Testing Equality of Two Mean Vectors with Monotone Missing Data

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Abstract

In this paper, testing the equality of two mean vectors is considered when each dataset has a monotone pattern of missing observations. A simplified Hotelling's T^2-type test statistic in a two-sample problem under monotone missing data is given. Approximation to the upper percentile of this simplified T^2 statistic and the transformed test statistics based on Bartlett adjustment in the case of the two-sample problem are derived. Furthermore, the approximate simultaneous confidence intervals for pairwise comparisons among mean vectors with monotone missing data are described. Finally, the accuracy of the approximations is investigated using a Monte Carlo simulation.

Key Words and Phrases: Asymptotic expansion; Hotelling's T^2-type test statistic; Monte Carlo simulation; Pairwise comparisons; Simultaneous confidence intervals; Transformed test statistic; Two-sample problem

1 Introduction

In testing for mean vectors, the Hotelling's T^2 test statistic is often used. Under multivariate normality, the Hotelling's T^2 test is a likelihood ratio test (LRT) and the null distribution is exactly distributed as F-distribution multiplied by a constant of sample size and dimension (see, e.g., Siotani, Hayakawa, and Fujikoshi (1985), Morrison (2005)). As for the null and non-null distributions of the Hotelling's T^2 statistic under nonnormality, an asymptotic expansion approximation has been given by Fujikoshi (1997), Kano (1995), Kakizawa and Iwashita (2008), and so on. However, under missing data, it is not easy to derive the null distribution of Hotelling's T^2 statistic even with the assumption of multivariate normality, because the Hotelling's T^2 statistic is defined by using the MLEs of the mean vector and the covariance matrix with the missing data. Srivastava and Carter (1986) and Shutoh, Kusumi, Morinaga, Yamada, and Seo (2010) calculated the Hotelling's T^2 statistic with missing data by using the numerical solution for likelihood equations by the Newton-Raphson method. However, since it is not easy to derive the exact upper percentile of the null distribution, the upper percentile of χ^2 distribution is used as an approximation in those papers. We note that χ^2 approximation is not a good approximation for a small sample size because the limiting distribution of the null distribution is χ^2 distribution. Recently, in a one-sample problem, the null distribution of the simplified Hotelling's T^2-type statistic with monotone missing data (k-step monotone missing data) has been discussed and a good approximation to the upper percentile of its null distribution was given by Yagi, Seo, and Hanusz (2018). This was derived by the decomposition of the test statistic and an asymptotic expansion of the distribution of each decomposed statistic by the perturbation method. Other approximations to its null distribution were discussed by Krishnamoorthy and Pannala (1999) and Yagi and Seo (2017), in the case of k-step monotone missing data. In the case of two-step monotone missing data, linear interpolation approximation for the null distribution of the Hotelling's T^2-type statistic was given by Seko, Yamazaki, and Seo (2012). Yu, Krishnamoorthy, and Pannala (2006) and Seko, Kawasaki, and Seo (2011) provided the extension for the two-sample problem. For related papers on LRT, see Krishnamoorthy and Pannala (1998), Yagi, Seo, and Srivastava (2017, 2018), among others. Although the type of missing data is a two-step monotone pattern of missing observations, an asymptotic expansion of the null distribution for the Hotelling's T^2-type statistic for a one-sample problem was derived recently by Kawasaki, Shutoh, and Seo (2018). In this paper, we consider the null distribution of the simplified T^2 statistic with k-step monotone missing data in a two-sample problem. The results of this paper are extensions of the one-sample problem in Yagi, Seo, and Hanusz (2018). The organization of this paper is as follows. In Section 2, an asymptotic expansion of the null distribution of the simplified T^2 statistic is derived for the three-step case. In the k-step case, an approximation to an asymptotic expansion of the null distribution of the statistic can be obtained by considering generalization of a three-step case. Further, two types of transformation for the simplified T^2 statistic with

Bartlett correction and Bartlett-type correction are presented in Section 3. In Section 4, simulation study on the upper percentiles of these test statistics and empirical type I errors are presented. Finally, concluding remarks are given in Section 5.

2 The simplified T^2 statistic and its distribution

Suppose two datasets with the same monotone pattern of missing observations are independent and distributed as multivariate normal distribution with common covariance matrix. Then, the ℓ-th dataset with a monotone pattern of missing observations is of the form

$$\boldsymbol{X}^{(\ell)} = \begin{pmatrix} \overbrace{\boldsymbol{X}_{11}^{(\ell)}}^{p_1} & \overbrace{\boldsymbol{X}_{12}^{(\ell)}}^{p_2} & \cdots & \overbrace{\boldsymbol{X}_{1,k-1}^{(\ell)}}^{p_{k-1}} & \overbrace{\boldsymbol{X}_{1k}^{(\ell)}}^{p_k} \\ \boldsymbol{X}_{21}^{(\ell)} & \boldsymbol{X}_{22}^{(\ell)} & \cdots & \boldsymbol{X}_{2,k-1}^{(\ell)} & * \\ \vdots & \vdots & & & \vdots \\ \boldsymbol{X}_{k-1,1}^{(\ell)} & \boldsymbol{X}_{k-1,2}^{(\ell)} & * & \cdots & * \\ \boldsymbol{X}_{k1}^{(\ell)} & * & \cdots & \cdots & * \end{pmatrix} \begin{matrix} \}n_1^{(\ell)} \\ \}n_2^{(\ell)} \\ \\ \}n_{k-1}^{(\ell)} \\ \}n_k^{(\ell)} \end{matrix}, \quad \ell = 1, 2, \tag{1}$$

where $\boldsymbol{X}_{ij}^{(\ell)}$ is a $n_i^{(\ell)} \times p_j$ block matrix ($i = 1, 2, \ldots, k; j = 1, 2, \ldots, k - i + 1$), and "$*$" indicates a missing part. Such a dataset is called a k-step monotone missing data pattern. Further, let $\boldsymbol{\mu}^{(\ell)}$ be the p dimensional mean vector for dataset $\boldsymbol{X}^{(\ell)}$, where $p = \sum_{i=1}^{k} p_i$. Then, the simplified T^2 statistic for the hypothesis

$$H_0 : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} \quad \text{vs.} \quad H_1 : \boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)}$$

is given by

$$Q = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})' \, \widetilde{\boldsymbol{\Gamma}}^{-1} (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}), \tag{2}$$

where $\widehat{\boldsymbol{\mu}}^{(\ell)}$ is the MLE of $\boldsymbol{\mu}^{(\ell)}$ ($\ell = 1, 2$), $\widetilde{\boldsymbol{\Gamma}}$ is a simple estimator of $\mathrm{Cov}[\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}]$. We note that Q in (2) is the same one as $\widetilde{T}_2^2$ in Section 3.2 of Yagi and Seo (2017) and the quantity $\widetilde{\boldsymbol{\Gamma}}$ is also defined there. In this paper, to illustrate the derivation of an asymptotic expansion approximation to the distribution of Q concisely, we use the new notations and definitions from our previous papers.

2.1 Three-step case ($k = 3$)

For illustration, we first consider the null distribution of the simplified T^2 statistic Q

in the case of three-step monotone missing data. Then, $\boldsymbol{X}^{(\ell)}$ can be reduced as

$$\boldsymbol{X}^{(\ell)} = \left(\begin{array}{ccc} \overbrace{\boldsymbol{X}_{11}^{(\ell)}}^{p_1} & \overbrace{\boldsymbol{X}_{12}^{(\ell)}}^{p_2} & \overbrace{\boldsymbol{X}_{13}^{(\ell)}}^{p_3} \\ \boldsymbol{X}_{21}^{(\ell)} & \boldsymbol{X}_{22}^{(\ell)} & * \\ \boldsymbol{X}_{31}^{(\ell)} & * & * \end{array} \right) \begin{array}{l} \}n_1^{(\ell)} \\ \}n_2^{(\ell)} \\ \}n_3^{(\ell)} \end{array}, \tag{3}$$

where rows of $\boldsymbol{X}^{(\ell)}$ (excluding the missing parts) are distributed as multivariate normal distributions. That is, there are $n_1^{(\ell)}$ observations available on the 1st $p(= p_1 + p_2 + p_3)$ components, $n_2^{(\ell)}$ observations available on the 2nd $p_{(12)}(= p_1 + p_2)$ components, and $n_3^{(\ell)}$ observations available on the 3rd p_1 components. For the dataset in (3), let

$$\boldsymbol{X}_{1(123)}^{(\ell)} = \left(\boldsymbol{X}_{11}^{(\ell)} \quad \boldsymbol{X}_{12}^{(\ell)} \quad \boldsymbol{X}_{13}^{(\ell)} \right), \quad \boldsymbol{X}_{2(12)}^{(\ell)} = \left(\boldsymbol{X}_{21}^{(\ell)} \quad \boldsymbol{X}_{22}^{(\ell)} \right),$$

where rows of $\boldsymbol{X}_{1(123)}^{(\ell)}$, $\boldsymbol{X}_{2(12)}^{(\ell)}$, and $\boldsymbol{X}_{31}^{(\ell)}$ are mutually independent and

$$\begin{aligned} \mathrm{vec}(\boldsymbol{X}_{1(123)}^{(\ell)}) &\sim N_{n_1^{(\ell)}p}(\mathrm{vec}(\mathbf{1}_{n_1^{(\ell)}}\boldsymbol{\mu}^{(\ell)\prime}), \boldsymbol{I}_{n_1^{(\ell)}} \otimes \boldsymbol{\Sigma}), \\ \mathrm{vec}(\boldsymbol{X}_{2(12)}^{(\ell)}) &\sim N_{n_2^{(\ell)}p_{(12)}}(\mathrm{vec}(\mathbf{1}_{n_2^{(\ell)}}\boldsymbol{\mu}_{(12)}^{(\ell)\prime}), \boldsymbol{I}_{n_2^{(\ell)}} \otimes \boldsymbol{\Sigma}_{(12)(12)}), \\ \mathrm{vec}(\boldsymbol{X}_{31}^{(\ell)}) &\sim N_{n_3^{(\ell)}p_1}(\mathrm{vec}(\mathbf{1}_{n_3^{(\ell)}}\boldsymbol{\mu}_1^{(\ell)\prime}), \boldsymbol{I}_{n_3^{(\ell)}} \otimes \boldsymbol{\Sigma}_{11}), \end{aligned}$$

and $\mathbf{1}_{n_i^{(\ell)}}$ is a vector on $n_i^{(\ell)}$ ones,

$$\boldsymbol{\mu}^{(\ell)} = \left(\begin{array}{c} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} \\ \hline \boldsymbol{\mu}_3^{(\ell)} \end{array} \right) \begin{array}{l} \}p_1 \\ \}p_2 \\ \}p_3 \end{array} = \left(\begin{array}{c} \boldsymbol{\mu}_{(12)}^{(\ell)} \\ \hline \boldsymbol{\mu}_3^{(\ell)} \end{array} \right),$$

and

$$\boldsymbol{\Sigma} = \left(\begin{array}{cc|c} \overbrace{\boldsymbol{\Sigma}_{11}}^{p_1} & \overbrace{\boldsymbol{\Sigma}_{12}}^{p_2} & \overbrace{\boldsymbol{\Sigma}_{13}}^{p_3} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} \\ \hline \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{array} \right) \begin{array}{l} \}p_1 \\ \}p_2 \\ \}p_3 \end{array} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\boldsymbol{\Sigma}_{(12)(12)}} & \begin{array}{c} \boldsymbol{\Sigma}_{13} \\ \boldsymbol{\Sigma}_{23} \end{array} \\ \hline \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{array} \right).$$

Indeed, $\boldsymbol{A} \otimes \boldsymbol{B}$ denotes the kronecker product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined by $\boldsymbol{A} \otimes \boldsymbol{B} = (a_{ij}\boldsymbol{B})$ and, for any matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_q)'$, we define $\mathrm{vec}(\boldsymbol{A}) = (\boldsymbol{a}_1', \boldsymbol{a}_2', \cdots, \boldsymbol{a}_q')'$.

Also, let

$$\begin{aligned} \overline{\boldsymbol{x}}_{1(123)}^{(\ell)} &= \frac{1}{N_1^{(\ell)}} \boldsymbol{X}_{1(123)}^{(\ell)\prime} \mathbf{1}_{N_1^{(\ell)}}, \\ \boldsymbol{S}_{1(123)}^{(\ell)} &= \frac{1}{N_1^{(\ell)} - 1} \left\{ \boldsymbol{X}_{1(123)}^{(\ell)\prime} \boldsymbol{X}_{1(123)}^{(\ell)} - N_1^{(\ell)} \overline{\boldsymbol{x}}_{1(123)}^{(\ell)} \overline{\boldsymbol{x}}_{1(123)}^{(\ell)\prime} \right\}, \end{aligned}$$

where $N_1^{(\ell)} = n_1^{(\ell)}$, then we partition $\overline{\boldsymbol{x}}_{1(123)}^{(\ell)}$ and $\boldsymbol{S}_{1(123)}^{(\ell)}$ as

$$\overline{\boldsymbol{x}}_{1(123)}^{(\ell)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{1(12)}^{(\ell)} \\ \overline{\boldsymbol{x}}_{13}^{(\ell)} \end{pmatrix} \begin{matrix} \}p_{(12)} \\ \}p_3 \end{matrix}, \quad \boldsymbol{S}_{1(123)}^{(\ell)} = \begin{pmatrix} \overbrace{\boldsymbol{S}_{1(123),11}^{(\ell)}}^{p_{(12)}} & \overbrace{\boldsymbol{S}_{1(123),12}^{(\ell)}}^{p_3} \\ \boldsymbol{S}_{1(123),21}^{(\ell)} & \boldsymbol{S}_{1(123),22}^{(\ell)} \end{pmatrix} \begin{matrix} \}p_{(12)} \\ \}p_3 \end{matrix} .$$

On the other hand, we define

$$\boldsymbol{X}_{(123)1}^{(\ell)} = \begin{pmatrix} \boldsymbol{X}_{11}^{(\ell)} \\ \boldsymbol{X}_{21}^{(\ell)} \\ \boldsymbol{X}_{31}^{(\ell)} \end{pmatrix}, \quad \boldsymbol{X}_{(12)(12)}^{(\ell)} = \begin{pmatrix} \boldsymbol{X}_{11}^{(\ell)} & \boldsymbol{X}_{12}^{(\ell)} \\ \boldsymbol{X}_{21}^{(\ell)} & \boldsymbol{X}_{22}^{(\ell)} \end{pmatrix}.$$

Furthermore, we define the sample mean vectors and the sample covariance matrices as follows:

$$\overline{\boldsymbol{x}}_{(123)1}^{(\ell)} = \frac{1}{N_3^{(\ell)}} \boldsymbol{X}_{(123)1}^{(\ell)\prime} \boldsymbol{1}_{N_3^{(\ell)}}, \quad \boldsymbol{S}_{(123)1}^{(\ell)} = \frac{1}{N_3^{(\ell)} - 1} \left\{ \boldsymbol{X}_{(123)1}^{(\ell)\prime} \boldsymbol{X}_{(123)1}^{(\ell)} - N_3^{(\ell)} \overline{\boldsymbol{x}}_{(123)1}^{(\ell)} \overline{\boldsymbol{x}}_{(123)1}^{(\ell)\prime} \right\},$$

$$\overline{\boldsymbol{x}}_{(12)(12)}^{(\ell)} = \frac{1}{N_2^{(\ell)}} \boldsymbol{X}_{(12)(12)}^{(\ell)\prime} \boldsymbol{1}_{N_2^{(\ell)}}, \quad \boldsymbol{S}_{(12)(12)}^{(\ell)} = \frac{1}{N_2^{(\ell)} - 1} \left\{ \boldsymbol{X}_{(12)(12)}^{(\ell)\prime} \boldsymbol{X}_{(12)(12)}^{(\ell)} - N_2^{(\ell)} \overline{\boldsymbol{x}}_{(12)(12)}^{(\ell)} \overline{\boldsymbol{x}}_{(12)(12)}^{(\ell)\prime} \right\},$$

where $N_i^{(\ell)} = \sum_{j=1}^{i} n_j^{(\ell)}, \ i = 2, 3, \ \ell = 1, 2$. Further, we partition

$$\overline{\boldsymbol{x}}_{(12)(12)}^{(\ell)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(12)1}^{(\ell)} \\ \overline{\boldsymbol{x}}_{(12)2}^{(\ell)} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \end{matrix}, \quad \boldsymbol{S}_{(12)(12)}^{(\ell)} = \begin{pmatrix} \overbrace{\boldsymbol{S}_{(12)(12),11}^{(\ell)}}^{p_1} & \overbrace{\boldsymbol{S}_{(12)(12),12}^{(\ell)}}^{p_2} \\ \boldsymbol{S}_{(12)(12),21}^{(\ell)} & \boldsymbol{S}_{(12)(12),22}^{(\ell)} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \end{matrix}.$$

Then, the simplified T^2 statistic in the case of three-step monotone missing data can be written as

$$Q = Q_1 + Q_2 + Q_3, \tag{4}$$

where

$$Q_1 = \frac{N_3^{(1)} N_3^{(2)}}{N_3^{(1)} + N_3^{(2)}} (\widehat{\boldsymbol{\eta}}_1^{(1)} - \widehat{\boldsymbol{\eta}}_1^{(2)})' \widehat{\boldsymbol{\Delta}}_{11}^{-1} (\widehat{\boldsymbol{\eta}}_1^{(1)} - \widehat{\boldsymbol{\eta}}_1^{(2)}), \tag{5}$$

$$Q_2 = \frac{N_2^{(1)} N_2^{(2)}}{N_2^{(1)} + N_2^{(2)}} (\widehat{\boldsymbol{\eta}}_2^{(1)} - \widehat{\boldsymbol{\eta}}_2^{(2)})' \widehat{\boldsymbol{\Delta}}_{22}^{-1} (\widehat{\boldsymbol{\eta}}_2^{(1)} - \widehat{\boldsymbol{\eta}}_2^{(2)}), \tag{6}$$

$$Q_3 = \frac{N_1^{(1)} N_1^{(2)}}{N_1^{(1)} + N_1^{(2)}} (\widehat{\boldsymbol{\eta}}_3^{(1)} - \widehat{\boldsymbol{\eta}}_3^{(2)})' \widehat{\boldsymbol{\Delta}}_{33}^{-1} (\widehat{\boldsymbol{\eta}}_3^{(1)} - \widehat{\boldsymbol{\eta}}_3^{(2)}), \tag{7}$$

and

$$\widehat{\boldsymbol{\eta}}_1^{(\ell)} = \overline{\boldsymbol{x}}_{(123)1}^{(\ell)},$$

$$\widehat{\boldsymbol{\eta}}_2^{(\ell)} = \overline{\boldsymbol{x}}_{(12)2}^{(\ell)} - \Big\{\sum_{\ell=1}^{2}(N_2^{(\ell)} - 1)\boldsymbol{S}_{(12)(12),21}^{(\ell)}\Big\}\Big\{\sum_{\ell=1}^{2}(N_2^{(\ell)} - 1)\boldsymbol{S}_{(12)(12),11}^{(\ell)}\Big\}^{-1}\overline{\boldsymbol{x}}_{(12)1}^{(\ell)},$$

$$\widehat{\boldsymbol{\eta}}_3^{(\ell)} = \overline{\boldsymbol{x}}_{13}^{(\ell)} - \Big\{\sum_{\ell=1}^{2}(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(123),21}^{(\ell)}\Big\}\Big\{\sum_{\ell=1}^{2}(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(123),11}^{(\ell)}\Big\}^{-1}\overline{\boldsymbol{x}}_{1(12)}^{(\ell)},$$

$$\widehat{\boldsymbol{\Delta}}_{11} = \frac{1}{N_3^{(1)} + N_3^{(2)}}\sum_{\ell=1}^{2}(N_3^{(\ell)} - 1)\boldsymbol{S}_{(123)1}^{(\ell)},$$

$$\widehat{\boldsymbol{\Delta}}_{22} = \frac{1}{N_2^{(1)} + N_2^{(2)}}\Big[\sum_{\ell=1}^{2}(N_2^{(\ell)} - 1)\boldsymbol{S}_{(12)(12),22}^{(\ell)} - \Big\{\sum_{\ell=1}^{2}(N_2^{(\ell)} - 1)\boldsymbol{S}_{(12)(12),21}^{(\ell)}\Big\}\Big\{\sum_{\ell=1}^{2}(N_2^{(\ell)} - 1)\boldsymbol{S}_{(12)(12),11}^{(\ell)}\Big\}^{-1}\Big\{\sum_{\ell=1}^{2}(N_2^{(\ell)} - 1)\boldsymbol{S}_{(12)(12),12}^{(\ell)}\Big\}\Big],$$

$$\widehat{\boldsymbol{\Delta}}_{33} = \frac{1}{N_1^{(1)} + N_1^{(2)}}\Big[\sum_{\ell=1}^{2}(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(123),22}^{(\ell)} - \Big\{\sum_{\ell=1}^{2}(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(123),21}^{(\ell)}\Big\}\Big\{\sum_{\ell=1}^{2}(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(123),11}^{(\ell)}\Big\}^{-1}\Big\{\sum_{\ell=1}^{2}(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(123),12}^{(\ell)}\Big\}\Big].$$

This decomposition in (4) is essentially based on Yu et al. (2006).

First, we consider a stochastic expansion of Q_1 in (5). Let $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \boldsymbol{\mu} = (\boldsymbol{\mu}_1' \ \boldsymbol{\mu}_2' \ \boldsymbol{\mu}_3')'$, where $\boldsymbol{\mu}_i$ is a $p_i \times 1$ vector. Further let $N_3^{(1)} = N_3$, $N_3^{(2)} = r_3 N_3$, where r_3 is a positive constant. We note that Q_1 is essentially the same as the Hotelling's T^2 statistic and $(N_3^{(1)} + N_3^{(2)})\widehat{\boldsymbol{\Delta}}_{11}$ is a Wishart matrix. Therefore, without loss of generality, we may assume that $\boldsymbol{\Sigma}_{11} = \boldsymbol{I}_{p_1}$. Then, let

$$\begin{cases} \overline{\boldsymbol{x}}_{(123)1}^{(1)} = \boldsymbol{\mu}_1 + \dfrac{1}{\sqrt{N_3}}\boldsymbol{z}^{(1)} \\ \boldsymbol{S}_{(123)1}^{(1)} = \boldsymbol{I}_{p_1} + \dfrac{1}{\sqrt{N_3 - 1}}\boldsymbol{Z}^{(1)} \end{cases}, \quad \begin{cases} \overline{\boldsymbol{x}}_{(123)1}^{(2)} = \boldsymbol{\mu}_1 + \dfrac{1}{\sqrt{r_3 N_3}}\boldsymbol{z}^{(2)} \\ \boldsymbol{S}_{(123)1}^{(2)} = \boldsymbol{I}_{p_1} + \dfrac{1}{\sqrt{r_3 N_3 - 1}}\boldsymbol{Z}^{(2)} \end{cases}.$$

Then, we can expand Q_1 as

$$Q_1 = \boldsymbol{z}'\boldsymbol{z} - \frac{1}{\sqrt{N_3}}\boldsymbol{z}'\boldsymbol{Z}\boldsymbol{z} + \frac{1}{N_3}\Big(\boldsymbol{z}'\boldsymbol{Z}^2\boldsymbol{z} + \frac{2}{1 + r_3}\boldsymbol{z}'\boldsymbol{z}\Big) + O_p(N_3^{-\frac{3}{2}}),$$

where

$$\boldsymbol{z} = \Big(\frac{r_3}{1 + r_3}\Big)^{\frac{1}{2}}\Big(\boldsymbol{z}^{(1)} - \frac{1}{\sqrt{r_3}}\boldsymbol{z}^{(2)}\Big), \quad \boldsymbol{Z} = \frac{1}{1 + r_3}\boldsymbol{Z}^{(1)} + \frac{\sqrt{r_3}}{1 + r_3}\boldsymbol{Z}^{(2)}.$$

We note that $\boldsymbol{z} \sim N_{p_1}(\boldsymbol{0}, \boldsymbol{I})$. Therefore, the characteristic function of Q_1 can be written as

$$\begin{aligned}\phi_1(t) =& \mathrm{E}[\exp(it\boldsymbol{z}'\boldsymbol{z})] + \frac{1}{\sqrt{N_3}}\mathrm{E}\Big[(-it)\boldsymbol{z}'\boldsymbol{Z}\boldsymbol{z}\exp(it\boldsymbol{z}'\boldsymbol{z})\Big]\\ &+ \frac{1}{N_3}\bigg\{\mathrm{E}\Big[it\boldsymbol{z}'\boldsymbol{Z}^2\boldsymbol{z}\exp(it\boldsymbol{z}'\boldsymbol{z})\Big] + \mathrm{E}\Big[it\frac{2}{1+r_3}\boldsymbol{z}'\boldsymbol{z}\exp(it\boldsymbol{z}'\boldsymbol{z})\Big]\\ &+ \mathrm{E}\Big[\frac{1}{2}(it)^2(\boldsymbol{z}'\boldsymbol{Z}\boldsymbol{z})^2\exp(it\boldsymbol{z}'\boldsymbol{z})\Big]\bigg\} + O(N_3^{-\frac{3}{2}}),\end{aligned}$$

where $i = \sqrt{-1}$. Calculating the expectations with respect to $\boldsymbol{z}$, $\boldsymbol{Z}^{(1)}$, and $\boldsymbol{Z}^{(2)}$, we obtain

$$\begin{aligned}\phi_1(t) =& u^{-\frac{1}{2}p_1} + \frac{1}{N_3}\Big\{-\frac{1}{4(1+r_3)}p_1(p_1+4)u^{-\frac{1}{2}p_1} + \frac{1}{2(1+r_3)}p_1u^{-\frac{1}{2}p_1-1}\\ &+ \frac{1}{4(1+r_3)}p_1(p_1+2)u^{-\frac{1}{2}p_1-2}\Big\} + O(N_3^{-2}),\end{aligned} \tag{8}$$

where $u = 1 - 2it$. Inverting the characteristic function, for large N_3, we have

$$\Pr(Q_1 \le x) = G_{p_1}(x) + \frac{1}{N_3}\sum_{j=0}^{2}\beta_{j,1}G_{p_1+2j}(x) + O(N_3^{-2}),$$

where $G_f(x)$ is the distribution function of a χ^2-variate with f degrees of freedom,

$$\beta_{0,1} = -\frac{1}{4(1+r_3)}p_1(p_1+4), \quad \beta_{1,1} = \frac{1}{2(1+r_3)}p_1, \quad \beta_{2,1} = \frac{1}{4(1+r_3)}p_1(p_1+2).$$

Second, we consider the stochastic expansion of Q_2 in (6). Let $N_2^{(1)} = N_2$, $N_2^{(2)} = r_2N_2$, where r_2 is a positive constant. As with Q_1, without loss of generality, we may assume that $\boldsymbol{\Sigma}_{(12)(12)} = \boldsymbol{I}_{p_{(12)}}$. Therefore let

$$\begin{cases}\overline{\boldsymbol{x}}^{(1)}_{(12)1} = \boldsymbol{\mu}_1 + \frac{1}{\sqrt{N_2}}\boldsymbol{u}_1^{(1)}, \quad \overline{\boldsymbol{x}}^{(1)}_{(12)2} = \boldsymbol{\mu}_2 + \frac{1}{\sqrt{N_2}}\boldsymbol{u}_2^{(1)}\\ \boldsymbol{S}^{(1)}_{(12)(12)} = \boldsymbol{I}_{p_{(12)}} + \frac{1}{\sqrt{N_2-1}}\boldsymbol{U}^{(1)}\end{cases},$$

$$\begin{cases}\overline{\boldsymbol{x}}^{(2)}_{(12)1} = \boldsymbol{\mu}_1 + \frac{1}{\sqrt{r_2N_2}}\boldsymbol{u}_1^{(2)}, \quad \overline{\boldsymbol{x}}^{(2)}_{(12)2} = \boldsymbol{\mu}_2 + \frac{1}{\sqrt{r_2N_2}}\boldsymbol{u}_2^{(2)}\\ \boldsymbol{S}^{(2)}_{(12)(12)} = \boldsymbol{I}_{p_{(12)}} + \frac{1}{\sqrt{r_2N_2-1}}\boldsymbol{U}^{(2)}\end{cases}.$$

Then, assigning

$$\boldsymbol{u}_i = \Big(\frac{r_2}{1+r_2}\Big)^{\frac{1}{2}}\Big(\boldsymbol{u}_i^{(1)} - \frac{1}{\sqrt{r_2}}\boldsymbol{u}_i^{(2)}\Big), \quad i = 1, 2,$$

$$\boldsymbol{U} = \frac{1}{1+r_2}\boldsymbol{U}^{(1)} + \frac{\sqrt{r_2}}{1+r_2}\boldsymbol{U}^{(2)}, \quad \boldsymbol{U} = \begin{pmatrix} \overbrace{\boldsymbol{U}_{11}}^{p_1} & \overbrace{\boldsymbol{U}_{12}}^{p_2}\\ \boldsymbol{U}_{21} & \boldsymbol{U}_{22}\end{pmatrix}\begin{matrix}\}p_1\\ \}p_2\end{matrix},$$

we can expand $\widehat{\boldsymbol{\eta}}_2^{(1)} - \widehat{\boldsymbol{\eta}}_2^{(2)}$ and $\widehat{\boldsymbol{\Delta}}_{22}^{-1}$ as

$$\begin{aligned}\widehat{\boldsymbol{\eta}}_2^{(1)} - \widehat{\boldsymbol{\eta}}_2^{(2)} =& \frac{1}{\sqrt{N_2}}\Big\{\boldsymbol{u}_2^{(1)} - \frac{1}{\sqrt{r_2}}\boldsymbol{u}_2^{(2)} - \Big(\frac{1}{\sqrt{N_2}}\boldsymbol{U}_{21} - \frac{1}{N_2}\boldsymbol{U}_{21}\boldsymbol{U}_{11}\Big)\Big(\boldsymbol{u}_1^{(1)} - \frac{1}{\sqrt{r_2}}\boldsymbol{u}_1^{(2)}\Big)\Big\} \\ &+ O_p(N_2^{-2}), \\ \widehat{\boldsymbol{\Delta}}_{22}^{-1} =& \boldsymbol{I}_{22} - \frac{1}{\sqrt{N_2}}\boldsymbol{U}_{22} + \frac{1}{N_2}\Big(\frac{2}{1+r_2}\boldsymbol{I}_{22} + \boldsymbol{U}_{21}\boldsymbol{U}_{12} + \boldsymbol{U}_{22}^2\Big) + O_p(N_2^{-\frac{3}{2}}),\end{aligned}$$

respectively, where $\boldsymbol{I}_{22}$ is a p_2 dimensional identity matrix. We note that $\boldsymbol{u}_i \sim N_{p_i}(\boldsymbol{0}, \boldsymbol{I})$, $i = 1, 2$. Since we can expand Q_2 as

$$\begin{aligned}Q_2 =& \boldsymbol{u}_2'\boldsymbol{u}_2 - \frac{1}{\sqrt{N_2}}(2\boldsymbol{u}_2'\boldsymbol{U}_{21}\boldsymbol{u}_1 + \boldsymbol{u}_2'\boldsymbol{U}_{22}\boldsymbol{u}_2) \\ &+ \frac{1}{N_2}\Big\{2(\boldsymbol{u}_2'\boldsymbol{U}_{21}\boldsymbol{U}_{11}\boldsymbol{u}_1 + \boldsymbol{u}_2'\boldsymbol{U}_{22}\boldsymbol{U}_{21}\boldsymbol{u}_1) + \boldsymbol{u}_1'\boldsymbol{U}_{12}\boldsymbol{U}_{21}\boldsymbol{u}_1 + \boldsymbol{u}_2'\boldsymbol{U}_{21}\boldsymbol{U}_{12}\boldsymbol{u}_2 \\ &\quad + \frac{2}{1+r_2}\boldsymbol{u}_2'\boldsymbol{u}_2 + \boldsymbol{u}_2'\boldsymbol{U}_{22}^2\boldsymbol{u}_2\Big\} + O_p(N_2^{-\frac{3}{2}}),\end{aligned}$$

after the calculation of the expectation with respect to $\boldsymbol{U}^{(1)}$ and $\boldsymbol{U}^{(2)}$, we obtain the characteristic function of Q_2 as

$$\begin{aligned}\phi_2(t) =& \mathrm{E}[\exp(it\boldsymbol{u}_2'\boldsymbol{u}_2)] + \frac{1}{N_2}\frac{1}{1+r_2}\Big[it\Big\{p_2\mathrm{E}\Big[\boldsymbol{u}_1'\boldsymbol{u}_1\exp(it\boldsymbol{u}_2'\boldsymbol{u}_2)\Big] \\ &+ (p_1+p_2+3)\mathrm{E}\Big[\boldsymbol{u}_2'\boldsymbol{u}_2\exp(it\boldsymbol{u}_2'\boldsymbol{u}_2)\Big]\Big\} \\ &+ (it)^2\Big\{2\mathrm{E}\Big[\boldsymbol{u}_1'\boldsymbol{u}_1\boldsymbol{u}_2'\boldsymbol{u}_2\exp(it\boldsymbol{u}_2'\boldsymbol{u}_2)\Big] + \mathrm{E}\Big[(\boldsymbol{u}_2'\boldsymbol{u}_2)^2\exp(it\boldsymbol{u}_2'\boldsymbol{u}_2)\Big]\Big\}\Big] + O(N_2^{-2}).\end{aligned}$$

Next, calculating the expectations with respect to $\boldsymbol{u}_1, \boldsymbol{u}_2$, the characteristic function of Q_2 can be written as

$$\begin{aligned}\phi_2(t) =& u^{-\frac{1}{2}p_2} + \frac{1}{N_2}\Big\{-\frac{1}{4(1+r_2)}p_2(4p_1+p_2+4)u^{-\frac{1}{2}p_2} \\ &+ \frac{1}{2(1+r_2)}p_2(2p_1+1)u^{-\frac{1}{2}p_2-1} + \frac{1}{4(1+r_2)}p_2(p_2+2)u^{-\frac{1}{2}p_2-2}\Big\} + O(N_2^{-2}). \quad (9)\end{aligned}$$

Inverting the characteristic function $\phi_2(t)$, for large N_2, we obtain

$$\Pr(Q_2 \leq x) = G_{p_2}(x) + \frac{1}{N_2}\sum_{j=0}^{2}\beta_{j,2}G_{p_2+2j}(x) + O(N_2^{-2}),$$

where

$$\begin{aligned}\beta_{0,2} &= -\frac{1}{4(1+r_2)}p_2(4p_1+p_2+4), \quad \beta_{1,2} = \frac{1}{2(1+r_2)}p_2(2p_1+1), \\ \beta_{2,2} &= \frac{1}{4(1+r_2)}p_2(p_2+2).\end{aligned}$$

Finally, we consider a stochastic expansion of Q_3 in (7). Let $\boldsymbol{\mu} = (\boldsymbol{\mu}'_{(12)}\ \boldsymbol{\mu}'_3)'$, where $\boldsymbol{\mu}_{(12)}$ is a $p_{(12)} \times 1$ vector. Further let $N_1^{(1)} = N_1$, $N_1^{(2)} = r_1 N_1$, where r_1 is a positive constant. Since, without loss of generality, we may assume that $\boldsymbol{\Sigma} = \boldsymbol{I}_p$, let

$$\begin{cases} \overline{\boldsymbol{x}}_{1(12)}^{(1)} = \boldsymbol{\mu}_{(12)} + \dfrac{1}{\sqrt{N_1}}\boldsymbol{v}_{(12)}^{(1)}, \quad \overline{\boldsymbol{x}}_{13}^{(1)} = \boldsymbol{\mu}_3 + \dfrac{1}{\sqrt{N_1}}\boldsymbol{v}_3^{(1)} \\ \boldsymbol{S}_{1(123)}^{(1)} = \boldsymbol{I}_p + \dfrac{1}{\sqrt{N_1 - 1}}\boldsymbol{V}^{(1)} \end{cases},$$

$$\begin{cases} \overline{\boldsymbol{x}}_{1(12)}^{(2)} = \boldsymbol{\mu}_{(12)} + \dfrac{1}{\sqrt{r_1 N_1}}\boldsymbol{v}_{(12)}^{(2)}, \quad \overline{\boldsymbol{x}}_{13}^{(2)} = \boldsymbol{\mu}_3 + \dfrac{1}{\sqrt{r_1 N_1}}\boldsymbol{v}_3^{(2)} \\ \boldsymbol{S}_{1(123)}^{(2)} = \boldsymbol{I}_p + \dfrac{1}{\sqrt{r_1 N_1 - 1}}\boldsymbol{V}^{(2)} \end{cases}.$$

After a great deal of calculation as with Q_1 and Q_2, the characteristic function of Q_3 can be written as

$$\begin{aligned} \phi_3(t) =& u^{-\frac{1}{2}p_3} + \frac{1}{N_1}\Big\{-\frac{1}{4(1+r_1)}p_3(4p_{(12)} + p_3 + 4)u^{-\frac{1}{2}p_3} \\ &+ \frac{1}{2(1+r_1)}p_3(2p_{(12)} + 1)u^{-\frac{1}{2}p_3 - 1} + \frac{1}{4(1+r_1)}p_3(p_3+2)u^{-\frac{1}{2}p_3 - 2}\Big\} + O(N_1^{-2}). \end{aligned} \tag{10}$$

Therefore, for large N_1

$$\Pr(Q_3 \le x) = G_{p_3}(x) + \frac{1}{N_1}\sum_{j=0}^{2}\beta_{j,3}G_{p_3+2j}(x) + O(N_1^{-2}),$$

where

$$\begin{aligned} \beta_{0,3} &= -\frac{1}{4(1+r_1)}p_3(4p_{(12)} + p_3 + 4), \quad \beta_{1,3} = \frac{1}{2(1+r_1)}p_3(2p_{(12)} + 1), \\ \beta_{2,3} &= \frac{1}{4(1+r_1)}p_3(p_3 + 2). \end{aligned}$$

We define $\nu_j = \sum_{\ell=1}^{2} n_j^{(\ell)}$, $j = 1, 2, 3$ and $s_j (= \nu_j/\nu_1)$ $j = 2, 3$, where s_j is a positive constant. Then, for large ν_1, using the results in (8), (9), and (10), we can expand $\prod_{i=1}^{3}\phi_i(t)$ as follows:

$$\prod_{i=1}^{3}\phi_i(t) = u^{-\frac{p}{2}} + \frac{1}{\nu_1}\sum_{j=0}^{2}\beta_j u^{-\frac{p}{2}-j} + O(\nu_1^{-2}),$$

where

$$
\begin{aligned}
\beta_0 &= -\frac{1}{4}\left\{\frac{1}{1+s_2+s_3}p_1(p_1+4)+\frac{1}{1+s_2}p_2(4p_1+p_2+4)+p_3(4p_{(12)}+p_3+4)\right\},\\
\beta_1 &= \frac{1}{2}\left\{\frac{1}{1+s_2+s_3}p_1+\frac{1}{1+s_2}p_2(2p_1+1)+p_3(2p_{(12)}+1)\right\},\\
\beta_2 &= \frac{1}{4}\left\{\frac{1}{1+s_2+s_3}p_1(p_1+2)+\frac{1}{1+s_2}p_2(p_2+2)+p_3(p_3+2)\right\}.
\end{aligned}
$$

Furthermore, since Q_i $(i = 1, 2, 3)$ are mutually and asymptotically independent, $\prod_{i=1}^{3}\phi_i(t)$ can be regarded as an approximation of $\mathrm{E}[\exp(itQ)]$. Therefore, we obtain

$$
\Pr(Q \le x) \simeq G_p(x) + \frac{1}{\nu_1}\sum_{j=0}^{2}\beta_j G_{p+2j}(x).
$$

Also, an approximation to the upper 100α percentile of Q is given by

$$
q_{\mathrm{AE}}(\alpha) = \chi_p^2(\alpha) - \frac{1}{\nu_1}\frac{2\chi_p^2(\alpha)}{p}\left\{\beta_0 - \frac{\beta_2}{p+2}\chi_p^2(\alpha)\right\},
$$

where $\chi_p^2(\alpha)$ is the upper 100α percentile of χ^2 distribution with p degrees of freedom.

2.2 General case ($k \ge 2$)

We consider the general case with k-step monotone missing data. Let, for $i = 1, 2, \ldots, k$,

$$
\boldsymbol{X}_{i(12\ldots,k-i+1)}^{(\ell)} = (\boldsymbol{X}_{i1}^{(\ell)}\ \boldsymbol{X}_{i2}^{(\ell)} \cdots \boldsymbol{X}_{i,k-i+1}^{(\ell)}),\ N_i^{(\ell)} = \sum_{j=1}^{i} n_j^{(\ell)},\ \nu_i = \sum_{\ell=1}^{2} n_i^{(\ell)},\ p_{(12\cdots i)} = \sum_{j=1}^{i} p_j.
$$

As with three-step case in Section 2.1, it is assumed that the rows of $\boldsymbol{X}_{i(12\ldots,k-i+1)}^{(\ell)}$ $(i = 1, 2, \ldots, k)$ are mutually independent and

$$
\mathrm{vec}(\boldsymbol{X}_{i(12\ldots,k-i+1)}^{(\ell)}) \sim N_{n_i^{(\ell)} p_{(12\ldots,k-i+1)}}(\mathrm{vec}(\boldsymbol{1}_{n_i^{(\ell)}}\boldsymbol{\mu}_{(12\ldots,k-i+1)}^{(\ell)\prime}), \boldsymbol{I}_{n_i^{(\ell)}} \otimes \boldsymbol{\Sigma}_{(12\ldots,k-i+1)(12\ldots,k-i+1)}),
$$
$$
i = 1, 2, \ldots, k,
$$

where

$$\underset{p_{(12\ldots,k-i+1)}\times 1}{\boldsymbol{\mu}^{(\ell)}_{(12\ldots,k-i+1)}} = \begin{pmatrix} \boldsymbol{\mu}^{(\ell)}_1 \\ \boldsymbol{\mu}^{(\ell)}_2 \\ \vdots \\ \boldsymbol{\mu}^{(\ell)}_{k-i+1} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \\ \\ \}p_{k-i+1} \end{matrix},$$

$$\underset{p_{(12,\ldots,k-i+1)}\times p_{(12,\ldots,k-i+1)}}{\boldsymbol{\Sigma}_{(12,\ldots,k-i+1)(12,\ldots,k-i+1)}} = \begin{pmatrix} \overbrace{\boldsymbol{\Sigma}_{11}}^{p_1} & \overbrace{\boldsymbol{\Sigma}_{12}}^{p_2} & \cdots & \overbrace{\boldsymbol{\Sigma}_{1,k-i+1}}^{p_{k-i+1}} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \cdots & \boldsymbol{\Sigma}_{2,k-i+1} \\ \vdots & & & \vdots \\ \boldsymbol{\Sigma}_{k-i+1,1} & \boldsymbol{\Sigma}_{k-i+1,2} & \cdots & \boldsymbol{\Sigma}_{k-i+1,k-i+1} \end{pmatrix} \begin{matrix} \}p_1 \\ \}p_2 \\ \\ \}p_{k-i+1} \end{matrix}.$$

Further, let

$$\boldsymbol{X}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)} = \begin{pmatrix} \boldsymbol{X}^{(\ell)}_{11} & \cdots & \boldsymbol{X}^{(\ell)}_{1i} \\ \vdots & & \vdots \\ \boldsymbol{X}^{(\ell)}_{k-i+1,1} & \cdots & \boldsymbol{X}^{(\ell)}_{k-i+1,i} \end{pmatrix},$$

where $\boldsymbol{X}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)}$ is a $N^{(\ell)}_{k-i+1} \times p_{(12\cdots i)}$ matrix. We then define the sample mean vectors and the sample covariance matrices. For $i = 1, 2, \ldots, k$,

$$\begin{aligned} \overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)} =& \frac{1}{N^{(\ell)}_{k-i+1}} \boldsymbol{X}^{(\ell)\prime}_{(12\cdots,k-i+1)(12\cdots i)} \mathbf{1}_{N^{(\ell)}_{k-i+1}}, \\ \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)} =& \frac{1}{N^{(\ell)}_{k-i+1}-1} \Big\{ \boldsymbol{X}^{(\ell)\prime}_{(12\cdots,k-i+1)(12\cdots i)} \boldsymbol{X}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)} \\ & - N^{(\ell)}_{k-i+1} \overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)} \overline{\boldsymbol{x}}^{(\ell)\prime}_{(12\cdots,k-i+1)(12\cdots i)} \Big\}, \end{aligned}$$

Further, we partition, for $i = 2, 3, \ldots, k$,

$$\overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)} = \begin{pmatrix} \overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots,i-1)} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots,k-i+1)i} \end{pmatrix} \begin{matrix} \}p_{(12\cdots,i-1)} \\ \}p_i \end{matrix},$$

$$\boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i)} = \begin{pmatrix} \overbrace{\boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),11}}^{p_{(12\cdots,i-1)}} & \overbrace{\boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),12}}^{p_i} \\ \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),21} & \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),22} \end{pmatrix} \begin{matrix} \}p_{(12\cdots,i-1)} \\ \}p_i \end{matrix}.$$

Then, the simplified T^2 statistic can be written as

$$Q = \sum_{i=1}^{k} Q_i, \tag{11}$$

where

$$Q_i = \frac{N^{(1)}_{k-i+1} N^{(2)}_{k-i+1}}{N^{(1)}_{k-i+1} + N^{(2)}_{k-i+1}} (\widehat{\boldsymbol{\eta}}^{(1)}_i - \widehat{\boldsymbol{\eta}}^{(2)}_i)' \widehat{\boldsymbol{\Delta}}^{-1}_{ii} (\widehat{\boldsymbol{\eta}}^{(1)}_i - \widehat{\boldsymbol{\eta}}^{(2)}_i), \tag{12}$$

$$\widehat{\boldsymbol{\eta}}^{(\ell)}_i = \begin{cases} \overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots k)1} \quad (i=1) \\ \overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots,k-i+1)i} - \Big\{ \sum_{\ell=1}^{2} (N^{(\ell)}_{k-i+1} - 1) \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),21} \Big\} \\ \times \Big\{ \sum_{\ell=1}^{2} (N^{(\ell)}_{k-i+1} - 1) \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),11} \Big\}^{-1} \overline{\boldsymbol{x}}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots,i-1)} \quad (i=2,3,\ldots,k) \end{cases},$$

$$\widehat{\boldsymbol{\Delta}}^{(\ell)}_{ii} = \begin{cases} \dfrac{1}{N^{(1)}_k + N^{(2)}_k} \sum_{\ell=1}^{2} (N^{(\ell)}_k - 1) \boldsymbol{S}^{(\ell)}_{(12\cdots k)1} \quad (i=1) \\ \dfrac{1}{N^{(1)}_{k-i+1} + N^{(2)}_{k-i+1}} \Big[\sum_{\ell=1}^{2} (N^{(\ell)}_{k-i+1} - 1) \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),22} \\ - \Big\{ \sum_{\ell=1}^{2} (N^{(\ell)}_{k-i+1} - 1) \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),21} \Big\} \Big\{ \sum_{\ell=1}^{2} (N^{(\ell)}_{k-i+1} - 1) \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),11} \Big\}^{-1} \\ \times \Big\{ \sum_{\ell=1}^{2} (N^{(\ell)}_{k-i+1} - 1) \boldsymbol{S}^{(\ell)}_{(12\cdots,k-i+1)(12\cdots i),12} \Big\} \Big] \quad (i=2,3,\ldots,k) \end{cases}.$$

As with the derivation of the three-step case, we have the following Theorem 1.

Theorem 1

For large $N^{(1)}_{k-i+1}$, the distribution of Q_i $(i=1,2,\ldots,k)$ in (12) can be expanded as

$$\Pr(Q_i \le x) = G_{p_i}(x) + \frac{1}{N^{(1)}_{k-i+1}} \sum_{j=0}^{2} \beta_{j,i} G_{p_i+2j}(x) + O(N^{(1)^{-2}}_{k-i+1}),$$

where

$$\beta_{0,i} = -\frac{1}{4(1+r_{k-i+1})} p_i \Big(4p_{(12\cdots,i-1)} + p_i + 4\Big), \quad \beta_{1,i} = \frac{1}{2(1+r_{k-i+1})} p_i \Big(2p_{(12\cdots,i-1)} + 1\Big),$$

$$\beta_{2,i} = \frac{1}{4(1+r_{k-i+1})} p_i(p_i+2), \quad r_{k-i+1} = \frac{N^{(2)}_{k-i+1}}{N^{(1)}_{k-i+1}},$$

and r_{k-i+1} is a positive constant. Furthermore, its upper 100α percentiles can be expanded as

$$q_i(\alpha) = \chi^2_{p_i}(\alpha) - \frac{1}{N^{(1)}_{k-i+1}} \left[\frac{2\chi^2_{p_i}(\alpha)}{p_i} \left\{ \beta_{0,i} - \frac{1}{p_i+2} \beta_{2,i} \chi^2_{p_i}(\alpha) \right\} \right] + O(N^{(1)^{-2}}_{k-i+1}).$$

Since, again, $\prod_{j=1}^{k} \mathrm{E}[\exp(itQ_j)]$ can be regarded as an approximation of $\mathrm{E}[\exp(itQ)]$, for large ν_1, we can obtain

$$\prod_{j=1}^{k} \mathrm{E}[\exp(itQ_j)] = u^{-\frac{p}{2}} + \frac{1}{\nu_1}\sum_{j=0}^{2} \beta_j u^{-\frac{p}{2}-j} + O(\nu_1^{-2}),$$

where

$$\begin{aligned}
\beta_0 &= -\frac{1}{4}\sum_{i=1}^{k}\left[\Big(1+\sum_{j=2}^{k-i+1} s_j\Big)^{-1} p_i\Big(4p_{(12\cdots,i-1)}+p_i+4\Big)\right],\\
\beta_1 &= \frac{1}{2}\sum_{i=1}^{k}\left[\Big(1+\sum_{j=2}^{k-i+1} s_j\Big)^{-1} p_i\Big(2p_{(12\cdots,i-1)}+1\Big)\right],\\
\beta_2 &= \frac{1}{4}\sum_{i=1}^{k}\left[\Big(1+\sum_{j=2}^{k-i+1} s_j\Big)^{-1} p_i(p_i+2)\right],
\end{aligned}$$

$s_i(=\nu_i/\nu_1)$, $(i=2,3,\ldots,k)$ is a positive constant. Therefore, an approximation to the upper 100α percentile of Q in (11) is given by

$$q_{\mathrm{AE}}(\alpha) = \chi_p^2(\alpha) - \frac{1}{\nu_1}\left[\frac{2\chi_p^2(\alpha)}{p}\left\{\beta_0 - \frac{\beta_2}{p+2}\chi_p^2(\alpha)\right\}\right]. \tag{13}$$

As a remark, in the case of a multi-sample problem, let $\boldsymbol{X}_{ij}^{(\ell)}$ be a $n_i^{(\ell)} \times p_j$ block matrix, where there are $n_i^{(\ell)}$ observations available on the i-th p_j components, $i=1,2,\ldots,k; j=1,2,\ldots,k-i+1; \ell=1,2,\ldots,m$. Then, we can construct the approximate simultaneous confidence intervals for pairwise comparisons among mean vectors by using the upper percentile of $Q^{(ab)}$, where $Q^{(ab)}$ is the same one as $\widetilde{T}_{ab}^2$ in Section 4.3 of Yagi and Seo (2017). Then, for $1 \le a < b \le m$, $Q^{(ab)}$ can be decomposed as follows:

$$Q^{(ab)} = \sum_{i=1}^{k} Q_i^{(ab)},$$

where

$$Q_i^{(ab)} = \frac{N_{k-i+1}^{(a)} N_{k-i+1}^{(b)}}{N_{k-i+1}^{(a)} + N_{k-i+1}^{(b)}} (\widehat{\boldsymbol{\eta}}_i^{(a)} - \widehat{\boldsymbol{\eta}}_i^{(b)})' \widehat{\boldsymbol{\Delta}}_{ii}^{[pl]^{-1}} (\widehat{\boldsymbol{\eta}}_i^{(a)} - \widehat{\boldsymbol{\eta}}_i^{(b)}), \tag{14}$$

$$\widehat{\boldsymbol{\eta}}_i^{(a)} = \begin{cases} \overline{\boldsymbol{x}}_{(12\cdots k)1}^{(a)} \quad (i=1), \\ \overline{\boldsymbol{x}}_{(12\cdots,k-i+1)i}^{(a)} - \left\{ \sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),21}^{(\ell)} \right\} \\ \times \left\{ \sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),11}^{(\ell)} \right\}^{-1} \overline{\boldsymbol{x}}_{(12\cdots,k-i+1)(12\cdots,i-1)}^{(a)} \quad (i = 2, 3, \ldots, k) \end{cases},$$

$$\widehat{\boldsymbol{\eta}}_i^{(b)} = \begin{cases} \overline{\boldsymbol{x}}_{(12\cdots k)1}^{(b)} \quad (i=1) \\ \overline{\boldsymbol{x}}_{(12\cdots,k-i+1)i}^{(b)} - \left\{ \sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),21}^{(\ell)} \right\} \\ \times \left\{ \sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),11}^{(\ell)} \right\}^{-1} \overline{\boldsymbol{x}}_{(12\cdots,k-i+1)(12\cdots,i-1)}^{(b)} \quad (i = 2, 3, \ldots, k) \end{cases},$$

$$\widehat{\boldsymbol{\Delta}}_{ii}^{(\ell)} = \begin{cases} \frac{1}{M_k} \sum_{\ell=1}^{m} (N_k^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots k)1}^{(\ell)} \quad (i=1) \\ \frac{1}{M_{k-i+1}} \Big[\sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),22}^{(\ell)} \\ - \left\{ \sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),21}^{(\ell)} \right\} \left\{ \sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),11}^{(\ell)} \right\}^{-1} \\ \times \left\{ \sum_{\ell=1}^{m} (N_{k-i+1}^{(\ell)} - 1) \boldsymbol{S}_{(12\cdots,k-i+1)(12\cdots i),12}^{(\ell)} \right\} \Big] \quad (i = 2, 3, \ldots, k) \end{cases},$$

$$M_i = \sum_{\ell=1}^{m} N_i^{(\ell)}, \quad i = 1, 2, \ldots, k.$$

As for the distribution of $Q_i^{(ab)}$, we have the following Corollary 2.

Corollary 2

For large $N_{k-i+1}^{(1)}$, the distribution of $Q_i^{(ab)}$ $(i = 1, 2, \ldots, k; 1 \le a < b \le m)$ in (14) can be expanded as

$$\Pr(Q_i^{(ab)} \le x) = G_{p_i}(x) + \frac{1}{N_{k-i+1}^{(1)}} \sum_{j=0}^{2} \beta_{j,i} G_{p_i+2j}(x) + O(N_{k-i+1}^{(1)^{-2}}),$$

where

$$\beta_{0,i} = -\frac{1}{4r_{k-i+1}} p_i \Big(4p_{(12\cdots,i-1)} + p_i + 2m\Big), \quad \beta_{1,i} = \frac{1}{2r_{k-i+1}} p_i \Big(2p_{(12\cdots,i-1)} + m - 1\Big),$$
$$\beta_{2,i} = \frac{1}{4r_{k-i+1}} p_i(p_i+2), \quad r_{k-i+1} = 1 + \sum_{\ell=2}^{m} r^{(\ell)}_{k-i+1}, \quad r^{(\ell)}_{k-i+1} = \frac{N^{(\ell)}_{k-i+1}}{N^{(1)}_{k-i+1}}, \quad \ell = 2,3,\ldots,m,$$

and $r^{(\ell)}_{k-i+1}$ *is a positive constant.*

Further, as with $q_{\mathrm{AE}}(\alpha)$ in (13), we can obtain an approximate upper 100α percentile of $Q^{(ab)}$ as follows:

$$q^{(ab)}_{\mathrm{AE}}(\alpha) = \chi^2_p(\alpha) - \frac{1}{\nu_{1\cdot m}} \frac{2\chi^2_p(\alpha)}{p} \left\{ \beta_0 - \frac{\beta_2}{p+2}\chi^2_p(\alpha) \right\},$$

where

$$\beta_0 = -\frac{1}{4}\sum_{i=1}^{k}\left[\Big(1 + \sum_{j=2}^{k-i+1} s_{j\cdot m}\Big)^{-1} p_i \Big(4\sum_{j=1}^{i-1} p_j + p_i + 2m\Big)\right],$$
$$\beta_2 = \frac{1}{4}\sum_{i=1}^{k}\left[\Big(1 + \sum_{j=2}^{k-i+1} s_{j\cdot m}\Big)^{-1} p_i(p_i+2)\right],$$

$p = \sum_{i=1}^{k} p_i$, $\nu_{i\cdot m} = \sum_{\ell=1}^{m} n^{(\ell)}_i$, $i = 1,2,\ldots,k$, and $s_{i\cdot m} (= \nu_{i\cdot m}/\nu_{1\cdot m})$, $(i = 2,3,\ldots,k)$ is a positive constant.

To obtain the approximate simultaneous confidence intervals for pairwise comparisons using Bonferroni's approximation, it is necessary that all the null distributions of $Q^{(ab)}$ are the same. That is, we assume that $n^{(1)}_i = n^{(2)}_i = \cdots = n^{(m)}_i$, $i = 1,2,\ldots,k$. Then, we can propose an approximate upper 100α percentile of the $\max\limits_{1\le a<b\le m} Q^{(ab)}$ statistic as $q^{(ab)}_{\mathrm{AE}}(\alpha_{\mathrm{p}})$ instead of $t^2_{\mathrm{YS\cdot L}m}(\alpha_{\mathrm{p}})$ as in Yagi and Seo (2017), where $\alpha_{\mathrm{p}} = 2\alpha/\{m(m-1)\}$.

3 Transformations with Bartlett adjustments

In this section, we consider transformation with Bartlett adjustment, which is an extension of Yagi, Seo, and Hanusz (2018). We derive two transformations for Q_i with $i = 1,2,\ldots,k$ in (11). The transformed test statistic with Bartlett correction is given by

$$Q^*_i = \left\{1 - \frac{1}{M_{k-i+1}}\Big(p_i + 2p_{(12\ldots,i-1)} + 3\Big)\right\} Q_i.$$

As an improvement of Q in (11), we propose

$$Q^* = \sum_{i=1}^{k} Q_i^*. \tag{15}$$

We note that $\mathrm{E}[Q_i^*] = p_i + O(M_{k-i+1}^{-2})$ and $\mathrm{E}[Q^*] = p + O(M_1^{-2})$. Furthermore, the transformed test statistics with Bartlett-type corrections are given by

$$Y_i = \left\{ M_{k-i+1} - \frac{1}{2}\Big(p_i + 4p_{(12\ldots,i-1)} + 4\Big) \right\} \log \left(1 + \frac{1}{M_{k-i+1}} Q_i \right)$$
$$\text{for} \quad M_{k-i+1} - \frac{1}{2}\Big(p_i + 4p_{(12\ldots,i-1)} + 4\Big) > 0.$$

We note that $\Pr(Y_i \leq x) = G_{p_i}(x) + O(M_{k-i+1}^{-2})$. Thus, we can propose a transformed test statistic as

$$Y = \sum_{i=1}^{k} Y_i. \tag{16}$$

As with the one-sample problem in Yagi, Seo, and Hanusz (2018), although the other transformations $Q^\dagger$, Z, $Y^\dagger$, and $Z^\dagger$ can be proposed as an approximation to Q, we will not deal with them in this paper, because our simulation studies indicated that the accuracy of the transformed test statistic Y in (16) is considerably higher than that of the other transformed test statistics in almost all cases.

4 Simulation studies

In order to investigate the accuracy of the approximations proposed in this paper, we compute the upper percentiles of the test statistic Q in (11) and the transformed test statistics Q^* in (15) and Y in (16) with k-step ($k = 2, 3$, and 5) monotone missing data using the Monte Carlo simulation. The simulation study of the two-sample problem in this paper corresponds to that of the one-sample problem in Yagi, Seo, and Hanusz (2018). The simulation results for the upper percentiles of the test statistics, their approximations, and their emprical type I errors in the case of two-step monotone missing data ($k = 2$) are summarized in Table 1 as follows: (i) q is the upper 100α percentile of Q by simulation, (ii) q_{Q^*} and α_{Q^*} are the upper 100α percentile of Q^* by simulation and its empirical type I error, respectively, (iii) q_Y and α_Y are the upper 100α percentile of Y by simulation and its empirical type I error, respectively, (iv) q_{AE} and α_{AE} are an asymptotic expansion approximation to the upper 100α percentile of Q and its empirical type I error, respectively, and (v) $q_{\rm YKP}$ and $\alpha_{\rm YKP}$ are an approximation to the upper 100α percentile of Q given by Yu et al. (2006) and its empirical type I error, respectively. As with Table 1, Tables 2 and 3 present the results in the case of three-step and five-step monotone missing data, respectively. It may be noted from the tables that both values of q_{Q^*} and q_Y converge to the upper percentile of the χ^2 limiting distribution at a very high speed. In addition, the value of $q_{\rm YKP}$ by Yu et al. (2006) is a very good approximation when the sample size is very small. We note from the tables that the value of $q_{\rm YKP}$ is closer to that of q than that of $q_{\rm AE}$ for most cases. Especially, in our simulation, the upper percentile of the transformed test statistic Y in (16), that is q_Y, has a very good χ^2-approximation.

5 Concluding remarks

In this paper, we considered the null distribution of the simplified Hotelling's T^2-type statistic for testing equality of two mean vectors when two datasets have general step monotone missing data. Using the same derivation procedure of a one-sample problem as in Yagi, Seo, and Hanusz (2018), an asymptotic expansion approximation for the upper percentile of the simplified T^2 statistic could be obtained in the general k-step monotone missing data pattern case. Furthermore, the approximation to the upper percentiles of the simplified T^2 statistic for the multi-sample case was given. Using this result and Bonferroni's approximation, the approximate upper percentile of $\max_{1\le a<b\le m} Q^{(ab)}$ and the approximate simultaneous confidence intervals for pairwise comparisons among mean vectors with monotone missing data were proposed. Finally, instead of using the original simplified T^2 statistic itself, we succeeded in deriving the transformed test statistics based on Bartlett adjustment in order to provide easy-to-handle statistic, whose null distribution is almost χ^2 distribution. The transformed test statistics with the Bartlett adjustment proposed in this paper are useful if the missing data are of monotone pattern in a two-sample problem.

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Table 1: The upper percentiles of test statistics and empirical type I errors for $(p_1, p_2) = (2, 2), (4, 4)$, and $\alpha = 0.05$.

$n_1^{(\ell)}$	$n_2^{(\ell)}$	q	q_{AE}	q_{YKP}	q_{Q^*}	q_Y	α_{χ^2}	α_{AE}	α_{YKP}	α_{Q^*}	α_Y
					$(p_1, p_2) = (2, 2)$						
10	10	14.95	12.63	14.90	9.98	9.45	16.17	8.11	5.05	5.86	4.93
20	20	11.52	11.06	11.48	9.75	9.52	9.35	5.76	5.07	5.51	5.06
30	30	10.77	10.53	10.71	9.69	9.54	7.69	5.41	5.11	5.40	5.10
40	40	10.40	10.27	10.37	9.62	9.51	6.94	5.23	5.05	5.28	5.05
50	50	10.20	10.12	10.17	9.60	9.51	6.50	5.17	5.06	5.23	5.05
100	100	9.81	9.80	9.82	9.52	9.48	5.69	5.01	4.98	5.08	4.99
200	200	9.64	9.64	9.65	9.50	9.48	5.33	4.99	4.98	5.03	4.98
400	400	9.60	9.57	9.57	9.53	9.52	5.23	5.06	5.06	5.09	5.06
10	5	15.34	12.87	15.19	10.09	9.50	16.95	8.24	5.16	6.04	5.02
20	10	11.71	11.18	11.61	9.82	9.55	9.78	5.86	5.15	5.64	5.14
30	15	10.87	10.62	10.79	9.71	9.55	7.94	5.44	5.13	5.44	5.12
40	20	10.48	10.33	10.43	9.65	9.53	7.09	5.26	5.09	5.33	5.09
50	25	10.25	10.16	10.22	9.61	9.51	6.62	5.16	5.05	5.24	5.05
100	50	9.86	9.83	9.84	9.56	9.51	5.79	5.07	5.04	5.13	5.04
200	100	9.66	9.66	9.66	9.51	9.49	5.36	5.01	5.00	5.05	5.00
400	200	9.56	9.57	9.57	9.48	9.47	5.14	4.97	4.96	4.99	4.96
10	20	14.58	12.39	14.64	9.89	9.42	15.48	8.04	4.93	5.71	4.86
20	40	11.38	10.94	11.35	9.71	9.50	9.08	5.76	5.04	5.45	5.03
30	60	10.64	10.45	10.62	9.64	9.51	7.48	5.34	5.03	5.30	5.04
40	80	10.32	10.21	10.30	9.60	9.50	6.78	5.21	5.03	5.22	5.02
50	100	10.14	10.07	10.12	9.58	9.50	6.40	5.13	5.03	5.19	5.04
100	200	9.80	9.78	9.79	9.53	9.50	5.65	5.04	5.01	5.09	5.02
200	400	9.63	9.63	9.64	9.50	9.48	5.29	4.99	4.99	5.02	4.99
400	800	9.58	9.56	9.56	9.51	9.50	5.19	5.04	5.03	5.05	5.03
					$(p_1, p_2) = (4, 4)$						
10	10	38.96	23.64	38.99	15.18	14.47	40.64	18.64	4.99	4.61	3.52
20	20	21.89	19.57	21.65	16.09	15.50	17.14	7.83	5.24	5.88	4.99
30	30	19.17	18.22	19.01	15.95	15.58	11.91	6.28	5.20	5.69	5.11
40	40	18.07	17.54	17.96	15.84	15.56	9.71	5.75	5.15	5.53	5.09
50	50	17.45	17.13	17.39	15.76	15.55	8.55	5.47	5.09	5.40	5.07
100	100	16.41	16.32	16.38	15.64	15.54	6.57	5.14	5.05	5.21	5.04
200	200	15.92	15.91	15.93	15.55	15.50	5.70	5.02	4.99	5.07	4.99
400	400	15.73	15.71	15.71	15.55	15.52	5.38	5.03	5.03	5.07	5.02
10	5	39.98	24.20	39.68	15.48	14.61	41.99	18.72	5.12	4.97	3.71
20	10	22.30	19.85	21.96	16.22	15.55	17.93	7.95	5.33	6.05	5.08
30	15	19.39	18.40	19.21	16.00	15.58	12.34	6.30	5.22	5.76	5.12
40	20	18.25	17.68	18.10	15.89	15.59	10.05	5.80	5.20	5.61	5.13
50	25	17.64	17.25	17.51	15.84	15.61	8.88	5.57	5.18	5.54	5.16
100	50	16.49	16.38	16.44	15.67	15.56	6.73	5.18	5.08	5.26	5.08
200	100	15.98	15.94	15.96	15.59	15.53	5.82	5.07	5.05	5.13	5.05
400	200	15.78	15.75	15.75	15.57	15.54	5.47	5.05	5.05	5.10	5.05
10	20	38.25	23.08	38.42	14.91	14.34	39.40	18.67	4.94	4.26	3.32
20	40	21.50	19.29	21.37	15.96	15.44	16.44	7.79	5.14	5.71	4.88
30	60	18.91	18.03	18.82	15.85	15.51	11.45	6.20	5.11	5.54	5.00
40	80	17.90	17.40	17.81	15.80	15.55	9.44	5.73	5.12	5.46	5.06
50	100	17.32	17.02	17.27	15.72	15.53	8.29	5.43	5.06	5.34	5.04
100	200	16.35	16.26	16.32	15.62	15.53	6.47	5.14	5.05	5.19	5.04
200	400	15.91	15.89	15.90	15.56	15.52	5.69	5.04	5.02	5.09	5.02
400	800	15.74	15.70	15.70	15.56	15.54	5.39	5.06	5.06	5.09	5.05

Note. $\chi_4^2(0.05) = 9.49$, $\chi_8^2(0.05) = 15.51$.

Table 2: The upper percentiles of test statistics and empirical type I errors for $(p_1, p_2, p_3) = (2, 2, 4)$ and $\alpha = 0.05$.

$n_1^{(\ell)}$	$n_2^{(\ell)}$	$n_3^{(\ell)}$	q	q_{AE}	q_{YKP}	q_{Q^*}	q_Y	α_{χ^2}	α_{AE}	α_{YKP}	α_{Q^*}	α_Y
10	10	10	38.69	23.34	38.77	15.01	14.47	40.25	18.89	4.97	4.41	3.51
20	20	20	21.72	19.42	21.51	16.01	15.50	16.96	7.89	5.21	5.77	4.98
30	30	30	19.06	18.12	18.91	15.89	15.57	11.71	6.28	5.18	5.60	5.09
40	40	40	17.97	17.46	17.88	15.78	15.55	9.58	5.72	5.12	5.44	5.07
50	50	50	17.42	17.07	17.33	15.74	15.55	8.44	5.49	5.12	5.37	5.08
100	100	100	16.41	16.29	16.35	15.65	15.56	6.57	5.19	5.09	5.23	5.09
200	200	200	15.94	15.90	15.91	15.57	15.53	5.73	5.06	5.04	5.11	5.04
400	400	400	15.73	15.70	15.71	15.55	15.53	5.38	5.05	5.04	5.08	5.05
800	800	800	15.61	15.61	15.61	15.53	15.52	5.18	5.01	5.01	5.03	5.01
10	5	5	39.71	23.86	39.41	15.26	14.61	41.56	19.01	5.11	4.72	3.75
20	10	10	22.12	19.68	21.80	16.13	15.56	17.72	8.01	5.32	5.93	5.09
30	15	15	19.32	18.29	19.10	15.97	15.62	12.18	6.38	5.27	5.72	5.17
40	20	20	18.19	17.60	18.02	15.87	15.60	9.98	5.85	5.23	5.57	5.16
50	25	25	17.54	17.18	17.44	15.78	15.58	8.71	5.54	5.15	5.44	5.12
100	50	50	16.48	16.34	16.40	15.67	15.57	6.70	5.21	5.12	5.25	5.10
200	100	100	15.98	15.92	15.94	15.59	15.55	5.81	5.09	5.06	5.14	5.06
400	200	200	15.72	15.72	15.72	15.53	15.51	5.36	5.01	5.01	5.04	5.00
800	400	400	15.61	15.61	15.61	15.52	15.51	5.18	5.00	5.00	5.02	5.00
10	20	20	38.03	22.85	38.26	14.78	14.34	39.16	18.94	4.91	4.13	3.34
20	40	40	21.35	19.18	21.27	15.86	15.40	16.19	7.77	5.08	5.54	4.82
30	60	60	18.84	17.96	18.75	15.82	15.51	11.23	6.20	5.11	5.48	5.01
40	80	80	17.82	17.34	17.76	15.74	15.53	9.25	5.68	5.08	5.38	5.04
50	100	100	17.28	16.98	17.23	15.70	15.52	8.22	5.45	5.08	5.31	5.02
100	200	200	16.35	16.24	16.30	15.62	15.54	6.46	5.16	5.07	5.19	5.06
200	400	400	15.88	15.87	15.89	15.54	15.51	5.64	5.01	4.99	5.06	5.00
400	800	800	15.72	15.69	15.69	15.55	15.53	5.35	5.04	5.04	5.07	5.04
800	1600	1600	15.61	15.60	15.60	15.53	15.52	5.18	5.02	5.02	5.04	5.02
20	10	40	21.97	19.56	21.68	16.07	15.54	17.39	8.03	5.29	5.84	5.05
30	15	60	19.15	18.21	19.02	15.88	15.54	11.94	6.27	5.16	5.58	5.06
40	20	80	18.10	17.53	17.96	15.83	15.59	9.79	5.81	5.19	5.53	5.14
50	25	100	17.49	17.13	17.39	15.76	15.57	8.58	5.54	5.15	5.41	5.10
100	50	200	16.41	16.32	16.38	15.62	15.53	6.60	5.15	5.05	5.19	5.05
200	100	400	15.96	15.91	15.93	15.58	15.54	5.77	5.07	5.04	5.12	5.05
400	200	800	15.73	15.71	15.71	15.54	15.52	5.38	5.03	5.03	5.06	5.02
800	400	1600	15.60	15.61	15.61	15.51	15.50	5.15	4.99	4.99	5.00	4.99
20	40	10	21.46	19.23	21.31	15.94	15.44	16.31	7.84	5.16	5.64	4.89
30	60	15	18.88	17.99	18.78	15.83	15.53	11.39	6.25	5.13	5.53	5.03
40	80	20	17.84	17.37	17.78	15.75	15.53	9.31	5.68	5.08	5.39	5.03
50	100	25	17.33	17.00	17.25	15.74	15.56	8.30	5.49	5.12	5.37	5.09
100	200	50	16.36	16.25	16.31	15.63	15.55	6.49	5.17	5.07	5.20	5.07
200	400	100	15.90	15.88	15.89	15.56	15.52	5.68	5.04	5.02	5.08	5.02
400	800	200	15.70	15.69	15.70	15.53	15.51	5.32	5.01	5.00	5.04	5.00
800	1600	400	15.60	15.60	15.60	15.51	15.51	5.16	5.00	5.00	5.01	5.00

Note. $\chi_8^2(0.05) = 15.51$.

Table 3: The upper percentiles of test statistics and empirical type I errors for $(p_1, p_2, p_3, p_4, p_5) = (15, 12, 9, 6, 3)$ and $\alpha = 0.05$.

n_1	$n_2^{(\ell)} = \cdots = n_5^{(\ell)}$	q	q_{AE}	q_{YKP}	q_{Q^*}	q_Y	α_{χ^2}	α_{AE}	α_{YKP}	α_{Q^*}	α_Y
20	10	42.92	33.15	41.90	24.41	24.06	35.47	14.97	5.62	4.36	3.93
30	10	34.76	30.93	33.89	25.44	25.04	22.17	9.15	5.74	5.54	5.06
40	10	31.96	29.70	31.23	25.57	25.25	16.88	7.55	5.72	5.73	5.33
50	10	30.46	28.90	29.85	25.56	25.30	14.07	6.78	5.63	5.73	5.40
100	10	27.64	27.13	27.36	25.34	25.21	9.02	5.62	5.33	5.45	5.26
200	10	26.34	26.12	26.18	25.21	25.13	6.95	5.27	5.20	5.28	5.18
400	10	25.69	25.58	25.59	25.13	25.09	5.97	5.15	5.13	5.18	5.13
20	20	40.59	31.86	40.09	24.21	23.99	31.19	14.23	5.31	4.11	3.83
30	20	33.30	30.07	32.74	25.21	24.90	19.38	8.65	5.51	5.26	4.87
40	20	30.90	29.07	30.44	25.35	25.09	14.96	7.10	5.46	5.45	5.13
50	20	29.65	28.42	29.27	25.38	25.16	12.65	6.44	5.42	5.49	5.22
100	20	27.42	26.95	27.16	25.31	25.19	8.69	5.59	5.33	5.42	5.27
200	20	26.27	26.06	26.12	25.20	25.13	6.83	5.26	5.19	5.27	5.18
400	20	25.67	25.56	25.57	25.12	25.09	5.92	5.15	5.13	5.17	5.12
20	50	38.51	30.47	38.45	23.95	23.86	26.99	13.79	5.04	3.78	3.64
30	50	31.68	29.01	31.46	24.97	24.76	16.35	8.15	5.21	4.97	4.70
40	50	29.71	28.23	29.44	25.16	24.98	12.74	6.76	5.27	5.21	4.98
50	50	28.68	27.74	28.47	25.20	25.04	10.88	6.13	5.24	5.26	5.06
100	50	26.94	26.63	26.80	25.19	25.09	7.91	5.39	5.17	5.25	5.13
200	50	26.12	25.94	25.98	25.16	25.11	6.61	5.23	5.17	5.22	5.15
400	50	25.58	25.52	25.53	25.08	25.05	5.82	5.09	5.07	5.10	5.06
20	20	40.59	31.86	40.09	24.21	23.99	31.19	14.23	5.31	4.11	3.83
30	30	32.48	29.57	32.12	25.07	24.81	17.88	8.36	5.33	5.09	4.77
40	40	29.95	28.43	29.66	25.20	24.99	13.20	6.81	5.30	5.26	4.99
50	50	28.68	27.74	28.47	25.20	25.04	10.88	6.13	5.24	5.26	5.06
100	100	26.61	26.37	26.53	25.12	25.05	7.37	5.30	5.10	5.16	5.06
200	200	25.79	25.68	25.72	25.08	25.05	6.09	5.13	5.08	5.11	5.07
400	400	25.35	25.34	25.35	25.02	25.00	5.50	5.02	5.00	5.03	5.00

Note. $\chi^2_{15}(0.05) = 25.00$.