A modified normalizing transformation statistic based on kurtosis for multivariate normality testing

Eri Kurita[a], Takashi Seo[a] and Zofia Hanusz[b]

[a] *Department of Applied Mathematics,*
Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

[b] *Department of Applied Mathematics and Computer Science*
University of Life Sciences
Gleboka 28, Lublin, 20-612 Lublin, Poland

Abstract

In this paper, we consider testing problem of multivariate normality (MVN). We deal with the kurtosis test statistic based on Mardia's multivariate kurtosis as one of MVN tests and give a modified normalizing transformation (NT) statistic. A standardized statistic based on the NT statistic is derived by using the exact expectation and variance of Mardia's multivariate sample kurtosis. Finally, we investigate the accuracy of the normal approximation of the proposed test statistic by a Monte Carlo simulation, and we provide a numerical example.

Keywords: Asymptotic expansion; Monte Carlo simulation; Multivariate kurtosis; Normal approximation; Standardized statistic.

1 Introduction

Assessing MVN of the data is an important and complicated problem, and many methods have been discussed from various perspectives. In particular, in many multivariate analysis, multivariate normality is assumed as the population distribution and underlies important techniques. From another perspective, there are discussions on the effect of non-normality or robustness if the multivariate normality of the population distribution does not hold, such as the distribution of the test statistic under a non-normal population (see e.g., Seo et al. (1994, 1995), Fujikoshi (2001), Wakaki et al. (2002), etc.). One of the MVN tests is the test using skewness and kurtosis, but an asymptotic result is used. For multivariate kurtosis, the normal approximation of the test statistic is discussed (see e.g., Mardia (1970), etc.). There are several definitions of multivariate kurtosis, including those by Mardia (1970), Srivastava(1984), and Koziol (1989), and their null distributions are given for large sample. As related to this study, an estimation of the kurtosis parameter, which is the fourth order moment under an elliptical distribution, is given by Seo and Toyama (1996). In this paper, we focus on the definition by Mardia (1970), which gives the multivariate sample kurtosis and the standardized test statistics from the expectation and variance. The asymptotic distributions follow a standard normal distribution, which are used for the MVN test. Recently, Enomoto et al. (2020) gave the normalizing NT statistic for Mardia's sample measure of multivariate kurtosis. In addition, the kurtosis tests under the assumption of a two-step monotone missing data discussed by Yamada et al. (2015) and Kurita and Seo (2022). Kurita and Seo (2022) gave a new sample measure of multivariate kurtosis available for the two-step monotone missing data and developed a test statistic with good normal approximation by asymptotically evaluating the expectation and variance using an asymptotic expansion procedure. In this paper, we give a modified NT statistic, which improves the normal approximation. The modified statistic is a standardized statistic that uses the exact expectation and variance of Mardia's multivariate sample kurtosis. The rest of this paper is organized as follows. Section 2 provides a definition of the sample measure of multivariate kurtosis and the test statistics by Mardia (1970). In Section 3, we describe the NT statistic by Enomoto et al. (2020) in order to derive a modified standardized test statistic by evaluation of the expectation and variance of the NT statistic. In Section 4, a simulation study is presented to investigate the accuracy of the normal approximation of the test statistic proposed in this paper. Section 5 gives a numerical example to illustrate the method, and Section 6 presents concluding remarks.

2 Multivariate sample kurtosis and kurtosis test statistic

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ be a random sample from a p-variate population with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Then the sample measure of multivariate kurtosis is defined as

$$b_{2,p} = \frac{1}{N}\sum_{i=1}^{N}\{(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \boldsymbol{S}^{-1}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})\}^2,$$

where

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_i, \ \ \boldsymbol{S} = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top.$$

This definition is due to Mardia (1970, 1974), and $\boldsymbol{S}$ is defined as the maximum likelihood of estimator of $\boldsymbol{\Sigma}$. In addition to MVN test statistic has been proposed as

$$ZM^* = \frac{b_{2,p} - \mu_{\mathrm{M}}}{\sigma_{\mathrm{M}}},$$

where

$$\mu_{\mathrm{M}} = p(p+2)\frac{N-1}{N+1}, \ \sigma_{\mathrm{M}}^2 = 8p(p+2)\frac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)},$$

respectively. Note that, under multivariate normality, it holds that $\mathrm{E}[b_{2,p}] = \mu_{\mathrm{M}}$, $\mathrm{Var}[b_{2,p}] = \sigma_{\mathrm{M}}^2$ and ZM^* test statistic is asymptotically distributed as $N(0,1)$ (e.g., see Siotani et al. (1985)). Using the asymptotic result, Mardia (1970) gave that

$$ZM = \frac{\sqrt{N}(b_{2,p} - \beta)}{\sigma}$$

is asymptotically distribution as $N(0,1)$, where $\beta = p(p+2)$ and $\sigma^2 = 8p(p+2)$. Note that as the sample size N increases, μ_{M} converges to β, and $N\sigma_{\mathrm{M}}^2$ converges to σ^2, respectively. This means the both ZM^* and ZM test statistics are the MVN test statistics using the fourth-order moment (kurtosis), magnitude of absolute values of these statistics leads to the rejection of the MVN hypothesis test. Mardia and Kanazawa (1983) used the third-order moment of multivariate sample kurtosis to give a chi-square approximation using the Wilson-Hilferty (WH) transformation for its null distribution. On the other hand, Enomoto et al. (2020) derived the NT statistic ZNT, and numerical comparisons have been made with ZM, ZM^*, ZNT, and WH transformation statistics.

3 NT statistic and modified statistic

Consider the distribution function for $\sqrt{N}(b_{2,p}-\beta)/\sigma$. Then its asymptotic expansion is given by

$$\Pr\left[\frac{\sqrt{N}(b_{2,p}-\beta)}{\sigma}\le z\right]=\Phi(z)-\frac{1}{\sqrt{N}}\left\{\frac{a_1}{\sigma}\Phi^{(1)}(z)+\frac{a_3}{\sigma^3}\Phi^{(3)}(z)\right\}+\mathrm{O}(N^{-1}),$$

where $\Phi(z)$ is the cumulative distribution function of $N(0,1)$, $\Phi^{(j)}(z)$ is the jth derivative of $\Phi(z)$, $j=1,\ 3$, and the coefficients a_1, σ^2 and a_3 are given by

$$a_1=-2p(p+2),\ \sigma^2=8p(p+2),\ a_3=\frac{32}{3}p(p+2)(p+8),$$

respectively. We note that a_1, σ^2 and a_3 are the coefficients corresponding to first, second and third cumulants of $Y=\sqrt{N}(b_{2,p}-\beta)$ such that

$$\begin{aligned}
\kappa_1(Y)&=\frac{a_1}{\sqrt{N}}+\mathrm{O}(N^{-\frac{3}{2}})\ (=\mathrm{E}[Y]),\\
\kappa_2(Y)&=\sigma^2+\mathrm{O}(N^{-1})\ (=\mathrm{E}[Y^2]-\{\mathrm{E}[Y]\}^2),\\
\kappa_3(Y)&=\frac{6}{\sqrt{N}}a_3+\mathrm{O}(N^{-\frac{3}{2}})\ (=\mathrm{E}[Y^3]-3\mathrm{E}[Y^2]\mathrm{E}[Y]+2\{\mathrm{E}[Y]\}^3),
\end{aligned}$$

respectively. This result was given in Enomoto et al. (2020). For a general discussion of an asymptotic expansion of the distribution function of some statistics and their normalizing transformations, see Konishi (1981) and others. The NT statistic is derived based on the result of the asymptotic expansion, and the outline of its derivation is given in order to obtain a modified NT statistic. Let $f(b_{2,p})$ be a function of $b_{2,p}$, then, under appropriate regularity conditions for a function $f(b_{2,p})$, the distribution function for $\sqrt{N}\{f(b_{2,p})-f(\beta)\}/\{f'(\beta)\sigma\}$ can be expanded for large N as

$$\Pr\left[\frac{\sqrt{N}\{f(b_{2,p})-f(\beta)\}}{f'(\beta)\sigma}\le z\right]=\Phi(z)-\frac{1}{\sqrt{N}}\{b_1\Phi^{(1)}(z)+b_3\Phi^{(3)}(z)\}+\mathrm{O}(N^{-1}),$$

where

$$b_i=\frac{a_i}{\sigma^i}+\frac{\sigma f''(\beta)}{2f'(\beta)},\ i=1,3.$$

See Siotani et al. (1985). Therefore, if there exists a function f such that the coefficient of the term of $1/\sqrt{N}$ is zero, then the distribution of the statistic using the function f converges to a standard normal distribution more quickly. The function that satisfies $b_1\Phi^{(1)}(y) + b_3\Phi^{(3)}(y) = 0$ is given by

$$f(b_{2,p}) = \gamma \exp\left(\frac{1}{\gamma} b_{2,p}\right),$$

where $\gamma = -3p(p+2)/(p+8)$. Furthermore, by bias correction for the term of $1/\sqrt{N}$, we obtain

$$\Pr[ZNT \le z] = \Phi(z) + \mathrm{O}(N^{-1}),$$

where

$$ZNT = \frac{\sqrt{N}}{f'(\beta)\sigma}\left\{f(b_{2,p}) - f(\beta) - \frac{c}{N}\right\},\ c = -\frac{2}{3}(3p^2 + 8p + 16)\exp\left(-\frac{p+8}{3}\right).$$

Thus, we obtain the NT statistic given by

$$ZNT = \frac{\sqrt{N}}{\sigma}\left[\frac{\exp\{\gamma(b_{2,p} - \beta)\} - 1}{\gamma} + \frac{2\beta(1-2\gamma)}{N}\right].$$

We note that this result coincides with Theorem 1 in Enomoto et al. (2020). Based on this result, we give a modified NT statistic by using exact expectation and variance of Mardia's multivariate sample kurtosis. For large N, we can assume that $b_{2,p}$ is distributed as $N(\mu_{\mathrm{M}}, \sigma^2_{\mathrm{M}})$, and therefore, approximately, we have

$$\mathrm{E}[f(b_{2,p})] = \gamma \exp\left(\frac{\mu_{\mathrm{M}}}{\gamma} + \frac{\sigma^2_{\mathrm{M}}}{2\gamma^2}\right),$$

$$\mathrm{Var}[f(b_{2,p})] = \exp\left(\frac{2\mu_{\mathrm{M}}}{\gamma}\right)\left\{\exp\left(\frac{2\sigma^2_{\mathrm{M}}}{\gamma^2}\right) - \exp\left(\frac{\sigma^2_{\mathrm{M}}}{\gamma^2}\right)\right\}.$$

Hence, we obtain approximation to $\mathrm{E}[ZNT]$ and $\mathrm{Var}[ZNT]$ as

$$\mu_{\mathrm{ZNT}} = \frac{\sqrt{N}}{\sigma}\exp\left(-\frac{\beta}{\gamma}\right)\left\{\gamma\exp\left(\frac{\mu_{\mathrm{M}}}{\gamma} + \frac{\sigma^2_{\mathrm{M}}}{2\gamma^2}\right) - \gamma\exp\left(\frac{\beta}{\gamma}\right) - \frac{c}{N}\right\},$$

$$\sigma^2_{\mathrm{ZNT}} = \frac{N\gamma^2}{\sigma^2}\exp\left\{\frac{2}{\gamma}(\mu_{\mathrm{M}} - \beta)\right\}\left\{\exp\left(\frac{2\sigma^2_{\mathrm{M}}}{\gamma^2}\right) - \exp\left(\frac{\sigma^2_{\mathrm{M}}}{\gamma^2}\right)\right\},$$

respectively. We note that $\mu_{\text{ZNT}} = 0$ and $\sigma^2_{\text{ZNT}} = 1$ when $N \to \infty$. Therefore, by calculating $(ZNT - \mu_{\text{ZNT}})/\sigma_{\text{ZNT}}$, we can propose the following ZNT^* test statistic as a modification of ZNT.

$$ZNT^* = \frac{\exp\left(\frac{1}{\gamma} b_{2,p}\right) - \exp\left(\frac{\mu_{\text{M}}}{\gamma} + \frac{\sigma^2_{\text{M}}}{2\gamma^2}\right)}{\exp\left(\frac{\mu_{\text{M}}}{\gamma}\right)\left\{\exp\left(\frac{2\sigma^2_{\text{M}}}{\gamma^2}\right) - \exp\left(\frac{\sigma^2_{\text{M}}}{\gamma^2}\right)\right\}^{\frac{1}{2}}}.$$

We note that ZNT^* is also asymptotically distributed as $N(0, 1)$. Furthermore, it seems that ZNT^* is closer to zero and one than ZNT with respect to expectation and variance.

4 A simulation study

In this section, we investigate the accuracy of the normal approximation of ZNT^* by a Monte Carlo simulation. Tables 1 and 2 give empirical expectation, variance, skewness, and kurtosis for ZM, ZM^*, ZNT, and ZNT^* computed through simulation of combinations of $p = 2,\ 4,\ 5,\ 7,\ 10,\ 15,\ 20$ and $N = 20,\ 50,\ 100,\ 200,\ 300,\ 500,\ 1000$ over 1,000,000 runs each. As all test statistics are invariant to affine transformation, without loss of generality, we assume that $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$. Tables 1 and 2 show that in many cases the expectation of ZNT^* is closer to zero than that of ZNT. In particular, we note that the expectations of all test statistics converge to zero as the sample size N becomes large. Furthermore, it can be seen from tables that even when N is small, the expectation of ZNT^* is close to zero. Focusing on the size of the dimension p, it can be seen that when N is fixed, the expectation of ZNT is further away from zero for larger dimensions, while that of ZNT^* is closer to zero for larger dimensions. In addition, we discuss the variance from Tables 1 and 2. It can be seen from tables that for any dimension, the values of variances of ZM^*, ZNT and ZNT^* converge to one as the sample size N becomes large. Comparing the variances of ZNT and ZNT^*, it can be seen from Table 2 that when

Table 1:
Empirical expectation, variance, skewness and kurtosis for ZM^*, ZNT and ZNT^* for $p = 2,\ 4,\ 5$.

	Expectation			Variance			Skewness		Kurtosis	
N	ZM^*	ZNT	ZNT^*	ZM^*	ZNT	ZNT^*	ZM^*	ZNT,ZNT^*	ZM^*	ZNT,ZNT^*
					$p = 2$					
20	(0.000)	0.120	**0.033**	(1.000)	**0.706**	0.600	1.109	**-0.295**	5.164	**2.741**
50	(0.003)	0.049	**0.022**	(1.003)	**0.814**	0.690	1.049	**-0.229**	5.335	**2.854**
100	(0.000)	0.020	**0.012**	(1.000)	**0.875**	0.788	0.862	**-0.147**	4.711	**2.912**
200	(0.001)	0.009	**0.006**	(1.003)	**0.926**	0.871	0.653	**-0.080**	3.999	**2.959**
300	(-0.001)	0.004	**0.003**	(0.999)	**0.944**	0.908	0.547	**-0.052**	3.693	**2.990**
500	(0.000)	**0.002**	**0.002**	(1.001)	**0.966**	0.937	0.434	**-0.029**	3.457	**3.000**
1000	(0.000)	**0.001**	**0.001**	(0.998)	**0.980**	0.971	0.310	**-0.012**	3.233	**3.010**
					$p = 4$					
20	(-0.002)	0.010	**0.009**	(0.998)	0.680	**0.808**	0.708	**-0.151**	3.794	**2.759**
50	(0.001)	0.014	**0.009**	(0.999)	0.836	**0.838**	0.697	**-0.119**	3.970	**2.881**
100	(0.000)	0.006	**0.003**	(1.001)	**0.904**	0.892	0.581	**-0.073**	3.734	**2.939**
200	(-0.001)	0.002	**0.001**	(0.998)	**0.944**	0.934	0.444	**-0.038**	3.440	**2.977**
300	(0.001)	0.002	**0.000**	(1.000)	**0.961**	0.952	0.381	**-0.021**	3.344	**3.001**
500	(0.000)	**0.000**	0.001	(1.002)	**0.978**	0.973	0.302	**-0.010**	3.210	**3.001**
1000	(-0.001)	**0.000**	0.001	(0.999)	**0.987**	0.984	0.213	**-0.007**	3.115	**3.010**
					$p = 5$					
20	(-0.001)	-0.038	**0.007**	(0.999)	0.646	**0.856**	0.617	**-0.100**	3.562	**2.752**
50	(0.002)	**0.002**	0.007	(1.002)	0.833	**0.869**	0.621	**-0.090**	3.749	**2.884**
100	(0.001)	**0.003**	**0.003**	(0.998)	0.902	**0.910**	0.521	**-0.052**	3.567	**2.946**
200	(0.000)	**0.001**	0.002	(0.999)	0.946	**0.947**	0.398	**-0.030**	3.357	**2.983**
300	(0.000)	**0.000**	0.002	(1.001)	**0.965**	**0.965**	0.336	**-0.018**	3.251	**2.988**
500	(0.000)	**0.000**	0.001	(1.000)	**0.978**	0.976	0.265	**-0.012**	3.163	**3.004**
1000	(0.000)	**0.000**	0.001	(1.000)	**0.988**	**0.988**	0.196	**-0.001**	3.089	**3.004**

Table 2:
Empirical expectation, variance, skewness and kurtosis for ZM^*, ZNT and ZNT^* for $p = 7,\ 10,\ 15,\ 20$.

	Expectation			Variance			Skewness		Kurtosis	
N	ZM^*	ZNT	ZNT^*	ZM^*	ZNT	ZNT^*	ZM^*	ZNT,ZNT^*	ZM^*	ZNT,ZNT^*
					$p = 7$					
20	(0.002)	-0.146	**0.002**	(1.002)	0.558	**0.905**	0.529	**-0.009**	3.369	**2.771**
50	(0.001)	-0.025	**0.003**	(1.003)	0.814	**0.909**	0.525	**-0.049**	3.498	**2.884**
100	(0.000)	-0.008	**0.002**	(1.000)	0.898	**0.938**	0.442	**-0.033**	3.384	**2.952**
200	(0.001)	**-0.001**	**-0.001**	(1.000)	0.947	**0.960**	0.341	**-0.018**	3.238	**2.976**
300	(0.001)	**-0.001**	**0.001**	(1.001)	0.964	**0.975**	0.290	**-0.011**	3.186	**2.994**
500	(0.000)	**-0.001**	**0.001**	(1.000)	0.978	**0.981**	0.233	**-0.003**	3.124	**3.002**
1000	(0.000)	**0.000**	-0.001	(1.000)	0.988	**0.992**	0.168	**-0.001**	3.069	**3.006**
					$p = 10$					
20	(0.000)	-0.360	**0.001**	(0.998)	0.400	**0.938**	0.498	**0.130**	3.326	**2.845**
50	(-0.001)	-0.074	**0.001**	(0.999)	0.767	**0.933**	0.448	**-0.012**	3.336	**2.895**
100	(0.000)	-0.024	**0.001**	(1.002)	0.884	**0.955**	0.381	**-0.017**	3.283	**2.952**
200	(0.000)	-0.008	**0.001**	(1.001)	0.940	**0.971**	0.300	**-0.005**	3.179	**2.975**
300	(0.000)	-0.005	**0.000**	(1.001)	0.961	**0.979**	0.248	**-0.008**	3.120	**2.987**
500	(0.001)	-0.001	**0.000**	(1.001)	0.976	**0.987**	0.200	**-0.002**	3.079	**2.989**
1000	(0.001)	**0.000**	0.001	(1.002)	0.989	**0.994**	0.147	**0.002**	3.041	**2.994**
					$p = 15$					
20	(-0.001)	-0.875	**0.000**	(0.998)	0.134	**0.961**	0.703	**0.525**	3.772	**3.338**
50	(0.000)	-0.182	**0.001**	(1.001)	0.676	**0.954**	0.385	**0.035**	3.217	**2.904**
100	(-0.001)	-0.061	**-0.001**	(1.002)	0.847	**0.968**	0.333	**0.008**	3.187	**2.946**
200	(0.000)	-0.021	**0.000**	(1.001)	0.926	**0.978**	0.253	**-0.006**	3.120	**2.973**
300	(0.001)	-0.010	**0.000**	(0.999)	0.948	**0.986**	0.220	**0.000**	3.096	**2.985**
500	(0.000)	-0.005	**-0.001**	(1.000)	0.970	**0.992**	0.171	**-0.004**	3.056	**2.991**
1000	(0.000)	-0.001	**0.000**	(1.001)	0.986	**0.996**	0.129	**0.002**	3.036	**2.998**
					$p = 20$					
20	—	—	—	—	—	—	—	—	—	—
50	(-0.001)	-0.333	**0.001**	(1.001)	0.568	**0.968**	0.369	**0.089**	3.192	**2.927**
100	(-0.001)	-0.110	**0.000**	(1.002)	0.804	**0.972**	0.302	**0.020**	3.141	**2.943**
200	(0.000)	-0.037	**0.001**	(1.001)	0.906	**0.983**	0.241	**0.008**	3.103	**2.977**
300	(0.001)	-0.019	**0.000**	(1.000)	0.938	**0.989**	0.202	**0.003**	3.068	**2.979**
500	(-0.001)	-0.010	**0.000**	(0.998)	0.962	**0.991**	0.159	**-0.001**	3.047	**2.991**
1000	(0.000)	-0.003	**0.001**	(0.997)	0.979	**0.996**	0.114	**-0.001**	3.016	**2.988**

$p = 2$, the variance of ZNT is closer to one for all cases of N. In the case of $p = 4$, it can be seen that the variance of ZNT is closer to one when $N \geq 100$. In the case of $p = 5$, it can be seen that the variance ZNT is closer to one when $N \geq 300$. Furthermore, when N is fixed and p is larger, it can be seen that the variance of ZNT goes away from one, while that of ZNT^* is closer to one. In summary, except the case of $p = 2$, when both the dimension and the sample size are small, the variance of ZNT^* is closer to one, and when the dimension is moderately large (specifically, when $p \geq 7$), the variance of ZNT^* is always close to one. As for skewness and kurtosis, if the statistic is distributed as a standard normal distribution, then its skewness and kurtosis are zero and three, and we can see that the values of skewness and kurtosis of ZNT^* are close to zero and three in all cases.

Next, we discuss the normal approximation of the proposed ZNT^* statistic. According to the simulation results of Enomoto et al. (2020), the type I error shows that ZM^* is much better than ZNT. The reason may be that the expectation and variance of ZM^* are exactly zero and one, respectively. However, the skewness and kurtosis are not close to zero and three, respectively. Therefore, the lower and upper percentiles of ZM^* are probably shifted from these of $N(0, 1)$. For example, when $p = 5$, $N = 20$ and $\alpha = 0.05$, the empirical type I error of ZM^* is 0.045. However, since $ZM^*_{1-\alpha/2} = -1.651 > -1.96$ and $ZM^*_{\alpha/2} = 2.233 > 1.96$, the normal approximation to the percentiles is not good, where $ZM^*_{1-\alpha/2}$ and $ZM^*_{\alpha/2}$ are the simulated values of the lower and upper $100(\alpha/2)$ percentiles of ZM^*, respectively. Therefore, the following criterion is hereby introduced to show that the ZNT^* improves on ZM^* under this criterion. As a criterion, the probability evaluation for the difference of percentiles is given by the following definition. For $U = ZM^*,\ ZNT,\ ZNT^*$, we define

$$\delta = \begin{cases} \left|\Pr(U < -z_{\frac{\alpha}{2}}) + \Pr(U > z_{\frac{\alpha}{2}}) - \alpha\right| & \text{for (1), (2),} \\ \left|\Pr(U < -z_{\frac{\alpha}{2}}) - \Pr(U > z_{\frac{\alpha}{2}})\right| & \text{for (3), (4),} \end{cases}$$

where

$$(1)\ u_{1-\frac{\alpha}{2}} > -z_{\frac{\alpha}{2}},\ u_{\frac{\alpha}{2}} < z_{\frac{\alpha}{2}},\quad (2)\ u_{1-\frac{\alpha}{2}} < -z_{\frac{\alpha}{2}},\ u_{\frac{\alpha}{2}} > z_{\frac{\alpha}{2}},$$

$$(3)\ u_{1-\frac{\alpha}{2}} > -z_{\frac{\alpha}{2}},\ u_{\frac{\alpha}{2}} > z_{\frac{\alpha}{2}},\quad (4)\ u_{1-\frac{\alpha}{2}} < -z_{\frac{\alpha}{2}},\ u_{\frac{\alpha}{2}} < z_{\frac{\alpha}{2}},$$

$u_{1-\alpha/2}$ and $u_{\alpha/2}$ are the lower and upper $100(\alpha/2)$ percentile of U, respectively. We note that the value of δ is non-negative, and the closer it is to zero, the better the normal approximation is. That is, the magnitude of values of δ is the measure of evaluation for the accuracy of normal approximation based on the lower and upper $100(\alpha/2)$ percentiles of the test statistic. From Tables 3 and 4, it may be noted that the value of δ for ZNT is close to zero for $p = 2$, but that for ZNT^* is close to zero for other large dimensions. In particular, when N is moderately large, the value of δ is almost zero, indicating that the accuracy of the normal approximation of ZNT^* is good. For the value of δ, the reason why ZNT^* is larger than ZM^* for $p = 2$ probably that the variance for ZNT^* does not improve when the dimension is small (see, Table 1).

Figures 1 $\sim$ 4 present histograms for ZM^*, ZNT and ZNT^* from the results of $1,000,000$ simulations in order to see the shape of the distribution of these test statistics as a normal approximation, where the curve in the figures indicates a standard normal density. Figures 1 through 4 show the results for $p = 2,\ 4,\ 10$ and 15, respectively, and for sample sizes of $N = 20,\ 50$ and 100 for each dimension. As can be seen from the figures, all cases fit the standard normal density as the sample size N becomes large. In particular, in all cases, the histogram for ZNT^* seems to overlap exactly with the standard normal density most often.

5 A numerical example

In this section, a numerical example is given to illustrate the use of the proposed test statistic. The head dimension data in millimeters handled here presents the six variables which were measured on heads of the members of army. This data is from Table 1.2 in

Table 3:

Value of δ for ZM, ZM^*, ZNT and ZNT^* ($p = 2,\ 4,\ 5$).

N	ZM	ZM^*	ZNT	ZNT^*
		$p = 2$		
20	0.045	0.044	**0.039**	0.043
50	0.031	0.040	**0.024**	0.035
100	0.021	0.035	**0.015**	0.025
200	0.016	0.030	**0.009**	0.015
300	0.014	0.026	**0.006**	0.010
500	0.012	0.021	**0.004**	0.007
1000	0.009	0.016	**0.002**	0.003
		$p = 4$		
20	0.047	0.036	0.037	**0.025**
50	0.030	0.033	**0.020**	**0.020**
100	0.018	0.028	**0.012**	0.013
200	0.010	0.022	**0.007**	0.008
300	0.007	0.019	**0.005**	**0.005**
500	0.004	0.016	**0.003**	**0.003**
1000	0.002	0.011	**0.001**	0.002
		$p = 5$		
20	0.043	0.033	0.039	**0.020**
50	0.025	0.031	0.020	**0.016**
100	0.015	0.026	0.012	**0.011**
200	0.010	0.020	0.007	**0.006**
300	0.009	0.017	**0.004**	**0.004**
500	0.006	0.014	**0.002**	0.003
1000	0.004	0.011	0.002	**0.001**

Table 4:

Value of δ for ZM, ZM^*, ZNT and ZNT^* ($p = 7,\ 10,\ 15,\ 20$).

N	ZM	ZM^*	ZNT	ZNT^*
		$p = 7$		
20	0.033	0.030	0.043	**0.014**
50	0.039	0.027	0.022	**0.012**
100	0.032	0.023	0.012	**0.008**
200	0.024	0.018	0.006	**0.005**
300	0.020	0.015	0.004	**0.003**
500	0.011	0.009	**0.001**	**0.001**
1000	0.011	0.009	**0.001**	**0.001**
		$p = 10$		
20	0.203	0.027	0.047	**0.010**
50	0.092	0.024	0.026	**0.009**
100	0.064	0.020	0.013	**0.006**
200	0.045	0.016	0.007	**0.004**
300	0.037	0.013	0.005	**0.002**
500	0.029	0.011	0.003	**0.002**
1000	0.020	0.008	**0.001**	**0.001**
		$p = 15$		
20	0.982	0.036	0.050	**0.029**
50	0.273	0.021	0.032	**0.006**
100	0.142	0.018	0.017	**0.004**
200	0.088	0.014	0.009	**0.003**
300	0.068	0.012	0.006	**0.002**
500	0.052	0.009	0.003	**0.001**
1000	0.035	0.007	**0.001**	**0.001**
		$p = 20$		
50	0.596	0.020	0.036	**0.001**
100	0.266	0.017	0.021	**0.004**
200	0.144	0.013	0.011	**0.002**
300	0.107	0.011	0.007	**0.001**
500	0.077	0.009	0.004	**0.001**
1000	0.051	0.006	0.002	**0.000**

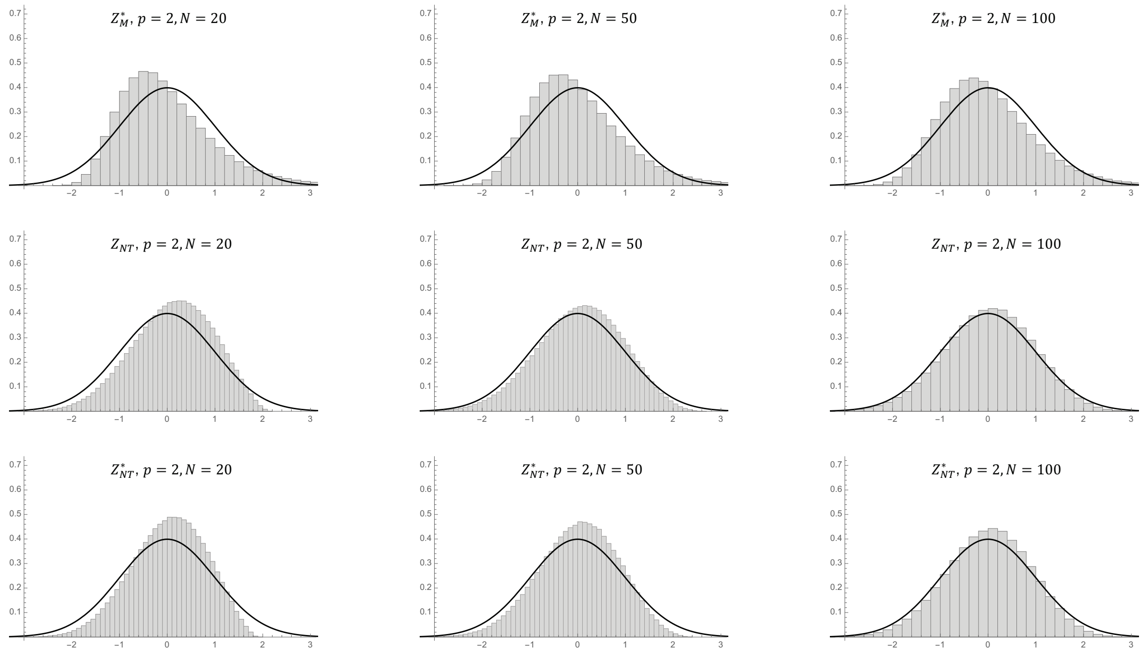

Figure 1: Histogram of ZM^*, ZNT and ZNT^* for $p = 2$ and $N = 20,\ 50,\ 100$.

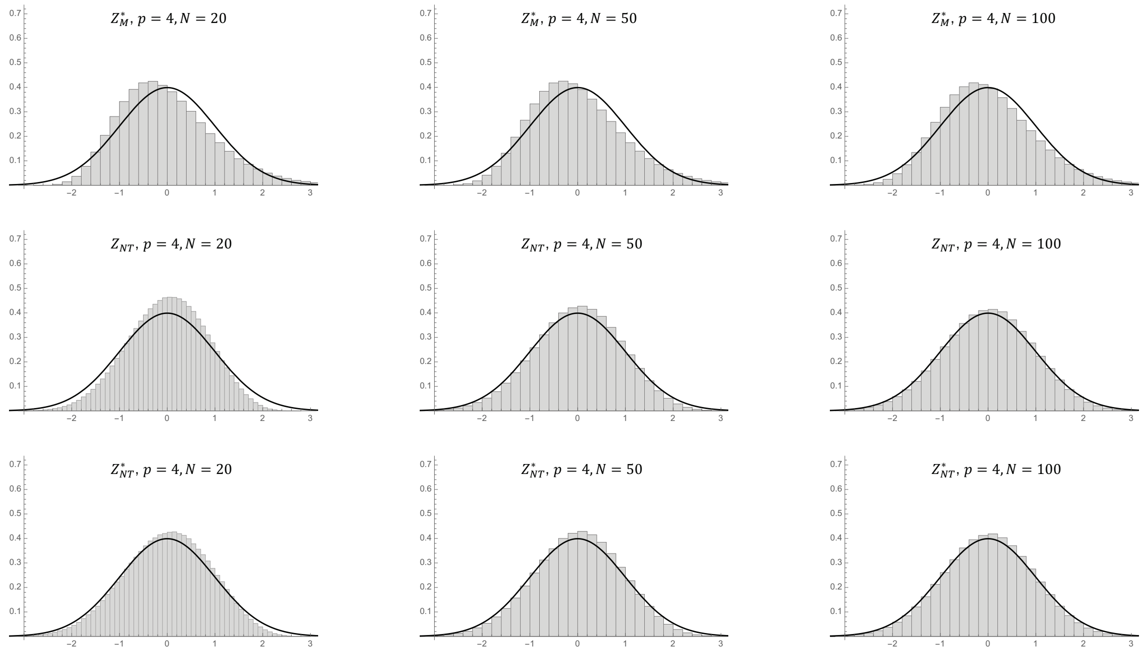

Figure 2: Histogram of ZM^*, ZNT and ZNT^* for $p = 4$ and $N = 20,\ 50,\ 100$.

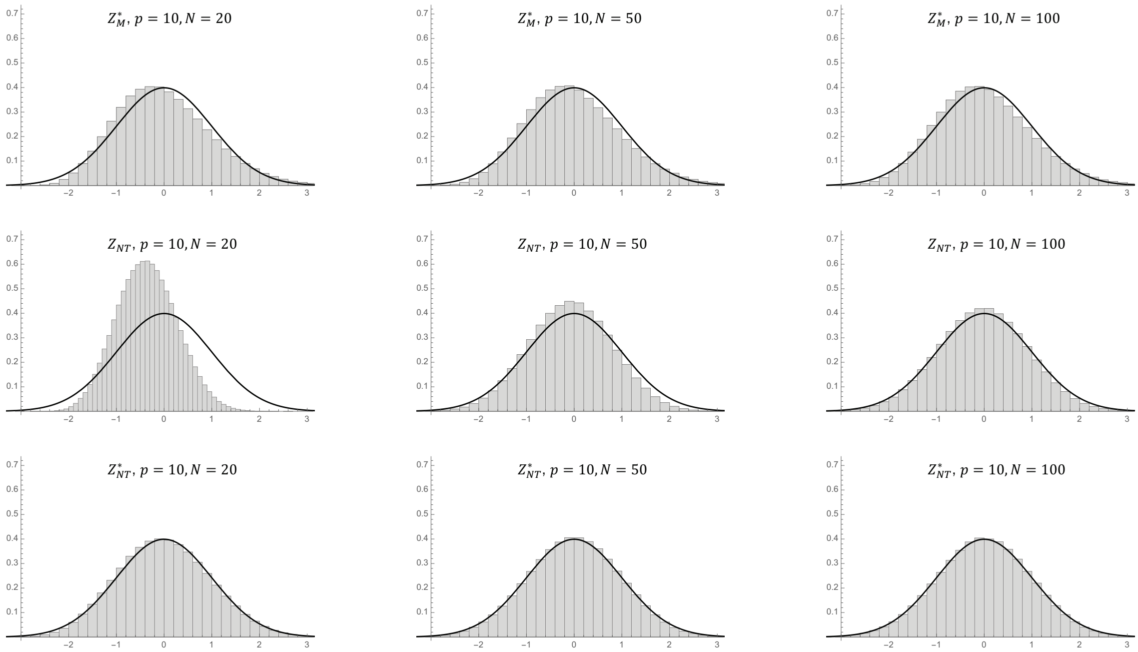

Figure 3: Histogram of ZM^*, ZNT and ZNT^* for $p = 10$ and $N = 20,\ 50,\ 100$.

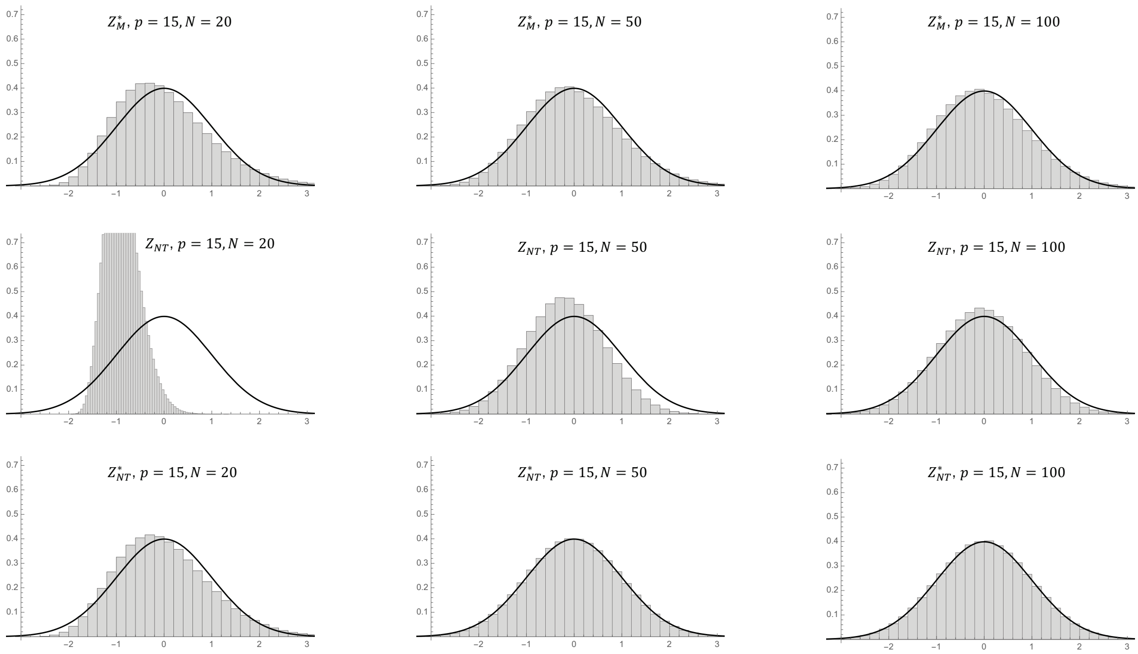

Figure 4: Histogram of ZM^*, ZNT and ZNT^* for $p = 15$ and $N = 20,\ 50,\ 100$.

Table 5: Value of test statistics, empirical and simulated p-value, and decision on MVN at significance level 0.05, p=6, N=20.

Statistic	Value	Empirical p-value	Decision	δ
ZM	-1.684	0.092	NR	0.034
ZM^*	-1.253	0.210	NR	0.031
ZNT	-1.154	0.249	NR	0.040
ZNT^*	-1.284	0.199	NR	0.016

Note. NR: Does not reject MVN, Simulated p-value is 0.179.

Flury (1997), where the sample size is N=200. Then, since p-values of all test statistics have almost one in this case, a numerical example was conducted here with N=20. Table 5 gives the values of the test statistics for ZM, ZM^*, ZNT and ZNT^*, their empirical and simulated p-values, and the test decision at significance level 0.05. It can be seen from Table 5 that the absolute values of all test statistics are less than 1.96, hence the MVN is not rejected. Furthermore, the p-values of the test statistics in Table 5 indicate that the empirical p-value of ZNT^* is closest to the simulated p-value.

6 Concluding remarks

In this study, we discussed the testing problem of multivariate normality based on Mardia's multivariate kurtosis. We proposed a modified normalizing transformation statistic by evaluating the expectation and variance of the normalizing transformation statistic. The null distribution of ZNT^* proposed in this paper has considerably good approximation to the standard normal distribution.

Acknowledgments

The second author was partly supposed by a Grant-in-Aid for Scientific Research (C) (JSPS, Japan KAKENHI Grant Number JP21K11795).

References

[1] Enomoto, R., Hanusz, Z., Hara, A. and Seo, T. (2020). Multivariate normality test using normalizing transformation for Mardia's multivariate kurtosis. *Communications in Statistics – Simulation and Computation*, **49**, 684–698.

[2] Flury, B. (1997). *A First Course in Multivariate Statistics.* New York: Springer-Verlag, Inc.

[3] Fujikoshi, Y. (2002). Asymptotic expansions for the distributions of multivariate basic statistics and one-way MANOVA tests under nonnormality. *Journal of Statistical Planning and Inference*, **108**, 263–282.

[4] Koziol, J. A. (1989). A note on measures of multivariate kurtosis. *Biometrical Journal*, **31**, 619–624.

[5] Konishi, S. (1981). Normalizing transformations of some statistics in multivariate analysis. *Biometrika*, **68**, 647–651.

[6] Kurita, E. and Seo, T. (2022). Multivariate normality test based on kurtosis with two-step monotone missing data. *Journal of Multivariate Analysis*, **188**, 104824.

[7] Mardia, K. V. (1970). Measures of multivariate skewness and kurtosis with applications. *Biometrika*, **57**, 519–530.

[8] Mardia, K. V. (1974). Applications of some measures of multivariate skewness and kurtosis in testing normality and robustness studies. *Sankhyà*, **36**, 115–128.

[9] Mardia, K. V. and Kanazawa, M. (1983). The null distribution of multivariate kurtosis. *Communications in Statistics – Simulation and Computation*, **12**, 569–576.

[10] Seo, T., Kanda, T. and Fujikoshi, Y. (1994). The effects on the distributions of sample canonical correlations under nonnormality. *Communications in Statistics -Theory and Methods*, **23**, 2615–2628.

[11] Seo, T., Kanda, T. and Fujikoshi, Y. (1995). The effects of nonnormality of tests for dimensionality in canonical correlation and MANOVA models. *Journal of Multivariate Analysis,*, **52**, 325–337.

[12] Siotani, M., Hayakawa, T. and Fujikoshi, Y. (1985). *Modern Multivariate Statistical Analysis: A Graduate Course and Handbook.* Columbus, OH: American Science Press, Inc.

[13] Srivastava, M. S. (1984). A measure of skewness and kurtosis and a graphical method for assessing multivariate normality. *Statistics & Probability Letters*, **2**, 263–267.

[14] Wakaki, H., Yanagihara, H. and Fujikoshi, Y. (2002). Asymptotic expansions of the null distributions of test statistics for multivariate linear hypothesis under nonnormality. *Hiroshima Math Journal*, **32**, 17–50.

[15] Yamada, T., Romer, M. M. and Richards, D. St. P. (2015). Kurtosis tests for multivariate normality with monotone incomplete data. *Test*, **24**, 532–557.