A Multivariate Normality Test Based on Kurtosis with Three-step Monotone Missing Data

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Abstract

A sample measure of multivariate kurtosis under monotone missing data is provided. To test a multivariate normality on the three-step monotone missing data, test statistics based on multivariate kurtosis are proposed by the evaluation of the expectation and variance of the sample measure of multivariate kurtosis. Finally, the normal approximation for the null distribution of the test statistics is investigated by a Monte Carlo simulation.

Keywords: Asymptotic expansion, Moment, Monte Carlo simulation, Multivariate kurtosis, Normal approximation.

1 Introduction

The statistical hypothesis testing problem of multivariate normality (MVN) is an important and difficult problem, and many statistical procedures have been proposed by many authors (for example, see Henze and Zirkler (1990), Thode (2002), Farrel et al. (2007), Kollo (2008), Hanusz et al. (2018)). In this paper, we consider the test statistics used in MVN tests base on multivariate kurtosis, which are defined by Mardia (1970). The measure of multivariate kurtosis is defines as $\beta_{2,p} = \mathrm{E}[\{(\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\}^2]$, where $\boldsymbol{x}$ be a random vector from a p-variate distribution with expectation $\boldsymbol{\mu}$ and non-singular covariance matrix $\boldsymbol{\Sigma}$, and a subscript $\top$ denotes a transpose. Then the sample measure of multivariate kurtosis is defines as $b_{2,p} = N^{-1} \sum_{i=1}^{N} \{(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \boldsymbol{S}^{-1} (\boldsymbol{x}_i - \overline{\boldsymbol{x}})\}^2$, where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ be observations (a random sample of N) from a p-variate distribution, $\overline{\boldsymbol{x}} = N^{-1} \sum_{i=1}^{N} \boldsymbol{x}_i$ and $\boldsymbol{S} = N^{-1} \sum_{i=1}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top$. In addition, note that multivariate kurtosis has other definitions (see, Srivastava (1984), Koziol (1989), Miyagawa et al. (2011) and so on). The Mardia's multivariate sample kurtosis has been applied as a kurtosis test statistic for an MVN test using the expectation and variance of $b_{2,p}$ under multivariate normality. Mardia (1970, 1974) proposed the test statistics $Z_{\rm M} = (b_{2,p} - p(p+2))/\sqrt{8p(p+2)/N}$ and $Z_{\rm M}^* = (b_{2,p} - \mu_{\rm M})/\sigma_{\rm M}$ where $\mu_{\rm M} = p(p+2)(N-1)/(N+1)$ and $\sigma_{\rm M}^2 = 8p(p+2)(N-3)(N-p-1)(N-p+1)/((N+1)^2(N+3)(N+5))$. Note that the null distributions of $Z_{\rm M}$ and $Z_{\rm M}^*$ are asymptotically a standard normal distribution. In addition, Henze (1994) discussed the asymptotic distribution of Mardia's kurtosis test statistic under nonnormality. An MVN test using a normalizing transformation for Mardia's multivariate kurtosis and its improvement were recently given by Enomoto et al. (2020) and Kurita et al. (2022, 2023). A test for multivariate kurtosis with two-step monotone missing data was discussed by Yamada et al. (2015) and Kurita and Seo (2022). In this paper, we give a new definition of multivariate sample measure of kurtosis with general monotone missing data. Furthermore, we give results for the test statistic and its null distribution in the case of three-step monotone missing data. Tests of the mean vectors or the covariance matrix were discussed by Hao and Krishnamoorthy (2001), Tsukada (2014), and Yagi et al. (2019), and so on. By decomposing the multivariate kurtosis, the sample analogue of multivariate kurtosis with general step monotone missing data can be defined. After a greate deal of calculation, we derive asymptotic results of the expectation and the variance under three-step monotone missing data using a perturbation method. For the perturbation method described in this paper, see Kawasaki and Seo (2016) and Kawasaki et al. (2018). The rest of this paper is organized as follows. In section 2, we give the definition of multivariate sample kurtosis for the case of three-step monotone missing data, as well as the definition of multivariate sample kurtosis for the general case. In Section 3, we derive the expectation and variance of the sample measure of multivariate kurtosis in the case of three-step monotone missing data. Using the above results, we give three test statistics. In Section 4, some simulation results for three-step monotone missing data are presented to investigate the accuracy of the normal approximation of the test statistics. Finally, some concluding remarks are given in Section 5.

2 Measure of multivariate sample kurtosis with three-step monotone missing data

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_1}$ be N_1 p-variate random sample vectors, $\boldsymbol{x}_{(12),N_1+1}, \ldots, \boldsymbol{x}_{(12),N_1+N_2}$ be N_2 (p_1+p_2)-variate random sample vectors and let $\boldsymbol{x}_{1,N_1+N_2+1}, \ldots, \boldsymbol{x}_{1,N}$ be N_3 p_1-variate random sample vectors. Such a dataset has three-step monotone missing data:

$$\begin{pmatrix} \boldsymbol{x}_{1,1}^\top & \boldsymbol{x}_{2,1}^\top & \boldsymbol{x}_{3,1}^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{x}_{1,N_1}^\top & \boldsymbol{x}_{2,N_1}^\top & \boldsymbol{x}_{3,N_1}^\top \\ \boldsymbol{x}_{1,N_1+1}^\top & \boldsymbol{x}_{2,N_1+1}^\top & * \\ \vdots & \vdots & \vdots \\ \boldsymbol{x}_{1,N_1+N_2}^\top & \boldsymbol{x}_{2,N_1+N_2}^\top & \vdots \\ \boldsymbol{x}_{1,N_1+N_2+1}^\top & * & \vdots \\ \vdots & \vdots & \vdots \\ \boldsymbol{x}_{1,N}^\top & * & * \end{pmatrix},$$

where $N = N_1 + N_2 + N_3$, $p = p_1 + p_2 + p_3$ and " $*$ " indicates a missing observation. Let $\boldsymbol{x}_i = (\boldsymbol{x}_{1,i}^\top, \boldsymbol{x}_{2,i}^\top, \boldsymbol{x}_{3,i}^\top)^\top$, $i = 1, \ldots, N_1$ be a random vector from a p-variate distribution with expectation $\boldsymbol{\mu}$ and nonsingular covariance matrix $\boldsymbol{\Sigma}$, and let $\boldsymbol{x}_{(12),i} = (\boldsymbol{x}_{1,i}^\top, \boldsymbol{x}_{2,i}^\top)^\top$, $i = N_1 + 1, \ldots, N_1 + N_2$ be a random vector from a $(p_1 + p_2)$-variate distribution with expectation $\boldsymbol{\mu}_{(12)}$ and nonsingular covariance matrix $\boldsymbol{\Sigma}_{(12)(12)}$. Furthermore, let $\boldsymbol{x}_{1,i}$, $i = N_1 + N_2 + 1, \ldots, N$ be a random vector from a p_1-variate distribution with expectation $\boldsymbol{\mu}_1$ and nonsingular covariance matrix $\boldsymbol{\Sigma}_{11}$,where the decomposition of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \boldsymbol{\mu}_3 \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \left(\begin{array}{cc|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} \\ \hline \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{(12)(12)} & \boldsymbol{\Sigma}_{(12)3} \\ \hline \boldsymbol{\Sigma}_{3(12)} & \boldsymbol{\Sigma}_{33} \end{array}\right),$$

respectively. Then the sample measure of multivariate kurtosis in the case of three-step monotone missing data can be defined as

$$b_{2,p_1,p_2,p_3} = \sum_{j=1}^{3} R_j^{(3)} + \sum_{\substack{j=1 \\ j<k}}^{3} \sum_{k=1}^{3} R_{jk}^{(3)}, \tag{1}$$

where

$$R_1^{(3)} = \frac{1}{N} \sum_{i=1}^{N} U_{1,i}^2, \; R_2^{(3)} = \frac{1}{N_1 + N_2} \sum_{i=1}^{N_1+N_2} U_{2\cdot 1,i}^2, \; R_3^{(3)} = \frac{1}{N_1} \sum_{i=i}^{N_1} U_{3\cdot 12,i}^2,$$

$$R_{12}^{(3)} = \frac{2}{N_1 + N_2} \sum_{i=1}^{N_1+N_2} U_{1,i} U_{2\cdot 1,i}, \; R_{13}^{(3)} = \frac{2}{N_1} \sum_{i=1}^{N_1} U_{1,i} U_{3\cdot 12,i}, \; R_{23}^{(3)} = \frac{2}{N_1} \sum_{i=1}^{N_1} U_{2\cdot 1,i} U_{3\cdot 12,i},$$

and

$$U_{1,i} = (\boldsymbol{x}_{1,i} - \widehat{\boldsymbol{\mu}}_1)^\top \widehat{\boldsymbol{\Sigma}}_{11}^{-1} (\boldsymbol{x}_{1,i} - \widehat{\boldsymbol{\mu}}_1), \tag{2}$$

$$U_{2\cdot 1,i} = (\boldsymbol{x}_{2\cdot 1,i} - \widehat{\boldsymbol{\mu}}_{2\cdot 1})^\top \widehat{\boldsymbol{\Sigma}}_{22\cdot 1}^{-1} (\boldsymbol{x}_{2\cdot 1,i} - \widehat{\boldsymbol{\mu}}_{2\cdot 1}), \tag{3}$$

$$U_{3\cdot 12,i} = (\boldsymbol{x}_{3\cdot 12,i} - \widehat{\boldsymbol{\mu}}_{3\cdot 12})^\top \widehat{\boldsymbol{\Sigma}}_{33\cdot 12}^{-1} (\boldsymbol{x}_{3\cdot 12,i} - \widehat{\boldsymbol{\mu}}_{3\cdot 12}), \tag{4}$$

$$\boldsymbol{x}_{2\cdot 1,i} = \boldsymbol{x}_{2,i} - \widehat{\boldsymbol{\Sigma}}_{21} \widehat{\boldsymbol{\Sigma}}_{11}^{-1} \boldsymbol{x}_{1,i},\ \ \boldsymbol{x}_{3\cdot 12,i} = \boldsymbol{x}_{3,i} - \widehat{\boldsymbol{\Sigma}}_{3(12)} \widehat{\boldsymbol{\Sigma}}_{(12)(12)}^{-1} \boldsymbol{x}_{(12),i}.$$

We note that $\widehat{\boldsymbol{\mu}}_1$, $\widehat{\boldsymbol{\Sigma}}_{11}$, $\widehat{\boldsymbol{\mu}}_{2\cdot 1}$, $\widehat{\boldsymbol{\Sigma}}_{22\cdot 1}$, $\widehat{\boldsymbol{\mu}}_{3\cdot 12}$, and $\widehat{\boldsymbol{\Sigma}}_{33\cdot 12}$ are MLEs of $\boldsymbol{\mu}_1$, $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\mu}_{2\cdot 1}$, $\boldsymbol{\Sigma}_{22\cdot 1}$, $\boldsymbol{\mu}_{3\cdot 12}$, and $\boldsymbol{\Sigma}_{33\cdot 12}$, respectively, where

$$\boldsymbol{\mu}_{2\cdot 1} = \boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\mu}_1,\ \ \boldsymbol{\mu}_{3\cdot 12} = \boldsymbol{\mu}_3 - \boldsymbol{\Sigma}_{3(12)} \boldsymbol{\Sigma}_{(12)(12)}^{-1} \boldsymbol{\mu}_{(12)},$$
$$\boldsymbol{\Sigma}_{22\cdot 1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12},\ \ \boldsymbol{\Sigma}_{33\cdot 12} = \boldsymbol{\Sigma}_{33} - \boldsymbol{\Sigma}_{3(12)} \boldsymbol{\Sigma}_{(12)(12)}^{-1} \boldsymbol{\Sigma}_{(12)3}.$$

The MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ for three-step monotone missing data under multivariate normal distribution are given by Kanda and Fujikoshi (1998). In this paper, we use the following definition as a notation.

$$\overline{\boldsymbol{x}}_{(1)} = \frac{1}{N_1} \sum_{i=1}^{N_1} \boldsymbol{x}_i = \begin{pmatrix} \overline{\boldsymbol{x}}_{(1),1} \\ \overline{\boldsymbol{x}}_{(1),2} \\ \overline{\boldsymbol{x}}_{(1),3} \end{pmatrix},\ \overline{\boldsymbol{x}}_{(2)} = \frac{1}{N_2} \sum_{i=N_1+1}^{N_1+N_2} \boldsymbol{x}_{(12),i} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(2),1} \\ \overline{\boldsymbol{x}}_{(2),2} \end{pmatrix},$$

$$\overline{\boldsymbol{x}}_{(3)} = \frac{1}{N_3} \sum_{i=N_1+N_2+1}^{N} \boldsymbol{x}_{1,i},\ \ \overline{\boldsymbol{x}}_{T(12)} = \frac{1}{N_1+N_2} \sum_{i=1}^{N_1+N_2} \boldsymbol{x}_{(12),i} = \begin{pmatrix} \overline{\boldsymbol{x}}_{T(12),1} \\ \overline{\boldsymbol{x}}_{T(12),2} \end{pmatrix},$$

$$\overline{\boldsymbol{x}}_T = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{x}_{1,i},$$

$$\boldsymbol{S}_{(1)} = \frac{1}{N_1} \sum_{i=1}^{N_1} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{(1)})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{(1)})^\top = \begin{pmatrix} \boldsymbol{S}_{(1),11} & \boldsymbol{S}_{(1),12} & \boldsymbol{S}_{(1),13} \\ \boldsymbol{S}_{(1),21} & \boldsymbol{S}_{(1),22} & \boldsymbol{S}_{(1),23} \\ \boldsymbol{S}_{(1),31} & \boldsymbol{S}_{(1),32} & \boldsymbol{S}_{(1),33} \end{pmatrix},$$

$$\boldsymbol{S}_{(2)} = \frac{1}{N_2} \sum_{i=N_1+1}^{N_1+N_2} (\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{(2)})(\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{(2)})^\top = \begin{pmatrix} \boldsymbol{S}_{(2),11} & \boldsymbol{S}_{(2),12} \\ \boldsymbol{S}_{(2),21} & \boldsymbol{S}_{(2),22} \end{pmatrix},$$

$$\boldsymbol{S}_{(3)} = \frac{1}{N_3} \sum_{i=N_1+N_2+1}^{N} (\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(3)})(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_{(3)})^\top,$$

$$\boldsymbol{S}_{T(12)} = \frac{1}{N_1+N_2}\sum_{i=1}^{N_1+N_2}(\boldsymbol{x}_{(12),i}-\overline{\boldsymbol{x}}_{T(12)})(\boldsymbol{x}_{(12),i}-\overline{\boldsymbol{x}}_{T(12)})^\top = \begin{pmatrix} \boldsymbol{S}_{T(12),11} & \boldsymbol{S}_{T(12),12} \\ \boldsymbol{S}_{T(12),21} & \boldsymbol{S}_{T(12),22} \end{pmatrix},$$

$$\boldsymbol{S}_T = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_{1,i}-\overline{\boldsymbol{x}}_T)(\boldsymbol{x}_{1,i}-\overline{\boldsymbol{x}}_T)^\top.$$

We note that

$$\begin{aligned}
\overline{\boldsymbol{x}}_T &= \tau_1\overline{\boldsymbol{x}}_{(1),1}+\tau_2\overline{\boldsymbol{x}}_{(2),1}+\tau_3\overline{\boldsymbol{x}}_{(3)},\\
\overline{\boldsymbol{x}}_{T(12)} &= \frac{\tau_1}{\tau_1+\tau_2}\overline{\boldsymbol{x}}_{(1),(12)}+\frac{\tau_2}{\tau_1+\tau_2}\overline{\boldsymbol{x}}_{(2)},\\
\boldsymbol{S}_T &= \tau_1\boldsymbol{S}_{(1),11}+\tau_2\boldsymbol{S}_{(2),11}+\tau_3\boldsymbol{S}_{(3)}+\tau_1\tau_2(\overline{\boldsymbol{x}}_{(1),1}-\overline{\boldsymbol{x}}_{(2),1})(\overline{\boldsymbol{x}}_{(1),1}-\overline{\boldsymbol{x}}_{(2),1})^\top\\
&\quad+\tau_1\tau_3(\overline{\boldsymbol{x}}_{(1),1}-\overline{\boldsymbol{x}}_{(3)})(\overline{\boldsymbol{x}}_{(1),1}-\overline{\boldsymbol{x}}_{(3)})^\top+\tau_2\tau_3(\overline{\boldsymbol{x}}_{(2),1}-\overline{\boldsymbol{x}}_{(3)})(\overline{\boldsymbol{x}}_{(2),1}-\overline{\boldsymbol{x}}_{(3)})^\top,\\
\boldsymbol{S}_{T(12)} &= \frac{\tau_1}{\tau_1+\tau_2}\boldsymbol{S}_{(1),(12)(12)}+\frac{\tau_2}{\tau_1+\tau_2}\boldsymbol{S}_{(2)}+\frac{\tau_1\tau_2}{\tau_1+\tau_2}(\overline{\boldsymbol{x}}_{(1),(12)}-\overline{\boldsymbol{x}}_{(2)})(\overline{\boldsymbol{x}}_{(1),(12)}-\overline{\boldsymbol{x}}_{(2)})^\top,
\end{aligned}$$

where τ_1,τ_2 and τ_3 are constants such that $N_1=\tau_1 N$, $N_2=\tau_2 N$, $N_3=\tau_3 N$, $0<\tau_1<1$, $0<\tau_2<1$ and $0<\tau_3<1$.Therefore, the MLEs in (2), (3) and (4) are given by

$$\begin{aligned}
\overline{\boldsymbol{x}}_{T(12)\cdot 1} &= \overline{\boldsymbol{x}}_{T(12),2}-\boldsymbol{S}_{T(12),21}\boldsymbol{S}_{T(12),11}^{-1}\overline{\boldsymbol{x}}_{T(12),1},\\
\overline{\boldsymbol{x}}_{(1)\cdot 12} &= \overline{\boldsymbol{x}}_{(1),3}-\boldsymbol{S}_{(1),3(12)}\boldsymbol{S}_{(1),(12)(12)}^{-1}\overline{\boldsymbol{x}}_{(1),(12)},\\
\boldsymbol{S}_{T(12)\cdot 1} &= \boldsymbol{S}_{T(12),22}-\boldsymbol{S}_{T(12),21}\boldsymbol{S}_{T(12),11}^{-1}\boldsymbol{S}_{T(12),12},\\
\boldsymbol{S}_{(1)\cdot 12} &= \boldsymbol{S}_{(1),33}-\boldsymbol{S}_{(1),3(12)}\boldsymbol{S}_{(1),(12)(12)}^{-1}\boldsymbol{S}_{(1),(12)3},
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{x}_{2\cdot 1,i} &= \boldsymbol{x}_{2,i}-\boldsymbol{S}_{T(12),21}\boldsymbol{S}_{T(12),11}^{-1}\boldsymbol{x}_{1,i},\\
\boldsymbol{x}_{3\cdot 12,i} &= \boldsymbol{x}_{3,i}-\boldsymbol{S}_{(1),3(12)}\boldsymbol{S}_{(1),(12)(12)}^{-1}\boldsymbol{x}_{(12),i}.
\end{aligned}$$

In general, we may give the definition in the case of k-step monotone missing data, that is,

$$b_{2,p_1,\ldots,p_k} = \sum_{j=1}^{k}\left\{\frac{1}{h_j}\sum_{i=1}^{h_j}(U_{j,i}^*)^2\right\}+\sum_{\substack{m=1\\m<\ell}}^{k}\sum_{\ell=1}^{k}\left(\frac{2}{\omega_{m\ell}}\sum_{i=1}^{\omega_{m\ell}}U_{m,i}^*U_{\ell,i}^*\right),$$

where

$$h_{k+1-u} = \sum_{v=1}^{u}N_v,\ u=1,\ldots,k,\ \omega_{m\ell}=\min(h_m,h_\ell),\ 1\le m<\ell\le k,$$

$$U_{1,i}^* = U_{1,i} = (\boldsymbol{x}_{1,i}-\widehat{\boldsymbol{\mu}}_1)^\top\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\boldsymbol{x}_{1,i}-\widehat{\boldsymbol{\mu}}_1),$$

and for $j = 2, \ldots, k$,

$$
\begin{aligned}
U^*_{j,i} = U_{j\cdot 1\ldots j-1,i} &= (\boldsymbol{x}_{j\cdot 1\ldots j-1,i} - \widehat{\boldsymbol{\mu}}_{j\cdot 1\ldots j-1})^\top \widehat{\boldsymbol{\Sigma}}^{-1}_{jj\cdot 1\ldots j-1}(\boldsymbol{x}_{j\cdot 1\ldots j-1,i} - \widehat{\boldsymbol{\mu}}_{j\cdot 1\ldots j-1}),\\
\boldsymbol{x}_{j\cdot 1\ldots j-1,i} &= \boldsymbol{x}_{j,i} - \widehat{\boldsymbol{\Sigma}}_{j(1\ldots j-1)}\widehat{\boldsymbol{\Sigma}}^{-1}_{(1\ldots j-1)(1\ldots j-1)}\boldsymbol{x}_{(1\ldots j-1),i},\\
\widehat{\boldsymbol{\mu}}_{j\cdot 1\ldots j-1} &= \widehat{\boldsymbol{\mu}}_j - \widehat{\boldsymbol{\Sigma}}_{j(1\ldots j-1)}\widehat{\boldsymbol{\Sigma}}^{-1}_{(1\ldots j-1)(1\ldots j-1)}\widehat{\boldsymbol{\mu}}_{(1\ldots j-1)},\\
\widehat{\boldsymbol{\Sigma}}_{jj\cdot 1\ldots j-1} &= \widehat{\boldsymbol{\Sigma}}_{jj} - \widehat{\boldsymbol{\Sigma}}_{j(1\ldots j-1)}\widehat{\boldsymbol{\Sigma}}^{-1}_{(1\ldots j-1)(1\ldots j-1)}\widehat{\boldsymbol{\Sigma}}_{(1\ldots j-1)j}.
\end{aligned}
$$

The above notation is not simple; for example, $U^*_{j,i}$ for $k = 4$ is expressed using the following notations for $\boldsymbol{x}_i$ and MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

$$
\boldsymbol{x}_i = \begin{pmatrix} \boldsymbol{x}_{1,i} \\ \boldsymbol{x}_{2,i} \\ \boldsymbol{x}_{3,i} \\ \boldsymbol{x}_{4,i} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{(12),i} \\ \boldsymbol{x}_{(34),i} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_{(123),i} \\ \boldsymbol{x}_{4,i} \end{pmatrix},
$$

$$
\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \\ \boldsymbol{\mu}_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \boldsymbol{\mu}_{(34)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(123)} \\ \boldsymbol{\mu}_4 \end{pmatrix},
$$

$$
\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13} & \boldsymbol{\Sigma}_{14} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23} & \boldsymbol{\Sigma}_{24} \\ \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33} & \boldsymbol{\Sigma}_{34} \\ \boldsymbol{\Sigma}_{41} & \boldsymbol{\Sigma}_{42} & \boldsymbol{\Sigma}_{43} & \boldsymbol{\Sigma}_{44} \end{pmatrix} = \left(\begin{array}{c|cc} \boldsymbol{\Sigma}_{(12)(12)} & \boldsymbol{\Sigma}_{(12)3} & \boldsymbol{\Sigma}_{(12)4} \\ \hline \boldsymbol{\Sigma}_{3(12)} & \multicolumn{2}{c}{\boldsymbol{\Sigma}_{3(34)}} \\ \boldsymbol{\Sigma}_{4(12)} & \multicolumn{2}{c}{\boldsymbol{\Sigma}_{4(34)}} \end{array}\right) = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{(123)(123)} & \boldsymbol{\Sigma}_{(123)4} \\ \hline \boldsymbol{\Sigma}_{4(123)} & \boldsymbol{\Sigma}_{44} \end{array}\right).
$$

Note that this result for $k = 2$ coincides with the definition for the two-step case given in Kurita and Seo (2022), and this result for the three-step case coincides with (1). In the next section, we give the expectation and variance of b_{2,p_1,p_2,p_3} under multivariate normality.

3 Test statistics for multivariate kurtosis

In this section, we proposed three test statistics with three-step monotone missing data,

$$Z_{MM}^{(3)} = \frac{b_{2,p_1,p_2,p_3} - p(p+2)}{\sqrt{\dfrac{(\sigma^{(3)})^2}{N}}}, \tag{5}$$

$$Z_{MM}^{(3)*} = \frac{b_{2,p_1,p_2,p_3} - \left\{p(p+2) - \dfrac{c^{(3)}}{N}\right\}}{\sqrt{\dfrac{(\sigma^{(3)})^2}{N}}}, \tag{6}$$

$$Z_{MM}^{(3)**} = \frac{b_{2,p_1,p_2,p_3} - \left\{p(p+2) - \dfrac{2}{N+1}p_1(p_1+2) - \dfrac{2}{\tau_1 N}p_2(2p_1+p_2+2)\right\}}{\nu^{(3)}}, \tag{7}$$

where

$$\begin{aligned}
c^{(3)} &= \left\{p_1(p_1+2) + \frac{2}{\tau_1+\tau_2}p_2(2p_1+p_2+2) + \frac{1}{\tau_1}p_3(2p_1+2p_2+p_3+2)\right\}, \\
(\sigma^{(3)})^2 &= 8\left\{p_1(p_1+2) + \frac{1}{\tau_1+\tau_2}p_2\{2p_1+p_2+\tau_3 p_1 p_2 - p_3^2+2\}\right. \\
&\quad \left. + \frac{1}{\tau_1}p_3\{2p_1+2p_2+p_3+(1-\tau_1)p_1p_3+p_2p_3+2\}\right\}, \\
(\nu^{(3)})^2 &= \nu_1^2 + \frac{8}{(\tau_1+\tau_2)N}p_2\left\{2p_1+p_2+\tau_3p_1p_2-p_3^2+2\right\} \\
&\quad + \frac{8}{\tau_1 N}p_3\left\{2p_1+2p_2+p_3+(1-\tau_1)p_1p_3+p_2p_3+2\right\}, \\
\nu_1^2 &= 8p_1(p_1+2)\frac{(N-3)(N-p_1-1)(N-p_1+1)}{(N+1)^2(N+3)(N+5)}.
\end{aligned}$$

Note that these test statistics are asymptotically distributed as $N(0,1)$. In order to derive $Z_{\mathrm{MM}}^{(3)}$, $Z_{\mathrm{MM}}^{(3)*}$ and $Z_{\mathrm{MM}}^{(3)**}$, we consider the first and second moments of b_{2,p_1,p_2,p_3} under multivariate normality. Using the Mardia (1974)'s result, we have

$$\mathrm{E}[R_1^{(3)}] = p_1(p_1+2)\frac{N-1}{N+1}. \tag{8}$$

Since a discussion of the exact theory is not easy, we give the asymptotic expansion. We give asymptotic expansions of $\mathrm{E}[R_j^{(3)}]$, $j = 2, 3$ and $\mathrm{E}[R_{jk}^{(3)}]$, $1 \le j < k \le 3$ using the perturbation expansion method, where $N_1 \to \infty$, $N_2 \to \infty$ and $N_3 \to \infty$ with $\tau_1\ (= N_1/N) \to \delta_1 \in (0,1)$, $\tau_2\ (= N_2/N) \to \delta_2 \in (0,1)$ and $\tau_3\ (= N_3/N) \to \delta_3 \in (0,1)$.

As a result, we obtain

$$\mathrm{E}[R_2^{(3)}] = \left(1 - \frac{2}{(\tau_1+\tau_2)N}\right)p_2(p_2+2) + \mathrm{O}(N^{-\frac{3}{2}}), \tag{9}$$

$$\mathrm{E}[R_3^{(3)}] = \left(1 - \frac{2}{\tau_1 N}\right)p_3(p_3+2) + \mathrm{O}(N^{-\frac{3}{2}}), \tag{10}$$

$$\mathrm{E}[R_{12}^{(3)}] = 2\left(1 - \frac{2}{(\tau_1+\tau_2)N}\right)p_1p_2 + \mathrm{O}(N^{-\frac{3}{2}}), \tag{11}$$

$$\mathrm{E}[R_{13}^{(3)}] = 2\left(1 - \frac{2}{\tau_1 N}\right)p_1p_3 + \mathrm{O}(N^{-\frac{3}{2}}), \tag{12}$$

$$\mathrm{E}[R_{23}^{(3)}] = 2\left(1 - \frac{2}{\tau_1 N}\right)p_2p_3 + \mathrm{O}(N^{-\frac{3}{2}}). \tag{13}$$

See Appendix A for details of the derivation.The above results can be obtained by using the results of the case of two-step monotone missing data. Therefore, the expectation of b_{2,p_1,p_2,p_3} is given by

$$\mathrm{E}[b_{2,p_1,p_2,p_3}] = p(p+2) - \frac{c^{(3)}}{N} + \mathrm{O}(N^{-\frac{3}{2}}),$$

where

$$c^{(3)} = 2\left\{p_1(p_1+2) + \frac{1}{\tau_1+\tau_2}p_2(2p_1+p_2+2) + \frac{1}{\tau_1}p_3(2p_1+2p_2+p_3+2)\right\}.$$

From this result, as an approximation, we have

$$\mathrm{E}[b_{2,p_1,p_2,p_3}] \fallingdotseq p(p+2) - \frac{c^{(3)}}{N} (= m_1^{(3)}). \tag{14}$$

Furthermore, because the exact expectation of $R_1^{(3)}$ is given by (8), as another approximation of $\mathrm{E}[b_{2,p_1,p_2,p_3}]$, we can also propose the following:

$$\begin{aligned}\mathrm{E}[b_{2,p_1,p_2,p_3}] \fallingdotseq\ & p(p+2) - \frac{2}{N+1}p_1(p_1+2) \\ & - \frac{2}{N}\left\{\frac{1}{\tau_1+\tau_2}p_2(2p_1+p_2+2) + \frac{1}{\tau_1}p_3(2p_1+2p_2+p_3+2)\right\}(= m_2^{(3)}).\end{aligned} \tag{15}$$

Next, we consider the variance of b_{2,p_1,p_2,p_3}, which is given by

$$\mathrm{Var}[b_{2,p_1,p_2,p_3}] = \sum_{j=1}^{3}\mathrm{Var}[R_j^{(3)}] + \sum_{\substack{j=1\\ j<k}}^{3}\sum_{k=1}^{3}\mathrm{Var}[R_{jk}^{(3)}] + \mathrm{O}(N^{-\frac{3}{2}})$$

From the result of Mardia (1974), we obtain

$$\mathrm{Var}[R_1^{(3)}] = 8p_1(p_1+2)\frac{(N-3)(N-p_1-1)(N-p_1+1)}{(N+1)^2(N+3)(N+5)}.$$

The variances of $R_j^{(3)}$ and $R_{jk}^{(3)}$ also provide asymptotic results using the same method as the derivation of the expectation. To summarize these results, variances are given by

$$\mathrm{Var}[R_2^{(3)}] = \frac{8}{(\tau_1+\tau_2)N} p_2(p_2+2) + \mathrm{O}(N^{-\frac{3}{2}}), \tag{16}$$

$$\mathrm{Var}[R_3^{(3)}] = \frac{8}{\tau_1 N} p_3(p_3+2) + \mathrm{O}(N^{-\frac{3}{2}}), \tag{17}$$

$$\mathrm{Var}[R_{12}^{(3)}] = \frac{8}{N} p_1 p_2 \left\{ \left(\frac{1}{\tau_1+\tau_2} - 1 \right) p_2 + \frac{2}{\tau_1+\tau_2} \right\} + \mathrm{O}(N^{-\frac{3}{2}}), \tag{18}$$

$$\mathrm{Var}[R_{13}^{(3)}] = \frac{8}{N} p_1 p_3 \left\{ \left(\frac{1}{\tau_1} - 1 \right) p_3 + \frac{2}{\tau_1} \right\} + \mathrm{O}(N^{-\frac{3}{2}}), \tag{19}$$

$$\mathrm{Var}[R_{23}^{(3)}] = \frac{8}{N} p_2 p_3 \left\{ \left(\frac{1}{\tau_1} - \frac{1}{\tau_1+\tau_2} p_3 \right) + \frac{2}{\tau_1} \right\} + \mathrm{O}(N^{-\frac{3}{2}}), \tag{20}$$

respectively. See Appendix B.2 for details of the derivation. Noting that the covariances between $R_j^{(3)}$ and $R_{jk}^{(3)}$, $1 \le j < k \le 3$, are $\mathrm{O}(N^{-3/2})$, the variance of b_{2,p_1,p_2,p_3} is given by

$$\mathrm{Var}[b_{2,p_1,p_2,p_3}] = \frac{1}{N} (\sigma^{(3)})^2 + \mathrm{O}(N^{-\frac{3}{2}}), \tag{21}$$

where

$$\begin{aligned}(\sigma^{(3)})^2 = 8\Big\{ & p_1(p_1+2) + \frac{1}{\tau_1+\tau_2} p_2 \{2p_1 + p_2 + \tau_3 p_1 p_2 - p_3^2 + 2\} \\ & + \frac{1}{\tau_1} p_3 \{2p_1 + 2p_2 + p_3 + (1-\tau_1) p_1 p_3 + p_2 p_3 + 2\} \Big\},\end{aligned}$$

Because the exact variance is given as in (**??**) for $R_1^{(3)}$, as another approximation, we have

$$\begin{aligned}\mathrm{Var}[b_{2,p_1,p_2,p_3}] \fallingdotseq\ & \nu_1^2 + \frac{8}{(\tau_1+\tau_2)N} p_2 \Big\{ 2p_1 + p_2 + \tau_3 p_1 p_2 - p_3^2 + 2 \Big\} \\ & + \frac{8}{\tau_1 N} p_3 \Big\{ 2p_1 + 2p_2 + p_3 + (1-\tau_1) p_1 p_3 + p_2 p_3 + 2 \Big\} (= (\nu^{(3)})^2),\end{aligned} \tag{22}$$

where

$$\nu_1^2 = 8p_1(p_1+2) \frac{(N-3)(N-p_1-1)(N-p_1+1)}{(N+1)^2(N+3)(N+5)}.$$

4 Simulation studies

In this section, the normal approximation for the three test statistics $Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$ and $Z_{MM}^{(3)**}$ in Section 3 is assessed based on a Monte Carlo simulation. For the simulations, 1,000,000 experiments were conducted for certain combinations of parameters. The parameter settings for the simulation are (p_1, p_2, p_3) =(2, 2, 2), (2, 2, 4), (2, 4, 2), (3, 3, 3), (5, 5, 5), (8, 2, 2), (8, 4, 2) for the dimensions and $(N_1, N_2, N_3) = (m, n, l)$, m =20, 30, 40, 50, 100, 200, 400, 1000, n =10, 20, l =10, 20 for the sample sizes with three-step monotone missing data. First, Table 1 list the simulated values and theoretical values for the expectation and variance of b_{2,p_1,p_2,p_3}. For the theoretical results, two approximations for the expectation are given. It can be seen from Table 1 that the simulated values and approximated values converge to $p(p+2)$, and in particular, the approximate values of $m_1^{(3)}$ and $m_2^{(3)}$ are highly accurate for all cases. Regarding the variance, Table 1 show that the simulated values and approximated values of the variance multiplied by N converges to $8p(p+2)$ as N_1 increases. $N(\nu^{(3)})^2$ is good approximation for the majority of cases. Next, as shown in Table 2, the empirical expectation, variance, skewness, and kurtosis of the test statistics, $Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$ and $Z_{MM}^{(3)**}$ are given. From Table 2, we can see that as the sample size increases, the expectation, variance, skewness, and kurtosis of any test statistics converge the corresponding values of the standard normal distribution of 0, 1, 0, and 3, respectively. In particular, it can be seen that the empirical expectation and variance of $Z_{MM}^{(3)**}$ converge to 0 and 1 more quickly than those of $Z_{MM}^{(3)}$ and $Z_{MM}^{(3)*}$, respectively. Finally, in Table 3, we give upper and lower percentiles and type I error for the three test statistics, where $z(\alpha/2)$ is the upper $100(\alpha/2)$ percentile of the standard normal distribution, $\alpha = 0.05$. From the results in Table 3, we can see that the empirical type I errors of test statistics are closer to 0.05, and upper and lower percentiles of test statistics are closer to 1.96 and -1.96 as N_1 increases. In particular, it can be seen that the empirical type I errors of $Z_{MM}^{(3)}$ are closer to standard normal distribution than the those of $Z_{MM}^{(3)}$ and $Z_{MM}^{(3)*}$ in most cases. On the other hand, type I error of $Z_{MM}^{(3)}$ may be closer to 0.05 than the others. However, the expectation and variance of $Z_{MM}^{(3)}$ are far from 0 and 1, and upper and lower percentiles also far from 1.96 and -1.96, then the $Z_{MM}^{(3)}$ is not necessarily better normal approximation. It can be seen that this result follows the same trend as Mardia's Z_M and Z_M^* for the complete data in Enomoto et al. (2020).

5 Conclusion

In this paper, we defined a new sample measure of multivariate kurtosis when the type of data has three-step and k-step monotone pattern of missing observations. This definition is based on Mardia's multivariate kurtosis, and we considered its sample version by decomposing the multivariate kurtosis. We then developed test statistics for an MVN test under three-step monotone missing data by asymptotically evaluating the expectation and variance using an asymptotic expansion procedure. In particular, in first term of the decomposition of multivariate kurtosis, we also provide the exact expectation and variance in their sample version. Hence, it was possible to give the test statistic ($Z_{\mathrm{MM}}^{(3)**}$) with a better approximation even when the sample size is moderately small. A future problem will involve extending the method to cases with more than a four-step monotone pattern and deriving a normalizing transformation statistic for the test statistic given in this paper.

Table 1: Expectations and variances of b_{2,p_1,p_2,p_3} where $m_1^{(3)}$ and $m_2^{(3)}$ are approximate values of $\mathrm{E}[b_{2,p_1,p_2,p_3}]$ given in (14) and (15), respectively. For the variance, $(\sigma^{(3)})^2$ is an asymptotic variance of $N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$ in (21) , and $N(\nu^{(3)})^2$ is an approximate variance in (22) multiplied by N.

			Simulation		Approximation			
N_1	N_2	N_3	$\mathrm{E}[b_{2,p_1,p_2,p_3}]$	$N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$	$m_1^{(3)}$	$m_2^{(3)}$	$(\sigma^{(3)})^2$	$N(\nu^{(3)})^2$
				(p_1, p_2, p_3) =(2,2,2)				
20	10	10	44.33	352.61	44.13	44.14	746.67	724.69
30	10	10	45.37	361.91	45.28	45.29	629.34	611.02
40	10	10	45.95	368.20	45.89	45.90	569.60	553.91
50	10	10	46.31	370.27	46.28	46.28	533.33	519.61
100	10	10	47.10	377.13	47.10	47.10	459.64	451.21
200	10	10	47.54	380.61	47.53	47.54	422.09	417.35
400	10	10	47.76	382.18	47.76	47.76	403.12	400.59
1000	10	10	47.91	382.60	47.90	47.90	391.67	390.61
20	20	10	44.67	452.17	44.48	44.49	896.00	877.69
30	20	10	45.57	437.67	45.49	45.50	729.60	713.91
40	20	10	46.09	428.34	46.04	46.04	645.33	631.61
50	20	10	46.41	423.23	46.38	46.39	594.29	582.10
100	20	10	47.14	406.55	47.13	47.13	490.67	482.85
200	20	10	47.55	396.95	47.55	47.55	437.82	433.27
400	20	10	47.77	390.20	47.77	47.77	411.05	408.57
1000	20	10	47.91	386.73	47.91	47.91	394.85	393.81
20	10	20	44.41	494.61	44.21	44.22	949.34	931.02
30	10	20	45.42	479.14	45.33	45.34	768.00	752.31
40	10	20	45.98	463.47	45.93	45.93	675.20	661.48
50	10	20	46.34	453.87	46.31	46.31	618.67	606.48
100	10	20	47.11	426.25	47.11	47.11	503.27	495.45
200	10	20	47.54	407.16	47.54	47.54	444.19	439.64
400	10	20	47.77	395.83	47.76	47.76	414.24	411.77
1000	10	20	47.91	389.09	47.90	47.90	396.13	395.09
20	20	20	44.72	585.53	44.53	44.54	1088.00	1072.31
30	20	20	45.62	548.56	45.53	45.53	861.87	848.14
40	20	20	46.11	520.45	46.07	46.07	746.67	734.48
50	20	20	46.44	502.97	46.41	46.41	676.57	665.61
100	20	20	47.15	454.69	47.14	47.14	533.33	526.04
200	20	20	47.55	423.54	47.55	47.55	459.64	455.27
400	20	20	47.77	405.10	47.77	47.77	422.10	419.67
1000	20	20	47.91	392.00	47.91	47.91	399.31	398.27
				(p_1, p_2, p_3) =(4,2,2)				
20	10	10	74.29	530.06	74.00	74.03	1173.33	1094.46
30	10	10	75.85	551.04	75.71	75.73	1002.67	936.53
40	10	10	76.73	567.06	76.64	76.65	915.20	858.30
50	10	10	77.29	577.47	77.23	77.24	861.87	811.96
100	10	10	78.54	605.00	78.52	78.53	752.87	722.00
200	10	10	79.24	620.54	79.23	79.23	696.99	679.50
400	10	10	79.61	630.19	79.61	79.61	668.64	659.27
1000	10	10	79.84	634.50	79.84	79.84	651.49	647.58

Table 1:(Continued)

N_1	N_2	N_3	Simulation $\mathrm{E}[b_{2,p_1,p_2,p_3}]$	$N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$	Approximation $m_1^{(3)}$	$m_2^{(3)}$	$(\sigma^{(3)})^2$	$N(\nu^{(3)})^2$
				(p_1, p_2, p_3) =(4,2,2)				
20	20	10	74.92	674.39	74.64	74.66	1376.00	1309.87
30	20	10	76.24	664.36	76.11	76.12	1139.20	1082.30
40	20	10	76.99	657.87	76.91	76.92	1018.67	968.76
50	20	10	77.49	652.19	77.43	77.44	945.37	900.93
100	20	10	78.60	645.49	78.59	78.59	795.73	767.05
200	20	10	79.26	640.70	79.25	79.25	718.84	702.07
400	20	10	79.62	641.90	79.61	79.61	679.70	670.54
1000	20	10	79.84	639.09	79.84	79.84	655.95	652.07
20	10	20	74.53	773.08	74.24	74.26	1482.67	1416.53
30	10	20	76.00	747.10	75.87	75.88	1216.00	1159.10
40	10	20	76.84	731.65	76.75	76.76	1078.40	1028.49
50	10	20	77.37	718.80	77.32	77.33	994.14	949.70
100	10	20	78.57	686.43	78.55	78.56	820.95	792.26
200	10	20	79.25	664.20	79.24	79.24	731.58	714.81
400	10	20	79.61	651.66	79.61	79.61	686.09	676.93
1000	10	20	79.84	644.90	79.84	79.84	658.51	654.63
20	20	20	75.07	898.87	74.80	74.81	1664.00	1607.10
30	20	20	76.35	841.32	76.22	76.23	1339.73	1289.82
40	20	20	77.08	809.61	77.00	77.01	1173.34	1128.90
50	20	20	77.55	785.36	77.50	77.51	1071.54	1031.50
100	20	20	78.63	722.85	78.62	78.62	861.87	835.09
200	20	20	79.26	684.48	79.26	79.26	752.87	736.77
400	20	20	79.62	663.81	79.62	79.62	696.99	688.04
1000	20	20	79.84	650.50	79.84	79.84	662.94	659.10
				(p_1, p_2, p_3) =(8,2,2)				
20	10	10	157.05	830.16	156.53	156.63	2218.67	1876.56
30	10	10	159.85	911.82	159.60	159.66	1941.34	1651.06
40	10	10	161.49	975.52	161.33	161.38	1798.40	1546.67
50	10	10	162.56	1022.34	162.46	162.49	1710.93	1488.88
100	10	10	165.01	1145.61	164.98	164.99	1531.34	1392.02
200	10	10	166.42	1230.95	166.41	166.42	1438.78	1359.12
400	10	10	167.18	1281.62	167.18	167.19	1391.69	1348.81
1000	10	10	167.67	1319.60	167.67	167.67	1363.15	1345.18
20	20	10	158.46	1087.18	158.00	158.06	2528.00	2237.72
30	20	10	160.76	1110.16	160.53	160.58	2150.40	1898.68
40	20	10	162.12	1135.21	161.98	162.01	1957.33	1735.28
50	20	10	163.03	1154.22	162.94	162.96	1839.54	1640.98
100	20	10	165.18	1218.26	165.14	165.15	1597.87	1468.25
200	20	10	166.47	1270.20	166.46	166.46	1472.87	1396.49
400	20	10	167.20	1302.44	167.20	167.20	1408.99	1367.08
1000	20	10	167.67	1323.86	167.67	167.67	1370.14	1352.34

Table 1:(Continued)

N_1	N_2	N_3	Simulation $\mathrm{E}[b_{2,p_1,p_2,p_3}]$	$N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$	Approximation $m_1^{(3)}$	$m_2^{(3)}$	$(\sigma^{(3)})^2$	$N(\nu^{(3)})^2$
				(p_1, p_2, p_3) =(8,2,2)				
20	10	20	157.85	1283.25	157.33	157.40	2741.34	2451.06
30	10	20	160.38	1273.31	160.13	160.18	2304.00	2052.28
40	10	20	161.87	1278.92	161.71	161.75	2076.80	1854.75
50	10	20	162.85	1279.42	162.75	162.77	1937.06	1738.51
100	10	20	165.11	1296.80	165.08	165.09	1648.30	1518.66
200	10	20	166.45	1316.24	166.44	166.45	1498.36	1421.98
400	10	20	167.19	1326.85	167.19	167.19	1421.77	1379.87
1000	10	20	167.67	1334.20	167.67	167.67	1375.26	1357.46
20	20	20	158.99	1495.12	158.53	158.58	3008.00	2756.27
30	20	20	161.14	1441.82	160.91	160.95	2487.46	2265.42
40	20	20	162.40	1414.02	162.27	162.29	2218.66	2020.11
50	20	20	163.25	1393.92	163.16	163.18	2053.49	1873.97
100	20	20	165.25	1360.54	165.23	165.24	1710.94	1589.74
200	20	20	166.50	1346.56	166.49	166.49	1531.35	1457.98
400	20	20	167.21	1343.45	167.21	167.21	1438.78	1397.80
1000	20	20	167.67	1345.87	167.67	167.67	1382.20	1364.57
				(p_1, p_2, p_3) =(8,4,2)				
20	10	10	209.23	1163.29	208.53	208.63	3157.33	2815.23
30	10	10	213.01	1264.72	212.67	212.73	2746.67	2456.39
40	10	10	215.22	1336.67	215.01	215.06	2528.00	2276.27
50	10	10	216.67	1395.49	216.54	216.57	2391.47	2169.41
100	10	10	219.99	1553.17	219.95	219.96	2103.85	1964.53
200	10	10	221.88	1653.29	221.88	221.88	1951.39	1871.73
400	10	10	222.91	1722.71	222.91	222.91	1872.62	1829.75
1000	10	10	223.56	1762.72	223.56	223.56	1824.48	1806.51
20	20	10	211.39	1448.93	210.80	210.86	3440.00	3149.72
30	20	10	214.36	1482.64	214.08	214.12	2944.00	2692.27
40	20	10	216.16	1520.62	215.98	216.01	2682.67	2460.61
50	20	10	217.37	1551.30	217.25	217.27	2519.77	2321.22
100	20	10	220.22	1632.66	220.18	220.19	2174.94	2045.30
200	20	10	221.95	1698.96	221.94	221.95	1989.53	1913.14
400	20	10	222.93	1741.81	222.93	222.93	1892.49	1850.59
1000	20	10	223.56	1773.13	223.56	223.56	1832.65	1814.85
20	10	20	210.03	1967.41	209.33	209.40	4106.67	3816.39
30	10	20	213.55	1919.93	213.20	213.24	3424.00	3172.27
40	10	20	215.60	1893.89	215.39	215.43	3056.00	2833.95
50	10	20	216.96	1876.23	216.83	216.85	2824.54	2625.98
100	10	20	220.09	1831.78	220.05	220.06	2332.51	2202.89
200	10	20	221.92	1815.26	221.91	221.91	2069.18	1992.80
400	10	20	222.92	1798.47	222.92	222.92	1932.45	1890.54
1000	10	20	223.56	1795.30	223.56	223.56	1848.64	1830.85

Table 1:(Continued)

			Simulation		Approximation			
N_1	N_2	N_3	$\mathrm{E}[b_{2,p_1,p_2,p_3}]$	$N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$	$m_1^{(3)}$	$m_2^{(3)}$	$(\sigma^{(3)})^2$	$N(\nu^{(3)})^2$
				(p_1, p_2, p_3) =(8,4,2)				
20	20	20	211.93	2148.29	211.33	211.38	4256.00	4004.27
30	20	20	214.75	2066.55	214.46	214.49	3541.34	3319.28
40	20	20	216.44	2016.74	216.27	216.29	3157.34	2958.78
50	20	20	217.59	1986.97	217.47	217.49	2914.74	2735.23
100	20	20	220.31	1906.24	220.27	220.28	2391.47	2270.28
200	20	20	221.98	1851.10	221.97	221.98	2103.85	2030.49
400	20	20	222.94	1823.18	222.94	222.94	1951.39	1910.41
1000	20	20	223.56	1804.89	223.56	223.56	1856.65	1839.02
				(p_1, p_2, p_3) =(3,3,3)				
20	10	10	91.36	752.63	90.95	90.97	1720.00	1674.69
30	10	10	93.54	763.24	93.35	93.36	1418.00	1380.13
40	10	10	94.75	768.71	94.63	94.64	1264.80	1232.28
50	10	10	95.51	774.67	95.43	95.44	1172.00	1143.52
100	10	10	97.15	782.33	97.13	97.13	984.00	966.45
200	10	10	98.04	787.04	98.04	98.04	888.57	878.65
400	10	10	98.51	792.34	98.51	98.51	840.44	835.13
1000	10	10	98.80	792.49	98.80	98.80	811.41	809.20
20	20	10	92.05	1007.18	91.65	91.66	2118.00	2080.13
30	20	10	93.96	954.98	93.78	93.79	1684.80	1652.28
40	20	10	95.03	928.36	94.92	94.93	1466.00	1437.52
50	20	10	95.71	905.67	95.64	95.65	1333.71	1308.38
100	20	10	97.21	855.43	97.20	97.20	1066.00	1049.70
200	20	10	98.07	828.63	98.06	98.06	930.00	920.49
400	20	10	98.52	810.13	98.52	98.52	861.29	856.10
1000	20	10	98.81	796.77	98.80	98.80	819.79	817.60
20	10	20	91.53	1129.00	91.10	91.11	2228.00	2190.13
30	10	20	93.64	1063.49	93.45	93.46	1764.00	1731.48
40	10	20	94.81	1023.30	94.70	94.71	1527.60	1499.12
50	10	20	95.56	997.99	95.49	95.49	1384.00	1358.66
100	10	20	97.17	911.01	97.15	97.15	1092.00	1075.70
200	10	20	98.05	858.63	98.05	98.05	943.14	933.63
400	10	20	98.52	827.96	98.51	98.51	867.88	862.69
1000	10	20	98.80	805.29	98.80	98.80	822.43	820.23
20	20	20	92.15	1355.40	91.75	91.76	2604.00	2571.48
30	20	20	94.03	1241.39	93.85	93.86	2017.60	1989.12
40	20	20	95.08	1166.38	94.98	94.98	1720.00	1694.66
50	20	20	95.76	1114.29	95.68	95.69	1539.43	1516.63
100	20	20	97.23	983.46	97.22	97.22	1172.00	1156.79
200	20	20	98.07	895.28	98.07	98.07	984.00	974.87
400	20	20	98.52	846.52	98.52	98.52	888.57	883.50
1000	20	20	98.81	813.34	98.80	98.80	830.78	828.60

Table 1:(Continued)

N_1	N_2	N_3	Simulation $\mathrm{E}[b_{2,p_1,p_2,p_3}]$	$N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$	Approximation $m_1^{(3)}$	$m_2^{(3)}$	$(\sigma^{(3)})^2$	$N(\nu^{(3)})^2$
				(p_1, p_2, p_3) =(5,5,5)				
20	10	10	235.18	2116.95	234.08	234.13	5346.68	5222.60
30	10	10	240.86	2061.16	240.35	240.38	4263.34	4158.99
40	10	10	243.98	2045.17	243.68	243.70	3716.00	3626.05
50	10	10	245.96	2048.44	245.77	245.78	3385.33	3306.32
100	10	10	250.22	2040.47	250.17	250.18	2717.82	2668.76
200	10	10	252.54	2042.03	252.52	252.52	2380.38	2352.53
400	10	10	253.75	2044.18	253.74	253.74	2210.59	2195.65
1000	10	10	254.50	2041.17	254.49	254.49	2108.33	2102.09
20	20	10	236.90	2962.51	235.85	235.88	6830.00	6725.65
30	20	10	241.90	2714.85	241.43	241.45	5256.00	5166.04
40	20	10	244.70	2583.70	244.42	244.43	4463.33	4384.32
50	20	10	246.47	2499.50	246.30	246.31	3985.14	3914.70
100	20	10	250.40	2304.69	250.35	250.35	3020.67	2975.08
200	20	10	252.59	2188.25	252.57	252.57	2532.90	2506.20
400	20	10	253.76	2114.56	253.76	253.76	2287.19	2272.59
1000	20	10	254.50	2070.50	254.50	254.50	2139.06	2132.88
20	10	20	235.58	3508.84	234.43	234.46	7113.35	7009.00
30	10	20	241.10	3175.10	240.58	240.60	5460.00	5370.04
40	10	20	244.15	2974.64	243.85	243.86	4622.00	4542.98
50	10	20	246.07	2840.33	245.89	245.90	4114.66	4044.23
100	10	20	250.27	2511.68	250.22	250.22	3087.63	3042.04
200	10	20	252.55	2302.97	252.54	252.54	2566.75	2540.05
400	10	20	253.75	2169.69	253.75	253.75	2304.17	2289.57
1000	10	20	254.50	2099.73	254.49	254.49	2145.86	2139.68
20	20	20	237.15	4261.54	236.08	236.10	8539.98	8450.04
30	20	20	242.09	3750.21	241.60	241.61	6418.66	6339.65
40	20	20	244.82	3468.45	244.54	244.55	5346.66	5276.23
50	20	20	246.58	3268.65	246.39	246.40	4698.29	4634.76
100	20	20	250.43	2773.93	250.38	250.39	3385.34	3342.75
200	20	20	252.60	2438.66	252.59	252.59	2717.81	2692.18
400	20	20	253.77	2249.68	253.76	253.76	2380.38	2366.10
1000	20	20	254.50	2127.16	254.50	254.50	2176.53	2170.41
				(p_1, p_2, p_3) =(2,4,2)				
20	10	10	74.03	545.69	73.73	73.74	1237.33	1215.35
30	10	10	75.68	570.23	75.55	75.55	1050.67	1032.35
40	10	10	76.62	583.57	76.53	76.54	953.60	937.91
50	10	10	77.21	592.51	77.16	77.16	893.87	880.15
100	10	10	78.52	615.71	78.50	78.50	770.33	761.90
200	10	10	79.23	628.93	79.23	79.23	706.13	701.38
400	10	10	79.61	634.91	79.61	79.61	673.33	670.79
1000	10	10	79.84	639.94	79.84	79.84	653.40	652.34

Table 1:(Continued)

			Simulation		Approximation			
N_1	N_2	N_3	$\mathrm{E}[b_{2,p_1,p_2,p_3}]$	$N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$	$m_1^{(3)}$	$m_2^{(3)}$	$(\sigma^{(3)})^2$	$N(\nu^{(3)})^2$
				$(p_1,p_2,p_3)=$(2,4,2)				
20	20	10	74.75	687.51	74.48	74.49	1424.00	1405.69
30	20	10	76.13	677.88	76.00	76.00	1177.60	1161.91
40	20	10	76.91	672.64	76.84	76.84	1050.67	1036.95
50	20	10	77.42	667.31	77.38	77.38	972.80	960.61
100	20	10	78.59	655.92	78.57	78.57	811.73	803.91
200	20	10	79.25	647.74	79.25	79.25	727.56	723.01
400	20	10	79.61	645.56	79.61	79.61	684.27	681.79
1000	20	10	79.84	641.33	79.84	79.84	657.83	656.78
20	10	20	74.13	808.44	73.81	73.82	1610.67	1592.35
30	10	20	75.75	785.02	75.60	75.60	1312.00	1296.31
40	10	20	76.66	765.61	76.57	76.57	1155.20	1141.48
50	10	20	77.24	752.40	77.19	77.19	1058.14	1045.94
100	10	20	78.52	708.73	78.51	78.51	855.86	848.04
200	10	20	79.23	679.14	79.23	79.23	749.87	745.32
400	10	20	79.61	661.21	79.61	79.61	695.45	692.98
1000	10	20	79.84	647.33	79.84	79.84	662.31	661.26
20	20	20	74.81	925.14	74.53	74.54	1760.00	1744.31
30	20	20	76.18	876.24	76.04	76.04	1416.53	1402.81
40	20	20	76.94	840.30	76.87	76.87	1237.34	1225.14
50	20	20	77.45	816.63	77.40	77.40	1126.40	1115.43
100	20	20	78.59	748.45	78.58	78.58	893.87	886.57
200	20	20	79.25	700.26	79.25	79.25	770.33	765.96
400	20	20	79.62	669.05	79.61	79.61	706.13	703.71
1000	20	20	79.84	655.51	79.84	79.84	666.70	665.67
				$(p_1,p_2,p_3)=$(2,2,4)				
20	10	10	73.31	681.31	72.93	72.94	1578.67	1556.69
30	10	10	75.32	677.00	75.15	75.15	1264.00	1245.69
40	10	10	76.40	672.41	76.29	76.30	1107.20	1091.51
50	10	10	77.06	669.48	77.00	77.00	1013.33	999.61
100	10	10	78.47	658.74	78.46	78.46	826.18	817.76
200	10	10	79.22	650.87	79.21	79.22	732.95	728.20
400	10	10	79.61	645.53	79.60	79.60	686.44	683.90
1000	10	10	79.84	644.04	79.84	79.84	658.57	657.51
20	20	10	73.67	969.05	73.28	73.29	2064.00	2045.69
30	20	10	75.53	895.90	75.36	75.36	1587.20	1571.51
40	20	10	76.53	848.83	76.44	76.44	1349.33	1335.61
50	20	10	77.16	820.22	77.10	77.11	1206.86	1194.66
100	20	10	78.50	744.40	78.49	78.49	922.67	914.85
200	20	10	79.23	695.28	79.23	79.23	781.09	776.54
400	20	10	79.61	669.76	79.61	79.61	710.48	708.00
1000	20	10	79.84	651.97	79.84	79.84	668.17	667.13

Table 1:(Continued)

			Simulation		Approximation			
N_1	N_2	N_3	$\mathrm{E}[b_{2,p_1,p_2,p_3}]$	$N\mathrm{Var}[b_{2,p_1,p_2,p_3}]$	$m_1^{(3)}$	$m_2^{(3)}$	$(\sigma^{(3)})^2$	$N(\nu^{(3)})^2$
				(p_1, p_2, p_3) =(2,2,4)				
20	10	20	73.41	980.34	73.01	73.02	2037.34	2019.02
30	10	20	75.37	912.41	75.20	75.20	1568.00	1552.31
40	10	20	76.43	868.18	76.33	76.33	1334.40	1320.68
50	10	20	77.10	838.06	77.03	77.03	1194.66	1182.48
100	10	20	78.48	759.08	78.47	78.47	916.36	908.55
200	10	20	79.22	705.74	79.22	79.22	777.90	773.35
400	10	20	79.60	673.07	79.60	79.60	708.88	706.40
1000	10	20	79.84	654.41	79.84	79.84	667.53	666.49
20	20	20	73.72	1263.89	73.33	73.34	2528.00	2512.31
30	20	20	75.57	1131.56	75.40	75.40	1894.40	1880.68
40	20	20	76.57	1047.13	76.47	76.47	1578.66	1566.48
50	20	20	77.18	985.58	77.13	77.13	1389.72	1378.75
100	20	20	78.52	840.46	78.50	78.50	1013.33	1006.04
200	20	20	79.23	750.63	79.23	79.23	826.18	821.81
400	20	20	79.61	698.36	79.61	79.61	732.95	730.53
1000	20	20	79.84	663.23	79.84	79.84	677.15	676.11

Table 2: Expectations, variances, skewness, and kurtosis for the $Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$, and $Z_{MM}^{(3)**}$ test statistics given in (5), (6), and (7).

			Expectation			Variance			
N_1	N_2	N_3	$Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	$Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Skewness	Kurtosis
					(p_1, p_2, p_3) =(2,2,2)				
20	10	10	-0.849	0.046	0.044	0.472	0.487	0.465	3.392
30	10	10	-0.740	0.026	0.025	0.575	0.592	0.486	3.454
40	10	10	-0.667	0.017	0.016	0.646	0.665	0.485	3.465
50	10	10	-0.611	0.012	0.011	0.694	0.713	0.478	3.465
100	10	10	-0.458	0.004	0.003	0.820	0.836	0.432	3.405
200	10	10	-0.334	0.001	0.001	0.902	0.912	0.343	3.268
400	10	10	-0.240	0.001	0.000	0.948	0.954	0.260	3.145
1000	10	10	-0.153	0.001	0.001	0.977	0.979	0.178	3.072
20	20	10	-0.787	0.045	0.044	0.505	0.515	0.479	3.412
30	20	10	-0.696	0.023	0.022	0.600	0.613	0.484	3.465
40	20	10	-0.630	0.016	0.015	0.664	0.678	0.481	3.456
50	20	10	-0.582	0.011	0.011	0.712	0.727	0.470	3.452
100	20	10	-0.444	0.003	0.003	0.829	0.842	0.420	3.370
200	20	10	-0.327	0.002	0.002	0.907	0.916	0.342	3.252
400	20	10	-0.237	0.002	0.001	0.949	0.955	0.266	3.167
1000	20	10	-0.152	0.001	0.001	0.979	0.982	0.178	3.075
20	10	20	-0.824	0.045	0.044	0.521	0.531	0.452	3.376
30	10	20	-0.720	0.026	0.025	0.624	0.637	0.457	3.405
40	10	20	-0.649	0.017	0.016	0.686	0.701	0.456	3.417
50	10	20	-0.597	0.012	0.011	0.734	0.748	0.450	3.408
100	10	20	-0.451	0.003	0.003	0.847	0.860	0.413	3.378
200	10	20	-0.332	0.001	0.000	0.917	0.926	0.335	3.249
400	10	20	-0.238	0.002	0.002	0.956	0.961	0.263	3.163
1000	10	20	-0.153	0.000	0.000	0.982	0.985	0.171	3.073
20	20	20	-0.770	0.044	0.043	0.538	0.546	0.464	3.401
30	20	20	-0.679	0.025	0.024	0.636	0.647	0.460	3.408
40	20	20	-0.618	0.015	0.014	0.697	0.709	0.458	3.431
50	20	20	-0.570	0.012	0.012	0.743	0.756	0.456	3.446
100	20	20	-0.437	0.004	0.003	0.853	0.864	0.410	3.375
200	20	20	-0.324	0.002	0.002	0.921	0.930	0.336	3.254
400	20	20	-0.237	0.000	0.000	0.960	0.965	0.259	3.143
1000	20	20	-0.153	0.000	0.000	0.982	0.984	0.170	3.074
					(p_1, p_2, p_3) =(4,2,2)				
20	10	10	-1.054	0.054	0.051	0.452	0.484	0.394	3.258
30	10	10	-0.928	0.031	0.028	0.550	0.588	0.410	3.296
40	10	10	-0.838	0.022	0.019	0.620	0.661	0.418	3.328
50	10	10	-0.773	0.015	0.013	0.670	0.711	0.421	3.349
100	10	10	-0.583	0.006	0.005	0.804	0.838	0.379	3.305
200	10	10	-0.429	0.001	0.001	0.890	0.913	0.307	3.213
400	10	10	-0.309	0.002	0.001	0.942	0.956	0.233	3.115
1000	10	10	-0.199	0.000	0.000	0.974	0.980	0.151	3.052

Table 2:(Continued)

N_1	N_2	N_3	Expectation $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Variance $Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Skewness	Kurtosis
						(p_1, p_2, p_3) =(4,2,2)			
20	20	10	-0.969	0.053	0.050	0.490	0.515	0.407	3.287
30	20	10	-0.863	0.030	0.028	0.583	0.614	0.417	3.319
40	20	10	-0.789	0.020	0.017	0.646	0.679	0.417	3.337
50	20	10	-0.731	0.016	0.014	0.690	0.724	0.409	3.323
100	20	10	-0.565	0.004	0.003	0.811	0.842	0.372	3.289
200	20	10	-0.421	0.002	0.001	0.891	0.913	0.294	3.179
400	20	10	-0.305	0.002	0.002	0.944	0.957	0.230	3.119
1000	20	10	-0.198	-0.001	-0.001	0.974	0.980	0.155	3.058
20	10	20	-1.004	0.054	0.052	0.521	0.546	0.378	3.259
30	10	20	-0.888	0.030	0.028	0.614	0.645	0.387	3.277
40	10	20	-0.806	0.021	0.019	0.678	0.711	0.391	3.290
50	10	20	-0.745	0.015	0.013	0.723	0.757	0.383	3.292
100	10	20	-0.571	0.004	0.003	0.836	0.866	0.358	3.272
200	10	20	-0.421	0.003	0.003	0.908	0.929	0.296	3.201
400	10	20	-0.306	0.002	0.002	0.950	0.963	0.233	3.134
1000	10	20	-0.195	0.003	0.003	0.979	0.985	0.153	3.054
20	20	20	-0.936	0.051	0.049	0.540	0.559	0.391	3.272
30	20	20	-0.835	0.028	0.027	0.628	0.652	0.390	3.302
40	20	20	-0.763	0.020	0.018	0.690	0.717	0.390	3.296
50	20	20	-0.709	0.015	0.014	0.733	0.761	0.389	3.309
100	20	20	-0.552	0.005	0.005	0.839	0.866	0.349	3.256
200	20	20	-0.415	0.002	0.001	0.909	0.929	0.295	3.193
400	20	20	-0.304	0.001	0.001	0.952	0.965	0.225	3.108
1000	20	20	-0.197	0.000	0.000	0.981	0.987	0.150	3.042
						(p_1, p_2, p_3) =(8,2,2)			
20	10	10	-1.470	0.070	0.061	0.374	0.442	0.348	3.184
30	10	10	-1.307	0.041	0.033	0.470	0.552	0.355	3.199
40	10	10	-1.190	0.028	0.021	0.542	0.631	0.357	3.216
50	10	10	-1.100	0.020	0.014	0.598	0.687	0.359	3.228
100	10	10	-0.837	0.008	0.005	0.748	0.823	0.324	3.198
200	10	10	-0.618	0.003	0.002	0.856	0.906	0.260	3.134
400	10	10	-0.449	0.000	-0.001	0.921	0.950	0.197	3.075
1000	10	10	-0.289	-0.002	-0.002	0.968	0.981	0.127	3.029
20	20	10	-1.342	0.064	0.059	0.430	0.486	0.349	3.200
30	20	10	-1.210	0.037	0.032	0.516	0.585	0.357	3.210
40	20	10	-1.111	0.027	0.022	0.580	0.654	0.354	3.223
50	20	10	-1.036	0.019	0.015	0.627	0.703	0.348	3.213
100	20	10	-0.805	0.010	0.008	0.762	0.830	0.314	3.181
200	20	10	-0.604	0.004	0.003	0.862	0.910	0.260	3.143
400	20	10	-0.442	0.001	0.001	0.924	0.953	0.203	3.091
1000	20	10	-0.285	0.001	0.001	0.966	0.979	0.133	3.038

Table 2:(Continued)

N_1	N_2	N_3	Expectation $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Variance $Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Skewness	Kurtosis
					(p_1, p_2, p_3) =(8,2,2)				
20	10	20	-1.371	0.070	0.065	0.468	0.524	0.320	3.157
30	10	20	-1.230	0.040	0.034	0.553	0.620	0.326	3.175
40	10	20	-1.126	0.028	0.023	0.616	0.690	0.336	3.188
50	10	20	-1.048	0.020	0.016	0.660	0.736	0.329	3.188
100	10	20	-0.811	0.008	0.006	0.787	0.854	0.307	3.183
200	10	20	-0.607	0.003	0.002	0.878	0.926	0.256	3.126
400	10	20	-0.443	0.001	0.000	0.933	0.962	0.198	3.081
1000	10	20	-0.285	0.001	0.001	0.970	0.983	0.127	3.036
20	20	20	-1.272	0.065	0.061	0.497	0.542	0.327	3.174
30	20	20	-1.152	0.037	0.033	0.580	0.636	0.333	3.192
40	20	20	-1.063	0.025	0.022	0.637	0.700	0.332	3.204
50	20	20	-0.994	0.020	0.016	0.679	0.744	0.326	3.196
100	20	20	-0.785	0.007	0.005	0.795	0.856	0.301	3.164
200	20	20	-0.594	0.004	0.003	0.879	0.924	0.243	3.111
400	20	20	-0.438	0.001	0.000	0.934	0.961	0.192	3.070
1000	20	20	-0.284	0.001	0.001	0.974	0.986	0.132	3.044
					(p_1, p_2, p_3) =(8,4,2)				
20	10	10	-1.662	0.078	0.071	0.368	0.413	0.279	3.108
30	10	10	-1.483	0.046	0.040	0.460	0.515	0.295	3.131
40	10	10	-1.353	0.032	0.026	0.529	0.587	0.308	3.151
50	10	10	-1.253	0.023	0.018	0.584	0.643	0.312	3.169
100	10	10	-0.959	0.009	0.007	0.738	0.791	0.289	3.147
200	10	10	-0.710	0.003	0.002	0.847	0.883	0.238	3.117
400	10	10	-0.516	0.001	0.000	0.920	0.942	0.189	3.070
1000	10	10	-0.331	0.001	0.000	0.966	0.976	0.124	3.038
20	20	10	-1.521	0.071	0.066	0.421	0.460	0.294	3.124
30	20	10	-1.376	0.040	0.036	0.504	0.551	0.303	3.143
40	20	10	-1.267	0.028	0.024	0.567	0.618	0.309	3.161
50	20	10	-1.182	0.022	0.018	0.616	0.668	0.307	3.158
100	20	10	-0.925	0.008	0.006	0.751	0.798	0.284	3.152
200	20	10	-0.696	0.003	0.002	0.854	0.888	0.237	3.107
400	20	10	-0.512	-0.001	-0.002	0.920	0.941	0.187	3.063
1000	20	10	-0.330	0.000	0.000	0.968	0.977	0.127	3.030
20	10	20	-1.541	0.077	0.073	0.479	0.516	0.248	3.089
30	10	20	-1.384	0.046	0.042	0.561	0.605	0.263	3.116
40	10	20	-1.271	0.031	0.027	0.620	0.668	0.273	3.132
50	10	20	-1.184	0.023	0.019	0.664	0.714	0.278	3.133
100	10	20	-0.924	0.009	0.007	0.785	0.832	0.263	3.137
200	10	20	-0.695	0.003	0.002	0.877	0.911	0.229	3.105
400	10	20	-0.510	0.001	0.000	0.931	0.951	0.181	3.066
1000	10	20	-0.330	-0.001	-0.001	0.971	0.981	0.127	3.039

Table 2:(Continued)

			Expectation			Variance			
N_1	N_2	N_3	$Z^{(3)}_{MM}$	$Z^{(3)*}_{MM}$	$Z^{(3)**}_{MM}$	$Z^{(3)}_{MM}$, $Z^{(3)*}_{MM}$	$Z^{(3)**}_{MM}$	Skewness	Kurtosis
					(p_1, p_2, p_3) =(8,4,2)				
20	20	20	-1.433	0.071	0.067	0.505	0.536	0.262	3.106
30	20	20	-1.300	0.041	0.038	0.584	0.623	0.272	3.120
40	20	20	-1.204	0.027	0.024	0.639	0.682	0.278	3.143
50	20	20	-1.126	0.022	0.019	0.682	0.726	0.279	3.140
100	20	20	-0.892	0.010	0.008	0.797	0.840	0.267	3.135
200	20	20	-0.683	0.002	0.001	0.880	0.912	0.226	3.102
400	20	20	-0.503	0.002	0.001	0.934	0.954	0.184	3.071
1000	20	20	-0.327	0.001	0.001	0.972	0.981	0.127	3.030
					(p_1, p_2, p_3) =(3,3,3)				
20	10	10	-1.165	0.063	0.061	0.438	0.449	0.315	3.167
30	10	10	-1.025	0.036	0.034	0.538	0.553	0.339	3.212
40	10	10	-0.926	0.025	0.024	0.608	0.624	0.353	3.229
50	10	10	-0.853	0.019	0.017	0.661	0.677	0.353	3.238
100	10	10	-0.647	0.006	0.005	0.795	0.809	0.336	3.234
200	10	10	-0.478	0.000	0.000	0.886	0.896	0.281	3.169
400	10	10	-0.344	0.000	0.000	0.943	0.949	0.220	3.109
1000	10	10	-0.220	0.001	0.001	0.977	0.979	0.145	3.049
20	20	10	-1.068	0.061	0.060	0.476	0.484	0.334	3.192
30	20	10	-0.951	0.034	0.032	0.567	0.578	0.347	3.224
40	20	10	-0.868	0.023	0.022	0.633	0.646	0.356	3.251
50	20	10	-0.807	0.015	0.015	0.679	0.692	0.352	3.254
100	20	10	-0.623	0.005	0.005	0.802	0.815	0.325	3.218
200	20	10	-0.465	0.003	0.003	0.891	0.900	0.279	3.165
400	20	10	-0.340	0.000	0.000	0.941	0.946	0.216	3.098
1000	20	10	-0.218	0.001	0.001	0.972	0.975	0.144	3.039
20	10	20	-1.119	0.064	0.063	0.507	0.515	0.299	3.153
30	10	20	-0.989	0.035	0.034	0.603	0.614	0.311	3.167
40	10	20	-0.896	0.024	0.023	0.670	0.683	0.319	3.206
50	10	20	-0.828	0.018	0.017	0.721	0.735	0.328	3.214
100	10	20	-0.632	0.007	0.006	0.834	0.847	0.312	3.214
200	10	20	-0.470	0.001	0.001	0.910	0.920	0.265	3.160
400	10	20	-0.341	0.001	0.001	0.954	0.960	0.214	3.110
1000	10	20	-0.221	-0.001	-0.001	0.979	0.982	0.152	3.044
20	20	20	-1.039	0.061	0.060	0.521	0.527	0.325	3.196
30	20	20	-0.925	0.034	0.033	0.615	0.624	0.325	3.197
40	20	20	-0.845	0.023	0.022	0.678	0.688	0.327	3.219
50	20	20	-0.784	0.018	0.017	0.724	0.735	0.323	3.223
100	20	20	-0.611	0.006	0.006	0.839	0.850	0.307	3.204
200	20	20	-0.461	0.000	0.000	0.910	0.918	0.257	3.135
400	20	20	-0.337	0.001	0.001	0.953	0.958	0.218	3.113
1000	20	20	-0.218	0.001	0.001	0.979	0.982	0.147	3.046

Table 2:(Continued)

N_1	N_2	N_3	Expectation $Z^{(3)}_{MM}$	$Z^{(3)*}_{MM}$	$Z^{(3)**}_{MM}$	Variance $Z^{(3)}_{MM}$, $Z^{(3)*}_{MM}$	$Z^{(3)**}_{MM}$	Skewness	Kurtosis
					(p_1, p_2, p_3) =(5,5,5)				
20	10	10	-1.714	0.095	0.093	0.396	0.405	0.183	3.045
30	10	10	-1.531	0.055	0.053	0.483	0.496	0.199	3.052
40	10	10	-1.401	0.037	0.035	0.550	0.564	0.216	3.085
50	10	10	-1.300	0.027	0.026	0.605	0.620	0.229	3.092
100	10	10	-1.005	0.009	0.008	0.751	0.765	0.243	3.122
200	10	10	-0.749	0.004	0.003	0.858	0.868	0.221	3.107
400	10	10	-0.545	0.002	0.002	0.925	0.931	0.176	3.059
1000	10	10	-0.351	0.001	0.001	0.968	0.971	0.124	3.026
20	20	10	-1.549	0.090	0.088	0.434	0.440	0.213	3.077
30	20	10	-1.399	0.050	0.048	0.517	0.526	0.214	3.080
40	20	10	-1.290	0.035	0.034	0.579	0.589	0.221	3.090
50	20	10	-1.209	0.024	0.023	0.627	0.638	0.236	3.108
100	20	10	-0.955	0.010	0.010	0.763	0.775	0.241	3.121
200	20	10	-0.727	0.004	0.004	0.864	0.873	0.214	3.092
400	20	10	-0.539	0.000	0.000	0.925	0.930	0.171	3.047
1000	20	10	-0.350	0.000	0.000	0.968	0.971	0.122	3.025
20	10	20	-1.628	0.096	0.094	0.493	0.501	0.180	3.044
30	10	20	-1.457	0.054	0.052	0.582	0.591	0.181	3.058
40	10	20	-1.336	0.037	0.035	0.644	0.655	0.185	3.057
50	10	20	-1.245	0.025	0.024	0.690	0.702	0.193	3.061
100	10	20	-0.971	0.010	0.010	0.813	0.826	0.210	3.088
200	10	20	-0.734	0.003	0.003	0.897	0.907	0.196	3.084
400	10	20	-0.541	0.000	0.000	0.942	0.948	0.168	3.064
1000	10	20	-0.349	0.001	0.001	0.979	0.981	0.117	3.026
20	20	20	-1.496	0.090	0.089	0.499	0.504	0.197	3.062
30	20	20	-1.348	0.051	0.050	0.584	0.592	0.198	3.063
40	20	20	-1.245	0.034	0.033	0.649	0.657	0.203	3.069
50	20	20	-1.166	0.025	0.024	0.696	0.705	0.204	3.078
100	20	20	-0.929	0.010	0.009	0.819	0.830	0.213	3.100
200	20	20	-0.714	0.003	0.003	0.897	0.906	0.197	3.083
400	20	20	-0.531	0.002	0.001	0.945	0.951	0.167	3.057
1000	20	20	-0.348	0.001	0.001	0.977	0.980	0.118	3.028
					(p_1, p_2, p_3) =(2,4,2)				
20	10	10	-1.073	0.054	0.053	0.441	0.449	0.349	3.229
30	10	10	-0.942	0.030	0.029	0.543	0.552	0.372	3.257
40	10	10	-0.848	0.022	0.021	0.612	0.622	0.388	3.297
50	10	10	-0.781	0.015	0.014	0.663	0.673	0.397	3.328
100	10	10	-0.585	0.007	0.007	0.799	0.808	0.366	3.289
200	10	10	-0.431	0.001	0.001	0.891	0.897	0.302	3.189
400	10	10	-0.310	0.001	0.001	0.943	0.947	0.233	3.117
1000	10	10	-0.197	0.001	0.001	0.979	0.981	0.150	3.048

Table 2:(Continued)

N_1	N_2	N_3	Expectation $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Variance $Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Skewness	Kurtosis
					(p_1, p_2, p_3) =(2,4,2)				
20	20	10	-0.984	0.051	0.050	0.483	0.489	0.380	3.274
30	20	10	-0.874	0.029	0.028	0.576	0.583	0.390	3.297
40	20	10	-0.798	0.019	0.018	0.640	0.649	0.397	3.316
50	20	10	-0.739	0.013	0.012	0.686	0.695	0.390	3.319
100	20	10	-0.566	0.006	0.006	0.808	0.816	0.365	3.282
200	20	10	-0.423	0.001	0.001	0.890	0.896	0.298	3.196
400	20	10	-0.306	0.001	0.001	0.943	0.947	0.226	3.108
1000	20	10	-0.198	0.000	0.000	0.975	0.976	0.154	3.054
20	10	20	-1.035	0.055	0.054	0.502	0.508	0.332	3.205
30	10	20	-0.909	0.032	0.031	0.598	0.606	0.352	3.247
40	10	20	-0.822	0.022	0.021	0.663	0.671	0.353	3.256
50	10	20	-0.759	0.014	0.014	0.711	0.719	0.361	3.274
100	10	20	-0.576	0.005	0.005	0.828	0.836	0.342	3.254
200	10	20	-0.425	0.002	0.002	0.906	0.911	0.288	3.185
400	10	20	-0.308	0.000	0.000	0.951	0.954	0.223	3.101
1000	10	20	-0.199	-0.001	-0.001	0.977	0.979	0.153	3.051
20	20	20	-0.958	0.051	0.050	0.526	0.530	0.354	3.230
30	20	20	-0.849	0.031	0.031	0.619	0.625	0.358	3.255
40	20	20	-0.777	0.019	0.019	0.679	0.686	0.365	3.270
50	20	20	-0.720	0.015	0.014	0.725	0.732	0.364	3.284
100	20	20	-0.557	0.005	0.005	0.837	0.844	0.336	3.254
200	20	20	-0.416	0.003	0.003	0.909	0.914	0.286	3.179
400	20	20	-0.302	0.003	0.003	0.947	0.951	0.224	3.108
1000	20	20	-0.199	-0.002	-0.002	0.983	0.985	0.156	3.062
					(p_1, p_2, p_3) =(2,2,4)				
20	10	10	-1.065	0.060	0.058	0.432	0.438	0.319	3.170
30	10	10	-0.931	0.034	0.033	0.536	0.543	0.348	3.225
40	10	10	-0.839	0.024	0.023	0.607	0.616	0.359	3.258
50	10	10	-0.774	0.015	0.015	0.661	0.670	0.367	3.270
100	10	10	-0.582	0.007	0.006	0.797	0.806	0.355	3.284
200	10	10	-0.429	0.001	0.001	0.888	0.894	0.299	3.190
400	10	10	-0.308	0.002	0.002	0.940	0.944	0.229	3.112
1000	10	10	-0.199	-0.001	-0.001	0.978	0.980	0.155	3.054
20	20	10	-0.985	0.061	0.060	0.470	0.474	0.344	3.206
30	20	10	-0.868	0.034	0.033	0.564	0.570	0.342	3.215
40	20	10	-0.790	0.021	0.021	0.629	0.636	0.351	3.245
50	20	10	-0.731	0.014	0.014	0.680	0.687	0.358	3.270
100	20	10	-0.562	0.005	0.004	0.807	0.814	0.342	3.253
200	20	10	-0.419	0.002	0.002	0.890	0.895	0.285	3.176
400	20	10	-0.305	0.001	0.001	0.943	0.946	0.223	3.108
1000	20	10	-0.196	0.001	0.001	0.976	0.977	0.155	3.058

Table 2:(Continued)

N_1	N_2	N_3	Expectation $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Variance $Z_{MM}^{(3)}$, $Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Skewness	Kurtosis
					(p_1, p_2, p_3) =(2,2,4)				
20	10	20	-1.033	0.062	0.061	0.481	0.486	0.326	3.194
30	10	20	-0.906	0.033	0.032	0.582	0.588	0.324	3.187
40	10	20	-0.818	0.023	0.022	0.651	0.657	0.342	3.247
50	10	20	-0.751	0.018	0.018	0.702	0.709	0.343	3.250
100	10	20	-0.571	0.007	0.006	0.828	0.835	0.330	3.235
200	10	20	-0.424	0.002	0.002	0.907	0.913	0.286	3.184
400	10	20	-0.308	0.000	0.000	0.949	0.953	0.225	3.117
1000	10	20	-0.197	0.001	0.001	0.980	0.982	0.154	3.054
20	20	20	-0.967	0.060	0.059	0.500	0.503	0.342	3.208
30	20	20	-0.852	0.032	0.032	0.597	0.602	0.340	3.224
40	20	20	-0.772	0.023	0.023	0.663	0.668	0.340	3.225
50	20	20	-0.717	0.015	0.015	0.709	0.715	0.343	3.246
100	20	20	-0.552	0.006	0.006	0.829	0.835	0.324	3.228
200	20	20	-0.413	0.003	0.003	0.909	0.913	0.274	3.169
400	20	20	-0.304	0.000	0.000	0.953	0.956	0.222	3.115
1000	20	20	-0.197	0.000	0.000	0.979	0.981	0.150	3.050

Table 3: Empirical type I error of the $Z^{(3)}_{MM}$, $Z^{(3)*}_{MM}$, and $Z^{(3)**}_{MM}$ test statistics given in (5), (6), and (7) for $\alpha = 0.05$.

			Empirical type I error			Percentiles					
N_1	N_2	N_3	$Z^{(3)}_{MM}$	$Z^{(3)*}_{MM}$	$Z^{(3)**}_{MM}$	$Z^{(3)}_{MM}$		$Z^{(3)*}_{MM}$		$Z^{(3)**}_{MM}$	
						Lower	Upper	Lower	Upper	Lower	Upper
					(p_1, p_2, p_3) =(2,2,2)						
20	10	10	0.037	0.008	0.009	-2.05	0.65	-1.15	1.54	-1.17	1.56
30	10	10	0.039	0.013	0.015	-2.06	0.91	-1.29	1.68	-1.31	1.70
40	10	10	0.040	0.018	0.019	-2.06	1.09	-1.38	1.77	-1.40	1.79
50	10	10	0.041	0.021	0.022	-2.06	1.20	-1.44	1.83	-1.46	1.85
100	10	10	0.044	0.031	0.032	-2.05	1.49	-1.59	1.96	-1.61	1.97
200	10	10	0.047	0.039	0.040	-2.04	1.68	-1.71	2.01	-1.72	2.02
400	10	10	0.048	0.044	0.045	-2.03	1.79	-1.79	2.03	-1.79	2.04
1000	10	10	0.049	0.047	0.047	-2.01	1.87	-1.86	2.02	-1.86	2.02
20	20	10	0.033	0.010	0.010	-2.02	0.76	-1.19	1.60	-1.20	1.61
30	20	10	0.037	0.015	0.016	-2.04	0.99	-1.32	1.71	-1.34	1.73
40	20	10	0.039	0.019	0.020	-2.05	1.14	-1.40	1.79	-1.42	1.81
50	20	10	0.040	0.022	0.023	-2.05	1.25	-1.46	1.84	-1.48	1.86
100	20	10	0.044	0.032	0.033	-2.05	1.52	-1.61	1.96	-1.62	1.98
200	20	10	0.046	0.039	0.040	-2.04	1.68	-1.71	2.01	-1.72	2.02
400	20	10	0.048	0.044	0.045	-2.03	1.79	-1.79	2.03	-1.80	2.04
1000	20	10	0.049	0.047	0.048	-2.01	1.87	-1.86	2.02	-1.86	2.03
20	10	20	0.043	0.010	0.011	-2.08	0.74	-1.22	1.61	-1.23	1.62
30	10	20	0.044	0.016	0.017	-2.10	0.99	-1.35	1.74	-1.37	1.76
40	10	20	0.045	0.020	0.022	-2.10	1.14	-1.43	1.81	-1.45	1.83
50	10	20	0.046	0.024	0.025	-2.10	1.26	-1.49	1.87	-1.51	1.88
100	10	20	0.047	0.033	0.034	-2.08	1.52	-1.63	1.98	-1.64	1.99
200	10	20	0.048	0.040	0.041	-2.06	1.69	-1.73	2.03	-1.74	2.04
400	10	20	0.049	0.044	0.045	-2.03	1.80	-1.79	2.04	-1.80	2.04
1000	10	20	0.050	0.048	0.048	-2.02	1.87	-1.86	2.02	-1.87	2.03
20	20	20	0.037	0.011	0.012	-2.05	0.82	-1.23	1.64	-1.24	1.65
30	20	20	0.041	0.017	0.018	-2.07	1.05	-1.37	1.75	-1.38	1.77
40	20	20	0.043	0.021	0.022	-2.08	1.19	-1.45	1.82	-1.46	1.84
50	20	20	0.044	0.024	0.025	-2.08	1.29	-1.50	1.88	-1.51	1.89
100	20	20	0.047	0.034	0.035	-2.08	1.54	-1.64	1.98	-1.65	2.00
200	20	20	0.048	0.041	0.042	-2.06	1.70	-1.73	2.03	-1.74	2.04
400	20	20	0.049	0.045	0.046	-2.04	1.80	-1.80	2.04	-1.81	2.04
1000	20	20	0.050	0.048	0.048	-2.02	1.87	-1.86	2.02	-1.87	2.02

Table 3:(Continued)

N_1	N_2	N_3	Empirical type I error $Z^{(3)}_{MM}$	$Z^{(3)*}_{MM}$	$Z^{(3)**}_{MM}$	Percentiles $Z^{(3)}_{MM}$ Lower	$Z^{(3)}_{MM}$ Upper	$Z^{(3)*}_{MM}$ Lower	$Z^{(3)*}_{MM}$ Upper	$Z^{(3)**}_{MM}$ Lower	$Z^{(3)**}_{MM}$ Upper
					(p_1, p_2, p_3) =(4,2,2)						
20	10	10	0.077	0.006	0.008	-2.24	0.38	-1.14	1.49	-1.18	1.54
30	10	10	0.069	0.011	0.013	-2.24	0.66	-1.28	1.62	-1.33	1.67
40	10	10	0.065	0.015	0.018	-2.23	0.85	-1.37	1.71	-1.41	1.77
50	10	10	0.062	0.019	0.022	-2.22	0.99	-1.43	1.78	-1.47	1.83
100	10	10	0.056	0.029	0.032	-2.18	1.33	-1.60	1.92	-1.63	1.96
200	10	10	0.053	0.038	0.040	-2.15	1.55	-1.71	1.98	-1.74	2.01
400	10	10	0.051	0.043	0.045	-2.11	1.70	-1.80	2.01	-1.81	2.02
1000	10	10	0.050	0.047	0.048	-2.06	1.81	-1.87	2.01	-1.87	2.01
20	20	10	0.066	0.008	0.010	-2.20	0.53	-1.18	1.55	-1.21	1.59
30	20	10	0.063	0.013	0.015	-2.21	0.78	-1.32	1.67	-1.35	1.71
40	20	10	0.061	0.017	0.019	-2.21	0.94	-1.40	1.75	-1.44	1.79
50	20	10	0.059	0.020	0.023	-2.20	1.05	-1.45	1.80	-1.49	1.84
100	20	10	0.055	0.030	0.033	-2.18	1.35	-1.61	1.92	-1.64	1.96
200	20	10	0.053	0.038	0.040	-2.14	1.56	-1.72	1.98	-1.74	2.00
400	20	10	0.052	0.044	0.045	-2.11	1.71	-1.80	2.01	-1.81	2.03
1000	20	10	0.050	0.047	0.047	-2.06	1.81	-1.86	2.01	-1.87	2.01
20	10	20	0.082	0.010	0.011	-2.29	0.54	-1.23	1.59	-1.26	1.63
30	10	20	0.075	0.015	0.017	-2.28	0.78	-1.36	1.70	-1.40	1.74
40	10	20	0.071	0.019	0.022	-2.27	0.96	-1.44	1.78	-1.48	1.82
50	10	20	0.068	0.022	0.025	-2.26	1.07	-1.50	1.83	-1.54	1.87
100	10	20	0.060	0.032	0.035	-2.21	1.37	-1.64	1.95	-1.67	1.98
200	10	20	0.055	0.040	0.042	-2.16	1.58	-1.74	2.00	-1.76	2.02
400	10	20	0.052	0.044	0.045	-2.11	1.71	-1.80	2.02	-1.81	2.03
1000	10	20	0.051	0.047	0.048	-2.06	1.81	-1.86	2.01	-1.87	2.02
20	20	20	0.070	0.011	0.012	-2.24	0.64	-1.25	1.62	-1.28	1.65
30	20	20	0.067	0.016	0.017	-2.25	0.86	-1.38	1.72	-1.41	1.75
40	20	20	0.065	0.020	0.022	-2.24	1.01	-1.46	1.80	-1.49	1.83
50	20	20	0.063	0.023	0.026	-2.23	1.12	-1.51	1.85	-1.54	1.88
100	20	20	0.058	0.033	0.035	-2.20	1.39	-1.64	1.94	-1.67	1.97
200	20	20	0.055	0.039	0.042	-2.15	1.58	-1.74	2.00	-1.76	2.02
400	20	20	0.052	0.044	0.045	-2.11	1.71	-1.81	2.01	-1.82	2.02
1000	20	20	0.051	0.048	0.048	-2.07	1.81	-1.87	2.01	-1.88	2.02

Table 3:(Continued)

N_1	N_2	N_3	Empirical type I error $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Percentiles $Z_{MM}^{(3)}$ Lower	$Z_{MM}^{(3)}$ Upper	$Z_{MM}^{(3)*}$ Lower	$Z_{MM}^{(3)*}$ Upper	$Z_{MM}^{(3)**}$ Lower	$Z_{MM}^{(3)**}$ Upper
					(p_1, p_2, p_3) =(8,2,2)						
20	10	10	0.217	0.003	0.006	-2.56	-0.18	-1.02	1.36	-1.13	1.47
30	10	10	0.170	0.006	0.010	-2.53	0.15	-1.18	1.50	-1.30	1.61
40	10	10	0.145	0.010	0.015	-2.51	0.38	-1.29	1.59	-1.40	1.71
50	10	10	0.128	0.013	0.019	-2.48	0.54	-1.36	1.66	-1.47	1.77
100	10	10	0.091	0.024	0.031	-2.40	0.98	-1.55	1.83	-1.63	1.92
200	10	10	0.071	0.034	0.039	-2.32	1.31	-1.70	1.93	-1.75	1.98
400	10	10	0.061	0.041	0.044	-2.24	1.52	-1.79	1.97	-1.82	2.00
1000	10	10	0.055	0.046	0.048	-2.16	1.70	-1.87	1.99	-1.88	2.00
20	20	10	0.173	0.005	0.008	-2.52	0.05	-1.11	1.45	-1.19	1.54
30	20	10	0.145	0.009	0.012	-2.49	0.32	-1.25	1.56	-1.34	1.65
40	20	10	0.127	0.012	0.017	-2.48	0.50	-1.34	1.64	-1.43	1.74
50	20	10	0.116	0.015	0.020	-2.46	0.64	-1.40	1.70	-1.49	1.79
100	20	10	0.087	0.025	0.032	-2.39	1.04	-1.57	1.85	-1.64	1.93
200	20	10	0.070	0.035	0.040	-2.31	1.33	-1.70	1.94	-1.75	1.99
400	20	10	0.061	0.041	0.044	-2.24	1.53	-1.79	1.98	-1.82	2.01
1000	20	10	0.054	0.046	0.047	-2.15	1.70	-1.87	1.99	-1.88	2.00
20	10	20	0.198	0.007	0.009	-2.60	0.07	-1.16	1.51	-1.24	1.59
30	10	20	0.162	0.010	0.015	-2.57	0.34	-1.30	1.61	-1.39	1.70
40	10	20	0.141	0.014	0.020	-2.54	0.53	-1.38	1.69	-1.47	1.78
50	10	20	0.126	0.017	0.023	-2.51	0.67	-1.45	1.74	-1.53	1.83
100	10	20	0.092	0.028	0.034	-2.42	1.05	-1.60	1.87	-1.67	1.95
200	10	20	0.073	0.036	0.041	-2.33	1.34	-1.72	1.95	-1.77	2.00
400	10	20	0.062	0.042	0.045	-2.24	1.54	-1.80	1.98	-1.83	2.01
1000	10	20	0.055	0.047	0.048	-2.16	1.71	-1.87	1.99	-1.88	2.01
20	20	20	0.164	0.008	0.010	-2.54	0.21	-1.21	1.55	-1.27	1.61
30	20	20	0.141	0.012	0.016	-2.52	0.45	-1.33	1.64	-1.40	1.72
40	20	20	0.126	0.016	0.020	-2.50	0.62	-1.41	1.71	-1.49	1.79
50	20	20	0.115	0.019	0.024	-2.48	0.74	-1.47	1.75	-1.54	1.83
100	20	20	0.089	0.028	0.034	-2.40	1.09	-1.61	1.88	-1.67	1.95
200	20	20	0.071	0.036	0.041	-2.32	1.35	-1.73	1.95	-1.77	2.00
400	20	20	0.061	0.042	0.045	-2.25	1.54	-1.81	1.98	-1.83	2.01
1000	20	20	0.055	0.047	0.048	-2.16	1.71	-1.87	2.00	-1.89	2.01

Table 3:(Continued)

N_1	N_2	N_3	Empirical type I error $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Percentiles $Z_{MM}^{(3)}$ Lower	$Z_{MM}^{(3)}$ Upper	$Z_{MM}^{(3)*}$ Lower	$Z_{MM}^{(3)*}$ Upper	$Z_{MM}^{(3)**}$ Lower	$Z_{MM}^{(3)**}$ Upper
					(p_1, p_2, p_3) =(8,4,2)						
20	10	10	0.325	0.003	0.004	-2.77	-0.40	-1.03	1.34	-1.10	1.41
30	10	10	0.249	0.006	0.008	-2.72	-0.07	-1.19	1.46	-1.26	1.54
40	10	10	0.205	0.009	0.012	-2.67	0.18	-1.29	1.56	-1.36	1.64
50	10	10	0.178	0.012	0.016	-2.64	0.35	-1.36	1.63	-1.43	1.70
100	10	10	0.118	0.023	0.028	-2.52	0.84	-1.56	1.81	-1.61	1.87
200	10	10	0.085	0.033	0.037	-2.41	1.20	-1.70	1.91	-1.74	1.95
400	10	10	0.068	0.041	0.043	-2.31	1.45	-1.79	1.97	-1.81	1.99
1000	10	10	0.057	0.046	0.047	-2.20	1.65	-1.87	1.98	-1.88	1.99
20	20	10	0.258	0.004	0.006	-2.70	-0.16	-1.11	1.43	-1.17	1.49
30	20	10	0.209	0.008	0.010	-2.66	0.11	-1.25	1.53	-1.31	1.59
40	20	10	0.180	0.011	0.014	-2.63	0.31	-1.34	1.61	-1.40	1.68
50	20	10	0.159	0.014	0.017	-2.60	0.47	-1.40	1.67	-1.46	1.74
100	20	10	0.112	0.024	0.029	-2.51	0.89	-1.57	1.82	-1.63	1.88
200	20	10	0.083	0.034	0.037	-2.40	1.22	-1.70	1.92	-1.74	1.95
400	20	10	0.068	0.041	0.043	-2.31	1.45	-1.80	1.96	-1.82	1.99
1000	20	10	0.057	0.046	0.047	-2.20	1.66	-1.87	1.99	-1.88	2.00
20	10	20	0.282	0.006	0.008	-2.81	-0.10	-1.20	1.51	-1.25	1.56
30	10	20	0.226	0.010	0.013	-2.76	0.17	-1.33	1.60	-1.38	1.66
40	10	20	0.193	0.014	0.018	-2.71	0.37	-1.41	1.68	-1.47	1.74
50	10	20	0.171	0.017	0.021	-2.67	0.52	-1.47	1.73	-1.53	1.78
100	10	20	0.119	0.027	0.032	-2.55	0.92	-1.62	1.85	-1.67	1.90
200	10	20	0.087	0.036	0.040	-2.43	1.24	-1.73	1.94	-1.76	1.97
400	10	20	0.068	0.042	0.044	-2.32	1.46	-1.81	1.97	-1.83	1.99
1000	10	20	0.058	0.046	0.048	-2.20	1.66	-1.87	1.99	-1.88	2.00
20	20	20	0.235	0.008	0.010	-2.74	0.05	-1.23	1.55	-1.27	1.59
30	20	20	0.196	0.012	0.014	-2.70	0.30	-1.35	1.64	-1.40	1.69
40	20	20	0.172	0.015	0.018	-2.67	0.47	-1.44	1.70	-1.49	1.75
50	20	20	0.155	0.019	0.022	-2.63	0.60	-1.49	1.75	-1.54	1.80
100	20	20	0.113	0.028	0.032	-2.53	0.97	-1.63	1.87	-1.67	1.92
200	20	20	0.085	0.036	0.040	-2.42	1.25	-1.74	1.94	-1.77	1.97
400	20	20	0.068	0.042	0.045	-2.31	1.48	-1.81	1.98	-1.83	2.00
1000	20	20	0.058	0.047	0.048	-2.20	1.66	-1.87	1.99	-1.88	2.00

Table 3:(Continued)

N_1	N_2	N_3	Empirical type I error $Z^{(3)}_{MM}$	$Z^{(3)*}_{MM}$	$Z^{(3)**}_{MM}$	Percentiles $Z^{(3)}_{MM}$ Lower	Upper	$Z^{(3)*}_{MM}$ Lower	Upper	$Z^{(3)**}_{MM}$ Lower	Upper
					(p_1, p_2, p_3) =(3,3,3)						
20	10	10	0.108	0.005	0.006	-2.36	0.23	-1.13	1.46	-1.15	1.47
30	10	10	0.093	0.010	0.011	-2.35	0.53	-1.29	1.59	-1.30	1.61
40	10	10	0.083	0.014	0.015	-2.32	0.73	-1.37	1.68	-1.39	1.70
50	10	10	0.078	0.018	0.019	-2.31	0.87	-1.44	1.75	-1.46	1.77
100	10	10	0.065	0.028	0.030	-2.26	1.24	-1.60	1.89	-1.62	1.91
200	10	10	0.058	0.037	0.038	-2.20	1.49	-1.72	1.96	-1.73	1.98
400	10	10	0.054	0.043	0.044	-2.15	1.66	-1.80	2.00	-1.81	2.01
1000	10	10	0.052	0.047	0.048	-2.09	1.79	-1.87	2.01	-1.87	2.01
20	20	10	0.090	0.007	0.008	-2.31	0.39	-1.18	1.52	-1.19	1.53
30	20	10	0.081	0.012	0.012	-2.30	0.64	-1.32	1.63	-1.33	1.64
40	20	10	0.076	0.016	0.016	-2.29	0.82	-1.40	1.71	-1.42	1.73
50	20	10	0.072	0.019	0.020	-2.29	0.94	-1.46	1.76	-1.48	1.78
100	20	10	0.062	0.029	0.030	-2.24	1.27	-1.61	1.90	-1.63	1.91
200	20	10	0.057	0.038	0.039	-2.19	1.51	-1.73	1.98	-1.74	1.99
400	20	10	0.054	0.043	0.044	-2.14	1.66	-1.80	2.00	-1.81	2.01
1000	20	10	0.051	0.047	0.047	-2.08	1.78	-1.86	2.00	-1.87	2.00
20	10	20	0.114	0.008	0.009	-2.41	0.37	-1.23	1.56	-1.24	1.57
30	10	20	0.098	0.013	0.014	-2.39	0.65	-1.37	1.67	-1.39	1.68
40	10	20	0.090	0.018	0.019	-2.38	0.83	-1.46	1.75	-1.47	1.76
50	10	20	0.084	0.022	0.023	-2.36	0.97	-1.52	1.81	-1.53	1.83
100	10	20	0.069	0.032	0.033	-2.29	1.29	-1.65	1.93	-1.67	1.94
200	10	20	0.060	0.040	0.041	-2.22	1.52	-1.75	1.99	-1.76	2.00
400	10	20	0.055	0.045	0.045	-2.16	1.67	-1.82	2.01	-1.82	2.02
1000	10	20	0.052	0.047	0.048	-2.09	1.79	-1.87	2.01	-1.87	2.01
20	20	20	0.093	0.009	0.010	-2.34	0.48	-1.24	1.58	-1.25	1.59
30	20	20	0.086	0.015	0.015	-2.34	0.73	-1.38	1.69	-1.39	1.70
40	20	20	0.080	0.019	0.020	-2.33	0.90	-1.47	1.76	-1.48	1.78
50	20	20	0.077	0.022	0.023	-2.32	1.01	-1.52	1.81	-1.53	1.82
100	20	20	0.067	0.032	0.033	-2.28	1.31	-1.66	1.93	-1.67	1.94
200	20	20	0.059	0.039	0.040	-2.21	1.52	-1.75	1.98	-1.76	1.99
400	20	20	0.055	0.044	0.045	-2.15	1.68	-1.81	2.01	-1.82	2.02
1000	20	20	0.052	0.047	0.047	-2.09	1.79	-1.87	2.01	-1.87	2.01

Table 3:(Continued)

N_1	N_2	N_3	Empirical type I error $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Percentiles $Z_{MM}^{(3)}$ Lower	Upper	$Z_{MM}^{(3)*}$ Lower	Upper	$Z_{MM}^{(3)**}$ Lower	Upper
					(p_1, p_2, p_3) =(5,5,5)						
20	10	10	0.357	0.003	0.003	-2.89	-0.43	-1.08	1.38	-1.10	1.40
30	10	10	0.275	0.006	0.006	-2.83	-0.10	-1.24	1.48	-1.26	1.50
40	10	10	0.230	0.009	0.010	-2.78	0.13	-1.34	1.56	-1.36	1.58
50	10	10	0.201	0.013	0.014	-2.74	0.31	-1.41	1.64	-1.43	1.65
100	10	10	0.133	0.024	0.025	-2.61	0.79	-1.59	1.81	-1.61	1.82
200	10	10	0.094	0.034	0.035	-2.47	1.16	-1.72	1.91	-1.73	1.93
400	10	10	0.072	0.041	0.042	-2.35	1.42	-1.80	1.97	-1.81	1.97
1000	10	10	0.059	0.046	0.047	-2.22	1.64	-1.87	1.99	-1.87	1.99
20	20	10	0.273	0.005	0.005	-2.77	-0.19	-1.13	1.45	-1.15	1.45
30	20	10	0.221	0.008	0.008	-2.74	0.08	-1.29	1.53	-1.30	1.54
40	20	10	0.191	0.011	0.012	-2.70	0.28	-1.38	1.60	-1.39	1.62
50	20	10	0.172	0.014	0.015	-2.67	0.43	-1.44	1.66	-1.45	1.68
100	20	10	0.122	0.025	0.026	-2.57	0.86	-1.60	1.82	-1.61	1.84
200	20	10	0.091	0.035	0.036	-2.46	1.19	-1.73	1.92	-1.74	1.93
400	20	10	0.072	0.041	0.042	-2.35	1.42	-1.81	1.96	-1.81	1.97
1000	20	10	0.059	0.046	0.046	-2.22	1.63	-1.87	1.98	-1.87	1.99
20	10	20	0.327	0.007	0.007	-2.94	-0.20	-1.22	1.53	-1.23	1.54
30	10	20	0.260	0.011	0.012	-2.88	0.10	-1.37	1.61	-1.39	1.62
40	10	20	0.221	0.015	0.016	-2.84	0.31	-1.47	1.68	-1.48	1.69
50	10	20	0.197	0.019	0.020	-2.79	0.46	-1.52	1.73	-1.54	1.74
100	10	20	0.136	0.030	0.031	-2.65	0.88	-1.67	1.87	-1.68	1.88
200	10	20	0.098	0.038	0.039	-2.50	1.21	-1.77	1.94	-1.77	1.95
400	10	20	0.074	0.043	0.044	-2.37	1.44	-1.83	1.98	-1.83	1.99
1000	10	20	0.061	0.047	0.047	-2.23	1.64	-1.88	1.99	-1.89	2.00
20	20	20	0.262	0.007	0.008	-2.81	-0.05	-1.23	1.54	-1.24	1.55
30	20	20	0.214	0.011	0.012	-2.77	0.22	-1.37	1.62	-1.38	1.63
40	20	20	0.189	0.016	0.016	-2.75	0.41	-1.47	1.69	-1.48	1.70
50	20	20	0.171	0.019	0.020	-2.72	0.55	-1.53	1.74	-1.54	1.75
100	20	20	0.126	0.031	0.032	-2.61	0.94	-1.68	1.88	-1.69	1.89
200	20	20	0.094	0.038	0.039	-2.48	1.23	-1.77	1.95	-1.77	1.96
400	20	20	0.074	0.044	0.044	-2.36	1.45	-1.83	1.98	-1.83	1.99
1000	20	20	0.060	0.047	0.047	-2.23	1.64	-1.88	1.99	-1.88	2.00

Table 3:(Continued)

N_1	N_2	N_3	Empirical type I error $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Percentiles $Z_{MM}^{(3)}$ Lower	Upper	$Z_{MM}^{(3)*}$ Lower	Upper	$Z_{MM}^{(3)**}$ Lower	Upper
					(p_1, p_2, p_3) =(2,4,2)						
20	10	10	0.081	0.006	0.006	-2.27	0.33	-1.14	1.46	-1.15	1.47
30	10	10	0.073	0.010	0.011	-2.26	0.63	-1.28	1.60	-1.30	1.61
40	10	10	0.067	0.014	0.015	-2.24	0.82	-1.37	1.69	-1.38	1.71
50	10	10	0.063	0.018	0.019	-2.23	0.96	-1.43	1.76	-1.44	1.77
100	10	10	0.056	0.029	0.030	-2.19	1.31	-1.60	1.91	-1.61	1.92
200	10	10	0.053	0.038	0.038	-2.14	1.55	-1.71	1.98	-1.72	1.99
400	10	10	0.051	0.043	0.044	-2.11	1.70	-1.80	2.01	-1.80	2.01
1000	10	10	0.051	0.047	0.047	-2.07	1.81	-1.87	2.01	-1.87	2.01
20	20	10	0.069	0.008	0.008	-2.22	0.50	-1.19	1.53	-1.20	1.54
30	20	10	0.064	0.012	0.013	-2.22	0.75	-1.32	1.65	-1.33	1.66
40	20	10	0.062	0.016	0.017	-2.22	0.92	-1.40	1.73	-1.41	1.74
50	20	10	0.060	0.020	0.020	-2.21	1.03	-1.46	1.78	-1.47	1.79
100	20	10	0.055	0.030	0.030	-2.18	1.34	-1.60	1.92	-1.61	1.92
200	20	10	0.053	0.037	0.038	-2.14	1.55	-1.72	1.98	-1.72	1.98
400	20	10	0.052	0.043	0.043	-2.11	1.70	-1.80	2.00	-1.80	2.01
1000	20	10	0.050	0.047	0.047	-2.06	1.81	-1.86	2.01	-1.87	2.01
20	10	20	0.088	0.008	0.009	-2.31	0.46	-1.22	1.55	-1.23	1.56
30	10	20	0.077	0.014	0.014	-2.30	0.73	-1.36	1.67	-1.37	1.68
40	10	20	0.072	0.018	0.018	-2.28	0.90	-1.44	1.75	-1.45	1.76
50	10	20	0.069	0.021	0.022	-2.27	1.03	-1.50	1.80	-1.51	1.81
100	10	20	0.060	0.031	0.032	-2.22	1.35	-1.64	1.93	-1.64	1.94
200	10	20	0.055	0.039	0.040	-2.16	1.57	-1.74	2.00	-1.74	2.00
400	10	20	0.052	0.044	0.045	-2.12	1.71	-1.81	2.01	-1.81	2.02
1000	10	20	0.051	0.047	0.047	-2.06	1.81	-1.87	2.01	-1.87	2.01
20	20	20	0.073	0.010	0.010	-2.26	0.58	-1.25	1.59	-1.26	1.60
30	20	20	0.069	0.015	0.015	-2.26	0.82	-1.38	1.70	-1.39	1.71
40	20	20	0.066	0.019	0.020	-2.25	0.98	-1.45	1.77	-1.46	1.78
50	20	20	0.064	0.023	0.023	-2.25	1.09	-1.51	1.83	-1.52	1.83
100	20	20	0.059	0.032	0.033	-2.21	1.38	-1.65	1.94	-1.66	1.95
200	20	20	0.055	0.040	0.040	-2.16	1.58	-1.74	2.00	-1.75	2.01
400	20	20	0.052	0.044	0.044	-2.11	1.71	-1.80	2.01	-1.81	2.02
1000	20	20	0.052	0.048	0.048	-2.07	1.82	-1.87	2.02	-1.87	2.02

Table 3:(Continued)

N_1	N_2	N_3	Empirical type I error $Z_{MM}^{(3)}$	$Z_{MM}^{(3)*}$	$Z_{MM}^{(3)**}$	Percentiles $Z_{MM}^{(3)}$ Lower	$Z_{MM}^{(3)}$ Upper	$Z_{MM}^{(3)*}$ Lower	$Z_{MM}^{(3)*}$ Upper	$Z_{MM}^{(3)**}$ Lower	$Z_{MM}^{(3)**}$ Upper
					(p_1, p_2, p_3) =(2,2,4)						
20	10	10	0.077	0.005	0.005	-2.25	0.32	-1.13	1.44	-1.14	1.45
30	10	10	0.069	0.010	0.010	-2.25	0.62	-1.28	1.59	-1.29	1.60
40	10	10	0.065	0.014	0.015	-2.24	0.82	-1.37	1.68	-1.38	1.69
50	10	10	0.062	0.018	0.018	-2.23	0.96	-1.44	1.75	-1.45	1.76
100	10	10	0.056	0.029	0.029	-2.19	1.31	-1.60	1.90	-1.61	1.91
200	10	10	0.052	0.037	0.038	-2.14	1.55	-1.71	1.98	-1.72	1.98
400	10	10	0.051	0.043	0.043	-2.10	1.70	-1.79	2.01	-1.80	2.01
1000	10	10	0.051	0.047	0.047	-2.07	1.81	-1.87	2.01	-1.87	2.01
20	20	10	0.067	0.007	0.007	-2.22	0.46	-1.17	1.51	-1.18	1.52
30	20	10	0.063	0.011	0.012	-2.22	0.72	-1.32	1.62	-1.33	1.63
40	20	10	0.060	0.015	0.016	-2.21	0.89	-1.40	1.70	-1.41	1.71
50	20	10	0.059	0.019	0.020	-2.21	1.02	-1.47	1.77	-1.47	1.77
100	20	10	0.055	0.030	0.030	-2.18	1.34	-1.62	1.91	-1.62	1.91
200	20	10	0.052	0.037	0.038	-2.15	1.55	-1.72	1.97	-1.73	1.98
400	20	10	0.051	0.043	0.044	-2.11	1.70	-1.80	2.00	-1.80	2.00
1000	20	10	0.050	0.047	0.047	-2.06	1.81	-1.86	2.01	-1.87	2.01
20	10	20	0.082	0.007	0.007	-2.28	0.43	-1.19	1.53	-1.20	1.53
30	10	20	0.075	0.012	0.013	-2.28	0.70	-1.34	1.64	-1.35	1.65
40	10	20	0.069	0.017	0.017	-2.27	0.89	-1.43	1.73	-1.44	1.74
50	10	20	0.066	0.021	0.021	-2.26	1.02	-1.49	1.79	-1.50	1.80
100	10	20	0.059	0.031	0.032	-2.21	1.35	-1.64	1.93	-1.64	1.93
200	10	20	0.055	0.039	0.040	-2.16	1.57	-1.74	1.99	-1.74	2.00
400	10	20	0.052	0.044	0.044	-2.11	1.71	-1.81	2.01	-1.81	2.02
1000	10	20	0.051	0.048	0.048	-2.07	1.81	-1.87	2.01	-1.87	2.01
20	20	20	0.070	0.008	0.008	-2.24	0.53	-1.21	1.56	-1.21	1.56
30	20	20	0.066	0.013	0.014	-2.24	0.78	-1.36	1.67	-1.36	1.67
40	20	20	0.063	0.018	0.018	-2.24	0.95	-1.44	1.75	-1.45	1.75
50	20	20	0.062	0.021	0.022	-2.23	1.07	-1.50	1.80	-1.51	1.81
100	20	20	0.057	0.032	0.032	-2.20	1.37	-1.64	1.93	-1.65	1.93
200	20	20	0.054	0.040	0.040	-2.16	1.58	-1.75	1.99	-1.75	2.00
400	20	20	0.053	0.044	0.045	-2.12	1.71	-1.81	2.01	-1.82	2.02
1000	20	20	0.051	0.048	0.048	-2.07	1.81	-1.87	2.01	-1.87	2.01

Appendix A. Derivation of $\mathrm{E}[b_{2,p_1,p_2,p_3}]$

We derive asymptotic expansions of $\mathrm{E}[R_2^{(3)}]$ and $\mathrm{E}[R_3^{(3)}]$ using the perturbation method as follows. To avoid the dependency between $\boldsymbol{x}_{1,i}$ and $\overline{\boldsymbol{x}}_T$ and between $\boldsymbol{x}_{1,i}$ and $\boldsymbol{S}_T$, we use

$$\overline{\boldsymbol{x}}_T^{(i)} = \frac{1}{N-1}\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{N}\boldsymbol{x}_{1,\alpha},\ \boldsymbol{S}_T^{(i)} = \frac{1}{N-1}\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{N}(\boldsymbol{x}_{1,\alpha}-\overline{\boldsymbol{x}}_T^{(i)})(\boldsymbol{x}_{1,\alpha}-\overline{\boldsymbol{x}}_T^{(i)})^\top,$$

$$\overline{\boldsymbol{x}}_{T(12)}^{(i)} = \frac{1}{N_1+N_2-1}\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{N_1+N_2}\boldsymbol{x}_{(12),\alpha},$$

$$\boldsymbol{S}_{T(12)}^{(i)} = \frac{1}{N_1+N_2-1}\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{N_1+N_2}(\boldsymbol{x}_{(12),\alpha}-\overline{\boldsymbol{x}}_{T(12)}^{(i)})(\boldsymbol{x}_{(12),\alpha}-\overline{\boldsymbol{x}}_{T(12)}^{(i)})^\top.$$

For $i = 1,\ldots,N_1$, we can then write

$$\begin{aligned}
\boldsymbol{x}_{1,i}-\overline{\boldsymbol{x}}_T &= \left(1-\frac{1}{N}\right)(\boldsymbol{x}_{1,i}-\overline{\boldsymbol{x}}_T^{(i)}),\\
\boldsymbol{x}_{(12),i}-\overline{\boldsymbol{x}}_{T(12)} &= \left(1-\frac{1}{N_1+N_2}\right)(\boldsymbol{x}_{(12),i}-\overline{\boldsymbol{x}}_{T(12)}^{(i)}),\\
\boldsymbol{S}_T^{-1} &= \left(1+\frac{1}{N}\right)(\boldsymbol{S}_T^{(i)})^{-1}\\
&\quad -\frac{1}{N}\left\{(\boldsymbol{S}_T^{(i)})^{-1}(\boldsymbol{x}_{1,i}-\overline{\boldsymbol{x}}_T^{(i)})(\boldsymbol{x}_{1,i}-\overline{\boldsymbol{x}}_T^{(i)})^\top(\boldsymbol{S}_T^{(i)})^{-1}\right\}+\mathrm{O}_p(N^{-2}),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{S}_{T(12)}^{-1} &= \left(1+\frac{1}{N_1+N_2}\right)(\boldsymbol{S}_{T(12)}^{(i)})^{-1}\\
&\quad -\frac{1}{N_1+N_2}(\boldsymbol{S}_{T(12)}^{(i)})^{-1}(\boldsymbol{x}_{(12),i}-\overline{\boldsymbol{x}}_{T(12)}^{(i)})(\boldsymbol{x}_{(12),i}-\overline{\boldsymbol{x}}_{T(12)}^{(i)})^\top(\boldsymbol{S}_{T(12)}^{(i)})^{-1}+\mathrm{O}_p(N^{-2}),
\end{aligned}$$

Then, $\overline{\boldsymbol{x}}_T^{(i)}$, $\boldsymbol{S}_T^{(i)}$, $\overline{\boldsymbol{x}}_{T(12)}^{(i)}$ and $\boldsymbol{S}_{T(12)}^{(i)}$ can be written as

$$\begin{aligned}
\overline{\boldsymbol{x}}_T^{(i)} &= \frac{N_1-1}{N-1}\overline{\boldsymbol{x}}_{(1),1}^{(i)}+\frac{N_2}{N-1}\overline{\boldsymbol{x}}_{(2)}+\frac{N_3}{N-1}\overline{\boldsymbol{x}}_{(3)},\\
\boldsymbol{S}_T^{(i)} &= \frac{N_1-1}{N-1}\boldsymbol{S}_{(1),11}^{(i)}+\frac{N_1-1}{N-1}(\overline{\boldsymbol{x}}_{(1),1}^{(i)}-\overline{\boldsymbol{x}}_T^{(i)})(\overline{\boldsymbol{x}}_{(1),1}^{(i)}-\overline{\boldsymbol{x}}_T^{(i)})^\top+\frac{N_2}{N-1}\boldsymbol{S}_{(2),11}\\
&\quad +\frac{N_2}{N-1}(\overline{\boldsymbol{x}}_{(2),1}-\overline{\boldsymbol{x}}_T^{(i)})(\overline{\boldsymbol{x}}_{(2),1}-\overline{\boldsymbol{x}}_T^{(i)})^\top+\frac{N_3}{N-1}\boldsymbol{S}_{(3)}\\
&\quad +\frac{N_3}{N-1}(\overline{\boldsymbol{x}}_{(3)}-\overline{\boldsymbol{x}}_T^{(i)})(\overline{\boldsymbol{x}}_{(3)}-\overline{\boldsymbol{x}}_T^{(i)})^\top,
\end{aligned}$$

$$
\begin{aligned}
\overline{\boldsymbol{x}}^{(i)}_{T(12)} &= \frac{N_1-1}{N_1+N_2-1}\overline{\boldsymbol{x}}^{(i)}_{(1),(12)} + \frac{N_2}{N_1+N_2-1}\overline{\boldsymbol{x}}_{(2)},\\
\boldsymbol{S}^{(i)}_{T(12)} &= \frac{N_1-1}{N_1+N_2-1}\boldsymbol{S}^{(i)}_{(1),(12)(12)} + \frac{N_1-1}{N_1+N_2-1}(\overline{\boldsymbol{x}}^{(i)}_{(1),(12)} - \overline{\boldsymbol{x}}^{(i)}_{T(12)})(\overline{\boldsymbol{x}}^{(i)}_{(1),(12)} - \overline{\boldsymbol{x}}^{(i)}_{T(12)})^\top\\
&\quad + \frac{N_2}{N_1+N_2-1}\boldsymbol{S}_{(2)} + \frac{N_2}{N_1+N_2-1}(\overline{\boldsymbol{x}}_{(2)} - \overline{\boldsymbol{x}}^{(i)}_{T(12)})(\overline{\boldsymbol{x}}_{(2)} - \overline{\boldsymbol{x}}^{(i)}_{T(12)})^\top.
\end{aligned}
$$

Herein, we use the perturbation method to expand the statistic $U_{1,i}$. Without a loss of generality, we can assume that $\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}_p$. Let

$$
\overline{\boldsymbol{x}}^{(i)}_{(1)} = \begin{pmatrix} \overline{\boldsymbol{x}}^{(i)}_{(1),1} \\ \overline{\boldsymbol{x}}^{(i)}_{(1),2} \\ \overline{\boldsymbol{x}}^{(i)}_{(1),3} \end{pmatrix} = \frac{1}{\sqrt{N_1-1}} \begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{z}_3 \end{pmatrix},
$$

$$
\begin{aligned}
\boldsymbol{S}^{(i)}_{(1)} &= \begin{pmatrix} \boldsymbol{S}^{(i)}_{(1),11} & \boldsymbol{S}^{(i)}_{(1),12} & \boldsymbol{S}^{(i)}_{(1),13} \\ \boldsymbol{S}^{(i)}_{(1),21} & \boldsymbol{S}^{(i)}_{(1),22} & \boldsymbol{S}^{(i)}_{(1),23} \\ \boldsymbol{S}^{(i)}_{(1),31} & \boldsymbol{S}^{(i)}_{(1),32} & \boldsymbol{S}^{(i)}_{(1),33} \end{pmatrix}\\
&= \left(1-\frac{1}{N_1-1}\right) \left\{ \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{p_2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{p_3} \end{pmatrix} + \frac{1}{\sqrt{N_1-1}} \begin{pmatrix} \boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12} & \boldsymbol{\Omega}_{13} \\ \boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_{22} & \boldsymbol{\Omega}_{23} \\ \boldsymbol{\Omega}_{31} & \boldsymbol{\Omega}_{32} & \boldsymbol{\Omega}_{33} \end{pmatrix} \right\},
\end{aligned}
$$

$$
\overline{\boldsymbol{x}}_{(2)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(2),1} \\ \overline{\boldsymbol{x}}_{(2),2} \end{pmatrix} = \frac{1}{\sqrt{N_2}} \begin{pmatrix} \boldsymbol{z}_4 \\ \boldsymbol{z}_5 \end{pmatrix},
$$

$$
\begin{aligned}
\boldsymbol{S}_{(2)} &= \begin{pmatrix} \boldsymbol{S}_{(2),11} & \boldsymbol{S}_{(2),12} \\ \boldsymbol{S}_{(2),21} & \boldsymbol{S}_{(2),22} \end{pmatrix}\\
&= \left(1-\frac{1}{N_2}\right) \left\{ \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{p_2} \end{pmatrix} + \frac{1}{\sqrt{N_2}} \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{pmatrix} \right\},
\end{aligned}
$$

$$
\overline{\boldsymbol{x}}_{(3)} = \frac{1}{\sqrt{N_3}}\boldsymbol{z}_6,\ \boldsymbol{S}_{(3)} = \left(1-\frac{1}{N_3}\right)\left(\boldsymbol{I}_{p_1} + \frac{1}{\sqrt{N_3}}\boldsymbol{W}\right),
$$

where

$$
\overline{\boldsymbol{x}}^{(i)}_{(1)} = \frac{1}{N_1-1}\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{N_1} \boldsymbol{x}_\alpha, \quad \boldsymbol{S}^{(i)}_{(1)} = \frac{1}{N_1-1}\sum_{\substack{\alpha=1\\ \alpha\neq i}}^{N_1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(i)}_{(1)})(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}^{(i)}_{(1)})^\top.
$$

Then, $U_{1,i}$, $i = 1, \ldots, N_1$, in (**??**) can be expanded as

$$U_{1,i} = \boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i} - \frac{1}{\sqrt{N}} A_1 + \frac{1}{N} A_2 + \mathrm{O}_p(N^{-\frac{3}{2}}), \tag{23}$$

where

$$\begin{aligned}
A_1 &= \sqrt{\tau_1}(2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_1 + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{11} \boldsymbol{x}_{1,i}) + \sqrt{\tau_2}(2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4 + \boldsymbol{x}_{1,i} \boldsymbol{\Phi}_{11} \boldsymbol{x}_{1,i}) + \sqrt{\tau_3}(2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_6 + \boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{x}_{1,i}), \\
A_2 &= 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i} - (\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i})^2 - (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_1)^2 - (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4)^2 - (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_6)^2 + \tau_1 \Bigg\{ \boldsymbol{z}_1^\top \boldsymbol{z}_1 + (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_1)^2 \\
&\quad + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{11} \boldsymbol{z}_1 + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{11}^2 \boldsymbol{x}_{1,i} \Bigg\} + \tau_2 \Bigg\{ \boldsymbol{z}_4^\top \boldsymbol{z}_4 + (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4)^2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11} \boldsymbol{z}_4 + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11}^2 \boldsymbol{x}_{1,i} \Bigg\} \\
&\quad + \tau_3 \Bigg\{ \boldsymbol{z}_6^\top \boldsymbol{z}_6 + (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_6)^2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{z}_6 + \boldsymbol{x}_{1,i}^\top \boldsymbol{W}^2 \boldsymbol{x}_{1,i} \Bigg\} + \sqrt{\tau_1 \tau_2} \Bigg\{ 2\boldsymbol{z}_1^\top \boldsymbol{z}_4 + 2\boldsymbol{x}_{1,i} \boldsymbol{\Phi}_{11} \boldsymbol{z}_1 \\
&\quad + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{11} \boldsymbol{z}_4 + \boldsymbol{x}_{1,i}^\top (\boldsymbol{z}_1 \boldsymbol{z}_4^\top + \boldsymbol{z}_4 \boldsymbol{z}_1^\top + \boldsymbol{\Omega}_{11} \boldsymbol{\Phi}_{11} + \boldsymbol{\Phi}_{11} \boldsymbol{\Omega}_{11}) \boldsymbol{x}_{1,i} \Bigg\} + \sqrt{\tau_1 \tau_3} \Bigg\{ 2\boldsymbol{z}_1^\top \boldsymbol{z}_6 \\
&\quad + 2\boldsymbol{x}_{1,i} \boldsymbol{\Omega}_{11} \boldsymbol{z}_6 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{z}_1 + \boldsymbol{x}_{1,i}^\top (\boldsymbol{z}_1 \boldsymbol{z}_6^\top + \boldsymbol{z}_6 \boldsymbol{z}_1^\top + \boldsymbol{\Omega}_{11} \boldsymbol{W} + \boldsymbol{W} \boldsymbol{\Omega}_{11}) \boldsymbol{x}_{1,i} \Bigg\} \\
&\quad + \sqrt{\tau_2 \tau_3} \Bigg\{ 2\boldsymbol{z}_4^\top \boldsymbol{z}_6 + 2\boldsymbol{x}_{1,i} \boldsymbol{W} \boldsymbol{z}_4 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11} \boldsymbol{z}_6 + \boldsymbol{x}_{1,i}^\top \Bigg(\boldsymbol{z}_4 \boldsymbol{z}_6^\top + \boldsymbol{z}_6 \boldsymbol{z}_4^\top \\
&\quad + \boldsymbol{\Phi}_{11} \boldsymbol{W} + \boldsymbol{W} \boldsymbol{\Phi}_{11} \Bigg) \boldsymbol{x}_{1,i} \Bigg\}
\end{aligned}$$

Next, we consider a stochastic expansion of $U_{2\cdot 1,i}$ in (**??**). Expanding in the same way as the perturbation expansion of $U_{1,i}$, we obtain as

$$U_{2\cdot 1,i} = \boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - \frac{1}{\sqrt{N}} B_1 + \frac{1}{N} B_2 + \mathrm{O}_p(N^{-\frac{3}{2}}), \tag{24}$$

where

$$
\begin{aligned}
B_1 &= \frac{1}{\tau_1+\tau_2}\{\sqrt{\tau_1}(2\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{12}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{22}\boldsymbol{x}_{2,i}) \\
&\quad + \sqrt{\tau_2}(2\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_5 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i})\}, \\
B_2 &= \frac{1}{(\tau_1+\tau_2)^2}\Big\{\tau_1(\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} + \boldsymbol{z}_2^\top \boldsymbol{z}_2 - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i}\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - (\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i})^2 - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4\boldsymbol{z}_5^\top \boldsymbol{x}_{2,i} \\
&\quad - (\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_5)^2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{11}\boldsymbol{\Omega}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{12}\boldsymbol{\Omega}_{22}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{12}\boldsymbol{\Omega}_{21}\boldsymbol{x}_{1,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{21}\boldsymbol{\Omega}_{12}\boldsymbol{x}_{2,i} \\
&\quad + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{22}^2\boldsymbol{x}_{2,i} + 2\boldsymbol{z}_1^\top \boldsymbol{\Omega}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{22}\boldsymbol{z}_2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{12}\boldsymbol{z}_2) \\
&\quad + \tau_2(\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} + \boldsymbol{z}_5^\top \boldsymbol{z}_5 - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i}\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - (\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i})^2 - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_1\boldsymbol{z}_2^\top \boldsymbol{x}_{2,i} - (\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_2)^2 \\
&\quad + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{21}\boldsymbol{x}_{1,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{21}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}^2\boldsymbol{x}_{2,i} \\
&\quad + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{21}\boldsymbol{z}_4 + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}\boldsymbol{z}_5 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{z}_5) \\
&\quad + \sqrt{\tau_1\tau_2}(2\boldsymbol{z}_2^\top \boldsymbol{z}_5 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_1\boldsymbol{z}_5^\top \boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4\boldsymbol{z}_2^\top \boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_2\boldsymbol{z}_5^\top \boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{11}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} \\
&\quad + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11}\boldsymbol{\Omega}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}\boldsymbol{\Omega}_{12}\boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{\Omega}_{22}\boldsymbol{x}_{2,i} \\
&\quad + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{12}\boldsymbol{\Phi}_{21}\boldsymbol{x}_{1,i} + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{\Omega}_{21}\boldsymbol{x}_{1,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{21}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{21}\boldsymbol{\Omega}_{12}\boldsymbol{x}_{2,i} \\
&\quad + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{22}\boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}\boldsymbol{\Omega}_{22}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{21}\boldsymbol{z}_1 + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}\boldsymbol{z}_2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{z}_2 \\
&\quad + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{21}\boldsymbol{z}_4 + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Omega}_{22}\boldsymbol{z}_5 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Omega}_{12}\boldsymbol{z}_5)\Big\}.
\end{aligned}
$$

Finally, we consider a stochastic expansion of $U_{3\cdot 12,i}$ in (**??**).

$$
U_{3\cdot 12,i} = \boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,i} - \frac{1}{\sqrt{N}}C_1 + \frac{1}{N}C_2 + \mathrm{O}(N^{-\frac{3}{2}}), \tag{25}
$$

where

$$
\begin{aligned}
C_1 &= \frac{1}{\sqrt{\tau_1}}\{2\boldsymbol{x}_{3,i}^\top \boldsymbol{z}_3 + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Omega}_{3(12)}\boldsymbol{x}_{(12),i} + \boldsymbol{x}_{3,i}^\top \boldsymbol{\Omega}_{33}\boldsymbol{x}_{3,i}\} \\
C_2 &= \frac{1}{\tau_1}\{\boldsymbol{z}_3^\top \boldsymbol{z}_3 - 2\boldsymbol{x}_{(12),i}^\top \boldsymbol{x}_{(12),i}\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,i} - (\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,i})^2 + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Omega}_{33}\boldsymbol{z}_3 + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Omega}_{3(12)}\boldsymbol{z}_{(12)} \\
&\quad + 2\boldsymbol{x}_{(12),i}^\top \boldsymbol{\Omega}_{(12)3}\boldsymbol{z}_3 + 2\boldsymbol{x}_{(12),i}^\top \boldsymbol{\Omega}_{(12)(12)}\boldsymbol{\Omega}_{(12)3}\boldsymbol{x}_{3,i} + 2\boldsymbol{x}_{(12),i}^\top \boldsymbol{\Omega}_{(12)3}\boldsymbol{\Omega}_{33}\boldsymbol{x}_{3,i} \\
&\quad + \boldsymbol{x}_{3,i}^\top \boldsymbol{\Omega}_{3(12)}\boldsymbol{\Omega}_{(12)3}\boldsymbol{x}_{3,i} + \boldsymbol{x}_{(12),i}^\top \boldsymbol{\Omega}_{(12)3}\boldsymbol{\Omega}_{3(12)}\boldsymbol{x}_{(12),i} + \boldsymbol{x}_{3,i}^\top \boldsymbol{\Omega}_{33}^2\boldsymbol{x}_{3,i}\}.
\end{aligned}
$$

Calculating the expectations of the expansion of $U_{2\cdot 1,i}^2$ and $U_{3\cdot 12i}^2$ by (24) and (25), and each of the two products of expanded results in (23), (24) and (25) with respect to $\boldsymbol{x}_{1,i}$, $\boldsymbol{x}_{2,i}$, $\boldsymbol{x}_{3,i}$, $\boldsymbol{z}_j$, $j = 1, 2, 3, 4, 5$, $\boldsymbol{\Phi}$, $\boldsymbol{W}$ and $\boldsymbol{\Omega}$, we obtain (9)$\sim$(13) in Section 3.

6 Appendix B Derivation of $\mathrm{Var}[b_{2,p_a,p_2,p_3}]$

First, for $i \neq j$, the second moments of $R_2^{(3)}$, $R_3^{(3)}$, $R_{12}^{(3)}$, $R_{13}^{(3)}$ and $R_{23}^{(3)}$ can be written as

$$\mathrm{E}[(R_2^{(3)})^2] = \frac{1}{N_1+N_2}\mathrm{E}[U_{2\cdot1,i}^4] + \left(1 - \frac{1}{N_1+N_2}\right)\mathrm{E}[U_{2\cdot1,i}^2 U_{2\cdot1,j}^2], \tag{26}$$

$$\mathrm{E}[(R_3^{(3)})^2] = \frac{1}{N_1}\mathrm{E}[U_{3\cdot12,i}^4] + \left(1 - \frac{1}{N_1}\right)\mathrm{E}[U_{3\cdot12,i}^2 U_{3\cdot12,j}^2], \tag{27}$$

$$\mathrm{E}[(R_{12}^{(3)})^2] = \frac{4}{N_1+N_2}\mathrm{E}[U_{1,i}^2 U_{2\cdot1,i}^2] + 4\left(1 - \frac{1}{N_1+N_2}\right)\mathrm{E}[U_{1,i}U_{2\cdot1,i}U_{1,j}U_{2\cdot1,j}], \tag{28}$$

$$\mathrm{E}[(R_{13}^{(3)})^2] = \frac{4}{N_1}\mathrm{E}[U_{1,i}^2 U_{3\cdot12,i}^2] + 4\left(1 - \frac{1}{N_1}\right)\mathrm{E}[U_{1,i}U_{3\cdot12,i}U_{1,j}U_{3\cdot12,j}], \tag{29}$$

$$\mathrm{E}[(R_{23}^{(3)})^2] = \frac{4}{N_1}\mathrm{E}[U_{2\cdot1,i}^2 U_{3\cdot12,i}^2] + 4\left(1 - \frac{1}{N_1}\right)\mathrm{E}[U_{2\cdot1,i}U_{3\cdot12,i}U_{2\cdot1,j}U_{3\cdot12,j}]. \tag{30}$$

For first terms on the right side of (26) $\sim$ (30), we obtain

$$\begin{aligned}
\mathrm{E}[U_{2\cdot1,i}^4] &= p_2(p_2+2)(p_2+4)(p_2+6) + \mathrm{O}(N^{-1}),\\
\mathrm{E}[U_{3\cdot12,i}^4] &= p_3(p_3+2)(p_3+4)(p_3+6) + \mathrm{O}(N^{-1}),\\
\mathrm{E}[U_{1,i}^2 U_{2\cdot1,i}^2] &= p_1p_2(p_1+2)(p_2+2) + \mathrm{O}(N^{-1}),\\
\mathrm{E}[U_{1,i}^2 U_{3\cdot12,i}^2] &= p_1p_3(p_1+2)(p_3+2) + \mathrm{O}(N^{-1}),\\
\mathrm{E}[U_{2\cdot1,i}^2 U_{3\cdot12,i}^2] &= p_2p_3(p_2+2)(p_3+2) + \mathrm{O}(N^{-1}).
\end{aligned}$$

Furthermore, in the second terms on the right side of (26) $\sim$ (30), to avoid dependence among random variables, let

$$\overline{\boldsymbol{x}}_T^{(i,j)} = \frac{1}{N-2}\sum_{\substack{\alpha=1\\ \alpha\neq i,j}}^{N}\boldsymbol{x}_{1,\alpha},\ \ \boldsymbol{S}_T^{(i,j)} = \frac{1}{N-2}\sum_{\substack{\alpha=1\\ \alpha\neq i,j}}^{N}(\boldsymbol{x}_{1,\alpha} - \overline{\boldsymbol{x}}_T^{(i,j)})(\boldsymbol{x}_{1,\alpha} - \overline{\boldsymbol{x}}_T^{(i,j)})^\top,$$

$$\overline{\boldsymbol{x}}_{T(12)}^{(i,j)} = \frac{1}{N_1+N_2-2}\sum_{\substack{\alpha=1\\ \alpha\neq i,j}}^{N_1+N_2}\boldsymbol{x}_{(12),\alpha},$$

$$\boldsymbol{S}_{T(12)}^{(i,j)} = \frac{1}{N_1+N_2-2}\sum_{\substack{\alpha=1\\ \alpha\neq i,j}}^{N_1+N_2}(\boldsymbol{x}_{(12),\alpha} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})(\boldsymbol{x}_{(12),\alpha} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})^\top.$$

That is, $\boldsymbol{x}_T^{(i,j)}$ and $\boldsymbol{S}_T^{(i,j)}$ are the sample mean vector and sample covariance matrix with $\boldsymbol{x}_{1,i}$ and $\boldsymbol{x}_{1,j}$ removed from $\boldsymbol{x}_{1,1}, \ldots, \boldsymbol{x}_{1,N}$. Therefore, we can write

$$\begin{aligned}
\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T &= \boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T^{(i,j)} - \frac{1}{N}(\boldsymbol{x}_{1,i} + \boldsymbol{x}_{1,j} - 2\overline{\boldsymbol{x}}_T^{(i,j)}), \\
\boldsymbol{S}_T^{-1} &= (1 + \frac{2}{N})\boldsymbol{S}_T^{(i,j)-1} - \frac{1}{N}\Big\{\boldsymbol{S}_T^{(i,j)-1}(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T^{(i,j)})(\boldsymbol{x}_{1,i} - \overline{\boldsymbol{x}}_T^{(i,j)})^\top \boldsymbol{S}^{(i,j)-1} \\
&\quad + \boldsymbol{S}_T^{(i,j)-1}(\boldsymbol{x}_{1,j} - \overline{\boldsymbol{x}}_T^{(i,j)})(\boldsymbol{x}_{1,j} - \overline{\boldsymbol{x}}_T^{(i,j)})^\top \boldsymbol{S}_T^{(i,j)-1}\Big\} + \mathrm{O}(N^{-2}), \\
\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{T(12)} &= \boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)} - \frac{1}{N_1 + N_2}(\boldsymbol{x}_{(12),i} + \boldsymbol{x}_{(12),j} - 2\overline{\boldsymbol{x}}_{T(12)}^{(i,j)}), \\
\boldsymbol{S}_{T(12)}^{-1} &= \left(1 + \frac{2}{N_1 + N_2}\right)\boldsymbol{S}_{T(12)}^{(i,j)-1} \\
&\quad - \frac{1}{N_1 + N_2}\{\boldsymbol{S}_{T(12)}^{(i,j)-1}(\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})(\boldsymbol{x}_{(12),i} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})^\top \boldsymbol{S}_{T(12)}^{(i,j)-1} \\
&\quad + \boldsymbol{S}_{T(12)}^{(i,j)-1}(\boldsymbol{x}_{(12),j} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})(\boldsymbol{x}_{(12),j} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})^\top \boldsymbol{S}_{T(12)}^{(i,j)-1}\} + \mathrm{O}(N^{-2}).
\end{aligned}$$

Furthermore, $\overline{\boldsymbol{x}}_T^{(i,j)}$, $\overline{\boldsymbol{x}}_{T(12)}^{(i,j)}$, $\boldsymbol{S}_T^{(i,j)}$ and $\boldsymbol{S}_{T(12)}^{(i,j)}$ can be written as

$$\begin{aligned}
\overline{\boldsymbol{x}}_T^{(i,j)} &= \frac{N_1 - 2}{N - 2}\overline{\boldsymbol{x}}_{(1),1}^{(i,j)} + \frac{N_2}{N - 2}\overline{\boldsymbol{x}}_{(2),1} + \frac{N_3}{N - 2}\overline{\boldsymbol{x}}_{(3)}, \\
\boldsymbol{S}_T^{(i,j)} &= \frac{N_1 - 2}{N - 2}\boldsymbol{S}_{(1),11}^{(i,j)} + \frac{N_1 - 2}{N - 2}(\overline{\boldsymbol{x}}_{(1),1}^{(i,j)} - \overline{\boldsymbol{x}}_T^{(i,j)})(\overline{\boldsymbol{x}}_{(1),1}^{(i,j)} - \overline{\boldsymbol{x}}_T^{(i,j)})^\top \\
&\quad + \frac{N_2}{N - 2}\boldsymbol{S}_{(2),11} + \frac{N_2}{N - 2}(\overline{\boldsymbol{x}}_{(2),1} - \overline{\boldsymbol{x}}_T^{(i,j)})(\overline{\boldsymbol{x}}_{(2),1} - \overline{\boldsymbol{x}}_T^{(i,j)})^\top \\
&\quad + \frac{N_3}{N - 2}\boldsymbol{S}_{(3)} + \frac{N_3}{N - 2}(\overline{\boldsymbol{x}}_{(3)} - \overline{\boldsymbol{x}}_T^{(i,j)})(\overline{\boldsymbol{x}}_{(3)} - \overline{\boldsymbol{x}}_T^{(i,j)})^\top, \\
\overline{\boldsymbol{x}}_{T(12)}^{(i,j)} &= \frac{N_1 - 2}{N_1 + N_2 - 2}\overline{\boldsymbol{x}}_{(1),(12)}^{(i,j)} + \frac{N_2}{N_1 + N_2 - 2}\overline{\boldsymbol{x}}_{(2)}, \\
\boldsymbol{S}_{T(12)}^{(i,j)} &= \frac{N_1 - 2}{N_1 + N_2 - 2}\boldsymbol{S}_{(1),(12)(12)}^{(i,j)} + \frac{N_1 - 2}{N_1 + N_2 - 2}(\overline{\boldsymbol{x}}_{(1),(12)}^{(i,j)} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})(\overline{\boldsymbol{x}}_{(1),(12)}^{(i,j)} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})^\top \\
&\quad + \frac{N_2}{N_1 + N_2 - 2}\boldsymbol{S}_{(2)} + \frac{N_2}{N_1 + N_2 - 2}(\overline{\boldsymbol{x}}_{(2)} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})(\overline{\boldsymbol{x}}_{(2)} - \overline{\boldsymbol{x}}_{T(12)}^{(i,j)})^\top.
\end{aligned}$$

and by using

$$\overline{\boldsymbol{x}}_{(1)}^{(i,j)} = \begin{pmatrix} \overline{\boldsymbol{x}}_{(1),1}^{(i,j)} \\ \overline{\boldsymbol{x}}_{(1),2}^{(i,j)} \\ \overline{\boldsymbol{x}}_{(1),3}^{(i,j)} \end{pmatrix} = \frac{1}{\sqrt{N_1-2}} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{pmatrix},$$

$$\boldsymbol{S}_{(1)}^{(i)} = \begin{pmatrix} \boldsymbol{S}_{(1),11}^{(i,j)} & \boldsymbol{S}_{(1),12}^{(i,j)} & \boldsymbol{S}_{(1),13}^{(i,j)} \\ \boldsymbol{S}_{(1),21}^{(i,j)} & \boldsymbol{S}_{(1),22}^{(i,j)} & \boldsymbol{S}_{(1),23}^{(i,j)} \\ \boldsymbol{S}_{(1),31}^{(i,j)} & \boldsymbol{S}_{(1),32}^{(i,j)} & \boldsymbol{S}_{(1),33}^{(i,j)} \end{pmatrix}$$

$$= \left(1 - \frac{1}{N_1-2}\right) \left\{ \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{p_2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{p_3} \end{pmatrix} + \frac{1}{\sqrt{N_1-2}} \begin{pmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} & \boldsymbol{\Psi}_{13} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} & \boldsymbol{\Psi}_{23} \\ \boldsymbol{\Psi}_{31} & \boldsymbol{\Psi}_{32} & \boldsymbol{\Psi}_{33} \end{pmatrix} \right\},$$

where

$$\overline{\boldsymbol{x}}_{(1)}^{(i,j)} = \frac{1}{N_1-2} \sum_{\substack{\alpha=1 \\ \alpha \neq i,j}}^{N_1} \boldsymbol{x}_\alpha,\ \boldsymbol{S}_{(1)}^{(i,j)} = \frac{1}{N_1-2} \sum_{\substack{\alpha=1 \\ \alpha \neq i,j}}^{N_1} (\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}_{(1)}^{(i,j)})(\boldsymbol{x}_\alpha - \overline{\boldsymbol{x}}_{(1)}^{(i,j)})^\top,$$

$U_{1,i}$, $U_{2\cdot 1,i}$ and $U_{3\cdot 12,i}$ are expanded as

$$\begin{aligned} U_{1,i} &= \boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i} - \frac{1}{\sqrt{N}} D_1 + \frac{1}{N} D_2 + \mathrm{O}_p(N^{-\frac{3}{2}}), \\ U_{2\cdot 1,i} &= \boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - \frac{1}{\sqrt{N}} E_1 + \frac{1}{N} E_2 + \mathrm{O}_p(N^{-\frac{3}{2}}), \\ U_{3\cdot 12,i} &= \boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,i} - \frac{1}{\sqrt{N}} F_1 + \frac{1}{N} F_2 + \mathrm{O}_p(N^{-\frac{3}{2}}), \end{aligned}$$

where

$$\begin{aligned} D_1 = {} & \sqrt{\tau_1}(2\boldsymbol{x}_{1,i}^\top \boldsymbol{u}_1 + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11} \boldsymbol{x}_{1,i}) + \sqrt{\tau_2}(2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4 + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11} \boldsymbol{x}_{1,i}) \\ & + \sqrt{\tau_3}(2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_6 + \boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{x}_{1,i}), \\ D_2 = {} & 3\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i} - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,j} - (\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i})^2 - (\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,j})^2 - (\boldsymbol{x}_{1,i}^\top \boldsymbol{u}_1)^2 - (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4)^2 \\ & - (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_6)^2 + \tau_1(\boldsymbol{u}_1^\top \boldsymbol{u}_1 + (\boldsymbol{x}_{1,i}^\top \boldsymbol{u}_1)^2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11} \boldsymbol{u}_1 + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11}^2 \boldsymbol{x}_{1,i}) \\ & + \tau_2(\boldsymbol{z}_4^\top \boldsymbol{z}_4 + (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4)^2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11} \boldsymbol{z}_4 + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11}^2 \boldsymbol{x}_{1,i}) + \tau_3(\boldsymbol{z}_6^\top \boldsymbol{z}_6 + (\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_6)^2 \\ & + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{z}_6 + \boldsymbol{x}_{1,i}^\top \boldsymbol{W}^2 \boldsymbol{x}_{1,i}) + \sqrt{\tau_1\tau_2}(2\boldsymbol{z}_4^\top \boldsymbol{u}_1 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{u}_1 \boldsymbol{z}_4^\top \boldsymbol{x}_{1,i} + \boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4 \boldsymbol{u}_1^\top \boldsymbol{x}_{1,i} \\ & + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11} \boldsymbol{z}_4 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11} \boldsymbol{u}_1 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11} \boldsymbol{\Phi}_{11} \boldsymbol{x}_{1,i}) + \sqrt{\tau_1\tau_3}(2\boldsymbol{z}_6^\top \boldsymbol{u}_1 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{u}_1 \boldsymbol{z}_6^\top \boldsymbol{x}_{1,i} \\ & + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{u}_1 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11} \boldsymbol{z}_6 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{\Psi}_{11} \boldsymbol{x}_{1,i}) + \sqrt{\tau_2\tau_3}(2\boldsymbol{z}_4^\top \boldsymbol{z}_6 \\ & + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{z}_4 \boldsymbol{z}_6^\top \boldsymbol{x}_{1,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{z}_4 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11} \boldsymbol{z}_6 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{W} \boldsymbol{\Phi}_{11} \boldsymbol{x}_{1,i}), \end{aligned}$$

$$
\begin{aligned}
E_1 &= \frac{1}{\tau_1+\tau_2}\{\sqrt{\tau_1}(2\boldsymbol{x}_{2,i}^\top \boldsymbol{u}_2 + 2\boldsymbol{x}_{1,i}\boldsymbol{\Psi}_{12}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{2,i}\boldsymbol{\Psi}_{22}\boldsymbol{x}_{2,i}) \\
&\quad + \sqrt{\tau_2}(2\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_5 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i})\}, \\
E_2 &= \frac{1}{(\tau_1+\tau_2)^2}\Big\{\tau_1(2\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - 2\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,j} + \boldsymbol{u}_2^\top \boldsymbol{u}_2 - (\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_5)^2 - (\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,j})^2 - (\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i})^2 \\
&\quad - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i}\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,j}\boldsymbol{x}_{2,j}^\top \boldsymbol{x}_{2,i} - 2\boldsymbol{x}_{2,i}^\top \boldsymbol{z}_5\boldsymbol{z}_4^\top \boldsymbol{x}_{1,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{12}\boldsymbol{u}_2 + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{21}\boldsymbol{u}_1 \\
&\quad + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{22}\boldsymbol{u}_2 + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{22}^2\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11}\boldsymbol{\Psi}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{12}\boldsymbol{\Psi}_{22}\boldsymbol{x}_{2,i} + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{12}\boldsymbol{\Psi}_{21}\boldsymbol{x}_{1,i} \\
&\quad + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{21}\boldsymbol{\Psi}_{12}\boldsymbol{x}_{2,i}) + \tau_2(2\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - 2\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,j} + \boldsymbol{z}_5^\top \boldsymbol{z}_5 - (\boldsymbol{x}_{2,i}^\top \boldsymbol{u}_2)^2 - (\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,j})^2 \\
&\quad - (\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i})^2 - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,i}\boldsymbol{x}_{2,i}^\top \boldsymbol{x}_{2,i} - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{x}_{1,j}\boldsymbol{x}_{2,j}^\top \boldsymbol{x}_{2,i} - 2\boldsymbol{x}_{1,i}^\top \boldsymbol{u}_1\boldsymbol{u}_2^\top \boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{z}_5 \\
&\quad + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{21}\boldsymbol{z}_4 + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}\boldsymbol{z}_5 + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}^2\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i} \\
&\quad + \boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{21}\boldsymbol{x}_{1,i} + \boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{21}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i}) + \sqrt{\tau_1\tau_2}(2\boldsymbol{u}_2^\top \boldsymbol{z}_5 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{u}_1\boldsymbol{z}_5^\top \boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}\boldsymbol{z}_4\boldsymbol{u}_2^\top \boldsymbol{x}_{2,i} \\
&\quad + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{u}_2\boldsymbol{z}_5^\top \boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{21}\boldsymbol{u}_1 + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{21}\boldsymbol{z}_4 + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Phi}_{22}\boldsymbol{u}_2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{u}_2 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{12}\boldsymbol{z}_5 \\
&\quad + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{22}\boldsymbol{z}_5 + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{11}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{11}\boldsymbol{\Psi}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Phi}_{12}\boldsymbol{\Psi}_{22}\boldsymbol{x}_{2,i} \\
&\quad + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{12}\boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{21}\boldsymbol{\Phi}_{12}\boldsymbol{x}_{2,i} + 2\boldsymbol{x}_{1,i}^\top \boldsymbol{\Psi}_{12}\boldsymbol{\Phi}_{21}\boldsymbol{x}_{1,i} + 2\boldsymbol{x}_{2,i}^\top \boldsymbol{\Psi}_{22}\boldsymbol{\Phi}_{22}\boldsymbol{x}_{2,i})\Big\}, \\
F_1 &= \frac{1}{\sqrt{\tau_1}}\Big\{2\boldsymbol{x}_{3,i}^\top \boldsymbol{u}_3 + \boldsymbol{x}_{3,i}^\top \boldsymbol{\Psi}_{33}\boldsymbol{x}_{3,i} + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Psi}_{3(12)}\boldsymbol{x}_{(12),i}\Big\}, \\
F_2 &= \frac{1}{\tau_1}\Big\{\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,i} - 2\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,j} + \boldsymbol{u}_3^\top \boldsymbol{u}_3 - (\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,i})^2 - (\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,j})^2 - 2\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,i}\boldsymbol{x}_{(12),i}^\top \boldsymbol{x}_{(12),i} \\
&\quad - 2\boldsymbol{x}_{3,i}^\top \boldsymbol{x}_{3,j}\boldsymbol{x}_{(12),j}^\top \boldsymbol{x}_{(12),i} + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Psi}_{33}\boldsymbol{u}_3 + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Psi}_{3(12)}\boldsymbol{u}_{(12)} + 2\boldsymbol{x}_{(12),i}^\top \boldsymbol{\Psi}_{(12)3}\boldsymbol{u}_3 \\
&\quad + \boldsymbol{x}_{3,i}^\top \boldsymbol{\Psi}_{3(12)}\boldsymbol{\Psi}_{(12)3}\boldsymbol{x}_{3,i} + \boldsymbol{x}_{(12),i}^\top \boldsymbol{\Psi}_{(12)3}\boldsymbol{\Psi}_{3(12)}\boldsymbol{x}_{(12),i} + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Psi}_{33}\boldsymbol{\Psi}_{3(12)}\boldsymbol{x}_{(12),i} \\
&\quad + 2\boldsymbol{x}_{3,i}^\top \boldsymbol{\Psi}_{3(12)}\boldsymbol{\Psi}_{(12)(12)}\boldsymbol{x}_{(12),i}\Big\}.
\end{aligned}
$$

Therefore, since the expectation of the second terms on the right side of (26) $\sim$ (30) can all be represented by random vectors and random matrices that are mutually independent, the following results can be obtained by calculating their expectation.

$$
\begin{aligned}
\mathrm{E}[U_{2\cdot 1,i}^2 U_{2\cdot 1,j}^2] &= p_2^2(p_2+2)^2 - \frac{4}{(\tau_1+\tau_2)N}p_2(p_2+2)^3 + \mathrm{O}(N^{-\frac{3}{2}}), \\
\mathrm{E}[U_{3\cdot 12,i}^2 U_{3\cdot 12,j}^2] &= p_3^2(p_3+2)^2 - \frac{4}{\tau_1 N}p_3(p_3+2)^3 + \mathrm{O}(N^{-\frac{3}{2}}), \\
\mathrm{E}[U_{1,i}U_{1,j}U_{2\cdot 1,i}U_{2\cdot 1,j}] &= p_1^2p_2^2 - \frac{2}{N}p_1p_2^2 - \frac{2}{(\tau_1+\tau_2)N}p_1^2p_2(2p_2+1) + \mathrm{O}(N^{-\frac{3}{2}}), \\
\mathrm{E}[U_{1,i}U_{3\cdot 12,i}U_{1,j}U_{3\cdot 12,j}] &= p_1^2p_3^2 - \frac{2}{N}p_1p_3^2 - \frac{2}{\tau_1 N}p_1^2p_3(2p_3+1) + \mathrm{O}(N^{-\frac{3}{2}}), \\
\mathrm{E}[U_{2\cdot 1,i}U_{3\cdot 12,i}U_{2\cdot 1,j}U_{3\cdot 12,j}] &= p_2^2p_3^2 - \frac{2}{(\tau_1+\tau_2)N}p_2p_3^2 - \frac{2}{\tau_1 N}p_2^2p_3(2p_3+1) + \mathrm{O}(N^{-\frac{3}{2}}).
\end{aligned}
$$

To summarize these, we obtain (16) $\sim$ (20).

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