}(X_1 + ... + X_{N_n})$. For the asymptotically normal statistics T_m the asymptotic expansion* (5) *and for the random size $N_n(s)$ with $s \ge s_0 > 0$ the asymptotic expansion* (31) *hold with $g_n = n$. There exists a constant $C(s) > 0$ such that*

$$\sup_x \left|\mathbb{P}\left(\sigma^{-1}\sqrt{n}(T_{N_n(s)} - \mu) \le x\right) - L_{1/\sqrt{s}\,;n}(x)\right| \le C(s)n^{-3/2}$$

for all $n \in \mathbb{N}$, where

$$L_{1/\sqrt{s}\,;n}(x) = L_{1/\sqrt{s}}(x) + n^{-1/2}A_1(x)\,l_{1/\sqrt{s}}(x) + n^{-1}\Big(A_{2,1}(x) + A_{2,2}(x)\Big)l_{1/\sqrt{s}}(x)$$

$$A_1(x) = \frac{\lambda_3}{6}\left(-x^2 + \frac{|x|}{\sqrt{2s}} + \frac{1}{2s}\right), \quad A_{2,2}(x) = \frac{x(1-s)}{8s}\,(\sqrt{2\,s}\,|x| + 1) \ \textit{and}$$

$$A_{2,1}(x) = \frac{x\,\lambda_4}{48\,s}\Big(3 - 2sx^2 + 3\sqrt{2s}|x|\Big) + \frac{x\,\lambda_3^2}{144\,s}\Big(20sx^2 - (2s)^{3/2}|x|^3 - 15\sqrt{2s}|x| - 15\Big).$$

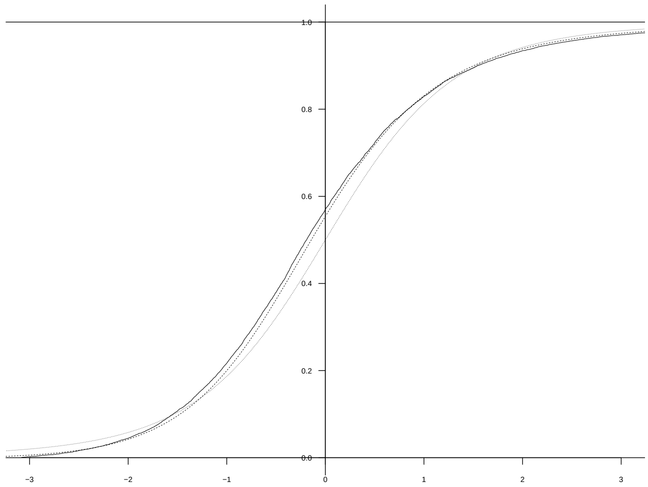

Figure 5: Empirical distribution of $\mathbb{P}\left(\sigma^{-1}\sqrt{n}(T_{N_n(s)}-\mu)\leq x\right)$ (solid line), Laplace approximation $L_{1/\sqrt{2}}(x)$ (dotted line) and second approximation $L_{1/\sqrt{2}\,;n}(x)$ (dashed line) for standardized random mean of $N_n(s)$ independent χ^2_1 random variables with $\lambda_3=\sqrt{8}$ and $\lambda_4=12$ for $n=10$ and $s=2$

Remark 11. Bening, Galieva and Korolev (2013) proved under the moment condition $\mathbb{E}|X|^{3+2\delta}$ for some $0<\delta<1/2$ as $n\to\infty$

$$\sup_x\left|\mathbb{P}\left(\sigma^{-1}\sqrt{n}(T_{N_n(s)}-\mu)<x\right)-L_{1/\sqrt{s}}(x)-\frac{\lambda_3 l_s^*(x)}{6\sqrt{n}}\right|=\mathcal{O}\left(\frac{1}{n^{1/2+\delta}}\right),$$

where

$$l_s^*(x)=\int_0^\infty\varphi(x\sqrt{y})\frac{1-x^2y}{\sqrt{y}}de^{-s/y}=l_{1/\sqrt{s}}(x)\left(\frac{|x|}{\sqrt{2s}}+\frac{1}{2s}-x^2\right).$$

The latter integral was calculated in Markov, Monakhov and Ulyanov (2016), where also a first order Cornish-Fisher expansion was presented (with some computational inaccuracies).

Using the second order Chebyshev-Edgeworth-expansion in Theorem 4 and transfer Proposition 2 we obtain the following statement:

Theorem 6. *Let $x=x_\alpha$ be α-quantile of standardized statistic $\sigma^{-1}\sqrt{n}\left(T_{N_n(s)}-\mu\right)$ and let $u=u_\alpha$ be α-quantile of the limit Laplace distribution $L_{1/\sqrt{s}}(x)$. Then for $u\neq 0$ with previous definitions the following asymptotic expansion holds*

$$x=u-\frac{\lambda_3}{6\sqrt{n}}\left(\frac{|u|}{\sqrt{2s}}+\frac{1}{2s}-u^2\right)+\frac{1}{n}B_2(u)+\mathcal{O}\left(n^{-3/2}\right)\quad \textit{as}\quad n\to\infty,$$

where

$$B_2(u) = -\frac{\lambda_3^2}{36}\left(\frac{\sqrt{2s}\,u}{2|u|}\Big(u^2-\frac{|u|}{\sqrt{2s}}-\frac{1}{2s}\Big)^2+\Big(u^2-\frac{|u|}{\sqrt{2s}}-\frac{1}{2s}\Big)\Big(2u-\frac{u}{\sqrt{2s}|u|}\Big)\right)$$
$$+\ \frac{u\,\lambda_3^2}{144\,s}\Big(20su^2-(2s)^{3/2}|u|^3-15\sqrt{2s}|u|-15\Big)$$
$$+\ \frac{u\,\lambda_4}{48\,s}\Big(3-2su^2+3\sqrt{2s}|u|\Big)\ +\ \frac{u(1-s)}{8s}\,(\sqrt{2\,s}\,|u|+1)$$

Figure 6 shows a Q–Q plot comparing the empirical quantiles of a randomly generated standardized random mean on the horizontal axis to the quantiles based on its Cornish-Fisher approximation on the vertical axis.

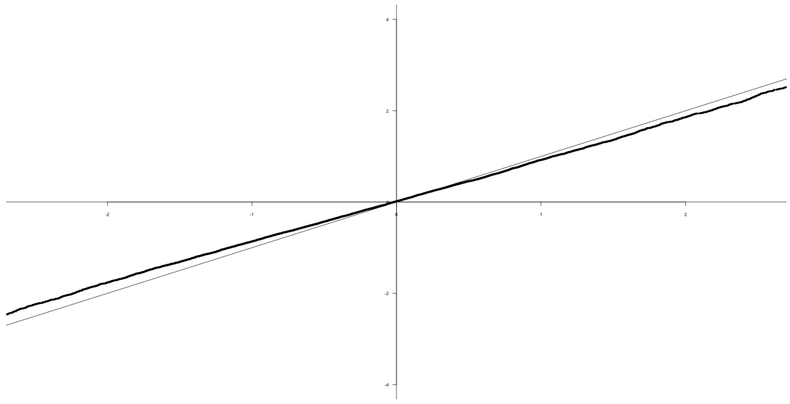

Figure 6: QQ-plot for quantiles of Cornish-Fisher approximation against the quantiles of the empirical quantiles of a standardized random mean of $N_{10}(2)$ independent χ_1^2 random variables with $\lambda_3 = \sqrt{8}$ and $\lambda_4 = 12$

5 Proofs

Proof of Proposition 1: Under the Condition 1 for the statistic $T_m(X_1, ..., X_m)$ and Condition 2 for the random sample size N_n Bening, Galieva and Korolev (2013, Theorem 3.1) proved in the setting of Proposition 1 the following Edgeworth-type expansion for the statistic $T_{N_n}(X_1, ..., X_{N_n})$:

$$\sup_x \big|\mathbb{P}\big(\sigma^{-1}\sqrt{g_n}(T_{N_n}-\mu)\le x\big)-G_{2,n}(x,1/g_n)\big| \le C_1\mathbb{E}(N_n^{-a})+(C_3^*+C_2M_n)n^{-b},$$

$\forall n \in \mathbb{N}$, where $C_3^* > 0$ is a constant independent of n, $0 < g_n \uparrow \infty$,

$$G_{2;n}(x,1/g_n) = \int_{1/g_n}^{\infty}\Phi(x\sqrt{y})dH(y)+\frac{1}{\sqrt{g_n}}\int_{1/g_n}^{\infty}\frac{f_1(x\sqrt{y})}{\sqrt{y}}dH(y)$$
$$+\ \frac{1}{g_n}\int_{1/g_n}^{\infty}\frac{f_2(x\sqrt{y})}{y}dH(y)+\frac{1}{n}\int_{1/g_n}^{\infty}\Phi(x\sqrt{y})dh_2(y),$$

$$M_n = \sup_x M_n(x) := \sup_x \int_{1/g_n}^{\infty}\left|\frac{\partial}{\partial y}\left(\Phi(x\sqrt{y})+\frac{f_1(x\sqrt{y})}{\sqrt{yg_n}}+\frac{f_2(x\sqrt{y})}{yg_n}\right)\right|dy,$$

$f_1(x) = -\frac{\lambda_3}{6} H_2(x)\varphi(x)$ and $f_2(x) = -\Big(\frac{\lambda_4}{24} H_3(x) + \frac{\lambda_3^2}{72} H_5(x)\Big)\varphi(x)$ with Chebyshev-Hermite polynomials $H_m(x)$ given in (6) .

To prove Proposition 1 we have to estimate $|G_{2;n}(x) - G_{2;n}(x, 1/g_n)|$ where $G_{2;n}(x)$ is given in (11) and M_n. Define

$$V_m(z) = H_m(z)\,\varphi(z) \text{ with Hermite polynomials } H_m(z), \text{ for } m = 2, 3, 5\,.$$

Since $V_m'(z_m) = 0$, $m = 2, 3$, with $z_2 = 0, \pm\sqrt{2}$ and $z_3 = \pm\sqrt{3 \pm \sqrt{6}}$, then $|V_2(z)| \le |V_2(0)| = 1/\sqrt{2\,\pi} \le 0.399$, $|V_3(z)| \le |V_3(\pm\sqrt{3 - \sqrt{6}})| \le 0.551$. Numerical calculus lead to $|V_5(z)| \le |V_5(0.380...)| \le 2.308$. Together with $\Phi(z) \le 1$ and (10) we find

$$|G_{2;n}(x) - G_{2;n}(x; 1/g_n)| \le C_0(2 + 0.067|\lambda_3| + 0.023\lambda_4 + 0.033\lambda_3^2) g_n^{-\gamma}.$$

To estimate M_n we consider three cases:

$$M_n = \sup_x |M_n(x)| = \max\{\sup_{x>0} |M_n(x)|, \sup_{x<0} |M_n(x)|, |M_n(0)|\}.$$

Let $x > 0$. Since $\frac{\partial}{\partial y}\Phi(x\sqrt{y}) = \frac{x\,\varphi(x\sqrt{y})}{2\sqrt{y}} \ge 0$ we find

$$\int_{1/g_n}^{\infty} \left|\frac{\partial}{\partial y}\Phi(x\sqrt{y})\right| dy = \int_{1/g_n}^{\infty} \frac{x\,\varphi(x\sqrt{y})}{2\sqrt{y}} dy = \Phi(\infty) - \Phi(x/\sqrt{g_n}) \le 1/2.$$

Consider now

$$\frac{\partial}{\partial y}\frac{V_2(x\sqrt{y})}{\sqrt{y}} = \frac{Q_2(x\sqrt{y})}{y^{3/2}} \quad \text{with} \quad Q_2(z) = \frac{1}{2}(1 + 2z^2 - z^4)\,\varphi(z)$$

and $|Q_2(z)| \le Q_2(\sqrt{3 - \sqrt{6}}) \le 0.273$,

$$\frac{\partial}{\partial y}\frac{V_3(x\sqrt{y})}{y} = \frac{Q_3(x\sqrt{y})}{y^2} \quad \text{with} \quad Q_3(z) = \frac{z}{2}(3 + 4z^2 - z^4)\,\varphi(z)$$

and numerical calculus lead to $|Q_3(z)| \le Q_3(1.210...) \le 0.780$. Next

$$\frac{\partial}{\partial y}\frac{V_5(x\sqrt{y})}{y} = \frac{Q_5(x\sqrt{y})}{y^2} \quad \text{with} \quad Q_5(z) = \frac{z}{2}(-15 - 25z^2 + 13z^4 - z^6)\,\varphi(z)$$

and $|Q_5(z)| \le |Q_5(1.061...)| \le 3.395$. Finally we obtain for $x > 0$

$$\begin{aligned} M_n(x) &\le \frac{1}{2} + \frac{2|\lambda_3|Q_2(\sqrt{3 - \sqrt{6}})}{6} + \frac{\lambda_4 Q_3(1.210...)}{24} + \frac{\lambda_3^2 Q_5(1.061...)}{72} \\ &\le 0.5 + 0.091|\lambda_3| + 0.033\lambda_4 + 0.032\lambda_3^2\,. \end{aligned}$$

For $x < 0$ we find the same upper bounds for the considered integrals as for $x > 0$. Finally, $M_n(0) = |\lambda_3|/(6\sqrt{2\,\pi}) \le 0.067\,|\lambda_3|$. Proposition 1 is proved. $\square$

Next we collect some mostly well-known inequalities used in different proofs.

Lemma 4.

$$\begin{aligned}
0 \ &\le\ g_1(t) := \ln(1+t) - t + t^2/2 \le t^3/3, \quad 0 \le t \le 1, && (32)\\
0 \ &\le (1+t)^{-a} - 1 + at \le a(a+1)t^2/2, \ \ 0 \le t \le 1, \ \ a > 0, && (33)\\
0 \ &\le (1+t)^{r} - 1 - rt \le (1/2)r(r-1)t^2 2^{|r-2|}, \ \ -1/2 < t \le 1, \ \ r > 1, && (34)\\
0 \ &\le\ \ g_2(t) = (1-t)^{-1} - 1 - t \le 3t^2/2\,, \quad 0 \le t \le 1/3\,, && (35)\\
0 \ &\le\ \ g_3(t) = (1-t)^{-2} - 1 \le 15t/4\,, \quad 0 \le t \le 1/3\,, && (36)\\
&i)\ |e^{-t} - 1| \le t\,, \ \ t > 0, \quad ii)\ 0 \le e^{t} - 1 - t \le t^2 e^{\max\{0,t\}}/2\,, \forall t\,, && (37)\\
&\{t^k e^{-at}\,,\, t^{-k} e^{-a/t}\} \le (k/a)^k e^{-k}\,, \quad t > 0 \quad \textit{with fixed} \quad a > 0, k > 0\,. && (38)
\end{aligned}$$

Proof of Lemma 4. Inequalities (32), (33), (34) and (37) may be obtained using Taylor's theorem with Lagrange form of the remainder in the given interval. The convergent geometric series leads to (35). Inequality (36) follows from $(1-t)^{-2} - 1 = 2t(1-t/2)(1-t)^{-2}$ and since $(1-t/2)(1-t)^{-2}$ is monotone increasing in the considered interval. We obtain (38) because both functions $t^k e^{-at}$ and $t^{-k} e^{-a/t}$ take the maxima at $t = k/a$ respectively $t = a/k$. $\square$

Proof of Theorem 1: We have to consider only the case $n \ge 2$, see Remark 4. Remember $g_n = \mathbb{E} N_n(r) = r(n-1) + 1$.
If $\ g_n x = (r(n-1)+1)x < r+1 \ $ then $\ 0 \le x < 2/(n-1) \le 4/n$,

$$\mathbb{P}\big(N_n(r) \le (r(n-1)+1)x\big) \le n^{-r} \sum\nolimits_{k=1}^{[r]+1} \frac{\Gamma(k+r-1)}{\Gamma(r)\,(k-1)!} \le c_1^*(r) n^{-r},$$

$$G_{r,r}(x) \le (r^{r-1}/\Gamma(r)) x^r \le c_2^*(r) n^{-r}, \quad g_{r,r}(x) \le r^r/\Gamma(r)) x^{r-1} \le c_3^*(r) n^{-r+1}$$

and (18) holds with $C_2(r) = c_1^*(r) + c_2^*(r) + c_3^*(r)$. We have to consider (15) only for $k \ge [r]+1 > r$.

Let now $r > 1$, $n \ge 2$ and $g_n x = (r(n-1)+1)x \ge r+1$, then

$$x \ge \frac{r+1}{g_n} \ge \frac{r+1}{r(n-1)+1} \ge \frac{r}{r(n-1)} \ge \frac{1}{n}\,. \tag{39}$$

Let us define $\tau = g_n\,x - [g_n\,x] \in [0\,,\,1)$ and introduce the abbreviations

$$m_{n,x} = g_n x + r - \tau, \ \ n_r = r(n-1), \quad x_r = x + r - \tau \ \text{ and } \ q = 1/(n-1). \tag{40}$$

Distribution function (15) of the discrete $N_n(r)$ jumps at $\{1, 2, 3, ...\}$, therefore

$$\mathbb{P}\big(N_n(r) \le g_n\,x\big) = \mathbb{P}\big(N_n(r) \le [g_n\,x]\big) = \mathbb{P}\big(N_n(r) \le g_n\,x - \tau\big)\,.$$

and we have to calculate (15) for k with $[r]+1 \le k = [g_n\,x] = g_n\,x - \tau$.

First we consider the second integral representation in (16) of the incomplete beta function $B_{1/n}(r,k)$ at the given point $k = g_n\,x - \tau$:

$$B_{1/n}(r,k) = \int_0^{1/(n-1)} t^{r-1}(1+t)^{-g_n x - r + \tau} dt = \int_0^q t^{r-1}(1+t)^{-m_{n,x}} dt. \tag{41}$$

Since $n \ge 2$, $t \le 1/(n-1)$, $n_r\,t \le r$, $t - t^2/2 \ge t/2$ for $0 \le t < 1$ and $x_r > 0$ we find for the second factor under the integral on the right-hand site in (41)

$$\begin{aligned}
(1+t)^{-m_{n,x}} \ &=\ e^{-m_{n,x}\ln(1+t)} \overset{(32)}{=} e^{-m_{n,x}(t-t^2/2)} - r_1\\
&=\ e^{-m_{n,x} t}\big(1 + n_r x t^2/2\big) + e^{-m_{n,x} t} x_r t^2/2 + r_2 - r_1\\
&=\ e^{-(n_r x + x_r)t}\big(1 + n_r x t^2/2\big) + r_3\\
&=\ e^{-n_r x t}\big(1 - x_r t + n_r t^2/2\big) + r_4 \qquad\qquad (42)
\end{aligned}$$

where

$$r_1 = e^{-m_{n,x}(t-t^2/2)}\big(1-e^{-m_{n,x}g_1(t)}\big) \overset{(37i),(32)}{\le} \frac{m_{n,x}t^3}{3}e^{-m_{n,x}t/2} \overset{(38)}{\le} \frac{4t^2}{3e}e^{-m_{n,x}t/4},$$

$$r_2 = \big(e^{m_{n,x}t^2/2} - 1 - m_{n,x}t^2/2\big) \overset{(37ii)}{\le} m_{n,x}^2 t^4 e^{-m_{n,x}xt/2}/8 \overset{(38)}{\le} 8e^{-2}t^2e^{-m_{n,x}t/4},$$

$$r_3 = e^{-m_{n,x}t}x_r t^2/2 + r_2 - r_1 \quad \text{with} \quad |r_3| \le \big(4/(3e) + 8e^{-2} + x_r\big)t^2e^{-m_{n,x}t/4}$$

and using $(37ii)$

$$\begin{aligned} r_4(t) &= r_3 + e^{-n_r xt}\big(x_r n_r x t^3/2 + (e^{-x_r t} - 1 + x_r t)(1 + n_r x t^2/2)\big) \\ &\overset{(38)}{\le} |r_3| + \big(x_r^2 + x_r\big)t^2e^{-n_r xt/2} \le \big(c_1(r) + c_2(r)x^2\big)t^2e^{-n_r xt/4}, \quad (43) \end{aligned}$$

where $c_1(r), c_2(r) > 0$ are constants independent of n, x, t.

It follows now from (41) and (42) that with $k = g_n\, x - \tau$ and $q = 1/(n-1)$

$$B_{1/n}(r,k) = \int_0^q \frac{1 - x_r t + n_r x t^2/2}{t^{1-r}\, e^{n_r xt}}dt + \int_0^q t^{r-1}r_4(t)dt =: J_1(x) + R_1(n). \quad (44)$$

Change of variable $s = (n-1)xt$ results in $0 \le s \le x$ and

$$\begin{aligned} I_j(x;r) &:= \int_0^{1/(n-1)} t^{r+j-1}e^{-n_r xt}dt = \frac{r^{r+j}}{(n_r\, x)^{r+j}}\int_0^x s^{r+j-1}e^{-rs}ds \\ &= \frac{\Gamma(r+j)}{(n_r\, x)^{r+j}}G_{r+j,r}(x) \quad \text{with} \quad j = 0,1,2, \quad (45) \end{aligned}$$

where $G_{\alpha,\beta}(x)$ is the gamma distribution function with density (17), we obtain

$$J_1(x) = \frac{\Gamma(r)}{(n_r x)^r}\left(G_{r,r}(x) - \frac{r\, x_r}{n_r x}G_{r+1,r}(x) + \frac{r(r+1)}{2n_r\, x}G_{r+2,r}(x)\right). \quad (46)$$

To calculate $R_1(x)$ we use (43) and (45):

$$\begin{aligned} R_1(n) &= \int_0^{1/(n-1)} t^{r-1}r_4(t)dt \le (c_1(r) + c_2(r)\, x^2)\int_0^{1/(n-1)} t^{r+1}e^{-n_r xt/4}dt \\ &\le c_1(r)(n-1)^{-2}I_0(x/4;r) + c_2(r)\, x^2 I_2(x/4;r) \\ &\le \frac{c_1(r)\Gamma(r)G_{r,r}(x/4)}{(n-1)^2(n_r x/4)^r} + \frac{c_2(r)\Gamma(r+2)G_{r+2,r}(x/4)}{(n_r/4)^2(n_r x/4)^r} \le \frac{\Gamma(r)R_2(n)}{(n_r x)^r} \quad (47) \end{aligned}$$

with

$$R_2(n) \le c_3(r)n^{-2}\big(G_{r,r}(x/4) + G_{r+2,r}(x/4)\big) \le 2\, c_3(r)n^{-2}, \quad (48)$$

where $c_3(r)$ and the following $c_4(r)$ till $c_{14}(r)$ are positive constants independent of n and x.

In the next step we estimate the beta function $B(r,k)$ involved in (15). Nemes (2015, Theorem 1.3) proved that for $z > 0$

$$\begin{aligned} \Gamma(z) &= \sqrt{2\,\pi}z^{z-1/2}e^{-z}(1 + \frac{1}{12z} + R_3(z)) \\ \frac{1}{\Gamma(z)} &= \frac{1}{\sqrt{2\,\pi}}z^{-z+1/2}e^{z}(1 - \frac{1}{12z} + R_4(z)) \end{aligned} \quad \text{with } |R_3(z)|, |R_4(z)| \le \frac{c}{z^2}$$

and $c = (\sqrt{2}+1)(1+\pi^2/6)/(16\pi^3)$. Hence, with $k = [g_n\, x] = g_n\, x - \tau$

$$\begin{aligned}\frac{1}{B(r,k)} &= \frac{\Gamma(r+k)}{\Gamma(r)\,\Gamma(k)} = \frac{(r+k)^{r+k-1/2}\, e^{-(r+k)}}{\Gamma(r)\, k^{k-1/2}\, e^{-k}}\big(1+R_5(k)\big) \\ &= \frac{e^{-r} k^r}{\Gamma(r)}\left(1+\frac{r}{k}\right)^{k+r-1/2}\big(1+R_5(k)\big) \qquad (49)\end{aligned}$$

with $\big(1+R_5(k)\big) = \left(1+\frac{1}{12(r+k)}+R_3(r+k)\right)\left(1-\frac{1}{12k}+R_4(k)\right)$. Since

$$\left(1+\frac{1}{12(r+k)}\right)\left(1-\frac{1}{12k}\right) = 1-\frac{12r+1}{144k(k+r)} \quad \text{we find} \quad R_5(k) \le \frac{c_4(r)}{k^2}.$$

Remember that $r \le k = [g_n\, x] = g_n\, x - \tau$, the case $r > k$ was considered at the beginning. With $(1+r/k) = \exp\{\ln(1+r/k)\}$ we obtain

$$\begin{aligned}(1+r/k)^{k+r-1/2} &\overset{(32)}{=} \exp\big\{(k+r-1/2)\big(r/k - r^2/(2k^2)\big)\big\} + R_6(k) \\ &= e^r\left(1+\frac{r^2-r}{2k}+R_7(k)\right)+R_6(k) \\ &= e^r\left(1+\frac{r^2-r}{2k}\right)+R_8(k)\,, \qquad (50)\end{aligned}$$

where utilizing $(k+r-1/2)g_1(r/k) \overset{(32)}{\le} (1/3)(k+r-1/2)r^3k^{-3} \le c_5(r)k^{-2}$

$$\begin{aligned}R_6(k) &= e^r\, e^{(r^2-r)/(2k)}\, e^{-(r-1/2)r^2/(2k^2)}\left(e^{(k+r-1/2)g_1(r/k)}-1\right) \\ &\le e^r\, e^{(r^2-r)/(2k)} c_5(r)k^{-2}e^{c_5(r)/k^2} \le c_6(r)k^{-2}\,,\end{aligned}$$

$$\begin{aligned}|R_7(k)| &= e^{(r^2-r)/(2k)}\big|e^{-(r-1/2)r^2/(2k^2)}-1\big| + e^{(r^2-r)/(2k)} - 1 - (r^2-r)/(2k) \\ &\overset{(37)}{\le} e^{(r^2-r)/2}\Big((r-1/2)r^2/(2k^2)+(r^2-r)^2/(8k^2)\Big) \le c_7(r)k^{-2}\end{aligned}$$

and

$$R_8(k) = R_6(k) + e^r|R_7(k)| \le c_8(r)k^{-2}\,.$$

It follows from (49) and (50) that

$$\frac{\Gamma(r)}{(n_r x)^r\, B(r,k)} = \frac{k^r}{(n_r x)^r}\left(1+\frac{r^2-r}{2k}+R_9(k)\right) \qquad (51)$$

with

$$R_9(k) = e^{-r}R_8(k) + R_5(k)\Big(1+(r^2-r)/(2k)+R_8(k)\Big) \le c_9(r)k^{-2}\,.$$

Having in mind $\tau \le 1 \le k = [g_n x] = [(n_r+1)x] = (n_r+1)x - \tau,$ we obtain

$$1+\frac{x-\tau}{n_r x} = \frac{k}{n_r x} = \frac{[(n_r+1)x]}{n_r x}\begin{cases} \le & \frac{(n_r+1)x}{n_r x} \le 1+\frac{1}{r(n-1)} < 2 \\ \ge & \frac{[n_r+1)x]}{(n_r+1)x} \ge \frac{k}{k+\tau} > \frac{1}{2}\end{cases}.$$

Hence $z = \frac{x-\tau}{n_r x} \in \left(-\frac{1}{2}, 1\right)$ and with $\left|\frac{1}{1+z} - 1\right| = \frac{|z|}{1+z} \le 2|z|$ we find

$$\frac{1}{k} = \frac{1}{n_r x}\left(1 + \frac{x-\tau}{n_r x}\right)^{-1} = \frac{1}{n_r x} + R_{10}(n,x) \quad \text{with} \quad |R_{10}(n,x)| \le \frac{2x+2}{(n_r x)^2}.$$

With $n_r = r(n-1) \ge r$ for $n \ge 2$ and $x \ge 1/(n_r+1)$ we find

$$i) \ \left|\frac{1}{k} - \frac{1}{n_r x}\right| \le \frac{2+2x}{(n_r x)^2}, \quad ii) \ \frac{1}{k^2} \le \frac{4}{(n_r x)^2} \quad \text{and} \quad iii) \ \frac{1}{n_r x} \le 1 + \frac{1}{r}. \tag{52}$$

On the other hand we have

$$\left(\frac{k}{n_r x}\right)^r = \left(\frac{n_r x + x - \tau}{n_r x}\right)^r = \left(1 + \frac{x-\tau}{n_r x}\right)^r = 1 + \frac{r(x-\tau)}{n_r x} + R_{11}(n,x) \tag{53}$$

where

$$\begin{aligned} |R_{11}(n,x)| &= \left|\left(1 + \frac{x-\tau}{n_r x}\right)^r - 1 - \frac{r(x-\tau)}{n_r x}\right| \overset{(34)}{\le} \frac{r(r-1)}{2}\left(\frac{x-\tau}{n_r x}\right)^2 2^{|r-2|} \\ &\le (c_{10}(r) + x^2 c_{11}(r))(n_r x)^{-2}. \end{aligned}$$

Combining (51), (53) and (52) we find

$$\frac{\Gamma(r)}{(n_r x)^r B(r,k)} = 1 + \frac{r(x-\tau)}{n_r x} + \frac{r^2 - r}{2 n_r x} + R_{12}(n,x) \tag{54}$$

with

$$\begin{aligned} |R_{12}(n,x)| \ &\le \ \frac{r|x-\tau|(r^2-r)}{2k\, n_r x} + |R_{11}(n,x)|\left(1 + \frac{r^2-r}{2k} + R_9(k)\right) \\ &\quad + \left(1 + \frac{r|x-\tau|}{n_r x}\right) R_9(k) \le \frac{c_{12}(r) + x^2 c_{13}(r)}{(n_r x)^2}. \end{aligned} \tag{55}$$

Taking together (15), (44), (46), (47), (48) and (54) we find

$$\begin{aligned} \frac{B_{1/n}(r,k)}{B(r\,,\,k)} \ &= \left(G_{r,r}(x) - \frac{r x_r}{n_r x} G_{r+1,r}(x) + \frac{r(r+1)}{2 n_r x} G_{r+2,r}(x) + R_2(n)\right) \\ &\quad \cdot \left(1 + \frac{r(x-\tau)}{n_r x} + \frac{r^2 - r}{2 n_r x} + R_{12}(n,x)\right) \\ &= G_{r,r}(x) + \frac{a_0 G_{r,r}(x) + a_1 G_{r+1,r}(x) + a_2 G_{r+2,r}(x)}{2(n-1)x} + R_{13}(n,x) \end{aligned}$$

with $a_0 = 2(x-\tau) + r - 1$, $a_1 = -2(x + r - \tau)$, $a_2 = r+1$ and

$$\begin{aligned} R_{13}(n,x) \ &= \ G_{r,r}(x)\, R_{12}(n,x) + R_2(n)\left(1 + \frac{a_0}{2(n-1)x} + R_{12}(n,x)\right) \\ &\quad + \left(\frac{a_1 G_{r+1,r}(x) + a_2 G_{r+2,r}(x)}{(n-1)x}\right)\left(\frac{a_0}{2(n-1)x} + R_{12}(n,x)\right). \end{aligned}$$

In order to obtain an uniform in $x > 0$ bound for $R_{13}(n,x)$ we use

$$G_{r+j,r}(x) \le 1 \quad \text{for} \quad j = 0,1,2, \quad G_{r,r}(x) \le r^r\, x^r / \Gamma(r+1), \quad \text{for} \quad 1 < r \le 2,$$

$$G_{r+j,r}(x) = \frac{(r+j)^{r+j}}{\Gamma(r+j)} \int_0^x y\, y^{r+j-2} e^{-ry} dy \leq \frac{(r+j)^{r+j}}{\Gamma(r+j)} \left(\frac{r+j-2}{e\, r}\right)^{r+j-2} \frac{x^2}{2}$$

for $j = 0$, $r > 2$ and $j = 1, 2$, $r > 1$ and also $x^{-1} G_{r+2,r}(x) \leq G_{r+1,r}(x) \leq 1$. Together with (47), (48), (55), $|a_0 a_1| \leq 12x^2 + 12r^2$, $|a_0 a_2| \leq 2(r+1)x + (r+1)^2$ and $(n-1)x \geq 1$ by (39) we find $|R_{13}(n,x)| \leq c_{14} n^{-\min\{r,2\}}$. Theorem 1 is proved with $C_2(r) = c_{14}(r)$. □

Proof of Lemma 1: If $n = 1$ then $\mathbb{P}(N_1(r) = 1) = 1$ and (23) holds with $C(r) = 1$. Let $n \geq 2$. From the definition of $N_n(r)$ by (14) for $r > 1$ we have

$$\mathbb{E}(N_n(r))^{-3/2} = \frac{1}{n^r} \left(\sum_{k=1}^{[r]} + \sum_{k=[r]+1}^{\infty} \right) \frac{\Gamma(k+r-1)}{k^{3/2}\, \Gamma(r)\Gamma(k)} \left(1 - \frac{1}{n}\right)^{k-1} =: \sum\nolimits_1 + \sum\nolimits_2,$$

where obviously $\sum_1 \leq c_1(r) n^{-r}$. To estimate $\sum_2$ we use the beta function $B(r\,,\,k)$ with $k > r > 1$ and the equations (49) and (50) with their corresponding bounds from the proof of Theorem 1, which leads to

$$\frac{\Gamma(k+r-1)}{\Gamma(r)\Gamma(k)} = \frac{1}{(k+r-1)\, B(r\,,\,k)} = \frac{k^{r-1}}{\Gamma(r)} \Big(1 + R_1(k)\Big), \ |R_1(k)| \leq \frac{c_2(r)}{k}. \quad (56)$$

For $r > 1$ and $x \geq k \geq 2$ using $(1 - 1/n)^x \leq e^{-x/n}$ we find

$$\frac{k^{r-1}(1-1/n)^{k-1}}{k^{3/2}} \leq \int_k^{k+1} \frac{x^{r-1}\,(1-1/n)^{x-2}}{(x-1)^{3/2}} dx \leq 8\sqrt{2} \int_k^{k+1} x^{r-5/2}\, e^{-x/n} dx.$$

Hence, with $c_3 = 8\sqrt{2}(1 + c_2)/\Gamma(r)$ we obtain

$$\sum\nolimits_2 \leq c_3(r) n^{-r} J_r(n) \quad \text{where} \quad J_r(n) = \int_1^{\infty} x^{r-5/2}\, e^{-x/n} dx$$

Since $J_r(n) \leq (3/2 - r)^{-1}$ for $1 < r < 3/2$, $J_r(n) \leq n^{r-3/2}\,\Gamma(r - 3/2)$ for $r > 3/2$ and $J_{3/2}(n) \leq \int_1^n \frac{1}{x} dx + \frac{1}{n} \int_n^{\infty} e^{-x/n} dx \leq \ln n + e^{-1}$ bound (23) is proved.

Let now $r = 3/2$. Consider (56), $0 \leq \sum_{k=2}^{\infty} k^{-1} |R_1(k)| \leq c_2(3/2)\pi^2/6 < \infty$, $\sum_{(n)} := \sum_{k=2}^{n-1} \frac{1}{k} \geq \ln n - 1$ and $\sum_{k=2}^{n-1} \frac{1 - (1-1/n)^{k-1}}{k} \leq \sum_{k=2}^{n-1} \frac{k-1}{k\, n} \leq 1$ we find a lower bound for $\mathbb{E}(N_n(3/2))^{-3/2}$:

$$\begin{aligned} \mathbb{E}(N_n(3/2))^{-3/2} &\geq n^{-3/2} \sum\nolimits_2 \geq (2/\sqrt{\pi}) n^{-3/2} \big(\sum\nolimits_{(n)} - c_2(3/2)\pi^2/6\big) \\ &\geq (2/\sqrt{\pi}) n^{-3/2} (\ln n - c_2(3/2)\pi^2/6 - 1). \end{aligned}$$ □

Proof of Theorem 2: Since in the transfer Proposition 1 the additional assumptions (10) for the limit gamma distribution $H(x) = G_{r,r}(x)$ of the normalized sample size $N_n(r)$ are satisfied with $\gamma = r > 1$, we have to calculate the integrals in (11). Remember $g_n = \mathbb{E}\big(N_n(r)\big) = r(n-1) + 1$ and define

$$\begin{aligned} J_1(x) &= \int_0^{\infty} \Phi(x\sqrt{y}) dG_{r,r}(y), \quad J_2(x) = \int_0^{\infty} \frac{H_2(x\sqrt{y})\, \varphi(x\sqrt{y})}{\sqrt{y}} dG_{r,r}(y) \\ J_{3,1}(x) &= \int_0^{\infty} \frac{H_3(x\sqrt{y})\, \varphi(x\sqrt{y})}{y} dG_{r,r}(y), \\ J_{3,2}(x) &= \int_0^{\infty} \frac{H_5(x\sqrt{y})\, \varphi(x\sqrt{y})}{y} dG_{r,r}(y) \ \text{and} \ J_4(x) = \int_0^{\infty} \Phi(x\sqrt{y}) dh_2(y) \end{aligned}$$

with $h_2(y) = \Big((y-1)(2-r) + 2Q_1\big((r(n-1)+1)y\big)\Big)\, g_{r,r}(y)/(2r)$, $Q_1(y) = 1/2 - (y - [y])$ and Chebyshev-Hermite polynomials $H_m(x)$ given in (6) . Then

$$G_{2;n}(x;0) = J_1(x) - \frac{\lambda_3\, J_2(x)}{6\sqrt{g_n}} \;-\; \frac{1}{g_n}\left(\frac{\lambda_4}{24} J_{3,1}(x) \;+\; \frac{\lambda_3^2}{72} J_{3,2}(x)\right) + \frac{J_4(x)}{n}\,. \quad (57)$$

Using formula 2.3.3.1 in Prudnikov, Brychkov and Marichev (2002, p. 322) with $\alpha = r-1/2$, $r+1/2$, $r+3/2$ and $p = 1 + x^2/(2\,r)$

$$K_\alpha(x) = \frac{r^r}{\Gamma(r)\,\sqrt{2\pi}} \int_0^\infty y^{\alpha-1} e^{-(r+x^2/2)y} dy = \frac{\Gamma(\alpha)\, r^{r-\alpha}}{\Gamma(r)\,\sqrt{2\,\pi}} \left(1 + x^2/(2r)\right)^{-\alpha} \quad (58)$$

we calculate the integrals occurring in (57). Consider

$$\begin{aligned} \frac{\partial}{\partial x} J_1(x) &= \int_0^\infty y^{1/2} \varphi(x\sqrt{y}) g_{r,r}(y) dy = \frac{r^r}{\Gamma(r)\,\sqrt{2\,\pi}} \int_0^\infty y^{r-1/2} e^{-(r+x^2/2)y} dy \\ &= K_{r+1/2}(x) = s_{2r}(x) \quad \text{and} \quad J_1(x) = S_{2r}(x)\,. \end{aligned}$$

The integrals $J_2(x)$, $J_{3,1}(x)$ and $J_{3,2}(x)$ in (57) we calculate again with (58) using $K_{r-1/2}(x) = s_{2r}(x)\,(2r + x^2)/(2r-1)$ and $K_{r+1/2}(x) = s_{2r}(x)$

$$\begin{aligned} J_2(x) &:= \int_0^\infty \frac{(x^2 y - 1)}{\sqrt{y}} \varphi(x\sqrt{y})\, g_{r,r}(y)\, dy \\ &= x^2 K_{r+1/2}(x) - K_{r-1/2}(x) = \frac{2}{2\,r-1}\left((r-1)x^2 - r\right)\, s_{2r}(x)\,, \end{aligned}$$

$$\begin{aligned} J_{3,1}(x) &:= \frac{r^r}{\sqrt{2\pi}\,\Gamma(r)} \int_0^\infty \frac{1}{y}\Big(x^3 y^{3/2} - 3\,x\,y^{1/2}\Big)\, y^{r-1}\, e^{-(r+x^2/2)y} dy \\ &= \big(x^3\, K_{r+1/2}(x) - 3\,x\, K_{r-1/2}(x)\big) = 2\,\frac{(r-2)x^2 - 3r}{2r-1}\, x\, s_{2r}(x) \end{aligned}$$

and together with $K_{r+3/2}(x) = \dfrac{2r+1}{2r+x^2}\, s_{2r}(x)$

$$\begin{aligned} J_{3,2}(x) &:= \frac{r^r}{\sqrt{2\pi}\,\Gamma(r)} \int_0^\infty \frac{1}{y}\Big(x^5 y^{5/2} - 10 x^3 y^{3/2} + 15 x y^{1/2}\Big)\, y^{r-1}\, e^{-(r+x^2/2)y} dy \\ &= x^5\, K_{r+3/2}(x) - 10\, x^3\, K_{r+1/2}(x) + 15\, x\, K_{r-1/2}(x) \\ &= \frac{(r-2)(r-3)x^4 + 10r(2-r)x^2 + 15r^2}{(2r-1)\,(2r+x^2)}\, 4\,x\, s_{2r}(x). \end{aligned}$$

Integration by parts in the integral $J_4(x)$ in (57) leads to

$$\begin{aligned} J_4(x) &= \int_0^\infty \Phi(x\sqrt{y}) dh_2(y) = -\frac{x}{2\,\sqrt{2\pi}} \int_0^\infty \frac{e^{-x^2 y/2}}{\sqrt{y}} h_2(y) dy \\ &= -\frac{x r^r}{4r\sqrt{2\pi}\Gamma(r)} \int_0^\infty y^{r-3/2} \left((y-1)(2-r) + 2Q_1(g_n y)\right) e^{-(r+x^2/2)y}\, dy \\ &= \frac{(r-2)x}{4r}\Big(K_{r+1/2}(x) - K_{r-1/2}(x)\Big) - J_4^*(x) \\ &= \frac{(2-r)x(x^2+1)}{4r(2r-1)}\, s_{2r}(x) - J_4^*(x). \end{aligned}$$

with

$$J_4^*(x) = \frac{x r^{r-1}}{2\sqrt{2\pi}\Gamma(r)} \int_0^\infty y^{r-3/2}\, Q_1\big(g_n y\big)\, e^{-(r+x^2/2)y}\, dy\,.$$

The function $Q_1(y)$ is periodic with period 1:

$$Q_1(y) = Q_1(y+1) \;\text{ for all }\; y \in \mathbb{R} \;\text{ and }\; Q_1(y) := 1/2 - y \;\text{ for }\; 0 \le y < 1\;, \tag{59}$$

it is right-continuous and has the jump 1 at every integer point y. The Fourier series expansion of $Q_1(y)$ at all non-integer points y is

$$Q_1(y) = 1/2 - (y - [y]) = \sum\nolimits_{k=1}^{\infty} \frac{\sin(2\,\pi\,k\,y)}{k\,\pi}\,, \quad y \ne [y], \tag{60}$$

see formula 5.4.2.9 in Prudnikov, Brychkov and Marichev (2002, p. 726) with $a = 0$. Now we may estimate the integral J_4^*. Using (60), interchange sum and integral and applying formula 2.5.31.4 in Prudnikov, Brychkov and Marichev (2002, p. 446) with $\alpha = r - 1/2$, $p = (r + x^2/2)$ and $b = 2\pi k g_n$

$$\begin{aligned}
J_4^*(x) &= \frac{x r^{r-1}}{2\sqrt{2\pi}\Gamma(r)} \int_0^\infty y^{r-3/2}\, e^{-(r+x^2/2)y} \left(\sum_{k=1}^{\infty} \frac{\sin\big(2\pi k g_n y\big)}{\pi k}\right) dy \\
&= \frac{x r^{r-1}}{2\pi\sqrt{2\pi}\Gamma(r)} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^\infty y^{r-3/2}\, e^{-(r+x^2/2)y} \sin\big(2\pi k g_n y\big) dy \\
&= \frac{x r^{r-1}\Gamma(r-1/2)}{2\pi\sqrt{2\pi}\Gamma(r)} \sum_{k=1}^{\infty} \frac{\sin\Big((r-1/2)\arctan\big(4\pi k g_n/(x^2+2r)\big)\Big)}{k\Big(\big(2\pi k g_n\big)^2 + (r+x^2/2)^2\Big)^{(r-1/2)/2}} \\
&=: \frac{r^{r-1}\Gamma(r-1/2)}{2\pi\sqrt{2\pi}\Gamma(r)} \sum_{k=1}^{\infty} \frac{a_k(x;n)}{k}.
\end{aligned}$$

Now we split the exponent $(r-1/2)/2 = (r-1)/2 + 1/4$ and obtain

$$\begin{aligned}
|a_k(x;n)| &\le \frac{|x|}{\Big(\big(2\pi k g_n\big)^2 + \big(r + x^2/2\big)^2\Big)^{(r-1)/2+1/4}} \\
&\le \frac{|x|}{(2\pi k g_n)^{r-1}\,(r+x^2/2)^{1/2}} \le \frac{\sqrt{2}}{(2\pi\,r\,k\,(n-1))^{r-1}}.
\end{aligned}$$

Since $r > 1$ and $n \ge 2$ we find uniform in x

$$n^{-1}\,|J_4^*| \le \frac{c(r)}{n^r} \sum\nolimits_{k=1}^{\infty} k^{-r} = \frac{c_1(r)}{n^r}\,.$$

Together with $|1/g_n - 1/(rn)| \le \max\{2, r\}(r-1)(rn)^{-2}$, Proposition 1 and Lemma 1 the estimate (24) is proved.

Consider now the case $r = 3/2$. If the random variable X has an additional moment condition $\mathbb{E}|X|^{5+\delta} < \infty$ with some $0 < \delta < 1$, then for the statistic T_m instead of (5) one can obtain a third order Edgeworth expansion with convergence rate $m^{-(3+\delta)/2}$. Like in the proof of Lemma 1 we find $\mathbb{E}(N_n(r))^{-(3+\delta)/2} \le C n^{-3/2}$. The additional term in Edgeworth expansion is

$$m^{-3/2} f_3(x) = -m^{-3/2}\varphi(x)\left(H_4(x)\lambda_5/5! + H_6(x)\lambda_3\lambda_4/(3!4!) + H_8(x)(\lambda_3/3!)^3\right),$$

where λ_k, $k = 3, 4, 5$ are normalized cumulants of X and $H_k(x)$ with absolute terms $H_k(0) > 0$ are Chebyshev-Hermite polynomials of even degree $k = 4, 6, 8$, see Petrov (1995, Theorem 5.19 with $r = 5 + \delta$).

The corresponding additional term in the expansion of $T_{N_n(3/2)}$ is $g_n^{-3/2} \int_{1/g_n}^{\infty} y^{-3/2} f_3(x\sqrt{y}) dG_{3/2,3/2}(y)$, see Bening, Galieva and Korolev (2013, Theorem 3.1). Therefore there are terms like $H_k(0) g_n^{-3/2} J(x)$ with

$$\begin{aligned} J(x) &= \int_{1/g_n}^{\infty} y^{-1} e^{-(x^2-3)y/2} dy \geq \int_{(x^2-3)y/(2g_n)}^{1} (y^{-1} - y^{-1}(1-e^{-y}))dy \\ &\geq -\ln((x^2-3)y/(2g_n)) - 1 = \ln(3(n-1)+2) - \ln(x^2+3) - 1. \end{aligned}$$

Hence, for $|x| \leq K < \infty$ in case $r = 3/2$ the factor $\ln n$ in (24) vanishes only if $\lambda_3 = \lambda_5 = 0$. $\square$

Proof of Theorem 4. Put again $\tau = nx - [nx] \in [0\,,\,1)$, i.e. $[nx] = nx - \tau$. We consider the case

$$0 \leq \frac{\max\{s, 1\}}{nx - \tau} < \frac{1}{3} \quad \text{including also} \quad 0 \leq \frac{s}{nx} < \frac{1}{3} \ \text{ and } \ 0 \leq \frac{\tau}{nx} < \frac{1}{3} \tag{61}$$

and offer preliminary estimates to prepare the proof of (31). Define and estimate

$$a(n,x) \ := \ -n\Big(\frac{s}{nx-\tau} - \frac{s^2}{2(nx-\tau)^2}\Big) \overset{(61)}{\leq} -\frac{5ns}{6(nx-\tau)} \leq -\frac{5s}{6x} \tag{62}$$

$$b(n,x) \ := \ -\frac{s}{x} + \frac{s^2 - 2s\tau}{2nx^2} \leq -\frac{s}{x} + \frac{s}{2x}\frac{s}{nx} \overset{(61)}{\leq} -\frac{5s}{6x}\,, \tag{63}$$

$$f(n,x) \ := \ -\frac{s}{x} g_2\Big(\frac{\tau}{nx}\Big) + \frac{s^2}{2nx^2} g_3\Big(\frac{\tau}{nx}\Big), \tag{64}$$

where $g_2(.) \geq 0$ and $g_3(.) \geq 0$ are defined in (35) and (36). Then

$$|f(n,x)| \leq \max\left\{\frac{s}{x} g_2\Big(\frac{\tau}{nx}\Big)\,,\, \frac{s^2}{2nx^2} g_3\Big(\frac{\tau}{nx}\Big)\right\} \overset{(35),(36)}{\leq} \frac{15 s \max\{1, s\}}{8n^2x^3} \overset{(61)}{\leq} \frac{5s}{24x}. \tag{65}$$

Now we are ready to prove (31):

$$\begin{aligned} \mathbb{P}\left(\frac{N_n(s)}{n} \leq x\right) &= \mathbb{P}(N_n(s) \leq n\,x) = \mathbb{P}(N_n(s) \leq [n\,x]) \overset{(27)}{=} \left(\frac{[nx]}{s + [nx]}\right)^n \\ &= \left(1 + \frac{s}{[nx]}\right)^{-n} = \exp\Big\{-\,n\,\ln\Big(1 + \frac{s}{nx-\tau}\Big)\Big\} \\ &\overset{(32),(62)}{=} \exp\Big\{-\,n g_1\Big(\frac{s}{nx-\tau}\Big) + a(n,x)\Big\} = e^{a(n,x)} - r_1(n,x) \\ &= \exp\left\{-\frac{s}{x}\Big(1 - \frac{\tau}{nx}\Big)^{-1} + \frac{s^2}{2nx^2}\Big(1 - \frac{\tau}{nx}\Big)^{-2}\right\} - r_1(n,x) \\ &\overset{(63)}{=} e^{b(n,x)} + r_2(n,x) - r_1(n,x) \\ &= e^{-s/x}\left(1 + \frac{s(s-2\tau)}{2\,n\,x^2}\right) + r_3(n,x) + r_2(n,x) - r_1(n,x) \end{aligned}$$

where

$$
\begin{aligned}
r_1(n,x) &:= \Big(1-\exp\Big\{-ng_1\Big(\frac{s}{nx-\tau}\Big)\Big\}\Big)e^{a(n,x)} \overset{(37)i,(62)}{\le} ng_1\Big(\frac{s}{nx-\tau}\Big)e^{-5s/(6x)}\\
&\overset{(32)}{\le} \frac{ns^3e^{-5s/(6x)}}{3(nx-\tau)^3} = \frac{s^3e^{-5s/(6x)}}{3n^2x^3(1-\tau/(nx))^3} \overset{(61)}{\le} \frac{9s^3e^{-s/(2x)}}{8n^2x^3} \overset{(38)}{\le} \frac{c_1(s)}{n^2}\,,\\
r_2(n,x) &\overset{(64)}{:=} \left|\exp\left\{-\frac{s}{x}g_2\Big(\frac{\tau}{nx}\Big)+\frac{s^2}{2nx^2}g_3\Big(\frac{\tau}{nx}\Big)\right\}-1\right|e^{b(n,x)}\\
&\overset{(37)i}{\le} |f(n,x)|e^{\{|f(n,x)|+b(n,x)\}} \overset{(65),(63)}{\le} \frac{15s\max\{1,s\}}{8n^2x^3}\exp\left\{\frac{5s}{24x}-\frac{5s}{6x}\right\}\\
&= \frac{15s\max\{1,s\}}{8n^2x^3}e^{-5s/(8x)} \overset{(38)}{\le} \frac{c_2(s)}{n^2}
\end{aligned}
$$

and with $s(\tau):=s^2-2s\tau$ using $|s(\tau)|\le\max\{s^2\,,\,2s\}$

$$
\begin{aligned}
r_3(n,x) &:= \left|e^{b(n,x)}-e^{-s/x}\Big(1+\frac{s(\tau)}{2nx^2}\Big)\right| = e^{-s/x}\left|\exp\left\{\frac{s(\tau)}{2nx^2}\right\}-1-\frac{s(\tau)}{2nx^2}\right|\\
&\overset{(37)ii}{\le} e^{-s/x}\frac{s^2(\tau)}{8n^2x^4}e^{|s(\tau)|/(2nx^2)} \overset{(61)}{\le} \frac{\max\{s^4,4s^2\}}{8n^2x^4}e^{-2s/(3x)} \overset{(38)}{\le} \frac{c_3(s)}{n^2}.
\end{aligned}
$$

Hence, (31) is proved for $s/(nx-\tau)\le 1/3$ with $C_2(s)=c_1(s)+c_2(s)+c_3(s)$.

Let now $s/(nx-\tau)=s/[nx]\ge 1/3$, which is satisfied only for the jump-points $x=k/n$ of $N_n(s)/n$ with $k=1,2,...,3s$. Then we have for $x\le 3s/n$

$$
\mathbb{P}\left(\frac{N_n(s)}{n}\le x\right)=\sum_{k=1}^{3s}\mathbb{P}\left(N_n(s)=k\right)\overset{(28)}{=}\left(\frac{3s}{4s}\right)^n=\frac{n^2}{n^2}\left(\frac{3}{4}\right)^n\overset{(38)}{\le}\frac{c_4}{n^2}.
$$

If $s/(nx-\tau)\ge 1/3$, then $1\le(3s+1)/(nx)$ and we find

$$
e^{-s/x}\left|1+\frac{s\,(s-2\,\tau)}{2\,x^2\,n}\right|\le\frac{(3s+1)^2}{n^2x^2}\left\{1+\frac{\max\{s^2,2s\}}{2\,x^2}\right\}e^{-s/x}\overset{(38)}{\le}\frac{c_5(s)}{n^2}
$$

and (31) is proved for $s/(nx-\tau)\ge 1/3$ with $C_2=c_4+c_5(s)$. Theorem 4 is proved. □

Proof of Lemma 2: Let $s>0$ and $n\ge 1$ be fixed. For $k\ge 1$ put $a_k=k/(s+k)$ and $b_k=(k-1)/(s+k-1)$. Then $a_k-b_k=s/\big((s+k)(s+k-1)\big)$, $a_k>b_k$ and $a_k^n-b_k^n=(a_k-b_k)\left(a_k^{n-1}+a_k^{n-2}b_k+...+a_kb_k^{n-2}+b_k^{n-1}\right)>(a_k-b_k)nb_k^{n-1}$. Further, for $k\ge s^*=2s+1$ we find $b_k\ge 2/3$, $a_k-b_k\ge(3k/2)^{-2}$ and

$$
\sum\nolimits_{k=s^*}^{\infty}k(a_k^n-b_k^n)\ge\sum\nolimits_{k=s^*}^{\infty}\frac{4n}{9k}\left(\frac{2}{3}\right)^{n-1}=\infty\,.\quad\text{Hence,}\quad\mathbb{E}N_n(s)=\infty\,.
$$

Remark 10 leads to $\mathbb{E}W(s)=s\int_0^\infty x^{-1}e^{-s/x}dx=s\int_0^\infty y^{-1}e^{-sy}dy=\infty$.

To investigate the absolute pseudo moment ν_1 we split the integration domain $\mathbb{R}_+=(0\,,\,\infty)=(\mathbb{R}_+\setminus\mathbb{M}_n)\cup\mathbb{M}_n$ for any fixed integer $n\ge 1$, where $\mathbb{M}_n=\{k/n:k\in\mathbb{N}\}$ is the set of discontinuity points of $\mathbb{P}(N_n(s)\le nx)$. Since the distribution function $e^{-s/x}\mathbb{I}_{\mathbb{R}_+}(x)$ is continuous and Lebesgue measure

$\lambda(\mathbb{M}_n)=0$ we find $\nu_1=\int_{\mathbb{R}_+} x\big|d\big(\mathbb{P}\big(N_n(s)\le n\,x\big)-e^{-s/x}\big)\big|\ge\int_{\mathbb{R}_+\setminus\mathbb{M}_n} x de^{-s/x}=\infty\,.$

If $k/n\le x<(k+1)/n$ then $0\le e^{-s/x}-1+s/x\overset{(37)ii}{\le} s^2/(2x^2)\le n^2s^2/(2k^2)$, $(1+s/k)^{-n}-1+ns/k\overset{(33)}{\le} n(n+1)s^2/(2k^2)$, $\quad 0\le ns/k-s/x\quad$ and

$\big|e^{-s/x}-P\big(N_n(s)\le nx\big)\big|\overset{(27)}{=}\big|e^{-s/x}-\big(1+s/k\big)^{-n}\big|\le ns/k-s/x+2n(n+1)s^2/k^2.$

Moreover, $\quad I_k=\displaystyle\int_{k/n}^{(k+1-0)/n}\big(ns/k-s/x\big)dx\le s/k-s\ln(1+1/k)\le s/(2k^2).$

Then for $k\ge s^*=2s+1$ and $1\le r<2$ we find

$$\begin{aligned}&\sum\nolimits_{k=s^*}^{\infty}\int_{k/n}^{(k+1-0)/n} x^{r-1}\big|e^{-s/x}-P\big(N_n(s)\le nx\big)\big|dx\\ \le\ &\sum\nolimits_{k=s^*}^{\infty}\Big(\frac{k+1}{n}\Big)^{r-1}\Big(I_k+\frac{2n(n+1)s^2}{k^2}\Big)\le c(s,n)\sum\nolimits_{k=s^*}^{\infty}\frac{(k+1)^{r-1}}{k^2}<\infty\end{aligned}$$

and $\chi_r<\infty$ for $1\le r<2$ is proved. $\square$

Proof of Lemma 3: Let $n\ge 2$. Proceeding as in Bening, Galieva and Korolev (2013) using

$$\mathbb{P}(N_n(s)=k)=\Big(\frac{k}{s+k}\Big)^n-\Big(\frac{k-1}{s+k-1}\Big)^n=s\,n\int_{k-1}^{k}\frac{x^{n-1}}{(s+x)^{n+1}}dx$$

and Formula 2.2.4.24 in Prudnikov, Brychkov and Marichev (2002, p. 298), then

$$\begin{aligned}\mathbb{E}(N_n^{-3/2})&=s\,n\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}\int_{k-1}^{k}\frac{x^{n-1}}{(s+x)^{n+1}}dx\le s\,n\sum_{k=1}^{\infty}\int_{k-1}^{k}\frac{x^{n-5/2}}{(s+x)^{n+1}}dx\\ &=s\,n\int_0^{\infty}\frac{x^{n-5/2}}{(s+x)^{n+1}}dx=s\,n\,B(5/2,n-3/2)\overset{(51)}{\le}\frac{sn}{\Gamma(5/2)(n-3/2)^{5/2}}.\end{aligned}$$

Moreover $\mathbb{E}(N_1^{-3/2})\le\sum_{k=1}^{\infty}k^{-3/2}=\zeta(3/2)=2.612....$ Lemma 3 is proved. $\square$

Proof of Theorem 5. In the transfer Proposition 1 the additional assumptions (10) for the limit inverse exponential distribution $H(x)=e^{-s/x}\,\mathbb{I}_{(0\,,\,\infty)}(x)$ of the normalized sample size $N_n(s)$, $s\ge s_0>0$ and $h_2(y)=e^{-s/y}\,s\,\big(s-1+2Q_1(y\,n)\big)/(2\,y^2)$, $Q_1(y)=1/2-(y-[y])$, $y>0$ are satisfied with $g_n=n$ and $\gamma=3/2$, where $h_2(0)=\lim_{y\downarrow 0}h_2(y)=0$. Hence, we have to calculate the integrals in (11). Define

$$\begin{aligned}J_1(x)&=\int_0^{\infty}\Phi(x\sqrt{y})de^{-s/x}(y),\quad J_2(x)=\int_0^{\infty}\frac{H_2(x\sqrt{y})\,\varphi(x\sqrt{y})}{\sqrt{y}}de^{-s/x}(y)\\ J_{3,1}(x)&=\int_0^{\infty}\frac{H_3(x\sqrt{y})\,\varphi(x\sqrt{y})}{y}de^{-s/x}(y),\\ J_{3,2}(x)&=\int_0^{\infty}\frac{H_5(x\sqrt{y})\,\varphi(x\sqrt{y})}{y}de^{-s/x}(y)\ \text{ and }\ J_4(x)=\int_0^{\infty}\Phi(x\sqrt{y})dh_2(y)\end{aligned}$$

with Chebyshev-Hermite polynomials $H_m(x)$ given in (6) . Then

$$G_{2;n}(x;0) = J_1(x) - \frac{\lambda_3\, J_2(x)}{6\sqrt{n}} \;-\; \frac{1}{n}\left(\frac{\lambda_4}{24}J_{3,1}(x) \;+\; \frac{\lambda_3^2}{72}J_{3,2}(x)\right) + \frac{J_4(x)}{n}\,. \quad (66)$$

Using formula 2.3.16.3 in Prudnikov, Brychkov and Marichev (2002, p. 444):

$$\int_0^\infty \frac{e^{-py-s/y}}{y^{m+1/2}}dy = (-1)^m \sqrt{\frac{\pi}{p}}\frac{\partial^m}{\partial s^m}e^{-2\sqrt{p\,s}},\;\; p>0,\;\; s>0,\;\; m=0,1,2,... \quad (67)$$

and Appendix II.1 in Prudnikov, Brychkov and Marichev (2002, p. 773):

$$\int_0^\infty y^{-m-1/2}e^{-s/y}dy = s^{-m+1/2}\int_0^\infty t^{m-3/2}e^{-t}dt = s^{-m+1/2}\,\Gamma(m-1/2)\,,$$

with $s>0$ and $m>1/2$. For $p=x^2/2$ and $s>0$ we find

$$K_m(x) := \int_0^\infty \frac{e^{-(x^2/2)y-s/y}}{\sqrt{2\,\pi}\,y^{m+1/2}}dy = \begin{cases} \dfrac{(-1)^m}{|x|}\,\dfrac{\partial^m}{\partial s^m}e^{-\sqrt{2\,s}\,|x|}, & x\neq 0, m=0,1,2 \\ s^{-m+1/2}\,\Gamma(m-1/2), & x=0,\;\; m=1,2. \end{cases} \quad (68)$$

Since $\Gamma(1/2)=\sqrt{\pi}$ and $\Gamma(3/2)=\sqrt{\pi}/2$ then for $x\neq 0$

$$\begin{aligned} K_0(x) &= \frac{1}{|x|}\,e^{-\sqrt{2s}\,|x|} = \frac{2}{\sqrt{2s}\,|x|}\; l_{1/\sqrt{s}}(x), \\ K_1(x) &= \frac{1}{\sqrt{2s}}\,e^{-\sqrt{2s}\,|x|} = \frac{1}{s}\,l_{1/\sqrt{s}}(x), \\ K_2(x) &= e^{-\sqrt{2s}\,|x|}\left(\frac{1}{(2s)^{3/2}} + \frac{|x|}{2s}\right) = \frac{2}{(2s)^2}(1+\sqrt{2s}|x|)\,l_{1/\sqrt{s}}(x) \end{aligned}$$

and $K_1(x)$, $K_2(x)$ and $|x|^\gamma K_0(x)$ with some $\gamma>1$ are continuous $\forall x\in\mathbb{R}$.

Let us now calculate the integrals occurring in (66). Consider

$$\begin{aligned} \frac{\partial}{\partial x}J_1(x) &= \frac{s}{\sqrt{2\,\pi}}\int_0^\infty y^{-3/2}e^{-(x^2/2)y-s/y}dy = s\;K_1(x) = l_{1/\sqrt{s}}(x) \\ \text{and}\quad J_1(x) &= L_{1/\sqrt{s}}(x)\,. \end{aligned}$$

The second and third integrals in (66) we calculate again with (68)

$$\begin{aligned} J_2(x) &:= \frac{s}{\sqrt{2\pi}}\int_0^\infty \frac{x^2y-1}{y^{5/2}}\,e^{-(x^2/2)y-s/y}dy = s\big(x^2K_1(x)-K_2(x)\big) \\ &= \left(x^2 - \frac{|x|}{\sqrt{2s}} \;-\frac{1}{2\,s}\right)\; l_{1/\sqrt{s}}(x)\,, \end{aligned}$$

$$J_{3,1}(x) \;:=\; s\big(x^3K_1(x)-3xK_2(x)\big) = \frac{x}{2\,s}\big(2sx^2 - 3\sqrt{2s}|x| - 3\big)\,l_{1/\sqrt{s}}(x)$$

and

$$\begin{aligned} J_{3,2}(x) &:= s\left(x^5K_0(x) - 10x^3K_1(x) + 15xK_2(x)\right) \\ &= \frac{x}{2\,s}\big((2s)^{3/2}\,|x|^3 - 20sx^2 + 15\sqrt{2s}\,|x| + 15\big)\,l_{1/\sqrt{s}}(x)\,. \end{aligned}$$

Integration by parts in the last integral of (66) leads to

$$
\begin{aligned}
J_4(x) \; &:= \; \int_0^\infty \Phi(x\sqrt{y})d(h_2(y)) = -\int_0^\infty \frac{x}{2\sqrt{y}}\varphi(x\sqrt{y})\, h_2(y)dy \\
&= \; -\frac{x\,s(s-1)}{4\sqrt{2\pi}}\int_0^\infty y^{-5/2}e^{-(x^2/2)y-s/y}dy + J_4^*(x) \\
&\overset{(68)}{=} \; \frac{xs(1-s)}{4}K_2(x) + J_4^*(x) = \frac{x(1-s)}{8s}\big(\sqrt{2s}|x|+1\big)l_{1/\sqrt{s}}(x) + J_4^*(x),
\end{aligned}
$$

where

$$
J_4^* = -\frac{x\,s}{2\sqrt{2\pi}}\int_0^\infty y^{-5/2}e^{-(x^2/2)y-s/y}\,Q_1(n\,y))dy.
$$

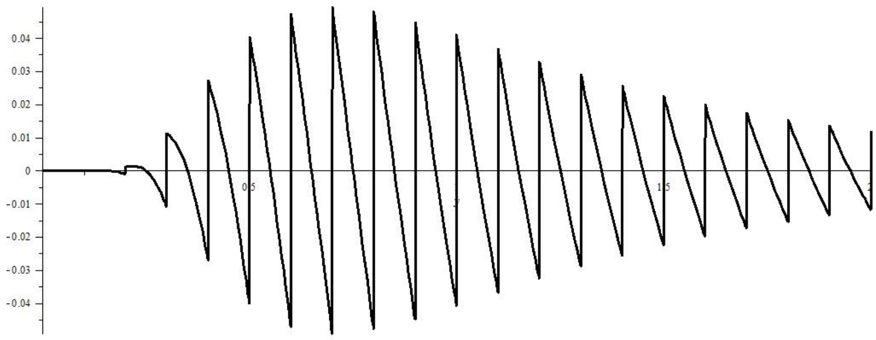

Figure 7: The function $h(y) = y^{-5/2}e^{-(x^2/2)y-s/y}\,Q_1(n\,y)$ under the integral J_4^* for $0 \leq y \leq 2$ and $x = 1$ with $s = 2$ and $n = 10$

Using the Fourier series expansion (60) of the periodic function $Q_1(y)$, given in (59), and interchange sum and integral, we find

$$
J_4^* = -\frac{s\,x}{2\sqrt{2\,\pi}}\sum_{k=1}^\infty \frac{1}{k}\int_0^\infty y^{-5/2}\,e^{-(x^2/2)y-s/y}\,\sin(2\pi\,k\,n\,y)dy. \tag{69}
$$

Let $p > 0$, $s > s_0/2 > 0$ and $b > 0$ be some real constants. Formula 2.5.37.3 in Prudnikov, Brychkov and Marichev (2002, p. 453) is

$$
\int_0^\infty y^{-3/2}\,e^{-p\,y-s/y}\,\sin(b\,y)dy = \frac{\sqrt{\pi}}{\sqrt{s}}\,e^{-\,2\,\sqrt{s}\,z_+}\,\sin(2\,\sqrt{s}\,z_-) \tag{70}
$$

with $2\,z_\pm^2 = \sqrt{p^2+b^2} \pm p$.

In order to estimate $J_4^*(x)$ in (69) we prove that we can apply Leibniz's integral rule and differentiate with respect to s under the integral sign in (70). Consider the function $f(s,y) = \sqrt{2\,p}\,y^{-3/2}\,e^{-p\,y-s/y}\,\sin(b\,y)$. For $p > 0$ we find

$$
\int_0^\infty |f(s,y)|dy \leq \sqrt{2\,p}\int_0^\infty y^{-3/2}\,e^{-p\,y-s/y}dy \overset{(67)}{=} \frac{\sqrt{2\,\pi\,p\,s}}{s}\,e^{-\,2\,\sqrt{p\,s}} \overset{(38)}{\leq} \frac{\sqrt{2\,\pi}}{e\,s_0}
$$

uniformly in $p > 0$, $b > 0$ and $s > s_0/2$. Moreover, the partial derivative $\frac{\partial}{\partial s}f(s,y) = \sqrt{2\,p}\,y^{-5/2}\,e^{-p\,y-s/y}\,\sin(b\,y)$ is continuous in s and y in the region

$(s_0/2, \infty) \times (0, \infty)$ and for $s > s_0/2$

$$\left|\frac{\partial}{\partial s} f(s,y)\right| \leq \sqrt{2\,p}\,\frac{e^{-p\,y - s/y}}{y^{5/2}} = \sqrt{2\,p}\,\frac{e^{-p\,y}}{y^{1/2}}\;\frac{e^{-s/y}}{y^2} \overset{(38)}{\leq} \sqrt{2\,p}\,\frac{e^{-p\,y}}{y^{1/2}}\,\frac{16}{e^2\,s_0^2} =: g(y)$$

with

$$\int_0^\infty g(y)dy = \frac{16}{e^2\,s_0^2}\sqrt{2\,p}\int_0^\infty \frac{e^{-p\,y}}{y^{1/2}}dy = \frac{16}{e^2\,s_0^2}\,\sqrt{2}\,\Gamma(1/2) < \infty$$

for all $p > 0$, $b > 0$ and $s > s_0/2$. Hence, differentiation with respect to s under the integral sign in (70) is allowed and we find

$$\begin{aligned}\int_0^\infty y^{-5/2}\,e^{-p\,y - s/y}\,\sin(b\,y)dy \quad &= \quad (\sqrt{\pi}/2)\;e^{-2\,\sqrt{s}\,z_+}\Big(s^{-3/2}\,\sin(2\sqrt{s}\,z_-) \\ &+ 2\,s^{-1}\,z_+\,\sin(2\sqrt{s}\,z_-) \quad - \quad 2\,s^{-1}\,z_-\,\cos(2\sqrt{s}\,z_-)\Big)\,.\end{aligned}$$

Consider $z_\pm$ defined in (70) with $p = x^2/2$, $b = 2\pi kn$, $k \geq 1$ and $n \geq 1$:

$$z_\pm = (1/\sqrt{2})\sqrt{\sqrt{x^4/4 + (2\pi kn)^2}\ \pm x^2/2}\,.$$

Then $0 < z_- \leq z_+$, $z_+ \geq |x|/2$ and $z_+ \geq \sqrt{\pi kn} \geq \sqrt{\pi}(\sqrt{k} + \sqrt{n})/2$, $(1 + z_+)\,e^{-\sqrt{s}\,z_+/2} \overset{(38)}{\leq} 1 + 2/(e\,\sqrt{s_0})$ and with $\sum_{k=1}^\infty e^{-\sqrt{\pi sk}/2}/k \leq C$ we obtain

$$\begin{aligned}|J_4^*| \quad &\leq \; C_1(s)|x|\sum\nolimits_{k=1}^\infty \frac{(1+z_+)e^{-2\sqrt{s}\,z_+}}{k} \leq C_2(s)|x|e^{-\sqrt{s}\,|x|/4}\sum\nolimits_{k=1}^\infty \frac{e^{-\sqrt{s}\,z_+}}{k} \\ &\leq \; C_3(s)\,e^{-\sqrt{\pi sn}/2}\sum\nolimits_{k=1}^\infty \frac{1}{k}\,e^{-\sqrt{\pi sk}/2} \leq C_4(s)\,e^{-\sqrt{\pi sn}/2} \leq C(s)n^{-3/2}\,.\end{aligned}$$

Note that the constant $C(s)$ may not be uniform bounded in $s > 0$, but there exist a constant $C^*(s_0) < \infty$ such that $C(s) \leq C^*(s_0)$ for $s \geq s_0$. Together with Proposition 1, (66) and Lemma 3 the statement of Theorem 5 is proved. $\square$

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