T^2 Type Test Statistic and Simultaneous Confidence Intervals for Sub-mean Vectors in k-sample Problem

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Abstract

In this paper, we consider tests for sub-mean vectors. In the one-sample and two-sample problems, we give the T^2 type test statistic and the simultaneous confidence intervals by using the approximate upper percentiles of T^2 type test statistic. In the k-sample problem, we give the simultaneous confidence intervals for pairwise multiple comparisons by using Bonferroni's approximation. Finally, we investigate the asymptotic behavior of the approximate upper percentiles of T^2 type statistic by Monte Carlo simulation, and we give an example to illustrate the simultaneous confidence intervals.

AMS 2000 Mathematics Subject Classification. 62H10, 62H15.

Key Words and Phrases: Bonferroni's approximation, Monte Carlo simulation, pairwise multiple comparisons, $T^2_{\max}$ statistic, type I error

1 Introduction

Consider the tests of mean when the partial mean vector is given. That is, letting $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')'$ be a mean vector, where $\boldsymbol{\mu}_1 = (\mu_1, \mu_2, \ldots, \mu_r)'$ and $\boldsymbol{\mu}_2 = (\mu_{r+1}, \mu_{r+2}, \ldots, \mu_p)'$, the problem is the testing of $\boldsymbol{\mu}_1$ when $\boldsymbol{\mu}_2$ is given. For the problem of sub-mean vectors, Eaton and Kariya (1983) derived tests for the independence of two normally distributed sub-mean vectors for the case that an additional random sample is available. Provost (1990) obtained explicit expressions for the case that the maximum likelihood estimators (MLEs) of all the parameters of the multi-normal random vector are given, and the likelihood ratio statistic for testing the independence between sub-mean vectors has been obtained. For the one-sample problem, Rao (1949) gave Rao's U-statistic and additional information. The null distribution of Rao's U-statistic has been introduced by Siotani et al. (1985). A test for sub-mean vectors with two-step monotone missing data was discussed by Kawasaki and Seo (2016). A test for sub-mean vectors in two-sample problem was introduced by Rencher (2012). For the k-sample problem, Fujikoshi et al. (2010) gave an asymptotic expansion of the distribution of the generalized U-statistic under normality. Gupta et al. (2006) gave an asymptotic expansion of the distribution of the generalized U-statistic under a general condition. However, the problem for sub-mean vectors in terms of simultaneous confidence intervals does not appear to have been discussed.

In this paper, we give the T^2 type test statistic and derive the simultaneous confidence intervals for sub-mean vectors in the one-sample and two-sample problems. For the k-sample problem, we give the simultaneous confidence intervals for pairwise multiple comparisons for sub-mean vectors. The remainder of this paper is organized. In Section 2, we propose the T^2 type test statistic for the one-sample case (T_1^2 test statistic), and its approximate upper percentiles and simultaneous confidence intervals. In Section 3, we propose the T^2 type test statistic for the two-sample case (T_{12}^2 test statistic), and its approximate upper percentiles and simultaneous confidence intervals. In Section 4, we present simultaneous confidence intervals for multiple comparisons in the k sample problem. In order to obtain the simultaneous confidence intervals, we derive the approximate upper percentiles of the $T_{\max}^2$ statistic by Bonferroni's approximation. In Section 5, we investigate the asymptotic behavior of the approximate upper percentiles of the T_{12}^2 test statistic and the $T_{\max}^2$ statistic by Monte Carlo simulation. In Section 6, we give an example to illustrate simultaneous confidence intervals.

2 One-sample problem

2.1 T^2 type test statistic for a sub-mean vector

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}, \Sigma)$. In this section, we consider the following hypothesis

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{01} \text{ given } \boldsymbol{\mu}_2 = \boldsymbol{\mu}_{02} \text{ vs. } H_1 : \boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_{01} \text{ given } \boldsymbol{\mu}_2 = \boldsymbol{\mu}_{02},$$

where $\boldsymbol{\mu}_0 = (\mu_{0,1}, \mu_{0,2}, \ldots, \mu_{0,p})' = (\boldsymbol{\mu}_{01}', \boldsymbol{\mu}_{02}')'$ are given values, and

$$\underset{p\times 1}{\boldsymbol{\mu}} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \ \underset{r\times 1}{\boldsymbol{\mu}_1} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_r \end{pmatrix}, \ \underset{s\times 1}{\boldsymbol{\mu}_2} = \begin{pmatrix} \mu_{r+1} \\ \mu_{r+2} \\ \vdots \\ \mu_p \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

We partition $\boldsymbol{x}_j$ into a $r \times 1$ random vector, a $s \times 1$ random vector as $\boldsymbol{x}_j = (\boldsymbol{x}_{1j}', \boldsymbol{x}_{2j}')'$, where $\boldsymbol{x}_{1j}$: $r \times 1$, $\boldsymbol{x}_{2j}$: $s \times 1$, $p = r + s$, $j = 1, 2, \ldots, N$.

This hypothesis test is then the same as the following hypothesis:

$$H_0' : \boldsymbol{\mu}_{1\cdot 2} = \boldsymbol{\mu}_{01\cdot 2} \text{ vs. } H_1' : \boldsymbol{\mu}_{1\cdot 2} \neq \boldsymbol{\mu}_{01\cdot 2},$$

where $\boldsymbol{\mu}_{1\cdot 2} = \boldsymbol{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_{01\cdot 2} = \boldsymbol{\mu}_{01} - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}_{02}$. We then derive the MLEs of $\boldsymbol{\mu}$ and Σ.

Employing the derivation of Siotani et al. (1985), we use the transformed parameters as follows:

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}_{02} \\ \boldsymbol{\mu}_{02} \end{pmatrix}, \ \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{11\cdot 2} & \Sigma_{12}\Sigma_{22}^{-1} \\ \Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. We note that $(\boldsymbol{\eta}, \Psi)$ is in one-to-one correspondence with $(\boldsymbol{\mu}, \Sigma)$. Using the transformed parameters $(\boldsymbol{\eta}, \Psi)$, the likelihood function is given by

$$\begin{aligned} L(\boldsymbol{\eta}, \Psi) =& (2\pi)^{-\frac{Np}{2}}|\Psi_{11}|^{-\frac{N}{2}}|\Psi_{22}|^{-\frac{N}{2}} \\ & \times \exp\left\{ -\frac{1}{2}\sum_{j=1}^{N}(\boldsymbol{x}_{1j} - \Psi_{12}\boldsymbol{x}_{2j} - \boldsymbol{\eta}_1)'\Psi_{11}^{-1}(\boldsymbol{x}_{1j} - \Psi_{12}\boldsymbol{x}_{2j} - \boldsymbol{\eta}_1) \right\} \\ & \times \exp\left\{ -\frac{1}{2}\sum_{j=1}^{N}(\boldsymbol{x}_{2j} - \boldsymbol{\mu}_{02})'\Psi_{22}^{-1}(\boldsymbol{x}_{2j} - \boldsymbol{\mu}_{02}) \right\}. \end{aligned}$$

Therefore, the logarithm of $L(\boldsymbol{\eta}, \Psi)$ can be expressed as

$$\begin{aligned} \log L(\boldsymbol{\eta}, \Psi) =& -\frac{Np}{2}\log(2\pi) - \frac{N}{2}\log|\Psi_{11}| - \frac{N}{2}\log|\Psi_{22}| \\ & -\frac{1}{2}\sum_{j=1}^{N}(\boldsymbol{x}_{1j} - \Psi_{12}\boldsymbol{x}_{2j} - \boldsymbol{\eta}_1)'\Psi_{11}^{-1}(\boldsymbol{x}_{1j} - \Psi_{12}\boldsymbol{x}_{2j} - \boldsymbol{\eta}_1) \\ & -\frac{1}{2}\sum_{j=1}^{N}(\boldsymbol{x}_{2j} - \boldsymbol{\mu}_{02})'\Psi_{22}^{-1}(\boldsymbol{x}_{2j} - \boldsymbol{\mu}_{02}). \end{aligned}$$

Differentiating $\log L(\boldsymbol{\eta}, \Psi)$ with respect to $\boldsymbol{\eta}$ and Ψ, we have

$$\begin{aligned} \frac{\partial \log L(\boldsymbol{\eta}, \Psi)}{\partial \boldsymbol{\eta}_1} &= \Psi_{11}^{-1}\sum_{j=1}^{N}(\boldsymbol{x}_{1j} - \Psi_{12}\boldsymbol{x}_{2j} - \boldsymbol{\eta}_1), \\ \frac{\partial \log L(\boldsymbol{\eta}, \Psi)}{\partial \Psi_{11}} &= -\frac{N}{2}\Psi_{11}^{-1} + \frac{1}{2}\sum_{j=1}^{N}\Psi_{11}^{-1}(\boldsymbol{x}_{1j} - \Psi_{12}\boldsymbol{x}_{2j} - \boldsymbol{\eta}_1)(\boldsymbol{x}_{1j} - \Psi_{12}\boldsymbol{x}_{2j} - \boldsymbol{\eta}_1)'\Psi_{11}^{-1}, \\ \frac{\partial \log L(\boldsymbol{\eta}, \Psi)}{\partial \Psi_{12}} &= \sum_{j=1}^{N}\Psi_{11}^{-1}\left\{(\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_1)(\boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_2)' - \Psi_{12}(\boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_2)(\boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_2)'\right\}, \\ \frac{\partial \log L(\boldsymbol{\eta}, \Psi)}{\partial \Psi_{22}} &= -\frac{N}{2}\Psi_{22}^{-1} + \frac{1}{2}\sum_{j=1}^{N}\Psi_{22}^{-1}(\boldsymbol{x}_{2j} - \boldsymbol{\mu}_{02})(\boldsymbol{x}_{2j} - \boldsymbol{\mu}_{02})'\Psi_{22}^{-1}. \end{aligned}$$

Solving the partial derivative of $\log L(\boldsymbol{\eta}, \Psi) = 0$, the MLEs of $\boldsymbol{\eta}$ and Ψ are given by

$$\widehat{\boldsymbol{\eta}}_1 = \overline{\boldsymbol{x}}_1 - \widehat{\Psi}_{12}\overline{\boldsymbol{x}}_2, \ \widehat{\Psi}_{11} = \frac{1}{N}V_{11\cdot 2}, \ \widehat{\Psi}_{12} = V_{12}V_{22}^{-1}\widehat{\Psi}_{22}, \ \widehat{\Psi}_{22} = \frac{1}{N}V_{22},$$

where

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_j = \begin{pmatrix} \overline{\boldsymbol{x}}_1 \\ \overline{\boldsymbol{x}}_2 \end{pmatrix}, \ V = \sum_{j=1}^{N}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})' = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \ V_{11\cdot 2} = V_{11} - V_{12}V_{22}^{-1}V_{21}.$$

Therefore, using the relation that $(\boldsymbol{\eta}, \Psi)$ is in one-to-one correspondence with $(\boldsymbol{\mu}, \Sigma)$, the MLEs of $\boldsymbol{\mu}$ and Σ are given by

$$\widehat{\boldsymbol{\mu}} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1 \\ \widehat{\boldsymbol{\mu}}_2 \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_1 - \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}(\overline{\boldsymbol{x}}_2 - \boldsymbol{\mu}_{02}) \\ \boldsymbol{\mu}_{02} \end{pmatrix},$$

$$\widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{21} & \widehat{\Sigma}_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{N}V_{11\cdot 2} + \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}\widehat{\Sigma}_{21} & V_{12}V_{22}^{-1}\widehat{\Sigma}_{22} \\ \widehat{\Sigma}_{22}V_{22}^{-1}V_{21} & \frac{1}{N}V_{22} \end{pmatrix}.$$

The T_1^2 test statistic for sub-mean vectors is then given by

$$T_1^2 = (\widehat{\boldsymbol{\mu}}_{1\cdot 2} - \boldsymbol{\mu}_{01\cdot 2})'\left\{\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}}_{1\cdot 2} - \boldsymbol{\mu}_{01\cdot 2})\right\}^{-1}(\widehat{\boldsymbol{\mu}}_{1\cdot 2} - \boldsymbol{\mu}_{01\cdot 2}),$$

where $\widehat{\boldsymbol{\mu}}_{1\cdot 2}$, $\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}}_{1\cdot 2} - \boldsymbol{\mu}_{01\cdot 2})$ are estimators of $\boldsymbol{\mu}_{1\cdot 2}$, $\mathrm{Cov}(\widehat{\boldsymbol{\mu}}_{1\cdot 2} - \boldsymbol{\mu}_{01\cdot 2})$. We note that under H_0', T_1^2 is approximately distributed as a F distribution with r and $N-p$ degrees of freedom. Using this result, the approximate upper 100α percentile of the T_1^2 test statistic is given by

$$t^2_{1\cdot \mathrm{app}}(\alpha) = \frac{(N-1)r}{N-p}F_{r,N-p}(\alpha),$$

where $F_{r,N-p}(\alpha)$ is the upper 100α percentile of the F distribution with r and $N-p$ degrees of freedom.

2.2 Simultaneous confidence intervals for sub-mean vectors

We consider the simultaneous confidence intervals for any and all linear compounds of the sub-mean. Using the approximate upper percentiles of T_1^2 from Section 2.1, for any nonnull vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_p)'$, the simultaneous approximate confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}_{1\cdot 2} - \boldsymbol{\mu}_{01\cdot 2})$ are given by

$$\boldsymbol{a}'\boldsymbol{u} - \sqrt{\frac{c_1}{N-1}}R_{\mathrm{app}} \le \boldsymbol{a}'(\boldsymbol{\mu}_{1\cdot 2} - \boldsymbol{\mu}_{01\cdot 2}) \le \boldsymbol{a}'\boldsymbol{u} + \sqrt{\frac{c_1}{N-1}}R_{\mathrm{app}},\ \forall \boldsymbol{a} \in \mathbb{R}^r - \{\boldsymbol{0}\},$$

where

$$\boldsymbol{u} = \overline{\boldsymbol{x}}_1 - \boldsymbol{\mu}_{01} - V_{12}V_{22}^{-1}(\overline{\boldsymbol{x}}_2 - \boldsymbol{\mu}_{02}),\ R_{\mathrm{app}} = (t^2_{1\cdot\mathrm{app}}(\alpha)\boldsymbol{a}'V_{11\cdot 2}\boldsymbol{a})^{\frac{1}{2}},\ c_1 = \frac{N(N-2)}{N-s-2}.$$

3 Two-sample problem

3.1 T^2 type test statistic for sub-mean vectors

Let $\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)}, \cdots, \boldsymbol{x}_{N^{(i)}}^{(i)} \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}^{(i)}, \Sigma),\ i = 1, 2,\ N = N^{(1)} + N^{(2)}$. In this section, we consider the following hypothesis

$$H_0 : \boldsymbol{\mu}_1^{(1)} = \boldsymbol{\mu}_1^{(2)} \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} \text{ vs. } H_1 : \boldsymbol{\mu}_1^{(1)} \ne \boldsymbol{\mu}_1^{(2)} \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)},$$

where

$$\underset{p\times 1}{\boldsymbol{\mu}^{(i)}} = \begin{pmatrix} \boldsymbol{\mu}_1^{(i)} \\ \boldsymbol{\mu}_2^{(i)} \end{pmatrix}, \ \underset{r\times 1}{\boldsymbol{\mu}_1^{(i)}} = \begin{pmatrix} \mu_1^{(i)} \\ \mu_2^{(i)} \\ \vdots \\ \mu_r^{(i)} \end{pmatrix}, \ \underset{s\times 1}{\boldsymbol{\mu}_2^{(i)}} = \begin{pmatrix} \mu_{r+1}^{(i)} \\ \mu_{r+2}^{(i)} \\ \vdots \\ \mu_p^{(i)} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

We partition $\boldsymbol{x}_j^{(i)}$ into a $r \times 1$ random vector and a $s \times 1$ random vector, as $\boldsymbol{x}_j^{(i)} = (\boldsymbol{x}_{1j}^{(i)\prime}, \boldsymbol{x}_{2j}^{(i)\prime})'$, where $j = 1, 2, \ldots, N^{(i)}$, and $p = r + s$.

This hypothesis test is then the same as the following hypothesis

$$H_0' : \boldsymbol{\mu}_{1\cdot 2}^{(1)} = \boldsymbol{\mu}_{1\cdot 2}^{(2)} \ \text{ vs. } H_1' : \boldsymbol{\mu}_{1\cdot 2}^{(1)} \neq \boldsymbol{\mu}_{1\cdot 2}^{(2)},$$

where $\boldsymbol{\mu}_{1\cdot 2}^{(i)} = \boldsymbol{\mu}_1^{(i)} - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)}$. We derive the MLEs of $\boldsymbol{\mu}$ and Σ as follows.

As for the one-sample case, we use the following transformed parameters $(\boldsymbol{\eta}^{(i)}, \Psi)$

$$\boldsymbol{\eta}^{(i)} = \begin{pmatrix} \boldsymbol{\eta}_1^{(i)} \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^{(i)} - \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \ \Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{11\cdot 2} & \Sigma_{12}\Sigma_{22}^{-1} \\ \Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, $i = 1, 2$. We note that $(\boldsymbol{\eta}^{(i)}, \Psi)$ is in one-to-one correspondence with $(\boldsymbol{\mu}^{(i)}, \Sigma)$. Using the transformed parameters $(\boldsymbol{\eta}^{(i)}, \Psi)$, the likelihood function is given by

$$\begin{aligned} L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi) =& (2\pi)^{-\frac{Np}{2}} |\Psi_{11}|^{-\frac{N}{2}} |\Psi_{22}|^{-\frac{N}{2}} \\ & \times \prod_{i=1}^{2} \left[\exp\left\{ -\frac{1}{2} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)})' \Psi_{11}^{-1} (\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)}) \right\} \right] \\ & \times \prod_{i=1}^{2} \left[\exp\left\{ -\frac{1}{2} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2)' \Psi_{22}^{-1} (\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2) \right\} \right]. \end{aligned}$$

Therefore, the logarithm of $L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)$ can be expressed as

$$\begin{aligned} \log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi) =& -\frac{Np}{2}\log(2\pi) - \frac{N}{2}\log|\Psi_{11}| - \frac{N}{2}\log|\Psi_{22}| \\ & - \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)})' \Psi_{11}^{-1} (\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)}) \\ & - \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2)' \Psi_{22}^{-1} (\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2). \end{aligned}$$

Differentiating $\log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)$ with respect to $\boldsymbol{\eta}^{(i)}$, Ψ, we have

$$\frac{\partial \log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)}{\partial \boldsymbol{\eta}_1^{(i)}} = \Psi_{11}^{-1} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)}),$$

$$\frac{\partial \log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)}{\partial \boldsymbol{\eta}_2} = \Psi_{22}^{-1} \sum_{i=1}^{2} \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2),$$

$$\frac{\partial \log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)}{\partial \Psi_{11}}$$

$$= -\frac{N}{2}\Psi_{11}^{-1} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{N^{(i)}} \Psi_{11}^{-1}(\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)})(\boldsymbol{x}_{1j}^{(i)} - \Psi_{12}\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_1^{(i)})'\Psi_{11}^{-1},$$

$$\frac{\partial \log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)}{\partial \Psi_{12}} = \sum_{i=1}^{2} \sum_{j=1}^{N^{(i)}} \Psi_{11}^{-1} \left\{ (\boldsymbol{x}_{1j}^{(i)} - \boldsymbol{\eta}_1^{(i)})\boldsymbol{x}_{2j}^{(i)\prime} - \Psi_{12}\boldsymbol{x}_2^{(i)}\boldsymbol{x}_2^{(i)\prime} \right\},$$

$$\frac{\partial \log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)}{\partial \Psi_{22}} = -\frac{N}{2}\Psi_{22}^{-1} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{N^{(i)}} \Psi_{22}^{-1}(\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2)(\boldsymbol{x}_{2j}^{(i)} - \boldsymbol{\eta}_2)'\Psi_{22}^{-1}.$$

Solving the partial derivative of $\log L(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi) = 0$, the maximum likelihood estimates of $\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}$, and Ψ are given by

$$\widehat{\boldsymbol{\eta}}_1^{(i)} = \overline{\boldsymbol{x}}_1^{(i)} - \widehat{\Psi}_{12}\overline{\boldsymbol{x}}_2^{(i)}, i = 1, 2,\ \widehat{\boldsymbol{\eta}}_2 = \overline{\boldsymbol{x}}_2,\ \widehat{\Psi}_{11} = \frac{1}{N}V_{11\cdot 2},\ \widehat{\Psi}_{12} = V_{12}V_{22}^{-1}\widehat{\Psi}_{22},$$

$$\widehat{\Psi}_{22} = \frac{1}{N}\left\{ V_{22} + \sum_{j=1}^{2} (\overline{\boldsymbol{x}}_2^{(j)} - \overline{\boldsymbol{x}}_2)(\overline{\boldsymbol{x}}_2^{(j)} - \overline{\boldsymbol{x}}_2)' \right\},$$

where

$$\overline{\boldsymbol{x}}^{(i)} = \frac{1}{N^{(i)}} \sum_{j=1}^{N^{(i)}} \boldsymbol{x}_j^{(i)} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{(i)} \\ \overline{\boldsymbol{x}}_2^{(i)} \end{pmatrix}, \overline{\boldsymbol{x}}_2 = \frac{1}{N} \sum_{i=1}^{2} N^{(i)}\overline{\boldsymbol{x}}_2^{(i)},$$

$$V^{(i)} = \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})' = \begin{pmatrix} V_{11}^{(i)} & V_{12}^{(i)} \\ V_{21}^{(i)} & V_{22}^{(i)} \end{pmatrix},$$

$$V = \sum_{i=1}^{2} V^{(i)} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},\ V_{11\cdot 2} = V_{11} - V_{12}V_{22}^{-1}V_{21}.$$

Therefore, using the relation that $(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}, \Psi)$ is in one-to-one correspondence with $(\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \Sigma)$, the MLEs of $\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}$, and Σ are given by

$$\widehat{\boldsymbol{\mu}}^{(i)} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1^{(i)} \\ \widehat{\boldsymbol{\mu}}_2 \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{(i)} - \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}(\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2) \\ \overline{\boldsymbol{x}}_2 \end{pmatrix}, i = 1, 2,$$

$$\widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{21} & \widehat{\Sigma}_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{N}V_{11\cdot 2} + \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}\widehat{\Sigma}_{21} & V_{12}V_{22}^{-1}\widehat{\Sigma}_{22} \\ \widehat{\Sigma}_{22}V_{22}^{-1}V_{21} & \frac{1}{N}\left\{ V_{22} + \sum_{i=1}^{2} (\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2)(\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2)' \right\} \end{pmatrix},$$

The T^2_{12} test statistic for the sub-mean vector is then given by

$$T^2_{12} = (\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2})' \{\widehat{\mathrm{Cov}}\left(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}\right)\}^{-1} (\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}),$$

where $\widehat{\boldsymbol{\mu}}^{(i)}_{1\cdot 2}$ and $\widehat{\mathrm{Cov}}\left(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}\right)$ are the estimators of $\boldsymbol{\mu}^{(i)}_{1\cdot 2}$ and $\mathrm{Cov}\left(\widehat{\boldsymbol{\mu}}^{(1)}_{1\cdot 2} - \widehat{\boldsymbol{\mu}}^{(2)}_{1\cdot 2}\right)$, respectively. We note that under H'_0, T^2_{12} is asymptotically distributed as a χ^2 distribution with r degrees of freedom. However, when the sample is not large, the χ^2 distribution is not a good approximation of the upper percentile of T^2_{12}.

Let

$$\boldsymbol{u}_{12} = \overline{\boldsymbol{x}}^{(1)}_1 - \overline{\boldsymbol{x}}^{(2)}_1 - V_{12}V_{22}^{-1}(\overline{\boldsymbol{x}}^{(1)}_2 - \overline{\boldsymbol{x}}^{(2)}_2), \ c_{12} = \frac{N(N-3)}{N^{(1)}N^{(2)}(N-s-3)}.$$

We can then rewrite T^2_{12} as

$$T^2_{12} = (N-2)c_{12}^{-1}\boldsymbol{u}'_{12}V_{11\cdot 2}^{-1}\boldsymbol{u}_{12} = (N-2)\boldsymbol{z}'W^{-1}\boldsymbol{z} = (N-2)\boldsymbol{z}'\boldsymbol{z}\frac{\boldsymbol{z}'W^{-1}\boldsymbol{z}}{\boldsymbol{z}'\boldsymbol{z}},$$

where $\boldsymbol{z} = c_{12}^{-\frac{1}{2}}\Sigma_{11\cdot 2}^{-\frac{1}{2}}\boldsymbol{u}_{12}$, $W = \Sigma_{11\cdot 2}^{-\frac{1}{2}}V_{11\cdot 2}\Sigma_{11\cdot 2}^{-\frac{1}{2}}$. We note that $\boldsymbol{u}_{12}$ is distributed on $N_r(\boldsymbol{\mu}^{(1)}_1 - \boldsymbol{\mu}^{(2)}_1, c_{12}\Sigma_{11\cdot 2})$ when $N^{(1)}, N^{(2)} \to \infty$. Therefore, the distribution of $\boldsymbol{z}'\boldsymbol{z}$ is a χ^2 distribution with r degrees of freedom. We note that under H'_0, $(N-p-s-1)T^2_{12}/(N-2)r$ is approximately distributed as a F distribution with r and $N-p-s-1$ degrees of freedom. Using this result, the approximate upper 100α percentile of the T^2_{12} statistic is given by

$$t^2_{12\cdot\mathrm{app}}(\alpha) = \frac{(N-2)r}{N-p-s-1}F_{r,N-p-s-1}(\alpha),$$

where $F_{r,N-p-s-1}(\alpha)$ is the upper 100α percentiles of the F distribution with r and $N-p-s-1$ degrees of freedom.

3.2 Simultaneous confidence intervals

We consider the simultaneous confidence intervals for any and all linear compounds of the sub-mean. Using the upper percentiles of T^2_{12} from Section 3.1, for any nonnull vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_p)'$, the simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2} - \boldsymbol{\mu}^{(2)}_{1\cdot 2})$ are given by

$$\boldsymbol{a}'\boldsymbol{u}_{12} - \sqrt{\frac{c_{12}}{N-2}}M \le \boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2} - \boldsymbol{\mu}^{(2)}_{1\cdot 2}) \le \boldsymbol{a}'\boldsymbol{u}_{12} + \sqrt{\frac{c_{12}}{N-2}}M, \ \forall \boldsymbol{a} \in \mathbb{R}^r - \{\boldsymbol{0}\},$$

where $M = (t_{12}^2(\alpha)\boldsymbol{a}'V_{11\cdot 2}\boldsymbol{a})^{\frac{1}{2}}$, and $t_{12}^2(\alpha)$ is the upper 100α percentiles of the T_{12}^2 test statistic. However, it is not easy to obtain $t_{12}^2(\alpha)$. Therefore, using the approximate upper 100α percentiles of the T_{12}^2 test statistic, $t_{12\cdot\mathrm{app}}^2(\alpha)$, the approximate simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}_{1\cdot 2}^{(1)} - \boldsymbol{\mu}_{1\cdot 2}^{(2)})$ can obtained by

$$\boldsymbol{a}'\boldsymbol{u}_{12} - \sqrt{\frac{c_{12}}{N-2}}M_{\mathrm{app}} \leq \boldsymbol{a}'(\boldsymbol{\mu}_{1\cdot 2}^{(1)} - \boldsymbol{\mu}_{1\cdot 2}^{(2)}) \leq \boldsymbol{a}'\boldsymbol{u}_{12} + \sqrt{\frac{c_{12}}{N-2}}M_{\mathrm{app}},\ \forall \boldsymbol{a} \in \mathbb{R}^r - \{\boldsymbol{0}\},$$

where $M_{\mathrm{app}} = (t_{12\cdot\mathrm{app}}^2(\alpha)\boldsymbol{a}'V_{11\cdot 2}\boldsymbol{a})^{\frac{1}{2}}$.

4 k-sample problem

4.1 The $T_{\max}^2$ type test statistic

In this section, we consider the $T_{\max}^2$ type test statistic for testing any two sub-mean vectors and propose the approximate upper 100α percentiles of this statistic with Bonferroni's approximation. Let $\boldsymbol{x}_1^{(i)}, \boldsymbol{x}_2^{(i)}, \cdots, \boldsymbol{x}_{N^{(i)}}^{(i)} \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}^{(i)}, \Sigma)$ for $i = 1, 2, \cdots, k$, $N = \sum_{i=1}^k N^{(i)}$. Gupta et al. (2006) has derived the likelihood ratio test (LRT) statistic for testing the hypothesis

$$H_0 : \boldsymbol{\mu}_1^{(1)} = \boldsymbol{\mu}_1^{(2)} = \cdots = \boldsymbol{\mu}_1^{(k)} \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \cdots = \boldsymbol{\mu}_2^{(k)}$$

$$\text{vs. } H_1 : \text{at least two } \boldsymbol{\mu}_1^{(i)}\text{s are an equal given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \cdots = \boldsymbol{\mu}_2^{(k)}.$$

When H_0 is rejected, our interest is pairwise comparisons of sub-mean vectors. Under the assumption of the common population covariance matrix, for fixed a; b, we can use the T^2 type statistic for the two-sample problem derived in Section3.1, that is,

$$T_{ab}^2 = (\widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(i)} - \widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(j)})'\{\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(i)} - \widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(j)})\}^{-1}(\widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(i)} - \widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(j)}),$$

where $\widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(i)}(= \widehat{\boldsymbol{\mu}}_1^{(i)} - \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}\widehat{\boldsymbol{\mu}}_2)$ and $\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(i)} - \widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(j)})$ are estimators of $\boldsymbol{\mu}_{1\cdot 2}^{(i)}$ and $\mathrm{Cov}(\widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(i)} - \widehat{\boldsymbol{\mu}}_{1\cdot 2}^{(j)})$, respectively. Similarly, as for the two-sample case, the MLEs are given by

$$\widehat{\boldsymbol{\mu}}^{(i)} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1^{(i)} \\ \widehat{\boldsymbol{\mu}}_2 \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{(i)} - \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}(\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2) \\ \overline{\boldsymbol{x}}_2 \end{pmatrix},$$

$$\widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} \\ \widehat{\Sigma}_{21} & \widehat{\Sigma}_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{N}V_{11\cdot 2} + \widehat{\Sigma}_{12}\widehat{\Sigma}_{22}^{-1}\widehat{\Sigma}_{21} & V_{12}V_{22}^{-1}\widehat{\Sigma}_{22} \\ \widehat{\Sigma}_{22}V_{22}^{-1}V_{21} & \frac{1}{N}\{V_{22} + \sum_{i=1}^k(\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2)(\overline{\boldsymbol{x}}_2^{(i)} - \overline{\boldsymbol{x}}_2)'\} \end{pmatrix},$$

where

$$\overline{\boldsymbol{x}}^{(i)} = \frac{1}{N^{(i)}} \sum_{j=1}^{N^{(i)}} \boldsymbol{x}_j^{(i)} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{(i)} \\ \overline{\boldsymbol{x}}_2^{(i)} \end{pmatrix}, \ \overline{\boldsymbol{x}}_2 = \frac{1}{N} \sum_{i=1}^{k} N^{(i)} \overline{\boldsymbol{x}}_2^{(i)},$$

$$V^{(i)} = \sum_{j=1}^{N^{(i)}} (\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})(\boldsymbol{x}_j^{(i)} - \overline{\boldsymbol{x}}^{(i)})' = \begin{pmatrix} V_{11}^{(i)} & V_{12}^{(i)} \\ V_{21}^{(i)} & V_{22}^{(i)} \end{pmatrix},$$

$$V = \sum_{i=1}^{k} V^{(i)} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \ V_{11\cdot 2} = V_{11} - V_{12} V_{22}^{-1} V_{21}.$$

Similarly, as Section 3.1, under the hypothesis that the two sub-mean vectors are equal, we have the following approximate upper 100α percentile of the T_{ab}^2 test statistic for fixed $i = a, b$:

$$t_{ab\cdot \mathrm{app}}^2(\alpha) = \frac{(N-k)r}{N-r-k(s+1)+1} F_{r,N-r-k(s+1)+1}(\alpha),$$

where $F_{r,N-r-k(s+1)+1}(\alpha)$ is the upper 100α percentile of the F distribution with r and $N-r-k(s+1)+1$ degrees of freedom.

Using the test statistic, the T^2 type test statistic for

$$H_0 : \boldsymbol{\mu}_1^{(a)} = \boldsymbol{\mu}_1^{(b)} \text{ for all } a, b, \ 1 \le a < b \le k \ \text{ given } \ \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \cdots = \boldsymbol{\mu}_2^{(k)}$$

$$\text{vs. } H_1 :\neq H_0 \text{ given } \ \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \cdots = \boldsymbol{\mu}_2^{(k)},$$

is given by

$$T_{\max}^2 = \max_{1 \le a < b \le k} T_{ab}^2 \ .$$

Generally, it is not easy to obtain the upper percentile of the $T_{\max}^2$ statistic. Therefore, in this section, we adopt Bonferroni's approximation, which is one of the solutions to this problem. We first consider the T_{ab}^2 test statistic. We consider the upper 100α percentile of the $T_{\max}^2$ statistic $t_{\max}^2(\alpha)$ when $N^{(1)} = N^{(2)} = \cdots = N^{(k)}$. An approximate upper percentile of $T_{\max}^2$, $t_{\mathrm{BON}}^2(\alpha)$, is a solution of the equation given by

$$\sum_{1 \le a < b \le k} \Pr(T_{ab}^2 > t_{\mathrm{BON}}^2(\alpha)) = \alpha.$$

Using Bonferroni's approximation, $t_{\mathrm{BON}}^2(\alpha)$ is given by

$$t_{\mathrm{BON}}^2(\alpha) = t_{ab}^2(\alpha^*),$$

where $\alpha^* = 2\alpha/k(k-1)$, $t^2_{ab}(\alpha^*)$ is the upper $100\alpha^*$ percentiles of the T^2_{ab} test statistic. However, it is not easy to obtain $t^2_{ab}(\alpha^*)$. Using $t^2_{ab\cdot\mathrm{app}}(\alpha)$, the approximate upper $100\alpha^*$ percentile of the $T^2_{\max}$ statistic is given by

$$t^2_{\mathrm{B}\cdot\mathrm{app}}(\alpha) = \frac{(N-k)r}{N-k(s+1)-r+1} F_{r,N-k(s+1)-r+1}(\alpha^*),$$

where the $F_{r,N-k(s+1)-r+1}(\alpha^*)$ is the upper approximate $100\alpha^*$ percentile of F distribution with r and $N-r-k(s+1)+1$ degrees of freedom.

4.2 Simultaneous confidence intervals for multiple comparisons among sub-mean vectors

In this section, we consider simultaneous confidence intervals for pairwise multiple comparisons among sub-mean vectors. Using the approximate upper percentile of $T^2_{\max}$, for any nonnull vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_p)'$, the approximate simultaneous confidence intervals for $\boldsymbol{a}'(\boldsymbol{\mu}^{(i)}_{1\cdot 2} - \boldsymbol{\mu}^{(j)}_{1\cdot 2})$, $1 \le a < b \le k$ are given by

$$\boldsymbol{a}'\boldsymbol{u}_{ab} - \sqrt{\frac{c_{ab}}{N-k}} L_{\mathrm{app}} \le \boldsymbol{a}'(\boldsymbol{\mu}^{(a)}_{1\cdot 2} - \boldsymbol{\mu}^{(b)}_{1\cdot 2}) \le \boldsymbol{a}'\boldsymbol{u}_{ab} + \sqrt{\frac{c_{ab}}{N-k}} L_{\mathrm{app}}, \ \forall \boldsymbol{a} \in \mathbb{R}^r - \{\boldsymbol{0}\}, \ 1 \le a < b \le k,$$

where

$$L_{\mathrm{app}} = \left(t^2_{\mathrm{B}\cdot\mathrm{app}}(\alpha)\boldsymbol{a}'V_{11\cdot 2}\boldsymbol{a}\right)^{\frac{1}{2}}, \ c_{ab} = \frac{(N^{(a)}+N^{(b)})(N-k-1)}{N^{(a)}N^{(b)}(N-k-s-1)}.$$

5 Simulation studies

In this section, we perform a Monte Carlo simulation (with 10^6 runs) in order to evaluate the asymptotic behavior of the F approximations and the accuracy of the approximate upper 100α percentiles of the T^2_{12} and $T^2_{\max}$ test statistic.

Tables 1a and 1b present the simulated upper 100α percentile of the T^2_{12} test statistic and the approximate upper 100α percentile of the T^2_{12} test statistic for the two-sample problem; $(p, r, s) = (4,1,3), (4,2,2), (4,3,1), (8,2,6), (8,4,4), (8,6,2)$; $\alpha = 0.05, 0.01$; and for the following two cases of $(N^{(1)}, N^{(2)})$:

$$(N^{(1)},\ N^{(2)}) = \begin{cases} (\ell,\ell),\ \ell = 20, 40, 100, 200, 400 \\ (\ell, 2\ell),\ \ell = 20, 40, 100, 200 \end{cases}.$$

Tables 1a and 1b present the type I errors for the upper 100α percentile of the χ^2 distribution with r degrees of freedom and the approximate upper 100α percentile of the T^2_{12} test statistic given by

$$\alpha_1 = \Pr(T^2_{12} > \chi^2_r(\alpha)),\ \alpha_2 = \Pr(T^2_{12} > t^2_{12\cdot\mathrm{app}}(\alpha)),$$

respectively. Table 2 gives the approximate upper 100α percentile of the $T^2_{\max}$ statistic, $t^2_{\text{BON}}(\alpha)$, and the approximate upper 100α percentile of the $T^2_{\max}$ statistic, $t^2_{\text{B}\cdot\text{app}}(\alpha)$, for the k-sample problem; $k = 3, 4, 5, 6, 10$; $(p, r, s) = (4, 2, 2)$; $\alpha = 0.05, 0.01$; $N^{(k)} = \ell$, where $\ell = 20, 40, 100, 200, 400$. Table 2 presents the type I error for the approximate upper 100α percentile of the $T^2_{\max}$ statistic:

$$\alpha_3 = \Pr(T^2_{\max} > t^2_{\text{B}\cdot\text{app}}(\alpha^*)),$$

where $\alpha^* = 2\alpha/k(k-1)$. It may be noted from Tables 1a and 1b that the simulated values approach closer to the upper percentile of the χ^2 distribution when both of the sample sizes $N^{(1)}$ and $N^{(2)}$ become large. In addition, it can be seen from both tables that the proposed approximation value is good for all cases even when the sample size is small. The results for the type I error of the proposed approximation value are closer than those of the χ^2 value for all cases.

In Table 2, the simulated values also approach the upper 100α percentile of the χ^2 distribution when both of the sample sizes $N^{(1)}$ and $N^{(2)}$ become large. The results for the type I error of the proposed approximation value are always lower than α for all cases. That is, it is conservative in terms of the type I error when k is large. Therefore, in terms of the type I error, it can be concluded that the proposed approximation values are more accurate than the simulated values for all cases. In addition, we perform a Monte Carlo simulation (with 10^6 runs) in order to evaluate the powers of the T^2_{12} test statistic.

Table 3a presents the powers of the T^2_{12} test statistic for the two-sample problem; $(p, r, s) = (4, 2, 2)$; $\alpha = 0.05, 0.01$; $\delta = |\boldsymbol{\mu}^{(1)}_1 - \boldsymbol{\mu}^{(2)}_1| = 0, 0.2, \ldots, 2$; and for the following cases of $(N^{(1)}, N^{(2)})$: $N^{(1)} = N^{(1)} = \ell = 10, 40, 50$. Table 3$b$ presents the powers of the T^2_{12} test statistic for the two-sample problem; $r = 2$, $(p, r, s) = (4, 2, 2), (5, 2, 3), (6, 2, 4), (10, 2, 8)$; $\alpha = 0.05, 0.01$; $\delta = |\boldsymbol{\mu}^{(1)}_1 - \boldsymbol{\mu}^{(2)}_1| = 0, 0.2, \ldots, 2$; $(N^{(1)}, N^{(2)}) = (50, 50)$. Tables 3$a$ and 3b present the powers of the T^2_{12} test statistic:

$$\beta_1 = \Pr(T^2_{12} > \chi^2_r(\alpha)|H_1),\ \beta_2 = \Pr(T^2_{12} > t^2_{12\cdot\text{app}}(\alpha)|H_1),$$

respectively. It may be noted from Table 3a that the power takes conservative values when both of the sample sizes $N^{(1)}$ and $N^{(2)}$ become small. In addition, it may be noted from Table 3b that the power takes values that are a little conservative when s become small. The reasons are that the power may depend on the noncentrality parameters only through $\theta^2 = \boldsymbol{\delta}'_{12}\Sigma^{-1}_{11\cdot2}\boldsymbol{\delta}_{12}/2c_{12}$, where $\boldsymbol{\delta}_{12} = \boldsymbol{\mu}^{(1)}_1 - \boldsymbol{\mu}^{(2)}_1$.

6 Numerical example

In this section, we discuss an example to illustrate the approximation of $t^2_{\mathrm{B\cdot app}}(\alpha)$ by comparing each of the simultaneous confidence intervals in Section 4.2. In this example, we utilize the data in the iris plant taken from Fisher (1936). Data are presented on three species of iris, setosa, versicolor, and virginica, comprising four different measurements: x_1: petal width, x_2: petal length, x_3: sepal width, and x_4: sepal length. The population mean vectors are $\boldsymbol{\mu}^{(i)} = (\boldsymbol{\mu}_1^{(i)\prime}, \boldsymbol{\mu}_2^{(i)\prime})' = (\mu_1^{(i)}, \mu_2^{(i)}, \mu_3^{(i)}, \mu_4^{(i)})'$, where $\mu_1^{(i)}$: mean of petal width, $\mu_2^{(i)}$: mean of petal length, $\mu_3^{(i)}$: mean of sepal width, $\mu_4^{(i)}$: mean of sepal length, $\boldsymbol{\mu}_1^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)})'$, and $\boldsymbol{\mu}_2^{(i)} = (\mu_3^{(i)}, \mu_4^{(i)})'$. We assume that these data are distributed normality, and $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \boldsymbol{\mu}_2^{(3)}$.

We consider the simultaneous approximate confidence intervals of pairwise comparisons for testing the sub-mean vectors hypothesis:

$$H_0 : \boldsymbol{\mu}_1^{(a)} = \boldsymbol{\mu}_1^{(b)} \text{ for all } a, b,\ 1 \le a < b \le 3 \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \boldsymbol{\mu}_2^{(3)}$$
$$\text{vs. } H_1 :\neq H_0 \text{ given } \boldsymbol{\mu}_2^{(1)} = \boldsymbol{\mu}_2^{(2)} = \boldsymbol{\mu}_2^{(3)}.$$

Table 4 presents the simultaneous confidence intervals obtained by using $t^2_{\max}(\alpha)$, $t^2_{\mathrm{BON}}(\alpha)$, and $t^2_{\mathrm{B\cdot app}}(\alpha)$. For example, let $\boldsymbol{a} = (0, 1)'$ and $\alpha = 0.05$; then the simultaneous confidence intervals obtained by using $t^2_{\max}(0.05)$, $t^2_{\mathrm{BON}}(0.05)$, and $t^2_{\mathrm{B\cdot app}}(0.05)$ for pairwise comparisons are constructed as $\boldsymbol{a}'(\boldsymbol{\mu}_{1\cdot 2}^{(1)} - \boldsymbol{\mu}_{1\cdot 2}^{(2)})$ obtained by using

$$t^2_{\max}(0.05) : (-2.274, -2.066),\ t^2_{\mathrm{BON}}(0.05) : (-2.275, -2.065),\ t^2_{\mathrm{B\cdot app}}(0.05) : (-2.277, -2.063),$$

respectively. These simultaneous confidence intervals present the petal length between the score of the setosa population and the score of the versicolor population. The simultaneous confidence intervals obtained by using the approximate upper 100α percentile, $t^2_{\mathrm{B\cdot app}}(0.05)$, are a little longer than those obtained by using $t^2_{\mathrm{BON}}(0.05)$ and $t^2_{\max}(0.05)$. The remaining results are the same. Therefore, it can be concluded that the simultaneous confidence intervals obtained by using $t^2_{\mathrm{B\cdot app}}(\alpha)$ are useful.

Table 1a $t^2_{12}(\alpha)$ and $t^2_{12\cdot\mathrm{app}}(\alpha)$ when $p = 4$

Sample Size		$\alpha = 0.05$				$\alpha = 0.01$			
$N^{(1)}$	$N^{(2)}$	$t^2_{12}(\alpha)$	$t^2_{12\cdot\mathrm{app}}(\alpha)$	α_1	α_2	$t^2_{12}(\alpha)$	$t^2_{12\cdot\mathrm{app}}(\alpha)$	α_1	α_2
$(r,s) = (1,3)$, $\chi^2_1(0.05) = 3.84$, $\chi^2_1(0.01) = 6.64$									
20	20	4.48	4.93	0.068	0.040	8.06	8.91	0.018	0.007
40	40	4.13	4.31	0.058	0.045	7.27	7.58	0.014	0.009
100	100	3.94	4.01	0.053	0.048	7.15	7.37	0.013	0.009
200	200	3.89	3.92	0.052	0.049	6.84	6.98	0.011	0.009
400	400	3.88	3.88	0.051	0.050	6.76	6.80	0.011	0.010
20	40	4.23	4.49	0.062	0.044	7.55	7.97	0.015	0.008
40	80	4.04	4.14	0.056	0.047	7.05	7.23	0.012	0.009
100	200	3.92	3.95	0.052	0.049	6.80	6.86	0.011	0.010
200	400	3.88	3.90	0.051	0.050	6.71	6.75	0.010	0.010
$(r,s) = (2,2)$, $\chi^2_2(0.05) = 5.99$, $\chi^2_2(0.01) = 9.21$									
20	20	7.10	7.57	0.077	0.042	11.45	12.23	0.022	0.008
40	40	6.50	6.67	0.062	0.046	10.20	10.49	0.015	0.009
100	100	6.19	6.24	0.055	0.049	9.60	9.68	0.012	0.010
200	200	6.07	6.11	0.052	0.049	9.38	9.44	0.011	0.010
400	400	6.03	6.05	0.051	0.049	9.30	9.32	0.010	0.010
20	40	6.68	6.94	0.067	0.045	10.63	11.01	0.018	0.009
40	80	6.32	6.43	0.058	0.048	9.86	10.02	0.013	0.009
100	200	6.12	6.16	0.053	0.049	9.46	9.52	0.011	0.010
200	400	6.05	6.07	0.051	0.049	9.32	9.36	0.011	0.010
$(r,s) = (3,1)$, $\chi^2_3(0.05) = 7.81$, $\chi^2_3(0.01) = 11.35$									
20	20	9.37	9.67	0.085	0.045	14.38	14.81	0.026	0.009
40	40	8.53	8.63	0.066	0.048	12.68	12.83	0.017	0.009
100	100	8.09	8.12	0.056	0.049	11.84	11.89	0.012	0.010
200	200	7.94	7.96	0.053	0.049	11.59	11.61	0.011	0.010
400	400	7.87	7.89	0.051	0.050	11.43	11.48	0.010	0.010
20	40	8.77	8.94	0.071	0.047	13.14	13.43	0.019	0.009
40	80	8.27	8.36	0.060	0.049	12.18	12.29	0.014	0.010
100	200	7.99	8.01	0.054	0.050	11.66	11.70	0.012	0.010
200	400	7.91	7.91	0.052	0.050	11.54	11.52	0.011	0.010

Table 1b $t^2_{12}(\alpha)$ and $t^2_{12\cdot\mathrm{app}}(\alpha)$ when $p = 8$

Sample Size		$\alpha = 0.05$				$\alpha = 0.01$			
$N^{(1)}$	$N^{(2)}$	$t^2_{12}(\alpha)$	$t^2_{12\cdot\mathrm{app}}(\alpha)$	α_1	α_2	$t^2_{12}(\alpha)$	$t^2_{12\cdot\mathrm{app}}(\alpha)$	α_1	α_2
$(r,s) = (2,6),\ \chi^2_2(0.05) = 5.99,\ \chi^2_2(0.01) = 9.21$									
20	20	8.13	10.29	0.103	0.025	13.36	16.93	0.035	0.004
40	40	6.87	7.53	0.072	0.038	10.82	11.87	0.019	0.007
100	100	6.31	6.52	0.058	0.045	9.75	10.11	0.013	0.009
200	200	6.13	6.24	0.054	0.047	9.49	9.64	0.011	0.009
400	400	6.06	6.11	0.052	0.049	9.35	9.42	0.011	0.010
20	40	7.24	8.26	0.082	0.034	11.55	13.17	0.024	0.006
40	80	6.53	6.93	0.064	0.042	10.17	10.82	0.015	0.008
100	200	6.19	6.33	0.055	0.047	9.60	9.79	0.012	0.009
200	400	6.08	6.16	0.052	0.048	9.38	9.49	0.011	0.009
$(r,s) = (4,4),\ \chi^2_4(0.05) = 9.49,\ \chi^2_4(0.01) = 13.3$									
20	20	13.16	15.36	0.129	0.028	19.69	23.11	0.048	0.005
40	40	11.00	11.68	0.082	0.040	15.80	16.82	0.023	0.007
100	100	10.02	10.25	0.061	0.046	14.15	14.49	0.014	0.009
200	200	9.75	9.85	0.550	0.048	13.74	13.85	0.012	0.010
400	400	9.61	9.67	0.052	0.049	13.52	13.56	0.011	0.010
20	40	11.62	12.68	0.096	0.036	16.92	18.50	0.030	0.006
40	80	10.43	10.84	0.067	0.043	14.89	15.44	0.018	0.008
100	200	9.84	9.98	0.057	0.047	13.91	14.06	0.013	0.009
200	400	9.65	9.73	0.530	0.048	13.56	13.66	0.011	0.010
$(r,s) = (6,2),\ \chi^2_6(0.05) = 12.59,\ \chi^2_6(0.01) = 16.8$									
20	20	17.79	19.12	0.152	0.038	25.49	27.51	0.062	0.007
40	40	14.71	15.15	0.091	0.044	20.27	20.86	0.027	0.008
100	100	13.35	13.50	0.064	0.048	17.98	18.22	0.015	0.009
200	200	12.95	13.03	0.057	0.049	17.38	17.49	0.011	0.010
400	400	12.80	12.80	0.054	0.050	17.07	17.14	0.010	0.010
20	40	15.59	16.27	0.109	0.042	21.74	22.69	0.036	0.008
40	80	13.93	14.18	0.076	0.046	18.95	19.30	0.020	0.009
100	200	13.10	13.18	0.059	0.049	17.57	17.72	0.013	0.009
200	400	12.80	12.80	0.054	0.050	17.20	17.26	0.012	0.010

Table 2 $t^2_{\text{BON}}(\alpha)$ and $t^2_{\text{B}\cdot\text{app}}(\alpha)$ when $(p,r,s)=(4,2,2)$

Sample Size	$\alpha=0.05$			$\alpha=0.01$		
$N^{(i)}$	$t^2_{\text{BON}}(\alpha)$	$t^2_{\text{B}\cdot\text{app}}(\alpha)$	α_3	$t^2_{\text{BON}}(\alpha)$	$t^2_{\text{B}\cdot\text{app}}(\alpha)$	α_3
			$k=3$			
20	9.01	10.14	0.033	13.26	14.61	0.006
40	8.43	9.04	0.038	12.15	12.79	0.008
100	8.15	8.51	0.043	11.67	11.92	0.009
200	8.05	8.34	0.044	11.43	11.66	0.009
400	7.99	8.27	0.044	11.36	11.53	0.009
			$k=4$			
20	10.21	11.68	0.028	14.32	15.99	0.005
40	9.69	10.50	0.035	13.45	14.19	0.007
100	9.39	9.92	0.039	12.93	13.31	0.008
200	9.32	9.74	0.041	12.77	13.05	0.009
400	9.26	9.66	0.042	12.68	12.92	0.009
			$k=5$			
20	11.06	12.77	0.025	15.10	16.98	0.005
40	10.62	11.56	0.033	14.32	15.21	0.007
100	10.33	10.96	0.035	13.88	14.33	0.008
200	10.26	10.77	0.040	13.72	14.07	0.009
400	10.22	10.68	0.040	13.64	13.94	0.009
			$k=6$			
20	11.77	13.63	0.024	15.78	17.76	0.004
40	11.33	12.40	0.032	15.03	16.01	0.006
100	11.11	11.78	0.037	14.63	15.14	0.008
200	11.03	11.59	0.039	14.51	14.88	0.008
400	10.99	11.50	0.040	14.48	14.75	0.009
			$k=10$			
20	13.63	15.93	0.019	17.47	19.89	0.004
40	13.32	14.65	0.028	16.97	18.19	0.006
100	13.16	14.00	0.035	16.70	17.34	0.007
200	13.09	13.80	0.036	16.60	17.08	0.008
400	13.07	13.70	0.038	16.57	16.95	0.008

Note : $\chi^2_2\left(\frac{0.05}{3}\right)=8.19,\ \chi^2_2\left(\frac{0.01}{3}\right)=11.41,\ \chi^2_2\left(\frac{0.05}{6}\right)=9.57,\ \chi^2_2\left(\frac{0.01}{6}\right)=12.79,$

$\chi^2_2\left(\frac{0.05}{10}\right)=10.60,\ \chi^2_2\left(\frac{0.01}{10}\right)=13.82,\ \chi^2_2\left(\frac{0.05}{15}\right)=11.41,\ \chi^2_2\left(\frac{0.01}{15}\right)=14.63,$

$\chi^2_2\left(\frac{0.05}{45}\right)=13.60,\ \chi^2_2\left(\frac{0.01}{45}\right)=16.82.$

Table 3a Power for T_{12}^2 test statistic when $(p, r, s) = (4, 2, 2)$

		$\alpha = 0.05$		$\alpha = 0.01$	
δ	$(N^{(1)}, N^{(2)})$	$1-\beta_1$	$1-\beta_2$	$1-\beta_1$	$1-\beta_2$
0	(10,10)	0.106	0.028	0.040	0.004
0.2		0.114	0.031	0.045	0.005
0.4		0.147	0.046	0.063	0.008
0.6		0.191	0.065	0.088	0.013
0.8		0.259	0.099	0.130	0.022
1.0		0.342	0.146	0.187	0.037
1.2		0.438	0.208	0.258	0.059
1.4		0.541	0.286	0.346	0.091
1.6		0.637	0.376	0.442	0.139
1.8		0.726	0.473	0.541	0.198
2.0		0.811	0.575	0.642	0.269
0	(40,40)	0.062	0.046	0.015	0.009
0.2		0.094	0.072	0.027	0.017
0.4		0.203	0.167	0.078	0.053
0.6		0.393	0.341	0.196	0.146
0.8		0.620	0.568	0.391	0.318
1.0		0.814	0.776	0.622	0.545
1.2		0.932	0.912	0.817	0.759
1.4		0.981	0.974	0.932	0.902
1.6		0.996	0.995	0.982	0.971
1.8		1.000	0.999	0.997	0.994
2.0		1.000	1.000	0.999	0.999
0	(50,50)	0.059	0.046	0.014	0.009
0.2		0.100	0.081	0.029	0.020
0.4		0.239	0.206	0.097	0.072
0.6		0.470	0.427	0.252	0.205
0.8		0.720	0.683	0.497	0.435
1.0		0.895	0.874	0.746	0.692
1.2		0.973	0.965	0.908	0.879
1.4		0.995	0.994	0.977	0.967
1.6		0.999	0.999	0.996	0.994
1.8		1.000	1.000	1.000	0.999
2.0		1.000	1.000	1.000	1.000

Note : $\delta = |\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}|$, $\beta_1 = \Pr(T_{12}^2 > \chi_r^2(\alpha)|H_1)$, $\beta_2 = \Pr(T_{12}^2 > t_{12\cdot\mathrm{app}}^2(\alpha)|H_1)$

Table 3b Power for T^2_{12} test statistic when $(N^{(1)}, N^{(2)}) = (50, 50)$, $r = 2$

		$\alpha = 0.05$		$\alpha = 0.01$	
δ	(p, r, s)	$1-\beta_1$	$1-\beta_2$	$1-\beta_1$	$1-\beta_2$
0	(4,2,2)	0.059	0.046	0.014	0.009
0.2		0.100	0.081	0.029	0.020
0.4		0.239	0.206	0.097	0.072
0.6		0.470	0.427	0.252	0.205
0.8		0.720	0.683	0.497	0.435
1.0		0.895	0.874	0.746	0.692
1.2		0.973	0.965	0.908	0.879
1.4		0.995	0.994	0.977	0.967
1.6		0.999	0.999	0.996	0.994
1.8		1.000	1.000	1.000	0.999
2.0		1.000	1.000	1.000	1.000
0	(5,2,3)	0.060	0.044	0.014	0.008
.2		0.102	0.079	0.030	0.019
0.4		0.239	0.198	0.097	0.068
0.6		0.470	0.416	0.254	0.197
0.8		0.718	0.670	0.497	0.421
1.0		0.894	0.866	0.744	0.678
1.2		0.972	0.963	0.908	0.871
1.4		0.995	0.993	0.977	0.964
1.6		0.999	0.999	0.996	0.993
1.8		1.000	1.000	1.000	0.999
2.0		1.000	1.000	1.000	1.000
0	(6,2,4)	0.062	0.043	0.015	0.008
0.2		0.102	0.075	0.030	0.017
0.4		0.241	0.192	0.098	0.064
0.6		0.469	0.405	0.256	0.189
0.8		0.720	0.661	0.499	0.408
1.0		0.893	0.860	0.745	0.665
1.2		0.972	0.959	0.907	0.862
1.4		0.995	0.992	0.978	0.960
1.6		0.999	0.999	0.996	0.992
1.8		1.000	1.000	1.000	0.999
2.0		1.000	1.000	1.000	1.000
0	(10,2,8)	0.067	0.035	0.017	0.006
0.2		0.110	0.063	0.034	0.014
0.4		0.247	0.164	0.103	0.050
0.6		0.472	0.360	0.261	0.153
0.8		0.715	0.610	0.498	0.349
1.0		0.889	0.824	0.740	0.602
1.2		0.970	0.944	0.903	0.818
1.4		0.995	0.988	0.971	0.941
1.6		0.999	0.998	0.996	0.987
1.8		1.000	1.000	0.999	0.998
2.0		1.000	1.000	1.000	1.000

Note : $\delta = |\boldsymbol{\mu}_1^{(1)} - \boldsymbol{\mu}_1^{(2)}|$, $\beta_1 = \Pr(T^2_{12} > \chi^2_r(\alpha)|H_1)$, $\beta_2 = \Pr(T^2_{12} > t^2_{12\cdot\mathrm{app}}(\alpha)|H_1)$

Table 4 : Simultaneous confidence intervals obtained by using $t^2_{\max}(0.05)$, $t^2_{\mathrm{BON}}(0.05)$, and $t^2_{\mathrm{B\cdot app}}(0.05)$

$\boldsymbol{a}$	$(1,0)'$	$(0,1)'$	$(1,1)'$
$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2}-\boldsymbol{\mu}^{(2)}_{1\cdot 2})$	$\mu^{(1)}_1-\mu^{(2)}_1$	$\mu^{(1)}_2-\mu^{(2)}_2$	$(\mu^{(1)}_1+\mu^{(1)}_2)-(\mu^{(2)}_1+\mu^{(2)}_2)$
$t^2_{\max}(0.05)$	$(-1.277,-1.070)$	$(-2.274,-2.066)$	$(-3.448,-3.240)$
$t^2_{\mathrm{BON}}(0.05)$	$(-1.277,-1.070)$	$(-2.275,-2.065)$	$(-3.449,-3.239)$
$t^2_{\mathrm{B\cdot app}}(0.05)$	$(-1.281,-1.067)$	$(-2.277,-2.063)$	$(-3.451,-3.237)$
$\boldsymbol{a}'(\boldsymbol{\mu}^{(1)}_{1\cdot 2}-\boldsymbol{\mu}^{(3)}_{1\cdot 2})$	$\mu^{(1)}_1-\mu^{(3)}_1$	$\mu^{(1)}_2-\mu^{(3)}_2$	$(\mu^{(1)}_1+\mu^{(1)}_2)-(\mu^{(3)}_1+\mu^{(3)}_2)$
$t^2_{\max}(0.05)$	$(-1.889,-1.681)$	$(-3.153,-2.945)$	$(-4.938,-4.730)$
$t^2_{\mathrm{BON}}(0.05)$	$(-1.890,-1.679)$	$(-3.155,-2.944)$	$(-4.939,-4.729)$
$t^2_{\mathrm{B\cdot app}}(0.05)$	$(-1.892,-1.678)$	$(-3.156,-2.942)$	$(-4.940,-4.727)$
$\boldsymbol{a}'(\boldsymbol{\mu}^{(2)}_{1\cdot 2}-\boldsymbol{\mu}^{(3)}_{1\cdot 2})$	$\mu^{(2)}_1-\mu^{(3)}_1$	$\mu^{(2)}_2-\mu^{(3)}_2$	$(\mu^{(2)}_1+\mu^{(2)}_2)-(\mu^{(3)}_1+\mu^{(3)}_2)$
$t^2_{\max}(0.05)$	$(-0.715,-0.507)$	$(-0.983,-0.775)$	$(-1.594,-1.386)$
$t^2_{\mathrm{BON}}(0.05)$	$(-0.716,-0.506)$	$(-0.984,-0.774)$	$(-1.595,-1.385)$
$t^2_{\mathrm{B\cdot app}}(0.05)$	$(-0.718,-0.504)$	$(-0.986,-0.772)$	$(-2.012,-0.968)$

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