# Simultaneous Testing of Mean Vector and Covariance Matrix with Three-step Monotone Missing Data

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**Abstract**

In this paper, we consider the problem of simultaneous testing of the mean vector and the covariance matrix in the case of a three-step monotone pattern. We provide the likelihood ratio test (LRT) statistic and propose a modified LRT statistic to improve the accuracy of the $\chi^2$ approximation. This modified LRT statistic is derived by decomposing of the likelihood ratio (LR) using the coefficient of the modified LRT statistic with complete data. Finally, we investigate the asymptotic behavior of the upper percentiles of these test statistics via Monte Carlo simulation.

*Key Words and Phrases.* Likelihood ratio test, linear interpolation, maximum likelihood estimator, modified likelihood ratio test statistic.

## 1 Introduction

Examining the problem of missing data is an important aspect of statistical data analysis. Several authors have developed various statistical methods for dealing with missing data. In this paper, we consider the problem of simultaneous testing of the mean vector and the covariance matrix in the case of a three-step monotone pattern. For the simultaneous test, the LRT statistic and the modified LRT statistic with Bartlett correction in the case of complete data were discussed by Muirhead (1982) and Srivastava (2002). Further, an LRT statistic and a modified LRT statistic for two-step monotone missing data was proposed by Hao and Krishnamoorthy (2001) and Hosoya and Seo (2015). In particular, Hosoya and Seo (2015) presented a modified LRT statistic by decomposing the LR; this paper is an extension of the work presented by Hosoya and Seo (2015). Indeed, an LRT statistic and a modified LRT statistic for general $k$-step monotone missing data, which is obtained by correcting only a part of the missing data, was given by Yagi, Yamaguchi, and Seo (2016).

The remainder of this paper is organized as follows. In Section 2, we describe the MLEs of the mean vector and covariance matrix and its LRT statistic in the case of three-step monotone missing data. MLEs for general $k$-step monotone missing data were discussed by Jinadasa and Tracy

(1992) and Kanda and Fujikoshi (1998). Section 3 describes the modified LRT statistic in the case of complete data, which is used to derive the modified LRT statistic in Section 4. In Section 4, we decompose the LR and propose two modified LRT statistics. We also propose a modified LRT statistic using linear interpolation. In Section 5, using Monte Carlo simulation, we investigate the $\chi^2$ approximation accuracy of the three modified LRT statistics proposed in Section 4. Finally, in Section 6, we state our conclusions.

## 2 Likelihood ratio with three-step monotone missing data

We suppose that the data is normally distributed as follows.

$$\begin{aligned} \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{N_1} &\overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \\ \boldsymbol{x}_{(12),N_1+1}, \boldsymbol{x}_{(12),N_1+2}, \ldots, \boldsymbol{x}_{(12),N_1+N_2} &\overset{i.i.d.}{\sim} N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}, \boldsymbol{\Sigma}_{(12)(12)}), \\ \boldsymbol{x}_{1,N_1+N_2+1}, \boldsymbol{x}_{1,N_1+N_2+2}, \ldots, \boldsymbol{x}_{1N} &\overset{i.i.d.}{\sim} N_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}), \end{aligned}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \boldsymbol{\mu}_3 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \left(\begin{array}{cc|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} \\ \hline \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{(12)(12)} & \boldsymbol{\Sigma}_{(12)3} \\ \hline \boldsymbol{\Sigma}_{3(12)} & \boldsymbol{\Sigma}_{33} \end{array}\right).$$

We partition $\boldsymbol{x}_j$ into a $p_1 \times 1$ random vector, a $p_2 \times 1$ random vector, and $p_3 \times 1$ random vector as $\boldsymbol{x}_j = (\boldsymbol{x}'_{1j}, \boldsymbol{x}'_{2j}, \boldsymbol{x}'_{3j})' (j = 1, \ldots, N_1)$. Additionally, let $\boldsymbol{x}_{(12),j} = (\boldsymbol{x}'_{1j}, \boldsymbol{x}'_{2j})' (j = N_1 + 1, \ldots, N_1 + N_2)$.

Such a dataset has three-step monotone missing data:

$$\begin{pmatrix} \overbrace{\boldsymbol{x}'_{11}}^{p_1} & \overbrace{\boldsymbol{x}'_{21}}^{p_2} & \overbrace{\boldsymbol{x}'_{31}}^{p_3} \\ \vdots & \vdots & \vdots \\ \boldsymbol{x}'_{1N_1} & \boldsymbol{x}'_{2N_1} & \boldsymbol{x}'_{3N_1} \\ \boldsymbol{x}'_{1,N_1+1} & \boldsymbol{x}'_{2,N_1+1} & * \cdots * \\ \vdots & \vdots & \vdots \quad \vdots \\ \boldsymbol{x}'_{1,N_1+N_2} & \boldsymbol{x}'_{2,N_1+N_2} & * \cdots * \\ \boldsymbol{x}'_{1,N_1+N_2+1} & * \cdots * & * \cdots * \\ \vdots & \vdots \quad \vdots & \vdots \quad \vdots \\ \boldsymbol{x}'_{1N} & * \cdots * & * \cdots * \end{pmatrix},$$

where $N = N_1 + N_2 + N_3, p = p_1 + p_2 + p_3$ and "$*$" indicates a missing observation.

To derive the LRT statistic of the simultaneous testing of the mean vector and the covariance matrix in the case of a three-step monotone missing data pattern, we present their MLEs, which are given by

$$
\begin{aligned}
\widehat{\boldsymbol{\mu}}_1 &= \frac{1}{N}(N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1} + N_3\overline{\boldsymbol{x}}_{(3)}),\\
\widehat{\boldsymbol{\mu}}_2 &= \frac{1}{N_1+N_2}(N_1\overline{\boldsymbol{x}}_{(1)2} + N_2\overline{\boldsymbol{x}}_{(2)2}) - \boldsymbol{F}'(\frac{N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1}}{N_1+N_2} - \widehat{\boldsymbol{\mu}}_1),\\
\widehat{\boldsymbol{\mu}}_3 &= \overline{\boldsymbol{x}}_{(1)3} - \boldsymbol{G}'\begin{pmatrix} \overline{\boldsymbol{x}}_{(1)1} - \widehat{\boldsymbol{\mu}}_1 \\ \overline{\boldsymbol{x}}_{(1)2} - \widehat{\boldsymbol{\mu}}_2 \end{pmatrix},\\
\widehat{\boldsymbol{\Sigma}}_{11} &= \frac{1}{N}(\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11} + \boldsymbol{W}_{(3)}), \quad \widehat{\boldsymbol{\Sigma}}_{12} = \widehat{\boldsymbol{\Sigma}}_{21}' = \widehat{\boldsymbol{\Sigma}}_{11}\boldsymbol{F},\\
\widehat{\boldsymbol{\Sigma}}_{22} &= \frac{1}{N_1+N_2}\boldsymbol{A}_{22\cdot 1} + \boldsymbol{F}'\widehat{\boldsymbol{\Sigma}}_{11}\boldsymbol{F}, \quad \widehat{\boldsymbol{\Sigma}}_{(12)3} = \widehat{\boldsymbol{\Sigma}}_{(3)12}' = \widehat{\boldsymbol{\Sigma}}_{(12)(12)}\boldsymbol{G},\\
\widehat{\boldsymbol{\Sigma}}_{33} &= \frac{1}{N_1}\boldsymbol{W}_{(1)33\cdot 12} + \boldsymbol{G}'\widehat{\boldsymbol{\Sigma}}_{(12)(12)}\boldsymbol{G},
\end{aligned}
$$

where

$$
\overline{\boldsymbol{x}}_{(1)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(1)1} \\ \overline{\boldsymbol{x}}_{(1)2} \\ \overline{\boldsymbol{x}}_{(1)3} \end{pmatrix}, \ \overline{\boldsymbol{x}}_{(1)1} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{1j}, \ \overline{\boldsymbol{x}}_{(1)2} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{2j}, \ \overline{\boldsymbol{x}}_{(1)3} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{3j},
$$

$$
\overline{\boldsymbol{x}}_{(2)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(2)1} \\ \overline{\boldsymbol{x}}_{(2)2} \end{pmatrix}, \ \overline{\boldsymbol{x}}_{(2)1} = \frac{1}{N_2}\sum_{j=N_1+1}^{N_1+N_2}\boldsymbol{x}_{1j}, \ \overline{\boldsymbol{x}}_{(2)2} = \frac{1}{N_2}\sum_{j=N_1+1}^{N_1+N_2}\boldsymbol{x}_{2j}, \ \overline{\boldsymbol{x}}_{(3)} = \frac{1}{N_3}\sum_{j=N_1+N_2+1}^{N}\boldsymbol{x}_{1j},
$$

and

$$
\begin{aligned}
\boldsymbol{F} &= (\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11})^{-1}(\boldsymbol{W}_{(1)12} + \boldsymbol{W}_{(2)12}), \quad \boldsymbol{G} = (\boldsymbol{W}_{(1),(12)(12)})^{-1}\boldsymbol{W}_{(1),(12)3},\\
\boldsymbol{W}_{(1)} &= \sum_{j=1}^{N_1}(\boldsymbol{x}_j - \overline{\boldsymbol{x}}_{(1)})(\boldsymbol{x}_j - \overline{\boldsymbol{x}}_{(1)})' = \left(\begin{array}{cc|c} \boldsymbol{W}_{(1)11} & \boldsymbol{W}_{(1)12} & \boldsymbol{W}_{(1)13} \\ \boldsymbol{W}_{(1)21} & \boldsymbol{W}_{(1)22} & \boldsymbol{W}_{(1)23} \\ \hline \boldsymbol{W}_{(1)31} & \boldsymbol{W}_{(1)32} & \boldsymbol{W}_{(1)33} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{W}_{(1),(12)(12)} & \boldsymbol{W}_{(1),(12)3} \\ \hline \boldsymbol{W}_{(1),3(12)} & \boldsymbol{W}_{(1)33} \end{array}\right),\\
\boldsymbol{W}_{(2)} &= \sum_{j=N_1+1}^{N_1+N_2}\begin{pmatrix} \boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_{(2)1} \\ \boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_{(2)2} \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_{(2)1} \\ \boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_{(2)2} \end{pmatrix}' + \frac{N_1N_2}{N_1+N_2}\begin{pmatrix} \overline{\boldsymbol{x}}_{(1)1} - \overline{\boldsymbol{x}}_{(2)1} \\ \overline{\boldsymbol{x}}_{(1)2} - \overline{\boldsymbol{x}}_{(2)2} \end{pmatrix}\begin{pmatrix} \overline{\boldsymbol{x}}_{(1)1} - \overline{\boldsymbol{x}}_{(2)1} \\ \overline{\boldsymbol{x}}_{(1)2} - \overline{\boldsymbol{x}}_{(2)2} \end{pmatrix}'\\
&= \begin{pmatrix} \boldsymbol{W}_{(2)11} & \boldsymbol{W}_{(2)12} \\ \boldsymbol{W}_{(2)21} & \boldsymbol{W}_{(2)22} \end{pmatrix},\\
\boldsymbol{W}_{(3)} &= \sum_{j=N_1+N_2+1}^{N}(\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_{(3)})(\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_{(3)})' + \frac{(N_1+N_2)N_3}{N}\left(\overline{\boldsymbol{x}}_{(3)} - \frac{1}{N_1+N_2}(N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1})\right)\\
&\qquad \times \left(\overline{\boldsymbol{x}}_{(3)} - \frac{1}{N_1+N_2}(N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1})\right)',\\
\boldsymbol{W}_{(1)33\cdot 12} &= \boldsymbol{W}_{(1)33} - \boldsymbol{W}_{(1),3(12)}\boldsymbol{W}_{(1),(12)(12)}^{-1}\boldsymbol{W}_{(1),(12)3},\\
\boldsymbol{A} &= \boldsymbol{W}_{(1),(12)(12)} + \boldsymbol{W}_{(2)}, \quad \boldsymbol{A}_{22\cdot 1} = \boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}.
\end{aligned}
$$

The results obtained are the same as those obtained in the work of Kanda and Fujikoshi (1998). For details, see Appendix. Now, consider the following hypothesis test when the dataset has a three-step monotone pattern.

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0, \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 \quad \text{vs.} \quad H_1 : \text{not} \quad H_0,$$

where $\boldsymbol{\mu}_0$ is a known vector and $\boldsymbol{\Sigma}_0$ is a known matrix. Without loss of generality, we can assume that $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. Then, the LR of the hypothesis test can be given by

$$\begin{aligned}
\lambda = & \left(\frac{e}{N}\right)^{\frac{1}{2}Np_1} \left|\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11} + \boldsymbol{W}_{(3)}\right|^{\frac{N}{2}} \left(\frac{e}{N_1+N_2}\right)^{\frac{1}{2}(N_1+N_2)p_2} \left|\boldsymbol{A}_{22\cdot 1}\right|^{\frac{N_1+N_2}{2}} \left(\frac{e}{N_1}\right)^{\frac{1}{2}N_1p_3} \left|\boldsymbol{W}_{(1)33\cdot 12}\right|^{\frac{N_1}{2}} \\
& \times \operatorname{etr}\left(-\frac{1}{2}\left(\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11} + \boldsymbol{W}_{(3)} + \frac{1}{N}(N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1} + N_3\overline{\boldsymbol{x}}_{(3)})(N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1} + N_3\overline{\boldsymbol{x}}_{(3)})'\right)\right) \\
& \times \operatorname{etr}\left(-\frac{1}{2}\left(\boldsymbol{A}_{22\cdot 1} + \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} + \frac{1}{N_1+N_2}(N_1\overline{\boldsymbol{x}}_{(1)2} + N_2\overline{\boldsymbol{x}}_{(2)2})(N_1\overline{\boldsymbol{x}}_{(1)2} + N_2\overline{\boldsymbol{x}}_{(2)2})'\right)\right) \\
& \times \operatorname{etr}\left(-\frac{1}{2}(\boldsymbol{W}_{(1)33\cdot 12} + \boldsymbol{W}_{(1),3(12)}\boldsymbol{W}_{(1),(12)(12)}^{-1}\boldsymbol{W}_{(1),(12)3} + N_1\overline{\boldsymbol{x}}_{(1)3}\overline{\boldsymbol{x}}_{(1)3}')\right).
\end{aligned}$$

This LR is essentially the same that obtained by Yagi, Yamaguchi, and Seo (2016). Thus, we obtain the test statistic $-2\log\lambda$. For example, Table 1 presents the simulated values of the upper $100\alpha$ percentiles of $-2\log\lambda$ and Type I error rate, $\alpha_1 = \Pr\{-2\log\lambda > \chi_f^2(\alpha)\}$ for the three-step monotone missing data case, where $\chi_f^2(\alpha)$ is the upper percentile of the $\chi^2$ distribution with $f$ degrees of freedom and $f = p(p+3)/2$.

Table 1: The upper percentile of $-2\log\lambda$ and type I error rates when $(p_1, p_2, p_3) = (3, 3, 3)$.

| Sample Size | | | Upper Percentile | Type I Error Rate |
|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $\alpha_1$ |
| | | | $\alpha = 0.05$ | |
| 10 | 10 | 10 | 155.129 | 0.907 |
| 20 | 10 | 10 | 87.775 | 0.282 |
| 40 | 10 | 10 | 78.955 | 0.127 |
| 80 | 10 | 10 | 75.419 | 0.082 |
| 200 | 10 | 10 | 73.431 | 0.061 |
| | | | $\alpha = 0.01$ | |
| 10 | 10 | 10 | 187.825 | 0.804 |
| 20 | 10 | 10 | 99.009 | 0.115 |
| 40 | 10 | 10 | 88.653 | 0.036 |
| 80 | 10 | 10 | 84.693 | 0.019 |
| 200 | 10 | 10 | 82.514 | 0.013 |

Note. $\chi_f^2(0.05) = 72.15$, $\chi_f^2(0.01) = 81.07$, $f = 54$.

As presented in Table 1, the accuracy of the $\chi^2$ approximation in this case is not desirable when the sample size is not large; therefore, a modified LRT statistic is needed to improve the accuracy of the $\chi^2$ approximation.

## 3 LRT statistic for complete data

We consider the LRT statistic and the modified LRT statistic with Bartlett correction in the case of complete data. These results are used in the next section. We first consider a simultaneous test for complete data as follows.

$$H_{01} : \boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\Sigma} = \boldsymbol{I} \quad \text{vs.} \quad H_{11} : \text{not} \quad H_{01}$$

In this case, the LR can be expressed as follows. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be independently distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and let $\lambda_S$ be the LR for the complete data. Then, the LR is given by

$$\lambda_S = e^{\frac{Np}{2}} \left| \frac{1}{N} \boldsymbol{W} \right|^{\frac{N}{2}} \text{etr}\left( -\frac{1}{2} \boldsymbol{W} \right) \exp\left( -\frac{1}{2} N \overline{\boldsymbol{x}}' \overline{\boldsymbol{x}} \right),$$

where

$$\boldsymbol{W} = \sum_{j=1}^{N} (\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})', \ \overline{\boldsymbol{x}} = \frac{1}{N} \sum_{j=1}^{N} \boldsymbol{x}_j.$$

Furthermore, the modified LRT statistic with Bartlett correction can be given by $-2\rho_1 \log \lambda_S$ (Muirhead (1982, p370)), where

$$\rho_1 = 1 - \frac{2p^2 + 9p + 11}{6N(p+3)}.$$

Next, we consider a covariance test for complete data as follows.

$$H_{02} : \boldsymbol{\Sigma} = \boldsymbol{I} \quad \text{vs.} \quad H_{12} : \text{not} \quad H_{02}$$

In this case, the LR, which is an unbiased test, can be expressed as follows.

$$\lambda_V = e^{\frac{(N-1)p}{2}} \left| \frac{1}{N-1} \boldsymbol{W} \right|^{\frac{N-1}{2}} \text{etr}\left( -\frac{1}{2} \boldsymbol{W} \right).$$

The modified LRT statistic with Bartlett correction $-2\rho_2 \log \lambda_V$ was provided by Muirhead (1982, p359), where

$$\rho_2 = 1 - \frac{2p^2 + 3p - 1}{6(N-1)(p+1)}.$$

## 4 Modified LRT statistic

We now decompose the LR to derive the modified LRT statistic. Let

$$\begin{aligned}
\omega_1 &= \exp\left(-\frac{1}{2N}(N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1} + N_3\overline{\boldsymbol{x}}_{(3)})'(N_1\overline{\boldsymbol{x}}_{(1)1} + N_2\overline{\boldsymbol{x}}_{(2)1} + N_3\overline{\boldsymbol{x}}_{(3)})\right), \\
\omega_2 &= \exp\left(-\frac{1}{2(N_1+N_2)}(N_1\overline{\boldsymbol{x}}_{(1)2} + N_2\overline{\boldsymbol{x}}_{(2)2})'(N_1\overline{\boldsymbol{x}}_{(1)2} + N_2\overline{\boldsymbol{x}}_{(2)2})\right), \\
\omega_3 &= \exp\left(-\frac{N_1}{2}\overline{\boldsymbol{x}}'_{(1)3}\overline{\boldsymbol{x}}_{(1)3}\right), \\
\omega_4 &= \left(\frac{e}{N}\right)^{\frac{1}{2}Np_1} \left|\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11} + \boldsymbol{W}_{(3)}\right|^{\frac{N}{2}} \mathrm{etr}\left(-\frac{1}{2}(\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11} + \boldsymbol{W}_{(3)})\right), \\
\omega_5 &= \left(\frac{e}{N_1+N_2}\right)^{\frac{1}{2}(N_1+N_2)p_2} |\boldsymbol{A}_{22\cdot1}|^{\frac{N_1+N_2}{2}} \mathrm{etr}\left(-\frac{1}{2}\boldsymbol{A}_{22\cdot1}\right), \\
\omega_6 &= \left(\frac{e}{N_1}\right)^{\frac{1}{2}N_1p_3} \left|\boldsymbol{W}_{(1)33\cdot12}\right|^{\frac{N_1}{2}} \mathrm{etr}\left(-\frac{1}{2}\boldsymbol{W}_{(1)33\cdot12}\right), \\
\omega_7 &= \mathrm{etr}\left(-\frac{1}{2}\boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\right)\mathrm{etr}\left(-\frac{1}{2}\boldsymbol{W}_{(1),3(12)}\boldsymbol{W}_{(1),(12)(12)}^{-1}\boldsymbol{W}_{(1),(12)3}\right).
\end{aligned}$$

Therefore,

$$\lambda = \prod_{i=1}^{7}\omega_i.$$

Then, $\omega_1\omega_4, \omega_2\omega_5, \omega_3\omega_6$ are of the form of LR for $H_{01}$ under non-missing normality. Hence, we can obtain the modified LRT statistic, $-2\rho_{14}\log\omega_1\omega_4, -2\rho_{25}\log\omega_2\omega_5, -2\rho_{36}\log\omega_3\omega_6$, where

$$\rho_{14} = 1 - \frac{2p_1^2 + 9p_1 + 11}{6N(p_1+3)},\ \rho_{25} = 1 - \frac{2p_2^2 + 9p_2 + 11}{6(N_1+N_2)(p_2+3)},\ \rho_{36} = 1 - \frac{2p_3^2 + 9p_3 + 11}{6N_1(p_3+3)}.$$

Thus, we propose a new modified LRT statistic given by $-2\log\tau$, where

$$\tau = (\omega_1\omega_4)^{\rho_{14}}(\omega_2\omega_5)^{\rho_{25}}(\omega_3\omega_6)^{\rho_{36}}\omega_7.$$

In addition, we denote

$$\begin{aligned}
\omega_4^* &= \left(\frac{e}{n}\right)^{\frac{1}{2}np_1} \left|\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11} + \boldsymbol{W}_{(3)}\right|^{\frac{n}{2}} \mathrm{etr}\left(-\frac{1}{2}(\boldsymbol{W}_{(1)11} + \boldsymbol{W}_{(2)11} + \boldsymbol{W}_{(3)})\right), \\
\omega_5^* &= \left(\frac{e}{n_1+n_2}\right)^{\frac{1}{2}(n_1+n_2)p_2} |\boldsymbol{A}_{22\cdot1}|^{\frac{n_1+n_2}{2}} \mathrm{etr}\left(-\frac{1}{2}\boldsymbol{A}_{22\cdot1}\right), \\
\omega_6^* &= \left(\frac{e}{n_1}\right)^{\frac{1}{2}n_1p_3} \left|\boldsymbol{W}_{(1)33\cdot12}\right|^{\frac{n_1}{2}} \mathrm{etr}\left(-\frac{1}{2}\boldsymbol{W}_{(1)33\cdot12}\right),
\end{aligned}$$

where

$$n = N-1,\ n_1 = N_1 - (p_1+p_2) - 1,\ n_1 + n_2 = N_1 + N_2 - p_1 - 1.$$

Then, since $\omega_4^*, \omega_5^*, \omega_6^*$ are of the form of LR for $H_{02}$ under non-missing normality, we can propose the modified LRT statistic as $-2\log\phi$, where

$$\phi = \omega_1\omega_2\omega_3(\omega_4^*)^{\rho_4^*}(\omega_5^*)^{\rho_5^*}(\omega_6^*)^{\rho_6^*}\omega_7$$

and

$$\rho_4^* = 1 - \frac{2p_1^2 + 3p_1 - 1}{6n(p_1+1)}, \ \rho_5^* = 1 - \frac{2p_2^2 + 3p_2 - 1}{6(n_1+n_2)(p_2+1)}, \ \rho_6^* = 1 - \frac{2p_3^2 + 3p_3 - 1}{6n_1(p_3+1)}.$$

Now, we propose the modified LRT statistic $-2\rho_m \log\lambda$ via linear interpolation, where

$$\rho_m = \left\{1 - \frac{(p_1+p_2)N_2 + p_1 N_3}{p(N_2+N_3)}\right\}\rho_{N_1} + \frac{(p_1+p_2)N_2 + p_1 N_3}{p(N_2+N_3)}\rho_N,$$

and

$$\rho_{N_1} = 1 - \frac{2p^2 + 9p + 11}{6N_1(p+3)}, \ \rho_N = 1 - \frac{2p^2 + 9p + 11}{6N(p+3)}.$$

# 5 Simulation studies

We evaluate the accuracy and the asymptotic behaviors of the $\chi^2$ approximations via Monte Carlo simulation ($10^6$ runs). Now let

$$\begin{aligned}
\alpha_1 &= \Pr\{-2\log\lambda > \chi_f^2(\alpha)\},\\
\alpha_{\rho_m} &= \Pr\{-2\rho_m\log\lambda > \chi_f^2(\alpha)\},\\
\alpha_\tau &= \Pr\{-2\log\tau > \chi_f^2(\alpha)\},\\
\alpha_\phi &= \Pr\{-2\log\phi > \chi_f^2(\alpha)\},
\end{aligned}$$

where $\chi_f^2(\alpha)$ is the upper percentile of the $\chi^2$ distribution with $f$ degrees of freedom.

In Tables 2, 4, 6, and 8, we provide the simulated upper $100\alpha$ percentiles and the actual type I error rates for the upper percentiles $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of $-2\log\lambda$, $-2\rho_m\log\lambda$, $-2\log\tau$ and $-2\log\phi$; $\alpha = 0.05, 0.01$; and for the following cases (Case I),

$$(N_1, N_2, N_3) = \begin{cases} (m, 10, 10), & \\ (m, 20, 20), & m = 10, 20, 30, 40, 80, 200, 400, \\ (m, 50, 50), & \end{cases}$$

where $(p_1, p_2, p_3)$ in Tables 2, 4, 6, and 8 are $(3,3,3), (4,4,4), (5,5,5), (6,6,6)$, respectively.

Similarly, Tables 3, 5, 7, and 9 show the results for the following cases (Case II),

$$(N_1, N_2, N_3) = \begin{cases} (m, m/2, m/2), & \\ (m, m, m), & m = 10, 20, 30, 40, 80, 200, 400, \\ (m, 2m, 2m), & \end{cases}$$

where $(p_1, p_2, p_3)$ in Tables 3, 5, 7, and 9 are $(3,3,3), (4,4,4), (5,5,5), (6,6,6)$ respectively.

It may be noted from above mentioned Tables that the simulated values are closer to the upper percentile of the $\chi^2$ distribution when the sample size increases. In addition, it can be seen that the upper percentile of $-2\log\phi$ is considerably better than that of $-2\log\lambda$ even for small sample sizes; while it is not as good as $-2\log\phi$, the upper percentile of $-2\rho_m\log\lambda$ is good.

Table 2: The simulated values for $-2\log\lambda, -2\rho_m \log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (3, 3, 3)$ for Case I.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 10 | 10 | 10 | 155.129 | 118.645 | 137.865 | 74.060 | 0.907 | 0.597 | 0.833 | **0.067** |
| 20 | 10 | 10 | 87.775 | 76.163 | 84.460 | 72.172 | 0.282 | 0.090 | 0.220 | **0.050** |
| 40 | 10 | 10 | 78.955 | 73.152 | 77.524 | 72.168 | 0.127 | 0.059 | 0.108 | **0.050** |
| 80 | 10 | 10 | 75.419 | 72.426 | 74.720 | 72.177 | 0.082 | 0.052 | 0.074 | **0.050** |
| 200 | 10 | 10 | 73.431 | 72.195 | 73.151 | 72.158 | 0.061 | **0.050** | 0.059 | **0.050** |
| 400 | 10 | 10 | 72.767 | 72.140 | 72.625 | 72.140 | 0.055 | **0.050** | 0.054 | **0.050** |
| 10 | 20 | 20 | 153.232 | 120.798 | 136.674 | 73.968 | 0.895 | 0.619 | 0.822 | **0.066** |
| 20 | 20 | 20 | 86.852 | 76.639 | 83.930 | 72.184 | 0.263 | 0.095 | 0.209 | **0.050** |
| 40 | 20 | 20 | 78.565 | 73.368 | 77.286 | 72.154 | 0.122 | 0.060 | 0.105 | **0.050** |
| 80 | 20 | 20 | 75.299 | 72.532 | 74.652 | 72.165 | 0.080 | 0.053 | 0.073 | **0.050** |
| 200 | 20 | 20 | 73.410 | 72.223 | 73.143 | 72.142 | 0.061 | 0.051 | 0.058 | **0.050** |
| 400 | 20 | 20 | 72.812 | 72.199 | 72.677 | 72.183 | 0.056 | **0.050** | 0.055 | **0.050** |
| 10 | 50 | 50 | 151.864 | 122.642 | 135.795 | 74.019 | 0.883 | 0.634 | 0.810 | **0.067** |
| 20 | 50 | 50 | 85.844 | 77.011 | 83.319 | 72.207 | 0.242 | 0.100 | 0.197 | **0.050** |
| 40 | 50 | 50 | 78.011 | 73.588 | 76.975 | 72.158 | 0.115 | 0.063 | 0.101 | **0.050** |
| 80 | 50 | 50 | 75.046 | 72.656 | 74.505 | 72.153 | 0.078 | 0.054 | 0.072 | **0.050** |
| 200 | 50 | 50 | 73.371 | 72.293 | 73.128 | 72.145 | 0.061 | 0.051 | 0.058 | **0.050** |
| 400 | 50 | 50 | 72.781 | 72.203 | 72.653 | 72.177 | 0.055 | **0.050** | 0.054 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 10 | 10 | 10 | 187.825 | 143.652 | 165.574 | 83.493 | 0.804 | 0.417 | 0.686 | **0.016** |
| 20 | 10 | 10 | 99.009 | 85.911 | 95.164 | 81.092 | 0.115 | 0.023 | 0.079 | **0.010** |
| 40 | 10 | 10 | 88.653 | 82.137 | 87.035 | 80.994 | 0.036 | 0.012 | 0.028 | **0.010** |
| 80 | 10 | 10 | 84.693 | 81.332 | 83.905 | 81.075 | 0.019 | 0.011 | 0.017 | **0.010** |
| 200 | 10 | 10 | 82.514 | 81.125 | 82.205 | 81.048 | 0.013 | **0.010** | 0.012 | **0.010** |
| 400 | 10 | 10 | 81.873 | 81.168 | 81.714 | 81.113 | 0.012 | **0.010** | 0.011 | **0.010** |
| 10 | 20 | 20 | 186.218 | 146.802 | 164.463 | 83.488 | 0.784 | 0.439 | 0.669 | **0.015** |
| 20 | 20 | 20 | 97.770 | 86.273 | 94.351 | 81.019 | 0.103 | 0.024 | 0.073 | **0.010** |
| 40 | 20 | 20 | 88.270 | 82.432 | 86.831 | 81.049 | 0.034 | 0.013 | 0.027 | **0.010** |
| 80 | 20 | 20 | 84.641 | 81.531 | 83.916 | 81.104 | 0.019 | 0.011 | 0.017 | **0.010** |
| 200 | 20 | 20 | 82.518 | 81.184 | 82.222 | 81.073 | 0.013 | **0.010** | 0.012 | **0.010** |
| 400 | 20 | 20 | 81.776 | 81.087 | 81.623 | 81.073 | 0.012 | **0.010** | 0.011 | **0.010** |
| 10 | 50 | 50 | 184.963 | 149.372 | 163.853 | 83.441 | 0.766 | 0.456 | 0.654 | **0.015** |
| 20 | 50 | 50 | 96.671 | 86.724 | 93.675 | 81.061 | 0.092 | 0.026 | 0.068 | **0.010** |
| 40 | 50 | 50 | 87.633 | 82.665 | 86.433 | 81.014 | 0.031 | 0.013 | 0.026 | **0.010** |
| 80 | 50 | 50 | 84.325 | 81.639 | 83.709 | 81.092 | 0.018 | 0.011 | 0.016 | **0.010** |
| 200 | 50 | 50 | 82.429 | 81.217 | 82.159 | 81.058 | 0.013 | **0.010** | 0.012 | **0.010** |
| 400 | 50 | 50 | 81.761 | 81.112 | 81.620 | 81.068 | 0.011 | **0.010** | 0.011 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 72.15$, $\chi^2_f(0.01) = 81.07$, $f = 54$.

Table 3: The simulated values for $-2\log\lambda, -2\rho_m\log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (3, 3, 3)$ for Case II.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 10 | 5 | 5 | 55.845 | 45.219 | 51.759 | 40.161 | 0.345 | 0.119 | 0.257 | **0.051** |
| 20 | 10 | 10 | 87.775 | 76.163 | 84.460 | 72.172 | 0.282 | 0.090 | 0.220 | **0.050** |
| 40 | 20 | 20 | 42.534 | 40.510 | 41.958 | 40.131 | 0.080 | 0.054 | 0.072 | **0.050** |
| 80 | 40 | 40 | 41.308 | 40.326 | 41.037 | 40.142 | 0.064 | 0.052 | 0.061 | **0.050** |
| 200 | 100 | 100 | 40.561 | 40.175 | 40.462 | 40.111 | 0.055 | 0.051 | 0.054 | **0.050** |
| 400 | 200 | 200 | 72.714 | 72.233 | 72.611 | 72.147 | 0.055 | 0.051 | 0.054 | **0.050** |
| 10 | 10 | 10 | 54.991 | 45.690 | 51.315 | 40.164 | 0.324 | 0.127 | 0.245 | **0.051** |
| 20 | 20 | 20 | 86.852 | 76.639 | 83.930 | 72.184 | 0.263 | 0.095 | 0.209 | **0.050** |
| 40 | 40 | 40 | 42.361 | 40.570 | 41.864 | 40.134 | 0.078 | 0.055 | 0.071 | **0.050** |
| 80 | 80 | 80 | 41.195 | 40.324 | 40.969 | 40.126 | 0.062 | 0.052 | 0.060 | **0.050** |
| 200 | 200 | 200 | 40.509 | 40.166 | 40.422 | 40.085 | 0.054 | 0.051 | 0.053 | **0.050** |
| 400 | 400 | 400 | 72.687 | 72.259 | 72.598 | 72.156 | 0.055 | 0.051 | 0.054 | **0.050** |
| 10 | 20 | 20 | 54.341 | 46.069 | 50.979 | 40.194 | 0.306 | 0.132 | 0.236 | **0.051** |
| 20 | 40 | 40 | 44.857 | 41.443 | 43.822 | 40.097 | 0.118 | 0.066 | 0.100 | **0.050** |
| 40 | 80 | 80 | 42.199 | 40.593 | 41.774 | 40.143 | 0.076 | 0.055 | 0.070 | **0.050** |
| 80 | 160 | 160 | 41.065 | 40.283 | 40.874 | 40.088 | 0.061 | 0.052 | 0.059 | **0.050** |
| 200 | 400 | 400 | 40.490 | 40.182 | 40.416 | 40.113 | 0.054 | 0.051 | 0.053 | **0.050** |
| 400 | 800 | 800 | 72.676 | 72.292 | 72.601 | 72.198 | 0.054 | 0.051 | 0.054 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 10 | 5 | 5 | 190.179 | 139.861 | 167.021 | 83.493 | 0.828 | 0.385 | 0.706 | **0.015** |
| 20 | 10 | 10 | 99.009 | 85.911 | 95.164 | 81.092 | 0.115 | 0.023 | 0.079 | **0.010** |
| 40 | 20 | 20 | 88.389 | 82.543 | 86.929 | 81.141 | 0.034 | 0.013 | 0.027 | **0.010** |
| 80 | 40 | 40 | 84.349 | 81.560 | 83.702 | 81.078 | 0.018 | 0.011 | 0.016 | **0.010** |
| 200 | 100 | 100 | 82.312 | 81.223 | 82.081 | 81.080 | 0.013 | **0.010** | 0.012 | **0.010** |
| 400 | 200 | 200 | 81.762 | 81.221 | 81.644 | 81.138 | 0.011 | **0.010** | 0.011 | **0.010** |
| 10 | 10 | 10 | 188.082 | 143.848 | 165.818 | 83.476 | 0.804 | 0.417 | 0.686 | **0.016** |
| 20 | 20 | 20 | 97.770 | 86.273 | 94.351 | 81.019 | 0.103 | 0.024 | 0.073 | **0.010** |
| 40 | 40 | 40 | 87.906 | 82.737 | 86.645 | 81.122 | 0.032 | 0.014 | 0.026 | **0.010** |
| 80 | 80 | 80 | 84.169 | 81.695 | 83.605 | 81.092 | 0.018 | 0.011 | 0.016 | **0.010** |
| 200 | 200 | 200 | 82.283 | 81.316 | 82.084 | 81.101 | 0.013 | **0.010** | 0.012 | **0.010** |
| 400 | 400 | 400 | 81.592 | 81.112 | 81.492 | 81.002 | 0.011 | **0.010** | 0.011 | **0.010** |
| 10 | 20 | 20 | 185.957 | 146.596 | 164.298 | 83.484 | 0.783 | 0.438 | 0.668 | **0.015** |
| 20 | 40 | 40 | 96.935 | 86.676 | 93.860 | 81.140 | 0.095 | 0.026 | 0.069 | **0.010** |
| 40 | 80 | 80 | 87.339 | 82.717 | 86.232 | 80.993 | 0.030 | 0.014 | 0.025 | **0.010** |
| 80 | 160 | 160 | 84.036 | 81.813 | 83.565 | 81.047 | 0.017 | 0.012 | 0.016 | **0.010** |
| 200 | 400 | 400 | 82.216 | 81.346 | 82.052 | 81.089 | 0.012 | 0.011 | 0.012 | **0.010** |
| 400 | 800 | 800 | 81.654 | 81.222 | 81.570 | 81.100 | 0.011 | **0.010** | 0.011 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 72.15$, $\chi^2_f(0.01) = 81.07$, $f = 54$.

Table 4: The simulated values for $-2\log\lambda, -2\rho_m\log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (4, 4, 4)$ for Case I.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 20 | 10 | 10 | 149.889 | 124.470 | 142.692 | 113.298 | 0.591 | 0.160 | 0.473 | **0.051** |
| 30 | 10 | 10 | 133.666 | 117.547 | 129.734 | 113.243 | 0.313 | 0.084 | 0.246 | **0.051** |
| 40 | 10 | 10 | 127.521 | 115.507 | 124.780 | 113.127 | 0.211 | 0.067 | 0.171 | **0.050** |
| 80 | 10 | 10 | 119.858 | 113.760 | 118.578 | 113.136 | 0.109 | 0.054 | 0.096 | **0.050** |
| 200 | 10 | 10 | 115.712 | 113.215 | 115.213 | 113.103 | 0.069 | **0.050** | 0.065 | **0.050** |
| 400 | 10 | 10 | 114.393 | 113.131 | 114.144 | 113.117 | 0.059 | **0.050** | 0.057 | **0.050** |
| 20 | 20 | 20 | 147.970 | 125.665 | 141.494 | 113.340 | 0.557 | 0.176 | 0.451 | **0.051** |
| 30 | 20 | 20 | 132.423 | 118.165 | 128.942 | 113.163 | 0.292 | 0.091 | 0.235 | **0.050** |
| 40 | 20 | 20 | 126.793 | 116.042 | 124.338 | 113.132 | 0.200 | 0.071 | 0.165 | **0.050** |
| 80 | 20 | 20 | 119.598 | 113.964 | 118.419 | 113.161 | 0.106 | 0.056 | 0.094 | **0.050** |
| 200 | 20 | 20 | 115.671 | 113.273 | 115.189 | 113.111 | 0.069 | 0.051 | 0.065 | **0.050** |
| 400 | 20 | 20 | 114.396 | 113.162 | 114.153 | 113.115 | 0.059 | **0.050** | 0.057 | **0.050** |
| 20 | 50 | 50 | 145.872 | 126.631 | 140.151 | 113.292 | 0.518 | 0.189 | 0.425 | **0.051** |
| 30 | 50 | 50 | 130.985 | 118.834 | 128.030 | 113.232 | 0.266 | 0.097 | 0.219 | **0.051** |
| 40 | 50 | 50 | 125.690 | 116.555 | 123.648 | 113.168 | 0.184 | 0.076 | 0.156 | **0.050** |
| 80 | 50 | 50 | 119.099 | 114.236 | 118.099 | 113.111 | 0.102 | 0.058 | 0.091 | **0.050** |
| 200 | 50 | 50 | 115.644 | 113.465 | 115.212 | 113.186 | 0.068 | 0.052 | 0.065 | **0.050** |
| 400 | 50 | 50 | 114.438 | 113.273 | 114.208 | 113.189 | 0.059 | 0.051 | 0.057 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 20 | 10 | 10 | 165.180 | 137.168 | 157.031 | 124.325 | 0.356 | 0.052 | 0.248 | **0.010** |
| 30 | 10 | 10 | 146.744 | 129.048 | 142.334 | 124.223 | 0.131 | 0.021 | 0.092 | **0.010** |
| 40 | 10 | 10 | 140.003 | 126.813 | 136.940 | 124.160 | 0.074 | 0.015 | 0.055 | **0.010** |
| 80 | 10 | 10 | 131.473 | 124.784 | 130.084 | 124.103 | 0.029 | 0.011 | 0.024 | **0.010** |
| 200 | 10 | 10 | 126.859 | 124.121 | 126.316 | 123.989 | 0.015 | **0.010** | 0.014 | **0.010** |
| 400 | 10 | 10 | 125.529 | 124.143 | 125.250 | 124.124 | 0.012 | **0.010** | 0.012 | **0.010** |
| 20 | 20 | 20 | 163.201 | 138.600 | 155.800 | 124.362 | 0.323 | 0.059 | 0.229 | **0.010** |
| 30 | 20 | 20 | 145.442 | 129.782 | 141.534 | 124.107 | 0.118 | 0.023 | 0.085 | **0.010** |
| 40 | 20 | 20 | 139.231 | 127.425 | 136.508 | 124.193 | 0.068 | 0.016 | 0.051 | **0.010** |
| 80 | 20 | 20 | 131.310 | 125.125 | 130.024 | 124.203 | 0.028 | 0.012 | 0.024 | **0.010** |
| 200 | 20 | 20 | 126.788 | 124.160 | 126.275 | 123.965 | 0.015 | **0.010** | 0.014 | **0.010** |
| 400 | 20 | 20 | 125.588 | 124.233 | 125.319 | 124.207 | 0.013 | **0.010** | 0.012 | **0.010** |
| 20 | 50 | 50 | 160.721 | 139.522 | 154.110 | 124.237 | 0.288 | 0.065 | 0.209 | **0.010** |
| 30 | 50 | 50 | 143.756 | 130.421 | 140.409 | 124.017 | 0.102 | 0.025 | 0.077 | **0.010** |
| 40 | 50 | 50 | 137.921 | 127.897 | 135.635 | 124.119 | 0.060 | 0.018 | 0.047 | **0.010** |
| 80 | 50 | 50 | 130.681 | 125.345 | 129.576 | 124.154 | 0.026 | 0.012 | 0.022 | **0.010** |
| 200 | 50 | 50 | 126.745 | 124.357 | 126.270 | 124.110 | 0.015 | **0.010** | 0.014 | **0.010** |
| 400 | 50 | 50 | 125.467 | 124.191 | 125.214 | 124.098 | 0.012 | **0.010** | 0.012 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 113.15$, $\chi^2_f(0.01) = 124.12$, $f = 90$.

Table 5: The simulated values for $-2\log\lambda, -2\rho_m \log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (4, 4, 4)$ for Case II.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 20 | 10 | 10 | 149.889 | 124.470 | 142.692 | 113.298 | 0.591 | 0.160 | 0.473 | **0.051** |
| 30 | 15 | 15 | 132.978 | 117.944 | 129.307 | 113.214 | 0.302 | 0.089 | 0.240 | **0.050** |
| 40 | 20 | 20 | 126.775 | 116.025 | 124.320 | 113.147 | 0.200 | 0.071 | 0.165 | **0.050** |
| 80 | 40 | 40 | 119.344 | 114.285 | 118.295 | 113.196 | 0.103 | 0.058 | 0.092 | **0.050** |
| 200 | 100 | 100 | 115.375 | 113.418 | 114.994 | 113.082 | 0.066 | 0.052 | 0.063 | **0.050** |
| 400 | 200 | 200 | 114.317 | 113.348 | 114.128 | 113.156 | 0.058 | 0.051 | 0.057 | **0.050** |
| 20 | 20 | 20 | 147.970 | 125.665 | 141.494 | 113.340 | 0.557 | 0.176 | 0.451 | **0.051** |
| 30 | 30 | 30 | 131.808 | 118.562 | 128.583 | 113.223 | 0.281 | 0.095 | 0.229 | **0.050** |
| 40 | 40 | 40 | 125.957 | 116.463 | 123.820 | 113.164 | 0.188 | 0.076 | 0.158 | **0.050** |
| 80 | 80 | 80 | 118.944 | 114.462 | 118.044 | 113.206 | 0.099 | 0.059 | 0.090 | **0.050** |
| 200 | 200 | 200 | 115.324 | 113.586 | 114.999 | 113.146 | 0.066 | 0.053 | 0.063 | **0.050** |
| 400 | 400 | 400 | 114.155 | 113.295 | 113.998 | 113.074 | 0.057 | 0.051 | 0.056 | **0.050** |
| 20 | 40 | 40 | 146.236 | 126.397 | 140.373 | 113.279 | 0.526 | 0.186 | 0.430 | **0.051** |
| 30 | 60 | 60 | 130.756 | 118.930 | 127.897 | 113.228 | 0.262 | 0.098 | 0.216 | **0.051** |
| 40 | 80 | 80 | 125.157 | 116.667 | 123.279 | 113.120 | 0.176 | 0.077 | 0.150 | **0.050** |
| 80 | 160 | 160 | 118.479 | 114.460 | 117.718 | 113.155 | 0.094 | 0.059 | 0.087 | **0.050** |
| 200 | 400 | 400 | 115.160 | 113.598 | 114.889 | 113.144 | 0.064 | 0.053 | 0.062 | **0.050** |
| 400 | 800 | 800 | 114.110 | 113.336 | 113.978 | 113.124 | 0.057 | 0.051 | 0.056 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 20 | 10 | 10 | 165.180 | 137.168 | 157.031 | 124.325 | 0.356 | 0.052 | 0.248 | **0.010** |
| 30 | 15 | 15 | 146.182 | 129.655 | 142.055 | 124.257 | 0.124 | 0.022 | 0.088 | **0.010** |
| 40 | 20 | 20 | 139.217 | 127.413 | 136.440 | 124.202 | 0.068 | 0.016 | 0.051 | **0.010** |
| 80 | 40 | 40 | 130.939 | 125.388 | 129.758 | 124.143 | 0.026 | 0.012 | 0.023 | **0.010** |
| 200 | 100 | 100 | 126.672 | 124.524 | 126.247 | 124.089 | 0.015 | 0.011 | 0.014 | **0.010** |
| 400 | 200 | 200 | 125.284 | 124.222 | 125.076 | 124.054 | 0.012 | **0.010** | 0.012 | **0.010** |
| 20 | 20 | 20 | 163.201 | 138.600 | 155.800 | 124.362 | 0.323 | 0.059 | 0.229 | **0.010** |
| 30 | 30 | 30 | 144.864 | 130.306 | 141.162 | 124.181 | 0.111 | 0.024 | 0.081 | **0.010** |
| 40 | 40 | 40 | 138.230 | 127.811 | 135.840 | 124.071 | 0.062 | 0.017 | 0.048 | **0.010** |
| 80 | 80 | 80 | 130.417 | 125.503 | 129.422 | 124.106 | 0.025 | 0.013 | 0.022 | **0.010** |
| 200 | 200 | 200 | 126.534 | 124.626 | 126.171 | 124.130 | 0.015 | 0.011 | 0.014 | **0.010** |
| 400 | 400 | 400 | 125.318 | 124.373 | 125.143 | 124.157 | 0.012 | **0.010** | 0.012 | **0.010** |
| 20 | 40 | 40 | 161.507 | 139.596 | 154.745 | 124.358 | 0.296 | 0.065 | 0.214 | **0.010** |
| 30 | 60 | 60 | 143.690 | 130.694 | 140.447 | 124.204 | 0.101 | 0.025 | 0.076 | **0.010** |
| 40 | 80 | 80 | 137.411 | 128.090 | 135.308 | 124.043 | 0.057 | 0.018 | 0.046 | **0.010** |
| 80 | 160 | 160 | 129.955 | 125.547 | 129.090 | 123.996 | 0.024 | 0.012 | 0.021 | **0.010** |
| 200 | 400 | 400 | 126.457 | 124.742 | 126.154 | 124.248 | 0.014 | 0.011 | 0.014 | **0.010** |
| 400 | 800 | 800 | 125.283 | 124.433 | 125.140 | 124.244 | 0.012 | 0.011 | 0.012 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 113.15$, $\chi^2_f(0.01) = 124.12$, $f = 90$.

Table 6: The simulated values for $-2\log\lambda, -2\rho_m\log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (5, 5, 5)$ for Case I.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 20 | 10 | 10 | 242.616 | 192.408 | 227.782 | 163.916 | 0.918 | 0.363 | 0.836 | **0.055** |
| 30 | 10 | 10 | 202.336 | 172.560 | 195.157 | 163.234 | 0.556 | 0.121 | 0.445 | **0.051** |
| 40 | 10 | 10 | 189.893 | 168.061 | 185.054 | 163.246 | 0.360 | 0.083 | 0.286 | **0.051** |
| 80 | 10 | 10 | 175.179 | 164.303 | 173.046 | 163.081 | 0.152 | 0.057 | 0.128 | **0.050** |
| 200 | 10 | 10 | 167.747 | 163.329 | 166.928 | 163.092 | 0.081 | 0.051 | 0.074 | **0.050** |
| 400 | 10 | 10 | 165.398 | 163.170 | 164.994 | 163.135 | 0.064 | **0.050** | 0.061 | **0.050** |
| 20 | 20 | 20 | 239.055 | 195.081 | 225.446 | 163.819 | 0.898 | 0.400 | 0.814 | **0.054** |
| 30 | 20 | 20 | 200.305 | 173.987 | 193.915 | 163.281 | 0.523 | 0.136 | 0.424 | **0.051** |
| 40 | 20 | 20 | 188.566 | 169.055 | 184.255 | 163.200 | 0.339 | 0.091 | 0.273 | **0.050** |
| 80 | 20 | 20 | 174.801 | 164.752 | 172.813 | 163.151 | 0.147 | 0.060 | 0.126 | **0.050** |
| 200 | 20 | 20 | 167.625 | 163.385 | 166.845 | 163.106 | 0.080 | 0.052 | 0.074 | **0.050** |
| 400 | 20 | 20 | 165.369 | 163.192 | 164.977 | 163.114 | 0.064 | **0.050** | 0.061 | **0.050** |
| 20 | 50 | 50 | 235.294 | 197.421 | 222.919 | 163.767 | 0.873 | 0.431 | 0.788 | **0.054** |
| 30 | 50 | 50 | 197.618 | 175.248 | 192.105 | 163.212 | 0.480 | 0.150 | 0.395 | **0.051** |
| 40 | 50 | 50 | 186.604 | 170.054 | 182.954 | 163.130 | 0.307 | 0.099 | 0.253 | **0.050** |
| 80 | 50 | 50 | 173.999 | 165.330 | 172.323 | 163.150 | 0.139 | 0.064 | 0.122 | **0.050** |
| 200 | 50 | 50 | 167.471 | 163.620 | 166.757 | 163.147 | 0.078 | 0.053 | 0.073 | **0.050** |
| 400 | 50 | 50 | 165.324 | 163.272 | 164.956 | 163.123 | 0.064 | 0.051 | 0.061 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 20 | 10 | 10 | 264.515 | 209.775 | 247.725 | 177.093 | 0.795 | 0.172 | 0.652 | **0.011** |
| 30 | 10 | 10 | 218.812 | 186.612 | 210.921 | 176.214 | 0.315 | 0.034 | 0.220 | **0.010** |
| 40 | 10 | 10 | 205.272 | 181.672 | 199.989 | 176.378 | 0.158 | 0.020 | 0.112 | **0.010** |
| 80 | 10 | 10 | 189.289 | 177.538 | 186.945 | 176.127 | 0.046 | 0.012 | 0.036 | **0.010** |
| 200 | 10 | 10 | 181.171 | 176.399 | 180.292 | 176.170 | 0.019 | **0.010** | 0.017 | **0.010** |
| 400 | 10 | 10 | 178.631 | 176.225 | 178.196 | 176.164 | 0.014 | **0.010** | 0.013 | **0.010** |
| 20 | 20 | 20 | 260.578 | 212.645 | 245.177 | 176.900 | 0.758 | 0.199 | 0.620 | **0.011** |
| 30 | 20 | 20 | 216.690 | 188.219 | 209.593 | 176.181 | 0.286 | 0.040 | 0.204 | **0.010** |
| 40 | 20 | 20 | 203.761 | 182.678 | 199.033 | 176.249 | 0.145 | 0.023 | 0.105 | **0.010** |
| 80 | 20 | 20 | 188.738 | 177.889 | 186.574 | 176.102 | 0.043 | 0.013 | 0.035 | **0.010** |
| 200 | 20 | 20 | 181.047 | 176.468 | 180.194 | 176.146 | 0.019 | **0.010** | 0.017 | **0.010** |
| 400 | 20 | 20 | 178.490 | 176.139 | 178.041 | 176.034 | 0.014 | **0.010** | 0.013 | **0.010** |
| 20 | 50 | 50 | 256.731 | 215.408 | 242.626 | 176.916 | 0.716 | 0.224 | 0.585 | **0.011** |
| 30 | 50 | 50 | 213.824 | 189.619 | 207.656 | 176.261 | 0.249 | 0.046 | 0.183 | **0.010** |
| 40 | 50 | 50 | 201.568 | 183.690 | 197.568 | 176.239 | 0.125 | 0.025 | 0.094 | **0.010** |
| 80 | 50 | 50 | 188.076 | 178.706 | 186.236 | 176.334 | 0.040 | 0.014 | 0.033 | **0.010** |
| 200 | 50 | 50 | 180.925 | 176.765 | 180.157 | 176.221 | 0.018 | 0.011 | 0.017 | **0.010** |
| 400 | 50 | 50 | 178.452 | 176.236 | 178.048 | 176.054 | 0.014 | **0.010** | 0.013 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 163.12$, $\chi^2_f(0.01) = 176.14$, $f = 135$.

Table 7: The simulated values for $-2\log\lambda, -2\rho_m\log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (5, 5, 5)$ for Case II.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 20 | 10 | 10 | 242.616 | 192.408 | 227.782 | 163.916 | 0.918 | 0.363 | 0.836 | **0.055** |
| 30 | 15 | 15 | 201.122 | 173.375 | 194.409 | 163.246 | 0.538 | 0.130 | 0.434 | **0.051** |
| 40 | 20 | 20 | 188.608 | 169.092 | 184.281 | 163.136 | 0.338 | 0.091 | 0.272 | **0.050** |
| 80 | 40 | 40 | 174.239 | 165.225 | 172.477 | 163.143 | 0.141 | 0.063 | 0.123 | **0.050** |
| 200 | 100 | 100 | 167.259 | 163.798 | 166.635 | 163.146 | 0.077 | 0.054 | 0.072 | **0.050** |
| 400 | 200 | 200 | 165.225 | 163.516 | 164.925 | 163.191 | 0.063 | 0.052 | 0.061 | **0.050** |
| 20 | 20 | 20 | 239.055 | 195.081 | 225.446 | 163.819 | 0.898 | 0.400 | 0.814 | **0.054** |
| 30 | 30 | 30 | 199.032 | 174.624 | 193.061 | 163.248 | 0.504 | 0.143 | 0.411 | **0.051** |
| 40 | 40 | 40 | 187.153 | 169.939 | 183.346 | 163.195 | 0.315 | 0.098 | 0.259 | **0.050** |
| 80 | 80 | 80 | 173.376 | 165.403 | 171.863 | 163.041 | 0.133 | 0.064 | 0.118 | **0.050** |
| 200 | 200 | 200 | 166.925 | 163.855 | 166.393 | 163.052 | 0.075 | 0.054 | 0.071 | **0.050** |
| 400 | 400 | 400 | 164.917 | 163.400 | 164.656 | 163.035 | 0.061 | 0.052 | 0.059 | **0.050** |
| 20 | 40 | 40 | 236.162 | 197.064 | 223.569 | 163.807 | 0.878 | 0.426 | 0.793 | **0.054** |
| 30 | 60 | 60 | 197.168 | 175.406 | 191.837 | 163.213 | 0.471 | 0.151 | 0.389 | **0.051** |
| 40 | 80 | 80 | 185.729 | 170.354 | 182.386 | 163.201 | 0.293 | 0.102 | 0.245 | **0.050** |
| 80 | 160 | 160 | 172.830 | 165.677 | 171.527 | 163.108 | 0.126 | 0.066 | 0.113 | **0.050** |
| 200 | 400 | 400 | 166.831 | 164.069 | 166.380 | 163.223 | 0.073 | 0.055 | 0.070 | **0.050** |
| 400 | 800 | 800 | 164.875 | 163.510 | 164.663 | 163.111 | 0.060 | 0.052 | 0.059 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 20 | 10 | 10 | 264.515 | 209.775 | 247.725 | 177.093 | 0.795 | 0.172 | 0.652 | **0.011** |
| 30 | 15 | 15 | 217.631 | 187.606 | 210.218 | 176.330 | 0.299 | 0.038 | 0.212 | **0.010** |
| 40 | 20 | 20 | 203.742 | 182.661 | 198.972 | 176.053 | 0.144 | 0.022 | 0.104 | **0.010** |
| 80 | 40 | 40 | 187.927 | 178.205 | 186.030 | 176.013 | 0.041 | 0.013 | 0.033 | **0.010** |
| 200 | 100 | 100 | 180.431 | 176.697 | 179.763 | 175.995 | 0.017 | 0.011 | 0.016 | **0.010** |
| 400 | 200 | 200 | 178.325 | 176.479 | 177.999 | 176.143 | 0.013 | **0.010** | 0.013 | **0.010** |
| 20 | 20 | 20 | 260.578 | 212.645 | 245.177 | 176.900 | 0.758 | 0.199 | 0.620 | **0.011** |
| 30 | 30 | 30 | 215.392 | 188.978 | 208.755 | 176.151 | 0.268 | 0.043 | 0.194 | **0.010** |
| 40 | 40 | 40 | 202.166 | 183.572 | 198.025 | 176.270 | 0.130 | 0.025 | 0.097 | **0.010** |
| 80 | 80 | 80 | 187.360 | 178.744 | 185.723 | 176.151 | 0.038 | 0.014 | 0.032 | **0.010** |
| 200 | 200 | 200 | 180.481 | 177.161 | 179.902 | 176.278 | 0.017 | 0.012 | 0.016 | **0.010** |
| 400 | 400 | 400 | 178.182 | 176.543 | 177.916 | 176.135 | 0.013 | 0.011 | 0.013 | **0.010** |
| 20 | 40 | 40 | 257.531 | 214.895 | 243.117 | 176.871 | 0.725 | 0.219 | 0.592 | **0.011** |
| 30 | 60 | 60 | 213.452 | 189.893 | 207.507 | 176.368 | 0.242 | 0.046 | 0.179 | **0.010** |
| 40 | 80 | 80 | 200.798 | 184.176 | 197.109 | 176.153 | 0.117 | 0.026 | 0.090 | **0.010** |
| 80 | 160 | 160 | 186.792 | 179.061 | 185.351 | 176.189 | 0.035 | 0.015 | 0.030 | **0.010** |
| 200 | 400 | 400 | 180.092 | 177.110 | 179.606 | 176.176 | 0.017 | 0.011 | 0.016 | **0.010** |
| 400 | 800 | 800 | 178.308 | 176.832 | 178.076 | 176.346 | 0.013 | 0.011 | 0.013 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 163.12$, $\chi^2_f(0.01) = 176.14$, $f = 135$.

Table 8: The simulated values for $-2\log\lambda, -2\rho_m\log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (6, 6, 6)$ for Case I.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 20 | 10 | 10 | 405.478 | 306.401 | 372.738 | 226.217 | 0.999 | 0.811 | 0.995 | **0.072** |
| 30 | 10 | 10 | 291.585 | 240.920 | 279.272 | 222.512 | 0.828 | 0.198 | 0.718 | **0.052** |
| 40 | 10 | 10 | 267.788 | 231.436 | 259.856 | 222.319 | 0.576 | 0.110 | 0.465 | **0.051** |
| 80 | 10 | 10 | 241.924 | 224.190 | 238.558 | 222.072 | 0.218 | 0.061 | 0.179 | **0.050** |
| 200 | 10 | 10 | 229.529 | 222.391 | 228.273 | 222.054 | 0.095 | 0.051 | 0.086 | **0.050** |
| 400 | 10 | 10 | 225.828 | 222.237 | 225.210 | 222.136 | 0.070 | 0.051 | 0.067 | **0.050** |
| 20 | 20 | 20 | 399.318 | 312.588 | 368.598 | 226.168 | 0.998 | 0.846 | 0.993 | **0.072** |
| 30 | 20 | 20 | 288.268 | 243.546 | 277.132 | 222.457 | 0.799 | 0.228 | 0.692 | **0.052** |
| 40 | 20 | 20 | 265.544 | 233.102 | 258.402 | 222.184 | 0.545 | 0.124 | 0.445 | **0.051** |
| 80 | 20 | 20 | 241.300 | 224.922 | 238.201 | 222.090 | 0.210 | 0.065 | 0.174 | **0.050** |
| 200 | 20 | 20 | 229.382 | 222.531 | 228.174 | 222.013 | 0.095 | 0.052 | 0.086 | **0.050** |
| 400 | 20 | 20 | 225.774 | 222.263 | 225.169 | 222.076 | 0.070 | 0.051 | 0.066 | **0.050** |
| 20 | 50 | 50 | 393.397 | 318.633 | 364.625 | 226.157 | 0.997 | 0.873 | 0.991 | **0.072** |
| 30 | 50 | 50 | 283.839 | 245.902 | 274.158 | 222.373 | 0.756 | 0.256 | 0.654 | **0.051** |
| 40 | 50 | 50 | 262.501 | 235.012 | 256.411 | 222.294 | 0.498 | 0.140 | 0.413 | **0.051** |
| 80 | 50 | 50 | 239.941 | 225.826 | 237.298 | 222.106 | 0.195 | 0.070 | 0.166 | **0.050** |
| 200 | 50 | 50 | 229.183 | 222.961 | 228.099 | 222.149 | 0.093 | 0.054 | 0.085 | **0.050** |
| 400 | 50 | 50 | 225.597 | 222.290 | 225.027 | 222.062 | 0.069 | 0.051 | 0.066 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 20 | 10 | 10 | 444.322 | 335.754 | 407.065 | 241.944 | 0.995 | 0.638 | 0.979 | **0.017** |
| 30 | 10 | 10 | 312.219 | 257.969 | 298.761 | 237.655 | 0.635 | 0.068 | 0.483 | **0.011** |
| 40 | 10 | 10 | 286.170 | 247.323 | 277.549 | 237.378 | 0.329 | 0.029 | 0.232 | **0.010** |
| 80 | 10 | 10 | 258.367 | 239.427 | 254.736 | 237.162 | 0.075 | 0.013 | 0.057 | **0.010** |
| 200 | 10 | 10 | 245.192 | 237.567 | 243.840 | 237.232 | 0.024 | 0.011 | 0.021 | **0.010** |
| 400 | 10 | 10 | 241.058 | 237.225 | 240.403 | 237.166 | 0.015 | **0.010** | 0.014 | **0.010** |
| 20 | 20 | 20 | 438.000 | 342.868 | 402.734 | 241.776 | 0.993 | 0.692 | 0.973 | **0.017** |
| 30 | 20 | 20 | 308.767 | 260.865 | 296.574 | 237.745 | 0.591 | 0.083 | 0.452 | **0.011** |
| 40 | 20 | 20 | 283.689 | 249.030 | 275.941 | 237.265 | 0.301 | 0.034 | 0.216 | **0.010** |
| 80 | 20 | 20 | 257.850 | 240.349 | 254.492 | 237.172 | 0.071 | 0.014 | 0.055 | **0.010** |
| 200 | 20 | 20 | 245.099 | 237.779 | 243.807 | 237.189 | 0.024 | 0.011 | 0.021 | **0.010** |
| 400 | 20 | 20 | 240.978 | 237.231 | 240.343 | 237.063 | 0.015 | **0.010** | 0.014 | **0.010** |
| 20 | 50 | 50 | 431.813 | 349.749 | 398.429 | 241.871 | 0.989 | 0.734 | 0.965 | **0.017** |
| 30 | 50 | 50 | 303.916 | 263.295 | 293.193 | 237.491 | 0.533 | 0.099 | 0.411 | **0.010** |
| 40 | 50 | 50 | 280.344 | 250.987 | 273.747 | 237.103 | 0.262 | 0.041 | 0.194 | **0.010** |
| 80 | 50 | 50 | 256.439 | 241.354 | 253.565 | 237.202 | 0.064 | 0.016 | 0.051 | **0.010** |
| 200 | 50 | 50 | 244.635 | 237.993 | 243.474 | 237.133 | 0.023 | 0.011 | 0.020 | **0.010** |
| 400 | 50 | 50 | 240.800 | 237.270 | 240.196 | 236.970 | 0.015 | **0.010** | 0.014 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 222.08$, $\chi^2_f(0.01) = 237.15$, $f = 189$.

Table 9: The simulated values for $-2\log\lambda, -2\rho_m\log\lambda, -2\log\tau$, and $-2\log\phi$ and type I error rates $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ when $(p_1, p_2, p_3) = (6, 6, 6)$ for Case II.

| Sample Size | | | Upper Percentile | | | | Type I Error Rate | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | $N_2$ | $N_3$ | $-2\log\lambda$ | $-2\rho_m\log\lambda$ | $-2\log\tau$ | $-2\log\phi$ | $\alpha_1$ | $\alpha_{\rho_m}$ | $\alpha_\tau$ | $\alpha_\phi$ |
| | | | | | $\alpha = 0.05$ | | | | | |
| 20 | 10 | 10 | 405.478 | 306.401 | 372.738 | 226.217 | 0.999 | 0.811 | 0.995 | **0.072** |
| 30 | 15 | 15 | 289.591 | 242.417 | 277.970 | 222.415 | 0.812 | 0.216 | 0.703 | **0.052** |
| 40 | 20 | 20 | 265.610 | 233.160 | 258.430 | 222.211 | 0.545 | 0.124 | 0.444 | **0.051** |
| 80 | 40 | 40 | 240.238 | 225.563 | 237.492 | 222.040 | 0.198 | 0.069 | 0.168 | **0.050** |
| 200 | 100 | 100 | 228.799 | 223.208 | 227.829 | 222.079 | 0.090 | 0.056 | 0.083 | **0.050** |
| 400 | 200 | 200 | 225.385 | 222.631 | 224.929 | 222.139 | 0.068 | 0.053 | 0.065 | **0.050** |
| 20 | 20 | 20 | 399.318 | 312.588 | 368.598 | 226.168 | 0.998 | 0.846 | 0.993 | **0.072** |
| 30 | 30 | 30 | 286.232 | 244.786 | 275.783 | 222.436 | 0.780 | 0.243 | 0.676 | **0.052** |
| 40 | 40 | 40 | 263.145 | 234.568 | 256.816 | 222.150 | 0.509 | 0.136 | 0.421 | **0.050** |
| 80 | 80 | 80 | 239.233 | 226.243 | 236.843 | 222.101 | 0.186 | 0.073 | 0.160 | **0.050** |
| 200 | 200 | 200 | 228.421 | 223.460 | 227.608 | 222.134 | 0.087 | 0.057 | 0.082 | **0.050** |
| 400 | 400 | 400 | 225.109 | 222.665 | 224.718 | 222.039 | 0.066 | 0.053 | 0.064 | **0.050** |
| 20 | 40 | 40 | 394.554 | 317.428 | 365.374 | 226.043 | 0.998 | 0.868 | 0.991 | **0.071** |
| 30 | 60 | 60 | 283.210 | 246.303 | 273.709 | 222.333 | 0.747 | 0.260 | 0.647 | **0.051** |
| 40 | 80 | 80 | 260.978 | 235.470 | 255.342 | 222.223 | 0.477 | 0.145 | 0.398 | **0.051** |
| 80 | 160 | 160 | 238.286 | 226.642 | 236.212 | 222.167 | 0.175 | 0.075 | 0.154 | **0.050** |
| 200 | 400 | 400 | 227.910 | 223.454 | 227.212 | 221.980 | 0.084 | 0.057 | 0.079 | **0.050** |
| 400 | 800 | 800 | 224.995 | 222.796 | 224.669 | 222.079 | 0.065 | 0.054 | 0.063 | **0.050** |
| | | | | | $\alpha = 0.01$ | | | | | |
| 20 | 10 | 10 | 444.322 | 335.754 | 407.065 | 241.944 | 0.995 | 0.638 | 0.979 | **0.017** |
| 30 | 15 | 15 | 310.127 | 259.608 | 297.356 | 237.521 | 0.609 | 0.077 | 0.465 | **0.010** |
| 40 | 20 | 20 | 283.953 | 249.262 | 276.140 | 237.261 | 0.302 | 0.035 | 0.218 | **0.010** |
| 80 | 40 | 40 | 256.799 | 241.112 | 253.832 | 237.241 | 0.066 | 0.016 | 0.052 | **0.010** |
| 200 | 100 | 100 | 244.292 | 238.323 | 243.276 | 237.170 | 0.022 | 0.011 | 0.020 | **0.010** |
| 400 | 200 | 200 | 240.587 | 237.648 | 240.100 | 237.115 | 0.015 | 0.011 | 0.014 | **0.010** |
| 20 | 20 | 20 | 438.000 | 342.868 | 402.734 | 241.776 | 0.993 | 0.692 | 0.973 | **0.017** |
| 30 | 30 | 30 | 306.312 | 261.959 | 294.826 | 237.543 | 0.563 | 0.091 | 0.431 | **0.011** |
| 40 | 40 | 40 | 281.334 | 250.782 | 274.396 | 237.165 | 0.271 | 0.040 | 0.199 | **0.010** |
| 80 | 80 | 80 | 255.449 | 241.578 | 252.883 | 237.186 | 0.060 | 0.017 | 0.049 | **0.010** |
| 200 | 200 | 200 | 243.761 | 238.467 | 242.874 | 236.965 | 0.021 | 0.012 | 0.019 | **0.010** |
| 400 | 400 | 400 | 240.595 | 237.982 | 240.176 | 237.388 | 0.015 | 0.011 | 0.014 | **0.010** |
| 20 | 40 | 40 | 433.129 | 348.463 | 399.361 | 241.846 | 0.990 | 0.727 | 0.967 | **0.017** |
| 30 | 60 | 60 | 303.122 | 263.620 | 292.685 | 237.401 | 0.523 | 0.101 | 0.403 | **0.010** |
| 40 | 80 | 80 | 279.136 | 251.854 | 272.958 | 237.344 | 0.244 | 0.043 | 0.183 | **0.010** |
| 80 | 160 | 160 | 254.460 | 242.025 | 252.214 | 237.208 | 0.055 | 0.017 | 0.045 | **0.010** |
| 200 | 400 | 400 | 243.474 | 238.714 | 242.728 | 237.213 | 0.020 | 0.012 | 0.019 | **0.010** |
| 400 | 800 | 800 | 240.380 | 238.031 | 240.028 | 237.312 | 0.014 | 0.011 | 0.014 | **0.010** |

Note. The closest to $\alpha$ in the values $\alpha_1, \alpha_{\rho_m}, \alpha_\tau$, and $\alpha_\phi$ of each row is in bold. $\chi^2_f(0.05) = 222.08$, $\chi^2_f(0.01) = 237.15$, $f = 189$.

## 6 Conclusions

We discussed the simultaneous test of the mean vector and the covariance matrix with three-step monotone missing data. We proposed two modified LRT statistics ($-2\log\tau, -2\log\phi$) by decomposing the LR and correcting it by extracting the LR of the simultaneous test and the test of variance for complete data. We also proposed a modified LRT statistic ($-2\rho_m \log\lambda$) via linear interpolation. From the simulation results, the modified LRT statistic $-2\log\phi$, which was modified only for the LR part of the test of variance for the complete data, gave the most accurate results. The focus of future works will be the extension to $k$-step monotone missing data and two-sample problem.

## Appendix

To derive the MLEs of the mean vector and the covariance matrix, we consider the following transformation matrix $\boldsymbol{Z}$:

$$\boldsymbol{Z} = \left(\begin{array}{cc|c} \boldsymbol{I}_{p_1} & \boldsymbol{O} & \\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{I}_{p_2} & \boldsymbol{O} \\ \hline \multicolumn{2}{c|}{-\boldsymbol{\Sigma}_{3(12)}\boldsymbol{\Sigma}_{(12)(12)}^{-1}} & \boldsymbol{I}_{p_3} \end{array}\right).$$

In this case, the transformed vector $\boldsymbol{y}_j$ is

$$\boldsymbol{y}_j = \boldsymbol{Z}\boldsymbol{x}_j = \begin{pmatrix} \boldsymbol{x}_{1j} \\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{x}_{1j} + \boldsymbol{x}_{2j} \\ -\boldsymbol{\Sigma}_{3(12)}\boldsymbol{\Sigma}_{(12)(12)}^{-1}\begin{pmatrix} \boldsymbol{x}_{1j} \\ \boldsymbol{x}_{2j} \end{pmatrix} + \boldsymbol{x}_{3j} \end{pmatrix}.$$

The transformed parameters are defined as

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\eta}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 \\ -\boldsymbol{\Sigma}_{3(12)}\boldsymbol{\Sigma}_{(12)(12)}^{-1}\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} + \boldsymbol{\mu}_3 \end{pmatrix},$$

$$\boldsymbol{\Delta} = \left(\begin{array}{cc|c} \boldsymbol{\Delta}_{11} & \boldsymbol{\Delta}_{12} & \boldsymbol{\Delta}_{13} \\ \boldsymbol{\Delta}_{21} & \boldsymbol{\Delta}_{22} & \boldsymbol{\Delta}_{23} \\ \hline \boldsymbol{\Delta}_{31} & \boldsymbol{\Delta}_{32} & \boldsymbol{\Delta}_{33} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Delta}_{(12)(12)} & \boldsymbol{\Delta}_{(12)3} \\ \hline \boldsymbol{\Delta}_{3(12)} & \boldsymbol{\Delta}_{33} \end{array}\right),$$

where

$$\begin{aligned} \boldsymbol{\Delta}_{11} &= \boldsymbol{\Sigma}_{11}, \\ \boldsymbol{\Delta}_{12} &= \boldsymbol{\Delta}_{21}' = \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}, \\ \boldsymbol{\Delta}_{22} &= \boldsymbol{\Sigma}_{22\cdot1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}, \\ \boldsymbol{\Delta}_{(12)3} &= \boldsymbol{\Delta}_{3(12)}' = \boldsymbol{\Sigma}_{(12)(12)}^{-1}\boldsymbol{\Sigma}_{(12)3}, \\ \boldsymbol{\Delta}_{33} &= \boldsymbol{\Sigma}_{33\cdot12} = \boldsymbol{\Sigma}_{33} - \boldsymbol{\Sigma}_{3(12)}\boldsymbol{\Sigma}_{(12)(12)}^{-1}\boldsymbol{\Sigma}_{(12)3}. \end{aligned}$$

We note that the pair $(\boldsymbol{\eta}, \boldsymbol{\Delta})$ is in one-to-one correspondence with $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. In this case, the likelihood function is

$$L(\boldsymbol{\eta}, \boldsymbol{\Delta}) = C|\boldsymbol{\Delta}_{11}|^{-\frac{1}{2}N}|\boldsymbol{\Delta}_{22}|^{-\frac{1}{2}(N_1+N_2)}|\boldsymbol{\Delta}_{33}|^{-\frac{1}{2}N_1}\exp\left\{-\frac{1}{2}\sum_{j=1}^{N}(\boldsymbol{y}_{1j}-\boldsymbol{\eta}_1)'\boldsymbol{\Delta}_{11}^{-1}(\boldsymbol{y}_{1j}-\boldsymbol{\eta}_1)\right\}$$
$$\times \exp\left\{-\frac{1}{2}\sum_{j=1}^{N_1+N_2}(\boldsymbol{y}_{2j}-\boldsymbol{\eta}_2)'\boldsymbol{\Delta}_{22}^{-1}(\boldsymbol{y}_{2j}-\boldsymbol{\eta}_2)\right\}\exp\left\{-\frac{1}{2}\sum_{j=1}^{N_1}(\boldsymbol{y}_{3j}-\boldsymbol{\eta}_3)'\boldsymbol{\Delta}_{33}^{-1}(\boldsymbol{y}_{3j}-\boldsymbol{\eta}_3)\right\},$$

where $C$ is a constant. The MLEs can be obtained by differentiating $\log L(\boldsymbol{\eta}, \boldsymbol{\Delta})$ with respect to $\boldsymbol{\eta}$ and $\boldsymbol{\Delta}$, respectively.

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