A Modified Normalizing Transformation Statistic Based on Kurtosis for Testing Multivariate Normality – Empirical Power –

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Abstract

This paper presents the results of empirical power of a modified normalizing transformation statistic based on kurtosis. The empirical power has been calculated as the ratio of all multivariate normality rejections of random samples generated from nonnormal alternative distributions to the number of all generations. Alternative distributions were chosen to represent different types of departure from multivariate normality. Moreover, to compare the empirical power of the modified normalizing transformation statistic, we consider a normalizing transformation statistic, the improved Mardia's test statistic, and the Henzer-Zirkler test statistic. All calculations were performed in Mathematica 11, using 10,000 data sets generated through Monte-Carlo simulation for each combination of fixed sample sizes and dimensions at the significance level of 0.05.

Keywords: Monte Carlo simulation; Multivariate kurtosis; Nonnormal alternative distribution; Power comparison.

1 Introduction

In a multivariate setting, checking the assumption of multivariate normality of multi-dimensional data remains an important problem. In the related literature, many test statistics have been proposed. The survey of some known test statistics can be found in Chen and Genton (2022), Farell et al. (2007), Henze (2002), Horswell and Looney (1992), Kim (2020), Mecklin and Mundfrom (2004, 2005) and Thode (2002), among others. One group of test statistics for multivariate normality is based on Mardia's multivariate kurtosis (Mardia (1970, 1974). Enomoto et al. (2020) proposed the normalizing transformation statistic for Mardia's sample measure of multivariate kurtosis. Kurita et al. (2022) proposed the modification of the normalizing transformation statistic. This test statistic improves the normal approximation by using the exact expectation and variance of Mardia's multivariate sample kurtosis. The accuracy of the proposed test statistic, that is, the expectation, variance, and normal approximation, are checked in simulation studies. In this paper we present the results of empirical power of a modified normalizing transformation statistic based on kurtosis. This power is compared with the power of the normalizing transformation statistic in Enomoto et al. (2020). Moreover, the power results are compared to a more accurate Mardia's test based on kurtosis (Mardia (1970, 1974)), and the Henze-Zirkler test which is recommended as a formal test for multivariate normality (Mecklin and Mundfrom (2004, 2005)). The empirical power is calculated via Monte-Carlo simulations for various alternative distributions that characterizes different deviations from multivariate normality. The remainder of this paper is organized as follows. In Section 2, we describe the test statistics considered. Section 3 presents the results of the empirical power for samples generated from the chosen alternative distributions. Finally, Section 4 provides some concluding remarks.

2 Description of the considered test statistics

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ be a random sample of size N from a p-dimensional population, and $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be a p-variate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Let us test the null hypothesis

$$H_0 : \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \text{ is the sample from } N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \text{ for some } \boldsymbol{\mu} \text{ and } \boldsymbol{\Sigma}.$$

To test the null hypothesis H_0, we consider three test statistics based on Mardia's multivariate kurtosis and the Henze and Zirkler test statistic as the counterpart.

2.1 More accurate Mardia's test

Mardia (1970, 1974) defined the sample measure of multivariate kurtosis as

$$b_{2,p} = \frac{1}{N}\sum_{i=1}^{N}\{(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^\top \boldsymbol{S}^{-1}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})\}^2,$$

where $\bar{\boldsymbol{x}} = (1/N)\sum_{i=1}^{N}\boldsymbol{x}_i$ and $\boldsymbol{S} = (1/N)\sum_{i=1}^{N}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^\top$ denote a sample mean vector and a maximum likelihood estimator of covariance matrix, respectively.

Moreover, Mardia (1970, 1974) and Mardia and Kanazawa (1983) gave the following exact mean and variance of $b_{2,p}$

$$\mathrm{E}[b_{2,p}] = p(p+2)\frac{N-1}{N+1},$$

$$\mathrm{Var}[b_{2,p}] = 8p(p+2)\frac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)}.$$

Hence, in power comparisons, we use the more accurate test statistic proposed by Mardia (1974) in the form

$$ZM^* = \frac{b_{2,p} - p(p+2)\dfrac{N-1}{N+1}}{\sqrt{8p(p+2)\dfrac{(N-3)(N-p-1)(N-p+1)}{(N+1)^2(N+3)(N+5)}}}. \tag{1}$$

Statistic ZM^* is asymptotically distributed as standard normal $N(0,1)$ (see, e.g., Siotani et al. 1985).

2.2 Normalizing transformation test statistic for Mardia's multivariate kurtosis

Enomoto et al. (2020) derived the normalizing transformation statistic for Mardia's measure of multivariate kurtosis $b_{2,p}$ in the form

$$ZNT = \frac{\sqrt{N}}{\sigma}\left[\gamma\left(\exp\left[\frac{b_{2,p}-\beta}{\gamma}\right]-1\right)+\frac{2\beta}{N}\left(1-\frac{2}{\gamma}\right)\right], \tag{2}$$

where $\beta = p(p+2)$, $\sigma^2 = 8p(p+2)$ and $\gamma = -3p(p+2)/(p+8)$. Under multivariate normality, statistic ZNT is asymptotically distributed as $N(0,1)$.

2.3 Modified normalizing transformation test statistic for Mardia's multivariate kurtosis

Kurita et al. (2022) modified the normalizing transformation statistic ZNT by using an exact expectation and variance of Mardia's multivariate kurtosis. The modified ZNT statistic was obtained as the standardization of the ZNT statistic, using the approximated formulas for expectation and variance of the ZNT statistic. This statistic is expressed as

$$ZNT^* = \frac{\exp\left(\dfrac{1}{\gamma}b_{2,p}\right) - \exp\left(\dfrac{\mu_{\mathrm{M}}}{\gamma}+\dfrac{\sigma_{\mathrm{M}}^2}{2\gamma^2}\right)}{\exp\left(\dfrac{\mu_{\mathrm{M}}}{\gamma}\right)\left\{\exp\left(\dfrac{2\sigma_{\mathrm{M}}^2}{\gamma^2}\right)-\exp\left(\dfrac{\sigma_{\mathrm{M}}^2}{\gamma^2}\right)\right\}^{\frac{1}{2}}}, \tag{3}$$

where $\mu_{\mathrm{M}} = \mathrm{E}[b_{2,p}]$, $\sigma_{\mathrm{M}}^2 = \mathrm{Var}[b_{2,p}]$ are the exact expectation and variance of Mardia's multivariate kurtosis given in Subsection 2.1, respectively. Statistic ZNT^* is asymptotically distributed as $N(0,1)$. Moreover, Kurita et al. (2022) investigated the accuracy of the normal approximation of the ZNT^* statistic.

2.4 Henze-Zirkler test statistic

The last test statistic considered in the study is the test proposed in Henze and Zirkler (1990), which is recommended as a formal test for multivariate normality (Ebner and Henze (2020), Kim (2020), Thode (2002)). The Henze-Zirkler test statistic is defined by

$$\begin{aligned} HZ &= \frac{1}{N}\sum_{i,j=1}^{N} \exp\left[-\frac{\eta^2}{2}(\boldsymbol{x}_i - \boldsymbol{x}_j)^\top \boldsymbol{S}^{-1}(\boldsymbol{x}_i - \boldsymbol{x}_j)\right] + (1+2\eta^2)^{-\frac{p}{2}} \\ &\quad -2(1+\eta^2)^{-\frac{p}{2}}\sum_{i=1}^{N} \exp\left[-\frac{\eta^2}{2(1+\eta^2)}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^\top \mathbf{S}^{-1}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})\right], \end{aligned} \tag{4}$$

where

$$\eta = \frac{1}{\sqrt{2}}\left(\frac{2p+1}{4}\right)^{\frac{1}{p+4}} N^{\frac{1}{p+4}}.$$

The Henze-Zirkler test statistic rejects multivariate normality if the absolute value of the test statistic HZ is greater than appropriate critical value given in Henze and Zirkler (1990).

3 Results of the simulation studies on tests power

In this section, we present the results of the simulation study on the power of ZNT, ZNT^*, ZM^*, and the Henze-Zirkler test statistics. We consider some chosen alternative distributions, characterized by different departures from multivariate normality. We consider representants of symmetric, skewed, mesocurtic, and platycurtic distributions. In the simulations, we fixed a dimension p =2, 3, 4, 5, 7, and 10 and a sample size N =20, 30, 40, 50, and 100. For each combination of p and N, $10,000$ random samples were generated from each of the considered alternative distribution. The power of the test statistics considered was calculated as the ratio of the number of multivariate normality rejections at the significance level of 0.05 to 10,000 number of generations. The power results in percent are presented in Tables 1-12. The largest empirical power of the test statistics for each combination of N and p considered are highlighted in bold.

3.1 Symmetric alternative distributions

In this subsection, we regard the power of test statistics under considerations, against the multivariate T distribution with 2 and 7 degrees of freedom, respectively. These alternative distributions represent a symmetric and mildly leptokurtic distribution. Simulation results are listed in Tables 1 and 2. Tables 1 and 2 demonstrate that the improved Mardia's test statistic is the most powerful for almost each p and N considered. The results in Table 2 indicate that the Henze-Zirkler test has lower empirical power than other tests considered for random sample generated from multivariate T distribution with 7 degrees of freedom for N =30, 40, 50, and 100. In general, the empirical power of all test statistics increases with sample size N and dimension p considered, except for the Henze-Zirkler test with $p = 7$ or 10, and for the ZNT test statistic with $p = 10$. We can also notice that the proposed ZNT^* test statistic has outstanding small empirical power for $p = 2$ and $N = 20$ or 30. However, for larger p and N considered, the empirical power of ZNT^* test statistic increases to have a good power for $N = 50$ and 100. For samples generated from multivariate T distribution with 2 degrees of freedom, all tests have satisfactory power (Table 1). For samples generated from multivariate T distribution with 7 degrees of freedom tests considered are powerful for $N = 100$ (Table 2).

3.2 Power for skewed alternative distributions

In this subsection, we consider a heavily skewed alternative distribution. As representatives we choose the log-multinormal distribution and marginals chi-squared distribution with 1 degree of freedom. The empirical power for random samples generated from these alternative distributions are listed in Tables 3 and 4, respectively. Based on the results presented in Table 3, we can conclude that the Henze-Zirkler test statistic is the most powerful for all sample sizes and dimensions considered almost everywhere. Moreover, for random samples generated from log-multinormal distribution, all test statistics considered for sample sizes $N = 40,\ 50$, and 100 rejected multivariate normality with a very high power. Similar conclusions to above were obtained for random samples generated from the marginal chi-squared distribution with 1 degree of freedom for the results presented in Table 4. The Henze-Zirkler test statistic is also the most powerful for all N and p considered, except for $p = 10$ and $N = 20$, where the ZM^*test statistic in slightly more powerful. The proposed ZNT^* test statistic is more powerful than the ZNT test statistic for p =4, 5, 7, and 10. We can also notice that all tests have a high empirical power for sample sizes 30, 40, 50, and 100.

3.3 Samples generated from symmetric and platykurtic alternative distributions

In this subsection, we regard the multivariate uniform distribution on a cube and p-dimensional elliptically contoured Pearson Type II distribution MPII(0) with shape parameter $m = 0$ (Johnson (1987)). Both alternative distributions represent a symmetric and platykurtic distribution. The simulated power results are listed in Tables 5 and 6.

The empirical power for the random samples generated from multivariate univariate distribution on the cube presented in Table 5 shows that the proposed ZNT^* is the most powerful for $p = 4,\ 5,\ 7$, and 10, except for $p = 10$ and $N = 20$, as the Henze-Zirkler test statistic is the most powerful. For $p = 2,\ 3$, the ZNT test statistic has a higher power than the others, except for $p = 3$ and $N = 20$, as the ZNT^* test statistic is more powerful. Let us note that the power of the Henze-Zirkler test in Table 5 is higher than that presented in Tables 6-9 of Ebner et al. (2022).

We obtained similar conclusions for the MPII (m=0) alternative distribution listed in Table 6. That is, the proposed ZNT^* test statistic is the most powerful for $p = 4,\ 5$. For $p = 2,\ 3$, the ZNT test is the most powerful, except for $p = 3$ and $N = 20$, as the ZNT^* test statistic has a higher empirical power. In the case of $p = 7,\ 10$, the ZNT, ZNT^*, and ZM^* test statistics have full power. Note that the Henze-Zirkler test is less powerful almost in each case considered.

3.4 Alternatives from the mixture of two multivariate normal distributions

In this subsection, we consider the empirical power of the ZNT, ZNT^*, ZM^*, and the Henze-Zirkler test statistic for random samples generated from the alternative distribution being the following mixture of two multivariate normal distributions:

$$\pi \mathrm{N}_p \left(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1\right) + (1-\pi) \, \mathrm{N}_p \left(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2\right),$$

where $\pi \in (0,1)$ is the probability of generation from distribution $\mathrm{N}_p \left(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1\right)$ and $(1-\pi)$ is the probability of generation from distribution $\mathrm{N}_p \left(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2\right)$.

In the paper of Mecklin and Mundfrom (2005, Table 7), three proportions π =0.9, 0.788675, and 0.5 are considered with different combinations of $\boldsymbol{\mu}_1 = \mathbf{0}_p$ (null vector) and $\boldsymbol{\mu}_2 = \mathbf{1}_p$ (unit vector) and correlation matrices with correlation coefficients of 0.2 and 0.5.

Moreover, Mecklin and Mundfrom (2005, Table 2) reported that

- for $\pi = 0.5$, the mixture is symmetric and platykurtic,
- for $\pi = 0.788675$, the mixture is skewed and leptokurtic,
- for $\pi = 0.9$, the mixture is skewed and mesokurtic.

In this study, we consider three types of mixture $\pi \mathrm{N}_p \left(\mathbf{0}, \boldsymbol{\Sigma}_1\right) + (1-\pi) \, \mathrm{N}_p \left(\mathbf{1}, \boldsymbol{\Sigma}_2\right)$, where $\boldsymbol{\Sigma}_i = \rho_i \mathbf{I}_p + (1-\rho_i)\mathbf{1}_p\mathbf{1}_p^\top$, $(i = 1, 2)$, with correlations $\rho_1 = 0.2$ and $\rho_2 = 0.5$. The empirical power for random samples generated from the above mixture of two multivariate normal distributions with proportions $\pi = 0.5$, 0.788675, and 0.9 are listed in Tables 7, 8, and 9, respectively.

The results in Tables 7-9 prove that the power of all test statistics is very low for each combination of n and p considered. These results are consistent with the results of Mecklin and Mundfrom (2005, Table 8) or Kim (2020, Tables 2-5). The empirical power for all proportions considered reveals that a minimal power has the ZM^* or Henze-Zirkler test statistic for considered sizes and dimensions. The empirical power of the proposed

ZNT^* test statistic is slightly higher than that of the ZNT test statistic for $p = 5,\ 7,\ 10$.

Moreover, for proportion $\pi = 0.5$ (Table 7), the Henze-Zirkler test is slighty more powerful for p =2 and 3, and $N > 20$, while for $p = 4$, 5, 7, 10, and $N > 20$, the ZM^* test statistic is more powerful. For proportion $\pi = 0.788675$ (Table 8), both ZM^* and the Henze-Zirkler test statistics are the most powerful, as well as for proportion $\pi = 0.9$ (Table 9).

We also regarded mixture of two multivariate normal distributions, considering new covariance matrix in the second distribution, namely, $\pi \mathrm{N}_p(\mathbf{0}, \mathbf{\Sigma}_1) + (1-\pi)\,\mathrm{N}_p(\mathbf{1}, \mathbf{\Sigma}_2^*)$, where $\mathbf{\Sigma}_2^* = 4\mathbf{\Sigma}_2$. The empirical power results for the same proportions $\pi = 0.5, 0.788675$, and 0.9 are listed in Tables 10-12, respectively. The empirical power presented in Tables 10-12 with considered proportions are much higher than that for mixtures in Tables 7, 8, and 9. The results for proportion $\pi = 0.5$ listed in Table 10 shows that the ZM^* test statistic is the most powerful for $p = 4,\ 5,\ 7$, and 10 for all N considered, as well as for $p = 2, 3$ and $N = 20$. Among cases considered, the Henze-Zirkler test is the most powerful. The empirical power of the proposed ZNT^* test statistic is higher than that for the ZNT test statistic for $p = 5,\ 7$, and 10 and all N considered. The empirical power for proportion $\pi = 0.788675$ and 0.9 listed in Tables 11 and 12 demonstrates that the improved Mardia's test is the most powerful in all the cases considered. The proposed ZNT^* test statistic has a high power only for larger sample sizes, namely, $N = 100$. All tests considered are more powerful for proportion $\pi = 0.788675$ than for proportion $\pi = 0.9$ or 0.5.

4 Conclusions

Based on the results of the Monte-Carlo simulations on the power of ZNT, ZNT^*, ZM^*, and the Henze-Zirkler test statistics for the combinations of dimentions $p = 2, 3, 4, 5, 7,$ and 10 and sample sizes $N = 20, 30, 40, 50,$ and 100 to test the multivariate normality, we conclude the following:

1. The improved Mardia's test statistic is the most powerful for multidimentional data sets from multivariate symmetric distribution.
2. When the data sets show skewness, the Henze-Zirkler test is the most powerful and should be used.
3. The ZNT or ZNT^* test statistics should be applied in the case of symmetric and platykurtic distributions as the most powerful.
4. When data sets come from a mixture of two multivariate normal distributions, the improved Mardia's test statistic is the most powerful and should be used.
5. The proposed ZNT^* test statistic should not be used to test the multivariate normality in the case of $p = 2$ and $N = 20$ as it has a very low power.

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Table 1:

Empirical power (in percent) of the test statistics for the multivariate T distribution with 2 degrees of freedom.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	51.39	77.84	90.53	96.38	99.99
	ZNT^*	0.00	59.02	83.95	94.17	99.97
	ZM^*	**78.24**	**91.58**	**96.45**	**98.79**	**100.**
	HZ	68.59	84.80	92.79	96.62	99.96
3	ZNT	71.62	92.04	97.96	99.60	**100.**
	ZNT^*	71.58	91.13	97.61	99.46	**100.**
	ZM^*	**96.86**	**93.45**	**99.25**	**99.87**	**100.**
	HZ	77.20	91.85	97.56	99.25	**100.**
4	ZNT	80.20	97.11	99.50	99.91	**100.**
	ZNT^*	84.88	97.48	99.54	99.91	**100.**
	ZM^*	**92.73**	**98.96**	**99.78**	**99.96**	**100.**
	HZ	81.00	95.73	98.93	99.70	**100.**
5	ZNT	84.37	98.60	99.86	**99.99**	**100.**
	ZNT^*	90.68	98.98	99.89	**99.99**	**100.**
	ZM^*	**94.79**	**99.55**	**99.97**	**99.99**	**100.**
	HZ	83.20	96.90	99.54	99.85	**100.**
7	ZNT	85.37	99.53	**100.**	**99.99**	**100.**
	ZNT^*	95.05	**99.76**	**100.**	**99.99**	**100.**
	ZM^*	**96.84**	97.30	**100.**	99.97	**100.**
	HZ	80.96	97.97	99.77	**99.99**	**100.**
10	ZNT	70.15	99.80	**100.**	**100.**	**100.**
	ZNT^*	96.36	99.94	**100.**	**100.**	**100.**
	ZM^*	**97.22**	**99.96**	**100.**	**100.**	**100.**
	HZ	72.15	97.73	99.93	**100.**	**100.**

Table 2:

Empirical power (in percent) of the test statistics for the multivariate T distribution with 7 degrees of freedom.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	5.01	12.75	21.62	30.67	61.93
	ZNT^*	0.12	3.91	12.67	22.20	57.79
	ZM^*	**23.63**	**33.23**	**41.97**	**48.97**	**73.87**
	HZ	13.06	16.21	18.84	23.00	38.12
3	ZNT	9.77	24.70	36.56	48.02	82.38
	ZNT^*	9.84	22.13	33.75	45.08	81.34
	ZM^*	**29.35**	**44.26**	**54.73**	**63.98**	**88.74**
	HZ	14.66	20.02	25.29	31.22	53.76
4	ZNT	12.15	33.41	50.35	63.09	**92.95**
	ZNT^*	17.80	35.49	50.80	62.99	92.87
	ZM^*	**34.89**	**53.09**	**66.19**	**75.64**	92.91
	HZ	16.18	23.82	30.99	38.13	66.51
5	ZNT	13.41	39.37	59.23	73.50	97.56
	ZNT^*	24.43	45.65	62.19	74.97	97.57
	ZM^*	**38.37**	**60.78**	**74.17**	**83.63**	**98.61**
	HZ	15.93	25.55	34.43	43.50	76.57
7	ZNT	11.42	48.35	73.67	86.83	99.72
	ZNT^*	33.37	60.39	79.35	89.34	99.75
	ZM^*	**43.36**	**70.28**	**85.76**	**92.92**	**99.90**
	HZ	16.66	28.02	39.87	52.81	88.73
10	ZNT	3.26	51.16	83.62	94.97	**100.**
	ZNT^*	36.65	74.03	90.46	96.91	**100.**
	ZM^*	**42.74**	**79.88**	**93.33**	**97.94**	**100.**
	HZ	15.62	26.97	42.58	57.97	95.57

Table 3:

Empirical power (in percent) of the test statistics for the multivariate lognormal alternative distribution.

p	MVN test	Sample size 20	30	40	50	100
2	ZNT	54.53	82.32	92.67	97.36	99.99
	ZNT^*	0.04	64.40	87.13	95.59	99.99
	ZM^*	80.61	93.01	97.34	99.10	**100.**
	HZ	**96.27**	**99.73**	**100.**	**100.**	**100.**
3	ZNT	71.56	92.54	98.25	99.50	**100.**
	ZNT^*	71.51	91.54	98.00	99.42	**100.**
	ZM^*	87.11	96.99	99.37	99.84	**100.**
	HZ	**97.83**	**99.91**	**100.**	**100.**	**100.**
4	ZNT	77.61	96.09	99.25	99.88	**100.**
	ZNT^*	82.73	96.33	99.27	99.88	**100.**
	ZM^*	90.57	98.47	99.76	99.93	**100.**
	HZ	**97.96**	**99.98**	**100.**	**100.**	**100.**
5	ZNT	79.90	97.39	99.74	99.99	**100.**
	ZNT^*	88.08	98.10	99.79	99.99	**100.**
	ZM^*	93.19	99.02	99.92	**100.**	**100.**
	HZ	**98.00**	**99.94**	**100.**	**100.**	**100.**
7	ZNT	76.87	98.45	99.92	99.99	**100.**
	ZNT^*	91.07	99.27	99.96	99.99	**100.**
	ZM^*	93.95	99.60	99.98	**100.**	**100.**
	HZ	**95.98**	**99.95**	**100.**	**100.**	**100.**
10	ZNT	50.93	98.60	99.93	**100.**	**100.**
	ZNT^*	89.86	99.58	99.99	**100.**	**100.**
	ZM^*	**92.10**	**99.75**	**100.**	99.88	**100.**
	HZ	84.16	99.53	**100.**	**100.**	**100.**

Table 4:

Empirical power (in percent) of the test statistics for the marginal chi-squared alternative distribution with 1 degree of freedom.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	45.52	72.20	87.85	94.39	99.90
	ZNT^*	0.03	48.77	78.56	90.53	99.88
	ZM^*	76.30	89.08	95.64	98.16	99.96
	HZ	**98.73**	**99.92**	**100.**	**100.**	**100.**
3	ZNT	62.22	86.17	95.49	98.60	99.99
	ZNT^*	62.16	84.43	94.57	98.25	99.99
	ZM^*	83.17	94.59	98.20	99.45	99.99
	HZ	**99.13**	**100.**	**100.**	**100.**	**100.**
4	ZNT	68.65	91.36	97.92	99.46	99.99
	ZNT^*	75.05	92.14	97.97	99.45	99.99
	ZM^*	86.91	96.67	99.20	99.76	**100.**
	HZ	**99.02**	**99.99**	**100.**	**100.**	**100.**
5	ZNT	70.15	93.62	98.68	99.71	**100.**
	ZNT^*	80.70	95.08	98.88	99.76	**100.**
	ZM^*	88.52	97.55	99.51	99.89	**100.**
	HZ	**98.52**	**100.**	**100.**	**100.**	**100.**
7	ZNT	63.21	95.02	99.49	99.95	**100.**
	ZNT^*	84.09	97.24	99.70	99.95	**100.**
	ZM^*	89.04	98.35	99.83	99.98	**100.**
	HZ	**96.39**	**99.98**	**100.**	**100.**	**100.**
10	ZNT	32.50	95.02	99.54	99.96	**100.**
	ZNT^*	81.65	97.24	99.81	99.98	**100.**
	ZM^*	**85.17**	98.35	99.89	**100.**	**100.**
	HZ	82.97	**99.98**	**99.98**	**100.**	**100.**

Table 5:

Empirical power (in percent) of the test statistics for the marginal uniform distribution on interval $[-\sqrt{3}, \sqrt{3}]$.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	**20.75**	**48.91**	**72.96**	**88.24**	**99.91**
	ZNT^{*}	17.91	42.06	66.47	84.09	99.88
	ZM^{*}	4.14	18.34	39.94	63.54	99.50
	HZ	15.92	33.62	51.51	66.88	97.70
3	ZNT	19.79	**48.23**	**73.39**	**88.83**	**99.95**
	ZNT^{*}	**23.16**	48.13	71.90	87.77	**99.94**
	ZM^{*}	9.25	26.99	52.29	74.44	99.81
	HZ	13.51	29.49	47.42	63.28	97.55
4	ZNT	15.75	45.09	72.13	88.00	**99.96**
	ZNT^{*}	**21.95**	**48.29**	**73.23**	**88.28**	**99.95**
	ZM^{*}	10.37	30.16	56.34	77.85	99.86
	HZ	10.93	24.98	41.43	57.78	96.64
5	ZNT	12.74	41.07	68.30	86.36	**99.94**
	ZNT^{*}	**20.30**	**46.65**	**71.30**	**87.39**	**99.94**
	ZM^{*}	10.49	30.17	56.24	77.72	99.86
	HZ	10.04	21.02	35.99	52.11	94.77
7	ZNT	6.55	30.97	60.40	81.41	**99.98**
	ZNT^{*}	**15.13**	**39.61**	**66.00**	**83.59**	**99.98**
	ZM^{*}	8.49	26.96	52.93	74.70	99.96
	HZ	9.01	16.69	27.73	40.28	88.89
10	ZNT	1.66	18.54	46.07	71.92	99.93
	ZNT^{*}	8.16	**28.02**	**54.34**	**76.66**	**99.95**
	ZM^{*}	5.33	19.90	43.24	67.83	99.86
	HZ	**9.91**	14.16	21.56	30.10	77.87

Table 6:

Empirical power (in percent) of the test statistics for the Pearson Type II (m=0) alternative.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	**30.63**	**67.99**	**89.92**	**97.36**	**100.**
	ZNT^*	26.98	61.40	86.27	96.11	**100.**
	ZM^*	7.93	32.89	65.94	86.64	99.99
	HZ	16.09	36.06	55.84	69.88	98.06
3	ZNT	35.80	**77.99**	**95.59**	**99.39**	**100.**
	ZNT^*	**40.65**	77.88	95.30	99.27	**100.**
	ZM^*	19.39	57.43	87.11	97.12	**100.**
	HZ	12.87	32.34	53.49	70.60	98.86
4	ZNT	36.57	81.41	97.38	99.73	**100.**
	ZNT^*	**45.16**	**83.79**	**97.57**	**99.76**	**100.**
	ZM^*	26.89	69.31	93.25	99.15	**100.**
	HZ	10.60	28.10	50.42	68.77	99.00
5	ZNT	32.94	82.09	97.9	99.81	**100.**
	ZNT^*	**45.30**	**85.52**	**98.29**	**99.84**	**100.**
5	ZM^*	28.65	73.81	95.67	99.59	**100.**
	HZ	9.55	24.34	46.77	66.55	99.11
7	ZNT	**99.99**	**100.**	**100.**	**100.**	**100.**
	ZNT^*	**99.99**	**100.**	**100.**	**100.**	**100.**
	ZM^*	**99.99**	**100.**	**100.**	**100.**	**100.**
	HZ	71.80	90.39	97.34	99.18	**100.**
10	ZNT	99.96	**100.**	**100.**	**100.**	**100.**
	ZNT^*	**99.97**	**100.**	**100.**	**100.**	**100.**
	ZM^*	**99.97**	**100.**	**100.**	**100.**	**100.**
	HZ	59.96	78.17	90.38	95.67	99.89

Table 7:

Empirical power (in percent) of the test statistics for mixture $\pi \mathrm{N}_p(\mathbf{0}, \mathbf{\Sigma}_1) + (1-\pi)\,\mathrm{N}_p(\mathbf{1}, \mathbf{\Sigma}_2)$, $\pi = 0.5$.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	1.19	1.92	2.49	2.76	3.60
	ZNT^*	0.77	0.92	1.28	1.51	2.77
	ZM^*	**4.78**	4.43	4.46	4.56	4.71
	HZ	4.66	**5.09**	**5.70**	**5.18**	**6.17**
3	ZNT	1.50	2.23	2.91	3.19	4.12
	ZNT^*	1.76	1.86	2.51	2.65	3.65
	ZM^*	**4.86**	5.45	5.21	5.67	6.56
	HZ	4.81	**5.52**	**5.91**	**5.79**	**7.43**
4	ZNT	1.46	2.36	3.18	3.20	5.00
	ZNT^*	2.83	2.82	3.37	3.27	4.84
	ZM^*	**5.67**	**6.43**	**6.72**	**6.37**	7.75
	HZ	4.99	5.57	5.95	6.18	**7.79**
5	ZNT	1.13	2.10	3.28	3.76	6.20
	ZNT^*	3.14	3.27	4.05	4.37	6.38
	ZM^*	**5.81**	**6.31**	**7.72**	**7.06**	**9.62**
	HZ	5.66	5.71	6.03	6.35	8.63
7	ZNT	0.73	2.16	3.75	4.54	9.69
	ZNT^*	4.02	4.96	5.90	6.16	10.63
	ZM^*	6.14	**8.19**	9.50	**10.28**	**14.79**
	HZ	**6.39**	6.43	**9.69**	6.88	9.42
10	ZNT	0.21	1.55	3.74	5.76	16.50
	ZNT^*	5.67	6.75	8.54	9.92	19.39
	ZM^*	7.01	**9.59**	**12.21**	**14.56**	**25.16**
	HZ	**8.91**	7.73	7.89	8.47	11.49

Table 8:

Empirical power (in percent) of the test statistics for mixture $\pi \mathrm{N}_p\left(\mathbf{0}, \boldsymbol{\Sigma}_1\right)+(1-\pi)\,\mathrm{N}_p\left(\mathbf{1}, \boldsymbol{\Sigma}_2\right)$, $\pi = 0.788675$.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	1.17	1.63	2.13	2.54	3.78
	ZNT^*	0.74	0.82	0.99	1.50	2.65
	ZM^*	**5.50**	5.20	5.69	5.68	6.16
	HZ	5.05	**5.48**	**5.70**	**6.06**	**7.53**
3	ZNT	1.28	2.28	2.67	3.00	3.89
	ZNT^*	1.48	2.00	2.32	2.48	3.40
	ZM^*	**5.38**	5.12	**6.04**	6.02	6.20
	HZ	4.70	**5.35**	6.01	**6.15**	**6.92**
4	ZNT	1.45	2.25	2.92	3.29	4.38
	ZNT^*	2.58	2.70	3.04	3.29	4.28
	ZM^*	**5.14**	**5.29**	**6.02**	**6.08**	6.52
	HZ	4.79	4.85	**6.02**	5.88	**7.33**
5	ZNT	1.09	2.10	3.16	3.11	4.60
	ZNT^*	3.25	3.25	3.81	3.51	4.75
	ZM^*	5.05	**5.61**	6.63	5.63	6.69
	HZ	**5.29**	5.54	**6.20**	**5.99**	**7.22**
7	ZNT	0.67	1.66	2.68	3.04	4.75
	ZNT^*	3.95	3.79	4.37	4.13	5.25
	ZM^*	5.08	5.66	5.95	6.05	**7.18**
	HZ	**5.79**	**6.46**	**6.21**	**6.38**	7.07
10	ZNT	0.23	1.23	2.15	3.23	5.55
	ZNT^*	4.68	5.16	5.08	5.53	6.70
	ZM^*	5.45	6.40	6.65	**7.63**	**8.79**
	HZ	**8.31**	**6.82**	**6.89**	6.93	7.95

Table 9:

Empirical power (in percent) of the test statistics for mixture $\pi \mathrm{N}_p\left(\mathbf{0}, \mathbf{\Sigma}_1\right) + (1-\pi)\,\mathrm{N}_p\left(\mathbf{1}, \mathbf{\Sigma}_2\right)$, $\pi = 0.9$.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	1.22	1.79	2.63	2.70	3.94
	ZNT^*	0.70	0.77	1.15	1.55	2.96
	ZM^*	**5.46**	**5.62**	**6.09**	6.04	**6.79**
	HZ	5.04	5.40	5.37	**6.38**	5.94
3	ZNT	1.49	2.12	2.96	3.35	4.56
	ZNT^*	1.82	1.93	2.47	2.81	3.99
	ZM^*	**5.80**	**5.66**	**6.11**	**6.43**	**7.10**
	HZ	4.90	5.52	5.42	5.73	5.47
4	ZNT	1.30	2.21	3.22	3.45	4.69
	ZNT^*	2.66	2.67	3.41	3.46	4.51
	ZM^*	**4.97**	**5.93**	**6.33**	**6.12**	**6.81**
	HZ	4.91	5.12	5.26	5.23	6.22
5	ZNT	1.21	2.16	2.79	3.61	4.66
	ZNT^*	3.25	2.96	3.44	4.11	4.76
	ZM^*	**5.14**	**5.57**	**5.91**	**6.52**	**7.00**
	HZ	4.91	5.02	5.28	5.70	6.06
7	ZNT	0.64	1.65	2.59	3.00	4.31
	ZNT^*	3.63	3.83	4.06	4.02	4.66
	ZM^*	5.01	5.43	**5.59**	**5.83**	**6.08**
	HZ	**5.58**	**5.67**	5.30	5.73	6.04
10	ZNT	0.22	1.05	1.84	2.34	4.16
	ZNT^*	4.98	4.18	4.21	4.55	5.20
	ZM^*	5.70	5.15	5.58	6.05	6.78
	HZ	**8.55**	**6.78**	**6.33**	**6.48**	**6.81**

Table 10:

Empirical power (in percent) of the test statistics for mixture $\pi \mathrm{N}_p\left(\mathbf{0}, \mathbf{\Sigma}_1\right)+(1-\pi)\,\mathrm{N}_p\left(\mathbf{1}, \mathbf{\Sigma}_2^*\right)$, $\pi = 0.5$.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	2.34	6.30	10.67	16.11	39.02
	ZNT^*	0.13	0.88	4.06	8.56	33.19
	ZM^*	**20.03**	25.46	31.46	36.18	57.03
	HZ	18.12	**26.31**	**35.41**	**42.59**	**73.87**
3	ZNT	5.42	12.63	20.02	27.52	59.45
	ZNT^*	5.45	10.65	17.00	24.57	57.03
	ZM^*	**23.64**	**31.96**	39.50	47.31	73.03
	HZ	19.43	29.15	**40.18**	**49.39**	**83.34**
4	ZNT	6.76	17.76	29.02	39.21	76.28
	ZNT^*	11.07	19.85	29.55	39.06	75.66
	ZM^*	**26.71**	**38.41**	**48.80**	**57.77**	**88.28**
	HZ	19.34	30.65	43.40	55.03	84.78
5	ZNT	7.79	22.57	37.04	49.65	87.63
	ZNT^*	17.01	28.91	40.78	52.20	87.79
	ZM^*	**30.64**	**44.77**	**57.60**	**67.22**	**92.78**
	HZ	19.05	32.52	45.21	57.74	91.86
7	ZNT	5.93	29.59	50.95	67.62	97.27
	ZNT^*	23.55	44.11	59.83	73.39	97.64
	ZM^*	**33.32**	**55.81**	**70.89**	**82.17**	**98.59**
	HZ	18.39	31.69	46.48	60.62	95.22
10	ZNT	1.65	30.92	62.87	83.22	99.82
	ZNT^*	29.11	58.08	77.25	89.66	99.85
	ZM^*	**34.89**	**66.38**	**83.47**	**92.78**	**99.90**
	HZ	17.22	27.67	44.10	60.60	96.87

Table 11:

Empirical power (in percent) of the test statistics for mixture $\pi \mathrm{N}_p\left(\mathbf{0}, \mathbf{\Sigma}_1\right) + (1-\pi)\,\mathrm{N}_p\left(\mathbf{1}, \mathbf{\Sigma}_2^*\right)$, $\pi = 0.788675$.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	5.81	16.03	26.87	36.86	75.15
	ZNT^*	0.17	3.85	14.33	26.42	70.56
	ZM^*	**28.52**	**40.76**	**50.51**	**59.24**	**85.66**
	HZ	19.95	27.77	34.06	40.14	66.37
3	ZNT	9.00	25.11	39.25	51.91	87.48
	ZNT^*	9.04	21.99	35.46	48.80	86.54
	ZM^*	**30.17**	**47.11**	**58.88**	**69.00**	**92.44**
	HZ	18.97	28.57	37.07	43.31	73.44
4	ZNT	9.96	29.39	47.96	62.24	93.41
	ZNT^*	15.91	31.71	48.53	62.18	93.26
	ZM^*	**32.33**	**51.00**	**65.38**	**76.03**	**96.32**
	HZ	17.87	27.94	36.89	46.20	77.32
5	ZNT	9.08	32.38	52.95	69.57	96.56
	ZNT^*	19.32	38.25	56.29	71.31	96.61
	ZM^*	**32.68**	**53.44**	**69.44**	**80.88**	**98.03**
	HZ	16.50	26.43	35.61	45.48	77.95
7	ZNT	5.53	33.16	59.99	76.32	99.14
	ZNT^*	21.95	47.01	68.00	80.10	99.23
	ZM^*	**31.24**	**58.01**	**76.72**	**86.49**	**99.51**
	HZ	13.52	21.16	30.81	40.71	78.11
10	ZNT	0.69	26.72	62.95	83.8	99.80
	ZNT^*	19.51	51.53	75.73	89.08	99.89
	ZM^*	**24.74**	**59.19**	**81.15**	**92.26**	**99.93**
	HZ	12.27	16.96	23.64	33.11	73.96

Table 12:

Empirical power (in percent) of the test statistics for mixture $\pi \mathrm{N}_p\left(\mathbf{0}, \boldsymbol{\Sigma}_1\right) + (1-\pi)\,\mathrm{N}_p\left(\mathbf{1}, \boldsymbol{\Sigma}_2^*\right)$, $\pi = 0.9$.

	MVN	Sample size				
p	test	20	30	40	50	100
2	ZNT	4.91	13.55	23.75	31.57	62.51
	ZNT^*	0.30	3.87	13.93	22.83	58.20
	ZM^*	**22.01**	**32.46**	**41.28**	**49.08**	**72.27**
	HZ	13.20	16.78	20.68	22.57	35.92
3	ZNT	7.37	20.42	31.62	41.71	74.79
	ZNT^*	7.51	18.21	28.88	39.17	73.53
	ZM^*	**23.48**	**36.84**	**46.49**	**55.63**	**81.53**
	HZ	12.22	16.88	20.73	24.71	39.71
4	ZNT	6.94	21.19	36.21	47.46	81.34
	ZNT^*	10.88	22.86	36.73	47.37	80.98
	ZM^*	**22.56**	**37.71**	**50.33**	**60.28**	**85.99**
	HZ	10.87	15.57	20.28	23.56	40.30
5	ZNT	5.90	21.88	37.75	50.99	86.52
	ZNT^*	12.92	26.78	40.56	52.68	86.67
	ZM^*	22.34	**38.81**	**52.52**	**63.00**	**89.87**
	HZ	10.35	14.37	18.04	21.89	39.67
7	ZNT	2.60	20.06	39.37	54.84	90.75
	ZNT^*	12.45	30.45	46.20	59.32	91.13
	ZM^*	**18.71**	**39.85**	**55.32**	**66.80**	**93.36**
	HZ	9.34	12.43	15.05	18.56	37.22
10	ZNT	0.34	11.69	35.45	56.16	95.12
	ZNT^*	10.84	28.68	48.84	64.04	95.70
	ZM^*	**14.11**	**35.00**	**55.87**	**69.71**	**96.69**
	HZ	9.44	9.66	11.53	14.07	29.93