Testing Equality of Multivariate Normal Populations with Three-step Monotone Missing Data

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Abstract

In this paper, we consider simultaneous tests of the mean vectors and the covariance matrices under three-step monotone missing data for a multi-sample problem, that is, the problem of testing the equality of multivariate normal populations with three-step monotone missing data. We provide the likelihood ratio test (LRT) statistic and propose test statistics for improving the accuracy of the χ^2 approximation. These test statistics are derived by decomposing of the likelihood ratio (LR) using the coefficients of the modified LRT statistics with complete data. As an alternative approach, we derive an approximate upper percentile of the LRT statistic with three-step monotone missing data using linear interpolation based on an asymptotic expansion of the LRT statistic with complete data. Finally, we investigate the asymptotic behavior of the upper percentiles of these test statistics and the accuracy of approximate upper percentiles via Monte Carlo simulation. In addition, we give an example for test statistics and approximate upper percentiles proposed in this paper.

Key Words and Phrases. Asymptotic expansion, likelihood ratio test, linear interpolation, maximum likelihood estimator, modified likelihood ratio test statistic.

1 Introduction

In this paper, we consider simultaneous tests of the mean vectors and the covariance matrices under three-step monotone missing data for a multi-sample problem. MLEs for general k-step monotone missing data were discussed by Jinadasa and Tracy (1992) and Kanda and Fujikoshi (1998). For the simultaneous test, the LRT statistic and the modified LRT statistic with Bartlett correction for the case of complete data were discussed by Muirhead (2005) and Srivastava (2002). Further, the LRT statistic and the test statistics for improving the accuracy of the χ^2 approximation for three-step monotone missing data were proposed by Hao and Krishnamoorthy (2001) and Hosoya and Seo (2015, 2016). In particular, Hosoya and Seo (2015, 2016) presented test statistics by decomposing the LR; this paper is an extension of the work presented by Hosoya and Seo (2016). Indeed, an LRT statistic and test statistics for general k-step monotone missing data, which are obtained by correcting only a part of the missing data, were given by Yagi, Yamaguchi, and Seo (2016).

The remainder of this paper is organized as follows. In Section 2, we describe the MLEs of the mean vectors and covariance matrices and its LRT statistic in the case of three-step monotone missing data for a multi-sample problem. In Section 3, the modified LRT statistics in the case of complete data are described, which are used to derive the test statistics in Section 4. In Section 4, we propose three test statistics for improving the accuracy of the χ^2 approximation using the coefficients of the modified LRT statistics with complete data. In Section 5, we derive an approximate upper percentile of the LRT statistic. In Section 6, using Monte Carlo simulation, we investigate the χ^2 approximation accuracy of the test statistics and the accuracy of approximate upper percentiles of the LRT statistic. In Section 7, the results are illustrated using an example. Finally, in Section 8, we state our conclusions.

2 Likelihood ratio with three-step monotone missing data

In this section, we will consider simultaneous tests of the mean vectors and the covariance matrices under three-step monotone missing data for a multi-sample problem. Let $\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{N_1^{(\ell)}}^{(\ell)}$ be independent p-dimensional sample vectors, $\boldsymbol{x}_{(12),N_1^{(\ell)}+1}^{(\ell)}, \boldsymbol{x}_{(12),N_1^{(\ell)}+2}^{(\ell)}, \ldots, \boldsymbol{x}_{(12),N_1^{(\ell)}+N_2^{(\ell)}}^{(\ell)}$ be independent (p_1+p_2)-dimensional sample vectors and $\boldsymbol{x}_{1,N_1^{(\ell)}+N_2^{(\ell)}+1}^{(\ell)}, \boldsymbol{x}_{1,N_1^{(\ell)}+N_2^{(\ell)}+2}^{(\ell)}, \ldots, \boldsymbol{x}_{1N^{(\ell)}}^{(\ell)}$ be independent p_1-dimensional sample vectors from the ℓth population ($\ell = 1, \ldots, m$). We suppose that the data is normally distributed as follows:

$$
\begin{aligned}
\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{N_1^{(\ell)}}^{(\ell)} &\overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)}),\\
\boldsymbol{x}_{(12),N_1^{(\ell)}+1}^{(\ell)}, \boldsymbol{x}_{(12),N_1^{(\ell)}+2}^{(\ell)}, \ldots, \boldsymbol{x}_{(12),N_1^{(\ell)}+N_2^{(\ell)}}^{(\ell)} &\overset{i.i.d.}{\sim} N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}^{(\ell)}, \boldsymbol{\Sigma}_{(12)(12)}^{(\ell)}),\\
\boldsymbol{x}_{1,N_1^{(\ell)}+N_2^{(\ell)}+1}^{(\ell)}, \boldsymbol{x}_{1,N_1^{(\ell)}+N_2^{(\ell)}+2}^{(\ell)}, \ldots, \boldsymbol{x}_{1N^{(\ell)}}^{(\ell)} &\overset{i.i.d.}{\sim} N_{p_1}(\boldsymbol{\mu}_1^{(\ell)}, \boldsymbol{\Sigma}_{11}^{(\ell)}),
\end{aligned}
$$

where

$$
\boldsymbol{\mu}^{(\ell)} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} \\ \boldsymbol{\mu}_3^{(\ell)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)}^{(\ell)} \\ \boldsymbol{\mu}_3^{(\ell)} \end{pmatrix},
$$

$$
\boldsymbol{\Sigma}^{(\ell)} = \left(\begin{array}{cc|c} \boldsymbol{\Sigma}_{11}^{(\ell)} & \boldsymbol{\Sigma}_{12}^{(\ell)} & \boldsymbol{\Sigma}_{13}^{(\ell)} \\ \boldsymbol{\Sigma}_{21}^{(\ell)} & \boldsymbol{\Sigma}_{22}^{(\ell)} & \boldsymbol{\Sigma}_{23}^{(\ell)} \\ \hline \boldsymbol{\Sigma}_{31}^{(\ell)} & \boldsymbol{\Sigma}_{32}^{(\ell)} & \boldsymbol{\Sigma}_{33}^{(\ell)} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{(12)(12)}^{(\ell)} & \boldsymbol{\Sigma}_{(12)3}^{(\ell)} \\ \hline \boldsymbol{\Sigma}_{3(12)}^{(\ell)} & \boldsymbol{\Sigma}_{33}^{(\ell)} \end{array}\right).
$$

and

$$
\begin{aligned}
\boldsymbol{x}_j^{(\ell)} &= (\boldsymbol{x}_{1j}^{(\ell)\prime}, \boldsymbol{x}_{2j}^{(\ell)\prime}, \boldsymbol{x}_{3j}^{(\ell)\prime})', j = 1, \ldots, N_1^{(\ell)}, \\
\boldsymbol{x}_{(12),j}^{(\ell)} &= (\boldsymbol{x}_{1j}^{(\ell)\prime}, \boldsymbol{x}_{2j}^{(\ell)\prime})', j = N_1^{(\ell)} + 1 \ldots, N_1^{(\ell)} + N_2^{(\ell)}, \\
N^{(\ell)} &= N_1^{(\ell)} + N_2^{(\ell)} + N_3^{(\ell)},\ p = p_1 + p_2 + p_3.
\end{aligned}
$$

Such a dataset has three-step monotone missing data for the ℓth population:

$$
\begin{pmatrix}
\overbrace{\qquad\qquad}^{p_1} & \overbrace{\qquad\qquad}^{p_2} & \overbrace{\qquad\quad}^{p_3} \\
\boldsymbol{x}_{11}^{(\ell)\prime} & \boldsymbol{x}_{21}^{(\ell)\prime} & \boldsymbol{x}_{31}^{(\ell)\prime} \\
\vdots & \vdots & \vdots \\
\boldsymbol{x}_{1N_1^{(\ell)}}^{(\ell)\prime} & \boldsymbol{x}_{2N_1^{(\ell)}}^{(\ell)\prime} & \boldsymbol{x}_{3N_1^{(\ell)}}^{(\ell)\prime} \\
\boldsymbol{x}_{1,N_1^{(\ell)}+1}^{(\ell)\prime} & \boldsymbol{x}_{2,N_1^{(\ell)}+1}^{(\ell)\prime} & * \cdots * \\
\vdots & \vdots & \vdots \quad \vdots \\
\boldsymbol{x}_{1,N_1^{(\ell)}+N_2^{(\ell)}}^{(\ell)\prime} & \boldsymbol{x}_{2,N_1^{(\ell)}+N_2^{(\ell)}}^{(\ell)\prime} & * \cdots * \\
\boldsymbol{x}_{1,N_1^{(\ell)}+N_2^{(\ell)}+1}^{(\ell)\prime} & * \cdots * & * \cdots * \\
\vdots & \vdots \quad \vdots & \vdots \quad \vdots \\
\boldsymbol{x}_{1N^{(\ell)}}^{(\ell)\prime} & * \cdots * & * \cdots *
\end{pmatrix},
$$

where "$*$" indicates a missing observation.

We consider the following hypothesis:

$$
H_0 : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \cdots = \boldsymbol{\mu}^{(m)},\ \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)} \quad \text{vs.} \quad H_1 : \text{not} \quad H_0
$$

To derive the MLEs of the mean vectors and the covariance matrices, we consider the following transformation matrix $\boldsymbol{Z}^{(\ell)}$:

$$
\boldsymbol{Z}^{(\ell)} = \left(\begin{array}{cc|c} \boldsymbol{I}_{p_1} & \boldsymbol{O} & \\ -\boldsymbol{\Sigma}_{21}^{(\ell)}\boldsymbol{\Sigma}_{11}^{(\ell)-1} & \boldsymbol{I}_{p_2} & \boldsymbol{O} \\ \hline \multicolumn{2}{c|}{-\boldsymbol{\Sigma}_{3(12)}^{(\ell)}\boldsymbol{\Sigma}_{(12)(12)}^{(\ell)-1}} & \boldsymbol{I}_{p_3} \end{array}\right).
$$

The transformed vector $\boldsymbol{y}_j^{(\ell)} = (\boldsymbol{y}_{1j}^{(\ell)'}, \boldsymbol{y}_{2j}^{(\ell)'}, \boldsymbol{y}_{3j}^{(\ell)'})'$ is

$$\begin{aligned}
\boldsymbol{y}_j^{(\ell)} &= \boldsymbol{Z}^{(\ell)}\boldsymbol{x}_j^{(\ell)} \\
&= \begin{pmatrix} \boldsymbol{x}_{1j}^{(\ell)} \\ -\boldsymbol{\Sigma}_{21}^{(\ell)}\boldsymbol{\Sigma}_{11}^{(\ell)-1}\boldsymbol{x}_{1j}^{(\ell)} + \boldsymbol{x}_{2j}^{(\ell)} \\ -\boldsymbol{\Sigma}_{3(12)}^{(\ell)}\boldsymbol{\Sigma}_{(12)(12)}^{(\ell)-1}\begin{pmatrix} \boldsymbol{x}_{1j}^{(\ell)} \\ \boldsymbol{x}_{2j}^{(\ell)} \end{pmatrix} + \boldsymbol{x}_{3j}^{(\ell)} \end{pmatrix}.
\end{aligned}$$

The transformed parameters $(\boldsymbol{\eta}^{(\ell)}, \boldsymbol{\Delta}^{(\ell)})$ are defined as

$$\boldsymbol{\eta}^{(\ell)} = \begin{pmatrix} \boldsymbol{\eta}_1^{(\ell)} \\ \boldsymbol{\eta}_2^{(\ell)} \\ \boldsymbol{\eta}_3^{(\ell)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ -\boldsymbol{\Sigma}_{21}^{(\ell)}\boldsymbol{\Sigma}_{11}^{(\ell)-1}\boldsymbol{\mu}_1^{(\ell)} + \boldsymbol{\mu}_2^{(\ell)} \\ -\boldsymbol{\Sigma}_{3(12)}^{(\ell)}\boldsymbol{\Sigma}_{(12)(12)}^{(\ell)-1}\begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} \end{pmatrix} + \boldsymbol{\mu}_3^{(\ell)} \end{pmatrix},$$

$$\boldsymbol{\Delta}^{(\ell)} = \left(\begin{array}{cc|c} \boldsymbol{\Delta}_{11}^{(\ell)} & \boldsymbol{\Delta}_{12}^{(\ell)} & \boldsymbol{\Delta}_{13}^{(\ell)} \\ \boldsymbol{\Delta}_{21}^{(\ell)} & \boldsymbol{\Delta}_{22}^{(\ell)} & \boldsymbol{\Delta}_{23}^{(\ell)} \\ \hline \boldsymbol{\Delta}_{31}^{(\ell)} & \boldsymbol{\Delta}_{32}^{(\ell)} & \boldsymbol{\Delta}_{33}^{(\ell)} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Delta}_{(12)(12)}^{(\ell)} & \boldsymbol{\Delta}_{(12)3}^{(\ell)} \\ \hline \boldsymbol{\Delta}_{3(12)}^{(\ell)} & \boldsymbol{\Delta}_{33}^{(\ell)} \end{array}\right),$$

where

$$\begin{aligned}
\boldsymbol{\Delta}_{11}^{(\ell)} &= \boldsymbol{\Sigma}_{11}^{(\ell)}, \\
\boldsymbol{\Delta}_{12}^{(\ell)} &= \boldsymbol{\Delta}_{21}^{(\ell)'} = \boldsymbol{\Sigma}_{11}^{(\ell)-1}\boldsymbol{\Sigma}_{12}^{(\ell)}, \\
\boldsymbol{\Delta}_{22}^{(\ell)} &= \boldsymbol{\Sigma}_{22\cdot 1}^{(\ell)} = \boldsymbol{\Sigma}_{22}^{(\ell)} - \boldsymbol{\Sigma}_{21}^{(\ell)}\boldsymbol{\Sigma}_{11}^{(\ell)-1}\boldsymbol{\Sigma}_{12}^{(\ell)}, \\
\boldsymbol{\Delta}_{(12)3}^{(\ell)} &= \boldsymbol{\Delta}_{3(12)}^{(\ell)'} = \boldsymbol{\Sigma}_{(12)(12)}^{(\ell)-1}\boldsymbol{\Sigma}_{(12)3}^{(\ell)}, \\
\boldsymbol{\Delta}_{33}^{(\ell)} &= \boldsymbol{\Sigma}_{33\cdot 12}^{(\ell)} = \boldsymbol{\Sigma}_{33}^{(\ell)} - \boldsymbol{\Sigma}_{3(12)}^{(\ell)}\boldsymbol{\Sigma}_{(12)(12)}^{(\ell)-1}\boldsymbol{\Sigma}_{(12)3}^{(\ell)}.
\end{aligned}$$

We note that the pair $(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)})$ is in one-to-one correspondence with $(\boldsymbol{\eta}^{(\ell)}, \boldsymbol{\Delta}^{(\ell)})$. Under H_1, we define the MLEs of $(\boldsymbol{\eta}^{(\ell)}, \boldsymbol{\Delta}^{(\ell)})$ as $(\widehat{\boldsymbol{\eta}}^{(\ell)}, \widehat{\boldsymbol{\Delta}}^{(\ell)})$,

$$\begin{aligned}
\widehat{\boldsymbol{\eta}}_1^{(\ell)} &= \frac{1}{N^{(\ell)}}(N_1^{(\ell)}\overline{\boldsymbol{x}}_{(1)1}^{(\ell)} + N_2^{(\ell)}\overline{\boldsymbol{x}}_{(2)1}^{(\ell)} + N_3^{(\ell)}\overline{\boldsymbol{x}}_{(3)}^{(\ell)}), \\
\widehat{\boldsymbol{\eta}}_2^{(\ell)} &= \frac{1}{N_1^{(\ell)} + N_2^{(\ell)}}\{N_1^{(\ell)}\overline{\boldsymbol{x}}_{(1)2}^{(\ell)} + N_2^{(\ell)}\overline{\boldsymbol{x}}_{(2)2}^{(\ell)} - \widehat{\boldsymbol{\Delta}}_{21}^{(\ell)}(N_1^{(\ell)}\overline{\boldsymbol{x}}_{(1)1}^{(\ell)} + N_2^{(\ell)}\overline{\boldsymbol{x}}_{(2)1}^{(\ell)})\}, \\
\widehat{\boldsymbol{\eta}}_3^{(\ell)} &= \overline{\boldsymbol{x}}_{(1)3}^{(\ell)} - \widehat{\boldsymbol{\Delta}}_{3(12)}^{(\ell)}\overline{\boldsymbol{x}}_{(1)(12)}^{(\ell)}, \\
\widehat{\boldsymbol{\Delta}}_{11}^{(\ell)} &= \frac{1}{N^{(\ell)}}(\boldsymbol{W}_{(1)11}^{(\ell)} + \boldsymbol{W}_{(2)11}^{(\ell)} + \boldsymbol{W}_{(3)}^{(\ell)}), \\
\widehat{\boldsymbol{\Delta}}_{22}^{(\ell)} &= \frac{1}{N_1^{(\ell)} + N_2^{(\ell)}}(\boldsymbol{W}_{(1),(12)(12)}^{(\ell)} + \boldsymbol{W}_{(2)}^{(\ell)})_{22\cdot 1}, \ \widehat{\boldsymbol{\Delta}}_{33}^{(\ell)} = \frac{1}{N_1^{(\ell)}}\boldsymbol{W}_{(1)33\cdot 12}^{(\ell)}, \\
\widehat{\boldsymbol{\Delta}}_{12}^{(\ell)} &= \widehat{\boldsymbol{\Delta}}_{21}^{(\ell)'} = (\boldsymbol{W}_{(1)11}^{(\ell)} + \boldsymbol{W}_{(2)11}^{(\ell)})^{-1}(\boldsymbol{W}_{(1)12}^{(\ell)} + \boldsymbol{W}_{(2)12}^{(\ell)}),
\end{aligned}$$

$$\widehat{\mathbf{\Delta}}^{(\ell)}_{(12)3} = \widehat{\mathbf{\Delta}}^{(\ell)\prime}_{3(12)} = (\boldsymbol{W}^{(\ell)}_{(1),(12)(12)})^{-1}\boldsymbol{W}^{(\ell)}_{(1),(12)3},$$

where

$$\overline{\boldsymbol{x}}^{(\ell)}_{(1)} = \begin{pmatrix} \overline{\boldsymbol{x}}^{(\ell)}_{(1)1} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(1)2} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(1)3} \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}^{(\ell)}_{(1)(12)} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(1)3} \end{pmatrix},$$

$$\overline{\boldsymbol{x}}^{(\ell)}_{(1)1} = \frac{1}{N^{(\ell)}_1}\sum_{j=1}^{N^{(\ell)}_1}\boldsymbol{x}^{(\ell)}_{1j},\ \overline{\boldsymbol{x}}^{(\ell)}_{(1)2} = \frac{1}{N^{(\ell)}_1}\sum_{j=1}^{N^{(\ell)}_1}\boldsymbol{x}^{(\ell)}_{2j},\ \overline{\boldsymbol{x}}^{(\ell)}_{(1)3} = \frac{1}{N^{(\ell)}_1}\sum_{j=1}^{N^{(\ell)}_1}\boldsymbol{x}^{(\ell)}_{3j},$$

$$\overline{\boldsymbol{x}}^{(\ell)}_{(2)} = \begin{pmatrix} \overline{\boldsymbol{x}}^{(\ell)}_{(2)1} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(2)2} \end{pmatrix},\ \overline{\boldsymbol{x}}^{(\ell)}_{(2)1} = \frac{1}{N^{(\ell)}_2}\sum_{j=N^{(\ell)}_1+1}^{N^{(\ell)}_1+N^{(\ell)}_2}\boldsymbol{x}^{(\ell)}_{1j},\ \overline{\boldsymbol{x}}^{(\ell)}_{(2)2} = \frac{1}{N^{(\ell)}_2}\sum_{j=N^{(\ell)}_1+1}^{N^{(\ell)}_1+N^{(\ell)}_2}\boldsymbol{x}^{(\ell)}_{2j},$$

$$\overline{\boldsymbol{x}}^{(\ell)}_{(3)} = \frac{1}{N^{(\ell)}_3}\sum_{j=N^{(\ell)}_1+N^{(\ell)}_2+1}^{N^{(\ell)}}\boldsymbol{x}^{(\ell)}_{1j},$$

$$\boldsymbol{W}^{(\ell)}_{(1)} = \sum_{j=1}^{N_1}(\boldsymbol{x}^{(\ell)}_j - \overline{\boldsymbol{x}}^{(\ell)}_{(1)})(\boldsymbol{x}^{(\ell)}_j - \overline{\boldsymbol{x}}^{(\ell)}_{(1)})' = \left(\begin{array}{cc|c} \boldsymbol{W}^{(\ell)}_{(1)11} & \boldsymbol{W}^{(\ell)}_{(1)12} & \boldsymbol{W}^{(\ell)}_{(1)13} \\ \boldsymbol{W}^{(\ell)}_{(1)21} & \boldsymbol{W}^{(\ell)}_{(1)22} & \boldsymbol{W}^{(\ell)}_{(1)23} \\ \hline \boldsymbol{W}^{(\ell)}_{(1)31} & \boldsymbol{W}^{(\ell)}_{(1)32} & \boldsymbol{W}^{(\ell)}_{(1)33} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{W}^{(\ell)}_{(1),(12)(12)} & \boldsymbol{W}^{(\ell)}_{(1),(12)3} \\ \hline \boldsymbol{W}^{(\ell)}_{(1),3(12)} & \boldsymbol{W}^{(\ell)}_{(1)33} \end{array}\right),$$

$$\begin{aligned} \boldsymbol{W}^{(\ell)}_{(2)} &= \sum_{j=N^{(\ell)}_1+1}^{N^{(\ell)}_1+N^{(\ell)}_2}\begin{pmatrix} \boldsymbol{x}^{(\ell)}_{1j} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)1} \\ \boldsymbol{x}^{(\ell)}_{2j} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)2} \end{pmatrix}\begin{pmatrix} \boldsymbol{x}^{(\ell)}_{1j} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)1} \\ \boldsymbol{x}^{(\ell)}_{2j} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)2} \end{pmatrix}' + \frac{N^{(\ell)}_1 N^{(\ell)}_2}{N^{(\ell)}_1 + N^{(\ell)}_2}\begin{pmatrix} \overline{\boldsymbol{x}}^{(\ell)}_{(1)1} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)1} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(1)2} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)2} \end{pmatrix}\begin{pmatrix} \overline{\boldsymbol{x}}^{(\ell)}_{(1)1} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)1} \\ \overline{\boldsymbol{x}}^{(\ell)}_{(1)2} - \overline{\boldsymbol{x}}^{(\ell)}_{(2)2} \end{pmatrix}' \\ &= \begin{pmatrix} \boldsymbol{W}^{(\ell)}_{(2)11} & \boldsymbol{W}^{(\ell)}_{(2)12} \\ \boldsymbol{W}^{(\ell)}_{(2)21} & \boldsymbol{W}^{(\ell)}_{(2)22} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{W}^{(\ell)}_{(3)} = \sum_{j=N^{(\ell)}_1+N^{(\ell)}_2+1}^{N^{(\ell)}}(\boldsymbol{x}^{(\ell)}_{1j} - \overline{\boldsymbol{x}}^{(\ell)}_{(3)})(\boldsymbol{x}^{(\ell)}_{1j} - \overline{\boldsymbol{x}}^{(\ell)}_{(3)})' + \frac{(N^{(\ell)}_1 + N^{(\ell)}_2)N^{(\ell)}_3}{N^{(\ell)}}\left(\overline{\boldsymbol{x}}^{(\ell)}_{(3)} - \frac{1}{N^{(\ell)}_1 + N^{(\ell)}_2}(N^{(\ell)}_1\overline{\boldsymbol{x}}^{(\ell)}_{(1)1} + N^{(\ell)}_2\overline{\boldsymbol{x}}^{(\ell)}_{(2)1})\right) \\ \times\left(\overline{\boldsymbol{x}}^{(\ell)}_{(3)} - \frac{1}{N^{(\ell)}_1 + N^{(\ell)}_2}(N^{(\ell)}_1\overline{\boldsymbol{x}}^{(\ell)}_{(1)1} + N^{(\ell)}_2\overline{\boldsymbol{x}}^{(\ell)}_{(2)1})\right)'. \end{aligned}$$

In contrast, under H_0, we define MLEs of $\boldsymbol{\eta}(= \boldsymbol{\eta}^{(1)} = \cdots = \boldsymbol{\eta}^{(m)}), \mathbf{\Delta}(= \mathbf{\Delta}^{(1)} = \cdots = \mathbf{\Delta}^{(m)})$ as $(\widetilde{\boldsymbol{\eta}}, \widetilde{\mathbf{\Delta}})$. Then, we obtain

$$\widetilde{\boldsymbol{\eta}}_1 = \frac{1}{N}\sum_{\ell=1}^{m}(N^{(\ell)}_1\overline{\boldsymbol{x}}^{(\ell)}_{(1)1} + N^{(\ell)}_2\overline{\boldsymbol{x}}^{(\ell)}_{(2)1} + N^{(\ell)}_3\overline{\boldsymbol{x}}^{(\ell)}_{(3)}),$$

$$\widetilde{\boldsymbol{\eta}}_2 = \frac{1}{N_1 + N_2}\sum_{\ell=1}^{m}\{N^{(\ell)}_1\overline{\boldsymbol{x}}^{(\ell)}_{(1)2} + N^{(\ell)}_2\overline{\boldsymbol{x}}^{(\ell)}_{(2)2} - \widetilde{\mathbf{\Delta}}_{21}(N^{(\ell)}_1\overline{\boldsymbol{x}}^{(\ell)}_{(1)1} + N^{(\ell)}_2\overline{\boldsymbol{x}}^{(\ell)}_{(2)1})\},$$

$$\widetilde{\boldsymbol{\eta}}_3 = \frac{1}{N_1}\sum_{\ell=1}^{m}N^{(\ell)}_1\{\overline{\boldsymbol{x}}^{(\ell)}_{(1)3} - \widetilde{\mathbf{\Delta}}_{3(12)}\overline{\boldsymbol{x}}^{(\ell)}_{(1)(12)}\},$$

$$\widetilde{\mathbf{\Delta}}_{11} = \frac{1}{N}\sum_{\ell=1}^{m}\sum_{j=1}^{N^{(\ell)}}(\boldsymbol{x}^{(\ell)}_{1j} - \widetilde{\boldsymbol{\eta}}_1)(\boldsymbol{x}^{(\ell)}_{1j} - \widetilde{\boldsymbol{\eta}}_1)',$$

$$\widetilde{\mathbf{\Delta}}_{22} = \frac{1}{N_1+N_2}\sum_{\ell=1}^{m}\sum_{j=1}^{N_1^{(\ell)}+N_2^{(\ell)}}(-\widetilde{\mathbf{\Delta}}_{21}\boldsymbol{x}_{1j}^{(\ell)}+\boldsymbol{x}_{2j}^{(\ell)}-\widetilde{\boldsymbol{\eta}}_2)(-\widetilde{\mathbf{\Delta}}_{21}\boldsymbol{x}_{1j}^{(\ell)}+\boldsymbol{x}_{2j}^{(\ell)}-\widetilde{\boldsymbol{\eta}}_2)',$$

$$\widetilde{\mathbf{\Delta}}_{33} = \frac{1}{N_1}\sum_{\ell=1}^{m}\sum_{j=1}^{N_1^{(\ell)}}(-\widetilde{\mathbf{\Delta}}_{3(12)}\boldsymbol{x}_{(12)j}^{(\ell)}+\boldsymbol{x}_{3j}^{(\ell)}-\widetilde{\boldsymbol{\eta}}_3)(-\widetilde{\mathbf{\Delta}}_{3(12)}\boldsymbol{x}_{(12)j}^{(\ell)}+\boldsymbol{x}_{3j}^{(\ell)}-\widetilde{\boldsymbol{\eta}}_3)',$$

$$\begin{aligned}
\widetilde{\mathbf{\Delta}}_{21} &= \widetilde{\mathbf{\Delta}}_{12}' \\
&= \sum_{\ell=1}^{m}\left[\sum_{j=1}^{N_1^{(\ell)}+N_2^{(\ell)}}\boldsymbol{x}_{2j}^{(\ell)}\boldsymbol{x}_{1j}^{(\ell)\prime}-\frac{1}{N_1+N_2}\left\{\sum_{k=1}^{m}(N_1^{(k)}\overline{\boldsymbol{x}}_{(1)2}^{(k)}+N_2^{(k)}\overline{\boldsymbol{x}}_{(2)2}^{(k)})\right\}(N_1^{(\ell)}\overline{\boldsymbol{x}}_{(1)1}^{(\ell)}+N_2^{(\ell)}\overline{\boldsymbol{x}}_{(2)1}^{(\ell)})'\right] \\
&\quad\times\sum_{\ell=1}^{m}\left[\sum_{j=1}^{N_1^{(\ell)}+N_2^{(\ell)}}\boldsymbol{x}_{1j}^{(\ell)}\boldsymbol{x}_{1j}^{(\ell)\prime}-\frac{1}{N_1+N_2}\left\{\sum_{k=1}^{m}(N_1^{(k)}\overline{\boldsymbol{x}}_{(1)1}^{(k)}+N_2^{(k)}\overline{\boldsymbol{x}}_{(2)1}^{(k)})\right\}(N_1^{(\ell)}\overline{\boldsymbol{x}}_{(1)1}^{(\ell)}+N_2^{(\ell)}\overline{\boldsymbol{x}}_{(2)1}^{(\ell)})'\right]^{-1},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathbf{\Delta}}_{3(12)} &= \widetilde{\mathbf{\Delta}}_{(12)3}' \\
&= \sum_{\ell=1}^{m}\left\{\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{x}_{3j}^{(\ell)}\boldsymbol{x}_{(12),j}^{(\ell)\prime}-N_1^{(\ell)}\left(\frac{1}{N_1}\sum_{k=1}^{m}N_1^{(k)}\overline{\boldsymbol{x}}_{(1)3}^{(\ell)}\right)\overline{\boldsymbol{x}}_{(1),(12)}^{(\ell)}\right\} \\
&\quad\times\sum_{\ell=1}^{m}\left\{\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{x}_{(12),j}^{(\ell)}\boldsymbol{x}_{(12),j}^{(\ell)\prime}-N_1^{(\ell)}\left(\frac{1}{N_1}\sum_{k=1}^{m}N_1^{(k)}\overline{\boldsymbol{x}}_{(1),(12)}^{(\ell)}\right)\overline{\boldsymbol{x}}_{(1),(12)}^{(\ell)}\right\}^{-1},
\end{aligned}$$

where $N=\sum_{\ell=1}^{m}N^{(\ell)},\ N_1=\sum_{\ell=1}^{m}N_1^{(\ell)},\ N_2=\sum_{\ell=1}^{m}N_2^{(\ell)}$. From the preceding MLEs, the LR can be given by

$$\lambda_m=\frac{\prod_{\ell=1}^{m}|\widehat{\mathbf{\Delta}}_{11}^{(\ell)}|^{\frac{N^{(\ell)}}{2}}|\widehat{\mathbf{\Delta}}_{22}^{(\ell)}|^{\frac{N_1^{(\ell)}+N_2^{(\ell)}}{2}}|\widehat{\mathbf{\Delta}}_{33}^{(\ell)}|^{\frac{N_1^{(\ell)}}{2}}}{|\widetilde{\mathbf{\Delta}}_{11}|^{\frac{N}{2}}|\widetilde{\mathbf{\Delta}}_{22}|^{\frac{N_1+N_2}{2}}|\widetilde{\mathbf{\Delta}}_{33}|^{\frac{N_1}{2}}}.$$

Thus, we obtain LRT statistic $-2\log\lambda_m$. $-2\log\lambda_m$ is asymptotically distributed as a χ^2 distribution with $f_m=p(p+3)(m-1)/2$ degrees of freedom. However, it is known that the accuracy of this approximation is not good for small samples. Therefore, we propose test statistics that are a good approximation to χ^2 distribution using several methods.

3 Complete data

In this section, we discuss the LRT statistic in the case of complete data and the modified LRT statistics with Bartlett correction. The results will be used to propose the test statistics in the next section. First, we consider a simultaneous test with complete data as follows:

$$H_{01}:\boldsymbol{\mu}^{(1)}=\boldsymbol{\mu}^{(2)}=\cdots=\boldsymbol{\mu}^{(m)},\ \mathbf{\Sigma}^{(1)}=\mathbf{\Sigma}^{(2)}=\cdots=\mathbf{\Sigma}^{(m)}\quad\text{vs.}\quad H_{11}:\text{not}\quad H_{01}$$

Let $\boldsymbol{x}_1^{(\ell)}, \boldsymbol{x}_2^{(\ell)}, \ldots, \boldsymbol{x}_{N^{(\ell)}}^{(\ell)}$ be independently distributed as $N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma}^{(\ell)})$, and let λ_{S_m} be the LR for the complete data. Then, the LR is given by

$$\lambda_{S_m} = \frac{\prod_{\ell=1}^m \left| \frac{1}{N^{(\ell)}} \boldsymbol{V}^{(\ell)} \right|^{\frac{1}{2}N^{(\ell)}}}{\left| \frac{1}{N}(\boldsymbol{V} + \boldsymbol{B}) \right|^{\frac{1}{2}N}},$$

where

$$\boldsymbol{V}^{(\ell)} = \sum_{j=1}^{N^{(\ell)}} (\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})(\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})', \quad \boldsymbol{V} = \sum_{\ell=1}^m \boldsymbol{V}^{(\ell)},$$

$$\boldsymbol{B} = \sum_{\ell=1}^m N^{(\ell)} (\overline{\boldsymbol{x}}^{(\ell)} - \overline{\boldsymbol{x}})(\overline{\boldsymbol{x}}^{(\ell)} - \overline{\boldsymbol{x}})',$$

$$\overline{\boldsymbol{x}}^{(\ell)} = \frac{1}{N^{(\ell)}} \sum_{j=1}^{N^{(\ell)}} \boldsymbol{x}_j^{(\ell)}, \quad \overline{\boldsymbol{x}} = \frac{1}{N} \sum_{\ell=1}^m N^{(\ell)} \overline{\boldsymbol{x}}^{(\ell)}, \ N = \sum_{\ell=1}^m N^{(\ell)}.$$

Furthermore, the modified LRT statistic with Bartlett correction can be given by $-2\rho_3 \log \lambda_{S_m}$ (Muirhead (2005, p.513)), where

$$\rho_3 = 1 - \frac{2p^2 + 9p + 11}{6N(p+3)(m-1)} \left(\sum_{\ell=1}^m \frac{N}{N^{(\ell)}} - 1 \right).$$

Next, we consider the covariance test in the case of complete data as follows:

$$H_{02} : \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(m)} \quad \text{vs.} \quad H_{12} : \text{not} \quad H_{02}$$

The modified LRT statistic $-2\rho_4 \log \lambda_{V_m}$ was provided by Muirhead (2005, p.308), where

$$\rho_4 = 1 - \frac{2p^2 + 3p - 1}{6(p+1)(m-1)} \left(\sum_{\ell=1}^m \frac{1}{n^{(\ell)}} - \frac{1}{n} \right), \ \lambda_{V_m} = \frac{\prod_{\ell=1}^m \left| \frac{1}{n^{(\ell)}} \boldsymbol{V}^{(\ell)} \right|^{\frac{n^{(\ell)}}{2}}}{\left| \frac{1}{n} \boldsymbol{V} \right|^{\frac{n}{2}}},$$

and

$$n^{(\ell)} = N^{(\ell)} - 1, \quad n = \sum_{\ell-1}^m n^{(\ell)}.$$

4 Test statistics

Using LR of the simultaneous test with complete data from the previous section, we propose test statistics by decomposing the LR λ_m with three-step monotone missing data. First, the LR can be decomposed as $\lambda_m = \xi_1 \xi_2 \xi_3$, where

$$\xi_1 = \frac{\prod_{\ell=1}^m \left| \widehat{\boldsymbol{\Delta}}_{11}^{(\ell)} \right|^{\frac{N^{(\ell)}}{2}}}{\left| \widetilde{\boldsymbol{\Delta}}_{11} \right|^{\frac{N}{2}}}, \ \xi_2 = \frac{\prod_{\ell=1}^m \left| \widehat{\boldsymbol{\Delta}}_{22}^{(\ell)} \right|^{\frac{N_1^{(\ell)} + N_2^{(\ell)}}{2}}}{\left| \widetilde{\boldsymbol{\Delta}}_{22} \right|^{\frac{N_1 + N_2}{2}}}, \ \xi_3 = \frac{\prod_{\ell=1}^m \left| \widehat{\boldsymbol{\Delta}}_{33}^{(\ell)} \right|^{\frac{N_1^{(\ell)}}{2}}}{\left| \widetilde{\boldsymbol{\Delta}}_{33} \right|^{\frac{N_1}{2}}}.$$

Because ξ_1 is of the form of LR for H_{01} in the case of without missing data, the modified LRT statistic $-2\rho_{\xi_1}\log\xi_1$ is given, where

$$\rho_{\xi_1} = 1 - \frac{2p_1^2 + 9p_1 + 11}{6N(p_1+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N}{N^{(\ell)}} - 1\right).$$

Next, ξ_2 can be decomposed as $\xi_2 = \xi_2^{\dagger}\xi_2^{\ddagger}$, where

$$\xi_2^{\dagger} = \frac{\prod_{\ell=1}^{m}\left|\widehat{\boldsymbol{\Delta}}_{22}^{(\ell)}\right|^{\frac{N_1^{(\ell)}+N_2^{(\ell)}}{2}}}{\left|\frac{1}{N_1+N_2}(\boldsymbol{V}_{p_2}+\boldsymbol{B}_{p_2})\right|^{\frac{N_1+N_2}{2}}},\quad \xi_2^{\ddagger} = \frac{\left|\frac{1}{N_1+N_2}(\boldsymbol{V}_{p_2}+\boldsymbol{B}_{p_2})\right|^{\frac{N_1+N_2}{2}}}{\left|\widetilde{\boldsymbol{\Delta}}_{22}\right|^{\frac{N_1+N_2}{2}}},$$

and

$$\boldsymbol{V}_{p_2}^{(\ell)} = \sum_{j=1}^{N_1^{(\ell)}+N_2^{(\ell)}}(\boldsymbol{x}_{2j}^{(\ell)} - \widehat{\boldsymbol{\Delta}}_{21}^{(\ell)}\boldsymbol{x}_{1j}^{(\ell)} - \widehat{\boldsymbol{\eta}}_2^{(\ell)})(\boldsymbol{x}_{2j}^{(\ell)} - \widehat{\boldsymbol{\Delta}}_{21}^{(\ell)}\boldsymbol{x}_{1j}^{(\ell)} - \widehat{\boldsymbol{\eta}}_2^{(\ell)})',\quad \boldsymbol{V}_{p_2} = \sum_{\ell=1}^{m}\boldsymbol{V}_{p_2}^{(\ell)},$$
$$\boldsymbol{B}_{p_2} = \sum_{\ell=1}^{m}(N_1^{(\ell)}+N_2^{(\ell)})(\widehat{\boldsymbol{\eta}}_2^{(\ell)} - \widetilde{\boldsymbol{\eta}}_2)(\widehat{\boldsymbol{\eta}}_2^{(\ell)} - \widetilde{\boldsymbol{\eta}}_2)'.$$

Because $\xi_2^{\dagger}$ is of the form of LR for H_{01} in the case of complete data, the modified LRT statistic $-2\rho_{\xi_2}\log\xi_2^{\dagger}$ is given, where

$$\rho_{\xi_2} = 1 - \frac{2p_2^2 + 9p_2 + 11}{6(N_1+N_2)(p_2+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N_1+N_2}{N_1^{(\ell)}+N_2^{(\ell)}} - 1\right).$$

Similarly, ξ_3 can be decomposed as $\xi_3 = \xi_3^{\dagger}\xi_3^{\ddagger}$, where

$$\xi_3^{\dagger} = \frac{\prod_{\ell=1}^{m}\left|\widehat{\boldsymbol{\Delta}}_{33}^{(\ell)}\right|^{\frac{N_1^{(\ell)}}{2}}}{\left|\frac{1}{N_1}(\boldsymbol{V}_{p_3}+\boldsymbol{B}_{p_3})\right|^{\frac{N_1}{2}}},\quad \xi_3^{\ddagger} = \frac{\left|\frac{1}{N_1}(\boldsymbol{V}_{p_3}+\boldsymbol{B}_{p_3})\right|^{\frac{N_1}{2}}}{\left|\widetilde{\boldsymbol{\Delta}}_{33}\right|^{\frac{N_1}{2}}},$$

and

$$\boldsymbol{V}_{p_3}^{(\ell)} = \sum_{j=1}^{N_1^{(\ell)}}(\boldsymbol{x}_{3j}^{(\ell)} - \widehat{\boldsymbol{\Delta}}_{3(12)}^{(\ell)}\boldsymbol{x}_{(12)j}^{(\ell)} - \widehat{\boldsymbol{\eta}}_3^{(\ell)})(\boldsymbol{x}_{3j}^{(\ell)} - \widehat{\boldsymbol{\Delta}}_{3(12)}^{(\ell)}\boldsymbol{x}_{(12)j}^{(\ell)} - \widehat{\boldsymbol{\eta}}_3^{(\ell)})',\quad \boldsymbol{V}_{p_3} = \sum_{\ell=1}^{m}\boldsymbol{V}_{p_3}^{(\ell)},$$
$$\boldsymbol{B}_{p_3} = \sum_{\ell=1}^{m}N_1^{(\ell)}(\widehat{\boldsymbol{\eta}}_3^{(\ell)} - \widetilde{\boldsymbol{\eta}}_3)(\widehat{\boldsymbol{\eta}}_3^{(\ell)} - \widetilde{\boldsymbol{\eta}}_3)'.$$

Because $\xi_3^{\dagger}$ is of the form of LR for H_{01} in the case of complete data, the modified LRT statistic $-2\rho_{\xi_3}\log\xi_3^{\dagger}$ is given, where

$$\rho_{\xi_3} = 1 - \frac{2p_3^2 + 9p_3 + 11}{6N_1(p_3+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N_1}{N_1^{(\ell)}} - 1\right).$$

Therefore, the decomposed form of $\lambda_m = \xi_1\xi_2\xi_3$ is $\lambda_m = \xi_1\xi_2^{\dagger}\xi_2^{\ddagger}\xi_3^{\dagger}\xi_3^{\ddagger}$. We give a correction only for $\xi_1, \xi_2^{\dagger}$, and $\xi_3^{\dagger}$. Thus, we give the test statistic $-2\log\tau_m$ for improving the accuracy of the χ^2 approximation, where

$$\tau_m = (\xi_1)^{\rho_{\xi_1}}(\xi_2^{\dagger})^{\rho_{\xi_2}}\xi_2^{\ddagger}(\xi_3^{\dagger})^{\rho_{\xi_3}}\xi_3^{\ddagger}.$$

Next, using LR of the covariance test with complete data from the previous section, we propose a test statistic by decomposing the LR λ_m with three-step monotone missing data. Let

$$\xi_{11}^* = \frac{\prod_{\ell=1}^m \left|\frac{1}{n^{(\ell)}}\boldsymbol{V}_{p_1}^{(\ell)}\right|^{\frac{n^{(\ell)}}{2}}}{\left|\frac{1}{n}\boldsymbol{V}_{p_1}\right|^{\frac{n}{2}}},\ \xi_{21}^* = \frac{\prod_{\ell=1}^m \left|\frac{1}{n_1^{(\ell)}+n_2^{(\ell)}}\boldsymbol{V}_{p_2}^{(\ell)}\right|^{\frac{n_1^{(\ell)}+n_2^{(\ell)}}{2}}}{\left|\frac{1}{n_1+n_2}\boldsymbol{V}_{p_2}\right|^{\frac{n_1+n_2}{2}}},\ \xi_{31}^* = \frac{\prod_{\ell=1}^m \left|\frac{1}{n_1^{(\ell)}}\boldsymbol{V}_{p_3}^{(\ell)}\right|^{\frac{n_1^{(\ell)}}{2}}}{\left|\frac{1}{n_1}\boldsymbol{V}_{p_3}\right|^{\frac{n_1}{2}}},$$

where

$$\boldsymbol{V}_{p_1}^{(\ell)} = \sum_{j=1}^{N^{(\ell)}}(\boldsymbol{x}_{1j}^{(\ell)} - \widehat{\boldsymbol{\eta}}_1^{(\ell)})(\boldsymbol{x}_{1j}^{(\ell)} - \widehat{\boldsymbol{\eta}}_1^{(\ell)})',\quad \boldsymbol{V}_{p_1} = \sum_{\ell=1}^m \boldsymbol{V}_{p_1}^{(\ell)},$$
$$n_1^{(\ell)} = N_1^{(\ell)} - (p_1+p_2) - 1,\quad n_1 = \sum_{\ell=1}^m n_1^{(\ell)},$$
$$n_1^{(\ell)} + n_2^{(\ell)} = N_1^{(\ell)} + N_2^{(\ell)} - p_1 - 1,\quad n_1+n_2 = \sum_{\ell=1}^m (n_1^{(\ell)} + n_2^{(\ell)}).$$

Because ξ_{11}^*, ξ_{21}^*, and ξ_{31}^* are of the form of LR for H_{02} in the case of complete data, the modified LRT statistics $-2\rho_{\xi_{11}^*}\log\xi_{11}^*, -2\rho_{\xi_{21}^*}\log\xi_{21}^*$, and $-2\rho_{\xi_{31}^*}\log\xi_{31}^*$ are given, where

$$\rho_{\xi_{11}^*} = 1 - \frac{2p_1^2+3p_1-1}{6(p_1+1)(m-1)}\left(\sum_{\ell=1}^m \frac{1}{n^{(\ell)}} - \frac{1}{n}\right),$$
$$\rho_{\xi_{21}^*} = 1 - \frac{2p_2^2+3p_2-1}{6(p_2+1)(m-1)}\left(\sum_{\ell=1}^m \frac{1}{n_1^{(\ell)}+n_2^{(\ell)}} - \frac{1}{n_1+n_2}\right),$$
$$\rho_{\xi_{31}^*} = 1 - \frac{2p_3^2+3p_3-1}{6(p_3+1)(m-1)}\left(\sum_{\ell=1}^m \frac{1}{n_1^{(\ell)}} - \frac{1}{n_1}\right).$$

Therefore, we propose the test statistic $-2\log\phi_m$ for improving the accuracy of the χ^2 approximation, where

$$\phi_m = (\xi_{11}^*)^{\rho_{\xi_{11}^*}}(\xi_{21}^*)^{\rho_{\xi_{21}^*}}(\xi_{31}^*)^{\rho_{\xi_{31}^*}}\frac{\lambda}{\xi_{11}\xi_{21}\xi_{31}},$$

and

$$\xi_{11} = \frac{\prod_{\ell=1}^m \left|\frac{1}{N^{(\ell)}}\boldsymbol{V}_{p_1}^{(\ell)}\right|^{\frac{N^{(\ell)}}{2}}}{\left|\frac{1}{N}\boldsymbol{V}_{p_1}\right|^{\frac{N}{2}}},\ \xi_{21} = \frac{\prod_{\ell=1}^m \left|\frac{1}{N_1^{(\ell)}+N_2^{(\ell)}}\boldsymbol{V}_{p_2}^{(\ell)}\right|^{\frac{N_1^{(\ell)}+N_2^{(\ell)}}{2}}}{\left|\frac{1}{N_1+N_2}\boldsymbol{V}_{p_2}\right|^{\frac{N_1+N_2}{2}}},\ \xi_{31} = \frac{\prod_{\ell=1}^m \left|\frac{1}{N_1^{(\ell)}}\boldsymbol{V}_{p_3}^{(\ell)}\right|^{\frac{N_1^{(\ell)}}{2}}}{\left|\frac{1}{N_1}\boldsymbol{V}_{p_3}\right|^{\frac{N_1}{2}}}.$$

Next, we propose the test statistic $-2\rho_{L_m}\log\lambda_m$ via linear interpolation, where

$$\rho_{L_m} = \left\{1 - \frac{(p_1+p_2)N_2 + p_1N_3}{p(N_2+N_3)}\right\}\rho_{N_1,m} + \frac{(p_1+p_2)N_2 + p_1N_3}{p(N_2+N_3)}\rho_{N,m}$$

and

$$\rho_{N,m} = 1 - \frac{2p^2+9p+11}{6N(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N}{N^{(\ell)}} - 1\right), \rho_{N_1,m} = 1 - \frac{2p^2+9p+11}{6N_1(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{N_1}{N_1^{(\ell)}} - 1\right).$$

5 Asymptotic expansion approximation

In this section, we give an approximate upper 100α percentile of $-2\log\lambda_m$ with three-step monotone missing data. The upper 100α percentile of $-2\log\lambda_{S_m}$ can be expanded as

$$\begin{aligned} q^*_{c_m}(\alpha) &= \chi^2_{f_m}(\alpha) + \frac{\nu}{N}\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right)\chi^2_{f_m}(\alpha) \\ &\quad + \frac{1}{N^2}\chi^2_{f_m}(\alpha)\left\{\nu^2\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right)^2 + \frac{2\gamma_1}{f_m} + \frac{2\gamma_1}{f_m(f_m+2)}\chi^2_{f_m}(\alpha)\right\} + O(N^{-3}), \end{aligned}$$

where

$$\nu = \frac{2p^2+9p+11}{6(p+3)(m-1)},\ k_1^{(\ell)} = \frac{N^{(\ell)}}{N},$$

$$\gamma_1 = \frac{1}{288}\left[6p(p+1)(p+2)(p+3)\left(\sum_{\ell=1}^{m}\frac{1}{\{k_1^{(\ell)}\}^2} - 1\right) - \frac{(2p^2+9p+11)^2(2p-1)}{p(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right)^2\right],$$

where $\chi^2_{f_m}(\alpha)$ is the upper 100α percentile of the χ^2 distribution with f_m degrees of freedom (Hosoya and Seo (2016)). Based on linear interpolation and letting $q^*_m(\alpha)$ be the upper 100α percentile of $-2\log\lambda_m$, the following can be obtained:

$$q^*_m(\alpha) = \left\{1 - \frac{(p_1+p_2)N_2 + p_1N_3}{p(N_2+N_3)}\right\}q_{N_1,m}(\alpha) + \frac{(p_1+p_2)N_2 + p_1N_3}{p(N_2+N_3)}q_{N,m}(\alpha)$$

where

$$q_{N,m}(\alpha) = \chi^2_{f_m}(\alpha) + \frac{\nu}{N}\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right)\chi^2_{f_m}(\alpha) + \frac{1}{N^2}\chi^2_{f_m}(\alpha)\left\{\nu^2\left(\sum_{\ell=1}^{m}\frac{1}{k_1^{(\ell)}} - 1\right)^2 + \frac{2\gamma_1}{f_m} + \frac{2\gamma_1}{f_m(f_m+2)}\chi^2_{f_m}(\alpha)\right\},$$

$$q_{N_1,m}(\alpha) = \chi^2_{f_m}(\alpha) + \frac{\nu}{N_1}\left(\sum_{\ell=1}^{m}\frac{1}{k_2^{(\ell)}} - 1\right)\chi^2_{f_m}(\alpha) + \frac{1}{N_1^2}\chi^2_{f_m}(\alpha)\left\{\nu^2\left(\sum_{\ell=1}^{m}\frac{1}{k_2^{(\ell)}} - 1\right)^2 + \frac{2\gamma_2}{f_m} + \frac{2\gamma_2}{f_m(f_m+2)}\chi^2_{f_m}(\alpha)\right\},$$

$$k_2^{(\ell)} = \frac{N_1^{(\ell)}}{N_1},$$

$$\gamma_2 = \frac{1}{288}\left[6p(p+1)(p+2)(p+3)\left(\sum_{\ell=1}^{m}\frac{1}{\{k_2^{(\ell)}\}^2} - 1\right) - \frac{(2p^2+9p+11)^2(2p-1)}{p(p+3)(m-1)}\left(\sum_{\ell=1}^{m}\frac{1}{k_2^{(\ell)}} - 1\right)^2\right].$$

6 Simulation studies

We evaluate the accuracy and the asymptotic behavior of the χ^2 approximation via Monte Carlo simulation (10^6 runs). Now let

$$\begin{aligned}
\alpha_m &= \Pr\{-2\log\lambda_m > \chi^2_{f_m}(\alpha)\},\\
\alpha_{\rho_{L_m}} &= \Pr\{-2\rho_{L_m}\log\lambda_m > \chi^2_{f_m}(\alpha)\},\\
\alpha_{\tau_m} &= \Pr\{-2\log\tau_m > \chi^2_{f_m}(\alpha)\},\\
\alpha_{\phi_m} &= \Pr\{-2\log\phi_m > \chi^2_{f_m}(\alpha)\},\\
\alpha_{q^*_m} &= \Pr\{-2\log\lambda_m > q^*_m(\alpha)\}.
\end{aligned}$$

In Tables 1-9, we provide the simulated upper 100α percentiles of $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m$, $-2\log\tau_m$, and $-2\log\phi_m$ and the approximate upper percentiles of $-2\log\lambda_m(q^*_m(\alpha))$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q^*_m}$; $\alpha = 0.05, 0.01$;

$$(N_1^{(\ell)}, N_2^{(\ell)}, N_3^{(\ell)}) = \begin{cases} (t,t,t), \\ (t,t/2,t/2), \quad t = 20, 40, 80, 160, 320, \\ (t,2t,2t), \end{cases}$$

where (p_1, p_2, p_3) in Tables 1-3, 4-6, and 7-9 are $(8,4,2), (4,4,4), (2,4,8)$, respectively.

The simulated values are closer to the upper percentile of the χ^2 distribution when the sample size increases. However, compared with one sample case (Sakai, Yagi, and Seo (2021)), the accuracy of the simulated values is not very good, even if the sample size is quite large. In addition, by comparing the Type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, the accuracy of the approximate percentile $(q^*_m(\alpha))$ is the best.

Table 1: Simulated values for $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m, -2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q_m^*}$ for $(p_1, p_2, p_3) = (8, 4, 2), m = 2$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)} = N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q_m^*(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q_m^*}$
					$\alpha = 0.05$						
20	20	216.28	172.21	205.92	198.27	192.15	.890	.347	.814	.735	**.222**
40	40	168.46	151.29	164.94	162.51	164.54	.318	.092	.261	.224	**.073**
80	80	155.45	147.53	153.83	152.80	153.94	.135	.063	.118	.107	**.059**
160	160	150.15	146.32	149.36	148.86	149.43	.083	.055	.077	.073	**.054**
320	320	147.74	145.86	147.36	147.12	147.38	.065	**.052**	.062	.060	**.052**
20	10	226.15	169.60	212.16	203.56	203.51	.942	.314	.869	.795	**.199**
40	20	172.28	150.74	167.29	164.52	169.07	.383	.088	.299	.254	**.068**
80	40	157.18	147.35	154.88	153.68	155.91	.156	.062	.129	.116	**.057**
160	80	151.01	146.29	149.89	149.32	150.35	.090	.055	.081	.077	**.054**
320	160	148.13	145.81	147.58	147.31	147.82	.067	**.052**	.063	.062	**.052**
20	40	208.35	173.62	200.62	193.78	184.39	.826	.364	.754	.675	**.228**
40	80	165.04	151.29	162.71	160.63	161.25	.263	.093	.228	.198	**.073**
80	160	153.91	147.49	152.86	151.98	152.44	.118	.063	.107	.099	**.059**
160	320	149.40	146.29	148.90	148.48	148.72	.077	.055	.073	.070	**.054**
320	640	147.37	145.83	147.11	146.91	147.03	.062	**.052**	.060	.059	**.052**
					$\alpha = 0.01$						
20	20	236.59	188.39	225.14	216.80	208.90	.746	.161	.624	.518	**.083**
40	40	182.90	164.27	179.09	176.45	178.61	.132	.023	.099	.080	**.017**
80	80	168.61	160.02	166.87	165.71	167.02	.038	.014	.032	.028	**.012**
160	160	163.05	158.90	162.21	161.66	162.11	.020	.012	.018	.017	**.011**
320	320	160.38	158.34	159.97	159.71	159.88	.014	**.011**	.013	.013	**.011**
20	10	247.21	185.39	231.92	222.51	221.35	.842	.137	.708	.596	**.069**
40	20	187.03	163.65	181.65	178.60	183.55	.174	.022	.120	.095	**.015**
80	40	170.51	159.85	168.02	166.73	169.17	.047	.013	.036	.031	**.012**
160	80	163.84	158.72	162.62	162.01	163.11	.022	**.011**	.019	.018	**.011**
320	160	160.66	158.15	160.07	159.77	160.36	.015	.011	.014	.013	**.010**
20	40	228.21	190.17	219.54	211.89	200.42	.644	.174	.545	.448	**.088**
40	80	179.16	164.23	176.64	174.36	175.03	.100	.023	.081	.067	**.016**
80	160	167.00	160.04	165.86	164.88	165.40	.032	.014	.028	.025	**.012**
160	320	162.15	158.77	161.60	161.14	161.34	.018	**.011**	.017	.016	**.011**
320	640	159.86	158.20	159.59	159.37	159.51	.013	**.011**	.013	.013	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q_m^*}$ of each row is in bold. $\chi^2_{f_m}(0.05) = 145.46$, $\chi^2_{f_m}(0.01) = 157.80$, $f_m = 119$.

Table 2: Simulated values for $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m, -2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q^*_m}$ for $(p_1, p_2, p_3) = (8, 4, 2), m = 3$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)} = N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q^*_m(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q^*_m}$
$\alpha = 0.05$											
20	20	386.37	316.39	369.89	353.72	350.57	.966	.428	.920	.835	**.266**
40	40	312.64	284.33	306.77	301.86	306.33	.415	.103	.337	.274	**.079**
80	80	291.58	278.38	288.88	286.76	289.05	.163	.066	.139	.121	**.061**
160	160	282.93	276.53	281.62	280.65	281.62	.094	.057	.085	.079	**.056**
320	320	278.88	275.72	278.23	277.75	278.20	.069	**.053**	.065	.063	**.053**
20	10	402.83	313.29	380.34	361.95	368.90	.988	.394	.956	.887	**.238**
40	20	319.14	283.67	310.81	305.06	313.75	.503	.098	.389	.314	**.073**
80	40	294.54	278.17	290.68	288.21	292.32	.192	.065	.155	.133	**.059**
160	80	284.25	276.35	282.39	281.24	283.14	.103	.056	.090	.082	**.055**
320	160	279.45	275.56	278.53	277.97	278.94	.072	.053	.067	.064	**.052**
20	40	373.02	317.75	360.99	346.78	337.93	.928	.441	.875	.778	**.269**
40	80	307.05	284.30	303.17	298.99	300.91	.339	.102	.289	.239	**.078**
80	160	289.08	278.37	287.31	285.51	286.56	.141	.066	.126	.112	**.061**
160	320	281.56	276.34	280.70	279.87	280.43	.085	.056	.079	.074	**.055**
320	640	278.30	275.72	277.88	277.47	277.62	.066	**.053**	.063	.061	**.053**
$\alpha = 0.01$											
20	20	411.79	337.21	394.14	376.70	372.32	.893	.215	.792	.645	**.105**
40	40	331.74	301.70	325.55	320.27	325.04	.194	.026	.141	.105	**.018**
80	80	309.18	295.18	306.31	304.07	306.62	.050	.014	.040	.033	**.013**
160	160	300.07	293.28	298.68	297.66	298.71	.023	**.012**	.020	.018	**.012**
320	320	295.59	292.24	294.91	294.40	295.09	.015	**.011**	.014	.013	**.011**
20	10	429.17	333.78	405.32	385.45	391.89	.954	.188	.867	.731	**.088**
40	20	338.64	301.01	329.83	323.77	332.94	.262	.025	.177	.128	**.017**
80	40	312.52	295.15	308.43	305.81	310.09	.062	.014	.046	.038	**.013**
160	80	301.40	293.02	299.42	298.23	300.33	.026	.012	.022	.019	**.011**
320	160	296.42	292.30	295.44	294.85	295.87	.016	**.011**	.015	.014	**.011**
20	40	397.83	338.88	384.80	369.44	358.85	.809	.226	.712	.566	**.108**
40	80	325.97	301.82	321.85	317.41	319.27	.143	.026	.113	.086	**.018**
80	160	306.52	295.17	304.66	302.77	303.98	.040	.014	.034	.029	.013
160	320	298.65	293.12	297.77	296.88	297.45	.020	.012	.018	.017	**.011**
320	640	295.08	292.35	294.64	294.19	294.47	.014	**.011**	.014	.013	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q^*_m}$ of each row is in bold. $\chi^2_{f_m}(0.05) = 274.99$, $\chi^2_{f_m}(0.01) = 291.68$, $f_m = 238$.

Table 3: Simulated values for $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m, -2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q^*_m}$ for $(p_1, p_2, p_3) = (8, 4, 2), m = 4$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)} = N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q^*_m(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q^*_m}$
					$\alpha = 0.05$						
20	20	551.12	457.54	529.02	504.47	503.92	.989	.500	.965	.894	**.308**
40	40	453.34	414.86	445.38	437.94	444.59	.491	.112	.398	.314	**.084**
80	80	424.69	406.66	420.98	417.79	421.22	.187	.069	.156	.133	**.063**
160	160	412.83	404.07	411.03	409.54	411.12	.100	.057	.090	.082	**.056**
320	320	407.35	403.03	406.46	405.74	406.46	.072	.054	.068	.064	**.053**
20	10	573.44	453.95	543.20	515.07	528.56	.997	.467	.984	.935	**.277**
40	20	462.11	413.97	450.73	442.04	454.63	.592	.107	.462	.361	**.078**
80	40	428.70	406.36	423.44	419.74	425.67	.223	.067	.176	.146	**.061**
160	80	414.75	403.95	412.20	410.46	413.20	.112	.057	.097	.087	**.056**
320	160	408.26	402.95	407.01	406.15	407.46	.076	**.053**	.070	.066	**.053**
20	40	533.12	459.06	517.10	495.51	486.83	.969	.512	.935	.846	**.311**
40	80	445.59	414.64	440.30	433.98	437.22	.401	.111	.341	.273	**.083**
80	160	421.29	406.66	418.87	416.18	417.83	.158	.069	.140	.121	**.063**
160	320	411.20	404.06	410.03	408.77	409.49	.091	.058	.085	.078	**.056**
320	640	406.47	402.94	405.89	405.28	405.66	.068	**.053**	.065	.062	**.053**
					$\alpha = 0.01$						
20	20	580.70	482.10	557.39	531.05	529.52	.955	.270	.887	.739	**.129**
40	40	476.28	435.85	467.89	459.97	466.86	.251	.030	.181	.127	**.020**
80	80	446.06	427.12	442.17	438.80	442.24	.059	.015	.046	.037	**.014**
160	160	433.53	424.33	431.65	430.07	431.60	.025	**.012**	.022	.020	**.012**
320	320	427.68	423.14	426.76	425.97	426.71	.016	**.011**	.015	.014	**.011**
20	10	604.22	478.32	572.34	542.27	555.52	.986	.241	.940	.818	**.109**
40	20	485.14	434.59	473.26	464.13	477.43	.340	.028	.227	.156	**.018**
80	40	450.12	426.67	444.61	440.75	446.92	.077	.015	.055	.042	**.013**
160	80	435.41	424.07	432.73	430.91	433.79	.030	**.012**	.024	.021	**.011**
320	160	428.58	423.00	427.26	426.38	427.76	.017	**.011**	.016	.014	**.011**
20	40	562.28	484.17	545.13	521.88	511.51	.899	.282	.821	.659	**.132**
40	80	468.11	435.60	462.52	455.79	459.10	.183	.030	.143	.103	**.020**
80	160	442.23	426.88	439.71	436.90	438.67	.047	.015	.040	.033	**.014**
160	320	431.91	424.41	430.69	429.34	429.90	.022	**.012**	.020	.018	**.012**
320	640	426.72	423.02	426.10	425.48	425.87	.015	**.011**	.014	.013	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q^*_m}$ of each row is in bold. $\chi^2_{f_m}(0.05) = 402.06$, $\chi^2_{f_m}(0.01) = 422.09$, $f_m = 357$.

Table 4: Simulated values for $-2\log\lambda_m$, $-2\rho_{L_m}\log\lambda_m$, $-2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q_m^*}$ for $(p_1, p_2, p_3) = (4, 4, 4), m = 2$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)} = N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q_m^*(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q_m^*}$
				$\alpha = 0.05$							
20	20	169.48	131.16	159.56	149.79	153.97	.842	.260	.740	.591	**.162**
40	40	133.02	117.98	129.97	126.58	129.75	.304	.090	.252	.197	**.072**
80	80	121.89	115.00	120.59	119.10	120.49	.133	.063	.117	.101	**.059**
160	160	117.27	113.95	116.66	115.96	116.58	.082	.055	.077	.071	**.054**
320	320	115.19	113.56	114.89	114.54	114.80	.065	.053	.062	.060	**.052**
20	10	172.43	128.56	161.33	151.16	159.08	.868	.220	.765	.617	**.138**
40	20	134.31	117.22	130.76	127.22	131.82	.329	.082	.266	.208	**.066**
80	40	122.47	114.68	120.96	119.42	121.41	.140	.061	.121	.104	**.057**
160	80	117.60	113.86	116.88	116.14	117.01	.085	.055	.079	.072	**.054**
320	160	115.29	113.46	114.94	114.57	115.01	.066	**.052**	.063	.060	**.052**
20	40	166.93	132.96	157.96	148.44	150.44	.815	.288	.716	.567	**.174**
40	80	131.87	118.45	129.21	125.94	128.22	.284	.093	.239	.188	**.074**
80	160	121.37	115.20	120.28	118.85	119.79	.126	.065	.113	.098	**.060**
160	320	117.06	114.08	116.55	115.87	116.25	.080	.057	.076	.070	**.055**
320	640	114.94	113.48	114.70	114.37	114.64	.063	**.052**	.061	.058	**.052**
				$\alpha = 0.01$							
20	20	186.80	144.56	175.63	164.85	169.37	.663	.103	.520	.355	**.050**
40	40	145.97	129.46	142.58	138.91	142.45	.125	.022	.095	.067	**.016**
80	80	133.76	126.20	132.33	130.70	132.21	.038	.014	.032	.026	**.013**
160	160	128.55	124.92	127.89	127.11	127.89	.020	**.011**	.018	.016	**.011**
320	320	126.42	124.63	126.10	125.73	125.94	.014	**.011**	.013	.013	**.011**
20	10	189.82	141.53	177.37	166.47	175.03	.705	.080	.552	.381	**.040**
40	20	147.47	128.71	143.57	139.72	144.74	.140	.020	.102	.071	**.014**
80	40	134.30	125.75	132.63	130.93	133.22	.041	.013	.034	.027	**.012**
160	80	128.97	124.86	128.18	127.37	128.37	.021	**.011**	.019	.017	**.011**
320	160	126.38	124.37	125.99	125.61	126.17	.014	**.010**	.014	.013	**.010**
20	40	183.96	146.53	173.80	163.41	165.47	.623	.120	.490	.332	**.057**
40	80	144.67	129.95	141.74	138.18	140.77	.112	.024	.087	.062	**.017**
80	160	133.14	126.37	131.92	130.37	131.44	.035	.014	.031	.025	**.013**
160	320	128.46	125.19	127.90	127.16	127.52	.019	.012	.018	.016	**.011**
320	640	126.08	124.47	125.81	125.45	125.76	.014	.011	.013	.012	**.010**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q_m^*}$ of each row is in bold. $\chi^2_{f_m}(0.05) = 113.15$, $\chi^2_{f_m}(0.01) = 124.12$, $f_m = 90$.

Table 5: Simulated values for $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m, -2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q_m^*}$ for $(p_1,p_2,p_3)=(4,4,4), m=3$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)}=N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q_m^*(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q_m^*}$
					$\alpha=0.05$						
20	20	301.38	240.80	285.82	265.97	277.83	.941	.325	.869	.684	**.192**
40	40	244.74	220.15	239.73	233.05	239.35	.397	.099	.324	.233	**.077**
80	80	226.75	215.36	224.60	221.67	224.40	.160	.066	.139	.112	**.062**
160	160	219.18	213.67	218.17	216.79	217.99	.092	.057	.085	.075	**.056**
320	320	215.62	212.91	215.13	214.46	215.06	.068	**.053**	.065	.061	**.053**
20	10	306.27	237.02	288.81	268.26	286.03	.957	.275	.890	.712	**.161**
40	20	246.91	219.00	241.11	234.07	242.74	.429	.090	.343	.245	**.070**
80	40	227.78	214.90	225.26	222.16	225.91	.171	.064	.145	.116	**.059**
160	80	219.62	213.42	218.44	216.98	218.71	.096	.056	.087	.077	**.055**
320	160	215.88	212.82	215.30	214.59	215.40	.070	.053	.066	.062	**.052**
20	40	297.14	243.39	283.18	264.08	272.10	.923	.358	.848	.657	**.207**
40	80	242.78	220.82	238.46	232.08	236.85	.367	.105	.306	.221	**.081**
80	160	225.76	215.55	223.94	221.16	223.24	.151	.068	.133	.108	**.063**
160	320	218.69	213.75	217.84	216.53	217.44	.088	.057	.082	.074	**.056**
320	640	215.41	212.98	215.00	214.36	214.79	.067	**.053**	.064	.061	**.053**
					$\alpha=0.01$						
20	20	323.22	258.25	306.29	284.86	297.63	.834	.139	.698	.445	**.064**
40	40	261.93	235.61	256.52	249.31	256.11	.182	.025	.134	.083	**.018**
80	80	242.55	230.36	240.26	237.09	240.02	.049	.015	.040	.030	**.013**
160	160	234.42	228.53	233.36	231.86	233.15	.023	**.012**	.020	.017	**.012**
320	320	230.61	227.71	230.09	229.37	230.01	.015	**.011**	.014	.013	**.011**
20	10	328.35	254.10	309.37	287.38	306.45	.870	.109	.734	.477	**.049**
40	20	264.16	234.30	257.95	250.46	259.74	.205	.022	.146	.089	**.016**
80	40	243.51	229.75	240.82	237.54	241.64	.053	.014	.042	.031	**.012**
160	80	234.91	228.27	233.63	232.06	233.91	.024	.012	.021	.018	**.011**
320	160	230.84	227.58	230.22	229.46	230.37	.016	**.011**	.014	.013	**.011**
20	40	318.91	261.22	303.60	283.02	291.47	.798	.161	.664	.415	**.072**
40	80	259.67	236.19	255.06	248.35	253.43	.163	.027	.123	.077	**.019**
80	160	241.46	230.54	239.52	236.61	238.78	.044	.015	.037	.028	**.014**
160	320	233.96	228.67	233.04	231.63	232.55	.021	**.012**	.020	.017	**.012**
320	640	230.35	227.75	229.92	229.26	229.71	.015	**.011**	.014	.013	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q_m^*}$ of each row is in bold. $\chi^2_{f_m}(0.05)=212.30$, $\chi^2_{f_m}(0.01)=227.06$, $f_m=180$.

Table 6: Simulated values for $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m, -2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q_m^*}$ for $(p_1,p_2,p_3) = (4,4,4), m = 4$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)}=N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q_m^*(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q_m^*}$
					$\alpha = 0.05$						
20	20	428.72	347.94	408.06	378.36	397.27	.977	.384	.933	.750	**.222**
40	40	353.35	320.06	346.59	336.56	345.88	.471	.108	.384	.261	**.083**
80	80	329.04	313.54	326.12	321.74	325.75	.183	.069	.156	.121	**.064**
160	160	318.65	311.14	317.27	315.20	317.07	.099	.058	.091	.078	**.056**
320	320	313.89	310.19	313.22	312.20	313.08	.071	.054	.068	.063	**.053**
20	10	435.55	343.22	412.30	381.39	408.27	.985	.329	.947	.777	**.186**
40	20	356.23	318.48	348.37	337.85	350.45	.508	.098	.407	.276	**.074**
80	40	330.33	312.83	326.91	322.27	327.80	.196	.066	.163	.125	**.060**
160	80	319.30	310.84	317.68	315.50	318.04	.104	.056	.093	.080	**.055**
320	160	314.26	310.10	313.47	312.41	313.55	.073	**.053**	.069	.064	**.053**
20	40	422.86	351.15	404.36	375.76	389.55	.967	.422	.918	.724	**.241**
40	80	350.62	320.89	344.80	335.29	342.50	.436	.114	.362	.248	**.086**
80	160	327.62	313.73	325.17	321.02	324.18	.170	.070	.148	.115	**.065**
160	320	317.98	311.24	316.83	314.88	316.32	.095	.058	.088	.077	**.057**
320	640	313.58	310.26	313.03	312.08	312.71	.070	.054	.067	.062	**.053**
					$\alpha = 0.01$						
20	20	454.38	368.76	432.11	400.68	420.46	.919	.177	.813	.520	**.078**
40	40	373.57	338.38	366.37	355.78	365.75	.235	.028	.172	.097	**.019**
80	80	347.92	331.53	344.82	340.20	344.38	.058	.015	.047	.033	**.014**
160	160	336.89	328.95	335.41	333.21	335.18	.025	**.012**	.022	.018	**.012**
320	320	331.74	327.83	331.03	329.95	330.96	.016	**.011**	.015	.014	**.011**
20	10	461.56	363.72	436.65	403.75	432.15	.943	.140	.843	.556	**.060**
40	20	376.70	336.77	368.39	357.35	370.60	.266	.025	.187	.105	**.017**
80	40	349.20	330.69	345.57	340.70	346.55	.064	.014	.050	.034	**.013**
160	80	337.52	328.57	335.80	333.51	336.21	.027	.012	.023	.019	**.011**
320	160	332.14	327.74	331.31	330.19	331.45	.017	**.011**	.015	.014	**.011**
20	40	448.33	372.30	428.28	397.83	412.26	.893	.205	.782	.487	**.088**
40	80	370.77	339.33	364.55	354.47	362.16	.209	.031	.157	.090	**.021**
80	160	346.48	331.79	343.85	339.51	342.71	.053	.016	.043	.031	**.014**
160	320	336.06	328.93	334.83	332.79	334.38	.024	**.012**	.021	.017	**.012**
320	640	331.41	327.89	330.80	329.79	330.56	.016	**.011**	.015	.013	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q_m^*}$ of each row is in bold. $\chi^2_{f_m}(0.05)=309.33$, $\chi^2_{f_m}(0.01)=326.98$, $f_m = 270$.

Table 7: Simulated values for $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m, -2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q_m^*}$ for $(p_1,p_2,p_3)=(2,4,8), m=2$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)}=N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q_m^*(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q_m^*}$
					$\alpha=0.05$						
20	20	254.40	174.29	219.28	192.99	226.10	.992	.393	.926	.697	**.229**
40	40	180.69	152.24	170.58	162.66	177.07	.527	.101	.355	.227	**.069**
80	80	160.66	148.01	156.47	153.06	159.09	.200	.067	.146	.109	**.059**
160	160	152.50	146.49	150.57	148.97	151.73	.104	.056	.087	.074	**.054**
320	320	148.89	145.96	147.95	147.18	148.46	.073	.053	.066	.061	**.052**
20	10	255.63	170.40	219.86	193.61	230.65	.993	.332	.930	.705	**.197**
40	20	181.17	150.97	170.71	162.73	178.88	.537	.090	.360	.230	**.062**
80	40	160.90	147.49	156.59	153.17	159.88	.204	.063	.148	.110	**.056**
160	80	152.65	146.28	150.66	149.04	152.10	.106	.055	.088	.075	**.053**
320	160	148.97	145.86	148.00	147.22	148.64	.074	**.052**	.066	.061	**.052**
20	40	253.18	177.21	218.69	192.51	223.00	.991	.442	.922	.690	**.251**
40	80	180.07	153.05	170.17	162.28	175.76	.517	.109	.349	.222	**.074**
80	160	160.36	148.33	156.29	152.91	158.49	.197	.069	.145	.108	**.061**
160	320	152.41	146.70	150.54	148.95	151.45	.103	.058	.086	.074	**.056**
320	640	148.85	146.06	147.95	147.19	148.32	.073	.054	.066	.061	**.053**
					$\alpha=0.01$						
20	20	236.59	188.39	225.14	216.80	208.90	.746	.161	.624	.518	**.083**
40	40	182.90	164.27	179.09	176.45	178.61	.132	.023	.099	.080	**.017**
80	80	168.61	160.02	166.87	165.71	167.02	.038	.014	.032	.028	**.012**
160	160	163.05	158.90	162.21	161.66	162.11	.020	.012	.018	.017	**.011**
320	320	160.38	158.34	159.97	159.71	159.88	.014	**.011**	.013	.013	**.011**
20	10	278.87	185.90	239.23	210.79	251.19	.971	.146	.813	.474	**.066**
40	20	196.61	163.83	185.22	176.58	194.30	.296	.022	.158	.082	**.013**
80	40	174.62	160.07	169.90	166.19	173.50	.069	.014	.044	.029	**.012**
160	80	165.60	158.70	163.43	161.69	165.01	.027	**.011**	.021	.017	**.011**
320	160	161.51	158.14	160.48	159.63	161.25	.017	.011	.015	.013	**.010**
20	40	276.16	193.29	237.93	209.48	242.81	.965	.225	.799	.457	**.093**
40	80	195.40	166.08	184.61	176.08	190.89	.279	.029	.151	.078	**.017**
80	160	173.96	160.91	169.56	165.91	171.99	.065	.015	.042	.028	**.013**
160	320	165.38	159.18	163.39	161.67	164.31	.027	**.012**	.021	.017	**.012**
320	640	161.47	158.44	160.50	159.67	160.91	.017	**.011**	.015	.013	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q_m^*}$ of each row is in bold. $\chi^2_{f_m}(0.05)=145.46$, $\chi^2_{f_m}(0.01)=157.80$, $f_m=119$.

Table 8: Simulated values for $-2\log\lambda_m$, $-2\rho_{L_m}\log\lambda_m$, $-2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q_m^*}$ for $(p_1, p_2, p_3) = (2, 4, 8), m = 3$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)} = N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q_m^*(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q_m^*}$
$\alpha = 0.05$											
20	20	386.37	316.39	369.89	353.72	350.57	.966	.428	.920	.835	**.266**
40	40	312.64	284.33	306.77	301.86	306.33	.415	.103	.337	.274	**.079**
80	80	291.58	278.38	288.88	286.76	289.05	.163	.066	.139	.121	**.061**
160	160	282.93	276.53	281.62	280.65	281.62	.094	.057	.085	.079	**.056**
320	320	278.88	275.72	278.23	277.75	278.20	.069	**.053**	.065	.063	**.053**
20	10	449.95	316.60	394.06	342.83	411.98	1.000	.440	.984	.768	**.240**
40	20	333.77	284.31	316.64	300.88	329.61	.688	.103	.472	.263	**.067**
80	40	300.73	278.45	293.54	286.73	298.83	.261	.067	.182	.121	**.058**
160	80	287.14	276.50	283.81	280.64	286.04	.124	.057	.099	.079	**.055**
320	160	280.91	275.71	279.30	277.77	280.30	.080	**.053**	.071	.063	**.053**
20	40	445.60	326.75	391.88	341.07	399.59	.999	.569	.981	.751	**.313**
40	80	331.94	287.67	315.70	300.14	324.47	.666	.128	.458	.254	**.083**
80	160	299.75	279.76	292.98	286.33	296.53	.250	.074	.177	.118	**.064**
160	320	286.62	277.07	283.51	280.39	284.96	.120	.059	.097	.077	**.057**
320	640	280.67	275.99	279.15	277.64	279.77	.079	**.054**	.070	.062	**.054**
$\alpha = 0.01$											
20	20	411.79	337.21	394.14	376.70	372.32	.893	.215	.792	.645	**.105**
40	40	331.74	301.70	325.55	320.27	325.04	.194	.026	.141	.105	**.018**
80	80	309.18	295.18	306.31	304.07	306.62	.050	.014	.040	.033	**.013**
160	160	300.07	293.28	298.68	297.66	298.71	.023	**.012**	.020	.018	**.012**
320	320	295.59	292.24	294.91	294.40	295.09	.015	**.011**	.014	.013	**.011**
20	10	479.07	337.10	418.85	364.41	438.00	.997	.220	.939	.544	**.087**
40	20	354.14	301.66	335.84	319.15	349.87	.443	.027	.236	.098	**.014**
80	40	319.12	295.48	311.46	304.29	317.03	.097	.015	.058	.033	**.012**
160	80	304.57	293.28	301.04	297.66	303.42	.034	.012	.025	.018	**.011**
320	160	297.90	292.38	296.20	294.55	297.31	.019	**.011**	.016	.014	**.011**
20	40	474.55	347.98	416.58	362.35	424.78	.996	.330	.932	.523	**.128**
40	80	352.26	305.29	334.99	318.57	344.40	.418	.036	.225	.093	**.019**
80	160	318.02	296.81	310.79	303.75	314.58	.091	.017	.055	.032	**.014**
160	320	304.05	293.91	300.72	297.42	302.26	.032	**.012**	.024	.018	**.012**
320	640	297.70	292.74	296.09	294.48	296.75	.018	**.011**	.016	.013	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q_m^*}$ of each row is in bold. $\chi^2_{f_m}(0.05) = 274.99$, $\chi^2_{f_m}(0.01) = 291.68$, $f_m = 238$.

Table 9: Simulated values for $-2\log\lambda_m, -2\rho_{L_m}\log\lambda_m, -2\log\tau_m$, and $-2\log\phi_m$ and the approximate values for $-2\log\lambda_m$ and the actual type I error rates $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$, and $\alpha_{q_m^*}$ for $(p_1,p_2,p_3) = (2,4,8), m=4$

Sample Size		Upper Percentile					Type I Error				
$N_1^{(\ell)}$	$N_2^{(\ell)}=N_3^{(\ell)}$	$-2\log\lambda_m$	$-2\rho_{L_m}\log\lambda_m$	$-2\log\tau_m$	$-2\log\phi_m$	$q_m^*(\alpha)$	α_m	$\alpha_{\rho_{L_m}}$	α_{τ_m}	α_{ϕ_m}	$\alpha_{q_m^*}$
				$\alpha=0.05$							
20	20	551.12	457.54	529.02	504.47	503.92	.989	.500	.965	.894	**.308**
40	40	453.34	414.86	445.38	437.94	444.59	.491	.112	.398	.314	**.084**
80	80	424.69	406.66	420.98	417.79	421.22	.187	.069	.156	.133	**.063**
160	160	412.83	404.07	411.03	409.54	411.12	.100	.057	.090	.082	**.056**
320	320	407.35	403.03	406.46	405.74	406.46	.072	.054	.068	.064	**.053**
20	10	636.66	459.77	562.66	487.33	586.06	1.000	.534	.996	.813	**.284**
40	20	481.98	415.02	458.76	435.23	475.99	.784	.113	.558	.287	**.070**
80	40	437.08	406.72	427.28	417.15	434.50	.309	.069	.210	.128	**.059**
160	80	418.51	403.98	413.94	409.21	417.15	.137	.057	.107	.081	**.055**
320	160	410.02	402.90	407.81	405.52	409.32	.085	.053	.074	.064	**.052**
20	40	630.73	473.01	559.58	485.05	569.37	1.000	.671	.995	.798	**.374**
40	80	479.47	419.52	457.49	434.28	469.02	.763	.144	.542	.277	**.090**
80	160	435.92	408.67	426.70	416.76	431.37	.295	.078	.203	.125	**.067**
160	320	417.95	404.89	413.67	409.03	415.67	.134	.061	.106	.080	**.058**
320	640	409.85	403.44	407.79	405.54	408.60	.084	.055	.074	.063	**.054**
				$\alpha=0.01$							
20	20	580.70	482.10	557.39	531.05	529.52	.955	.270	.887	.739	**.129**
40	40	476.28	435.85	467.89	459.97	466.86	.251	.030	.181	.127	**.020**
80	80	446.06	427.12	442.17	438.80	442.24	.059	.015	.046	.037	**.014**
160	160	433.53	424.33	431.65	430.07	431.60	.025	**.012**	.022	.020	**.012**
320	320	427.68	423.14	426.76	425.97	426.71	.016	**.011**	.015	.014	**.011**
20	10	670.19	483.99	591.51	512.03	616.33	1.000	.296	.981	.604	**.111**
40	20	506.09	435.79	481.65	456.91	499.97	.559	.030	.307	.111	**.016**
80	40	458.83	426.96	448.54	437.94	456.21	.123	.015	.070	.035	**.012**
160	80	439.34	424.08	434.51	429.51	437.95	.039	.012	.028	.019	**.011**
320	160	430.54	423.06	428.23	425.82	429.72	.020	**.011**	.017	.014	**.011**
20	40	664.17	498.09	588.50	509.64	598.73	1.000	.431	.977	.582	**.167**
40	80	503.28	440.36	480.13	455.79	492.63	.530	.042	.293	.105	**.021**
80	160	457.65	429.04	447.95	437.51	452.92	.115	.018	.067	.034	**.015**
160	320	438.56	424.85	434.07	429.17	436.39	.037	.013	.027	.018	**.012**
320	640	430.26	423.53	428.10	425.74	428.96	.020	**.011**	.017	.014	**.011**

Note: The closest value to α from among $\alpha_m, \alpha_{\rho_{L_m}}, \alpha_{\tau_m}, \alpha_{\phi_m}$ and $\alpha_{q_m^*}$ of each row is in bold. $\chi^2_{f_m}(0.05)=402.06$, $\chi^2_{f_m}(0.01)=422.09$, $f_m=357$.

7 Numerical example

In this section, we give an example for test statistics and approximate upper percentiles proposed in this paper. The data consisted of cholesterol values measured during treatment at five time points (baseline, 6 months, 12 months, 20 months and 24 months) of a placebo group and a high dose group (Wei and Lachin (1984)). To construct the three-step monotone missing data, we used data with values available for up to 24 months, data with values available for up to 20 months, and data with values available for up to 12 months. That is $m = 2, p_1 = 3, p_2 = 1, p_3 = 1$. For the placebo group ($\ell = 1$), $N_1^{(1)} = 31, N_2^{(1)} = 4, N_3^{(1)} = 3$, and for the high dose group ($\ell = 2$), $N_1^{(2)} = 36, N_2^{(2)} = 7, N_3^{(2)} = 12$. Then, LRT statistic and test statistics are

$$-2\log\lambda_m = 82.201,\ -2\rho_{L_m}\log\lambda_m = 75.425,$$
$$-2\log\tau_m = 66.542,\ -2\log\phi_m = 80.527.$$

And, approximate upper percentile is

$$q_m^*(0.05) = 34.422,\ q_m^*(0.01) = 41.196,$$

and $\chi^2_{20}(0.05) = 31.410, \chi^2_{20}(0.01) = 37.566$. Thus, the null hypothesis is rejected for all test statistics and approximate upper percentile.

8 Conclusions

We discussed simultaneous tests for mean vectors and covariance matrices with three-step monotone missing data for a multi-sample problem. We proposed two test statistics ($-2\log\tau_m, -2\log\phi_m$) by decomposing the LR and correcting it by extracting the LR of the simultaneous test and the test of the variance in the case of complete data. We also proposed a test statistic ($-2\rho_{L_m}\log\lambda_m$) via linear interpolation. In addition, we provided an approximate upper 100α percentile ($q_m^*(\alpha)$). Further, based on the simulation results, an approximate upper 100α percentile $q_m^*(\alpha)$ is the most accurate. Finally, we gave an example of the proposed test statistics. The results of this paper can be extended to the k-step monotone missing data. We are currently investigating this problem.

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