An Asymptotic Expansion of the Null Distribution for Sphericity Test Statistic with Monotone Missing Data

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Abstract

In this paper, we consider the sphericity test problem under monotone missing data. Specifically, we derive the likelihood ratio and asymptotic expansion of the null distribution for the likelihood ratio test (LRT) statistic and modified LRT statistic. We also propose approximate upper percentiles and an unbiased estimator, and estimate its mean squared error. Finally, numerical evaluations using Monte Carlo simulation and numerical examples are presented under monotone missing data.

Key Words and Phrases: Approximate accuracy, chi-square distribution, modified likelihood ratio test statistic.

1 Introduction

In this study, we consider the sphericity test problem, which is one of the tests of variance-covariance matrices under monotone missing data. Under complete data, Muirhead (1982) discussed this problem, and under two-step monotone missing data, Chang and Richards (2010) discussed the likelihood ratio (LR) for this problem and its null moment, and the limiting null distribution of the likelihood ratio test (LRT) statistic was provided in Choi (2005). Under general k-step monotone missing data, the LR for the sphericity test problem was provided in Batsidis and Zografos (2006). Here, we consider the multivariate normal distribution, while Batsidis and Zografos (2006) have investigated the elliptical distribution. Maximum likelihood estimators (MLEs) of the mean vector and variance-covariance matrix under monotone missing data were reported by Kanda and Fujikoshi (1998), and Box (1949) introduced a general distribution theory for a class of likelihood criteria. Additionally, Chang and Richards (2009) proposed the Hotelling's T^2-statistic under two-step monotone missing data, and Romer and Richards (2013) showed that the

Hotelling's T^2-statistic under two-step monotone missing data is affine invariant.

The remainder of this paper is organized as follows. In Section 2, we provide the LR under monotone missing data. In Section 3, we derive an asymptotic expansion of the distribution functions for the LRT statistic and modified LRT statistic using the general distribution proposed by Box (1949) when the null hypothesis is true; further, the upper percentiles are provided. In Section 4, we discuss an unbiased estimator and its mean squared error (MSE). In Section 5, we discuss affine transformation in the LR. In Section 6, we provide a numerical evaluation of the upper percentiles of the LRT statistic and modified LRT statistic, expectation of an unbiased estimator, and the corresponding MSE using Monte Carlo simulations. In Section 7, we present numerical examples. Finally, in Section 8, we present our conclusions.

2 LR under monotone missing data

In this section, we consider the LR in the sphericity test problem under monotone missing data:

$$H_0 : \boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}_p \text{ vs. } H_1 : \boldsymbol{\Sigma} \neq \sigma^2 \boldsymbol{I}_p,$$

where σ^2 is unknown and $\sigma^2 > 0$.

2.1 Two-step case

In the case of two-step monotone missing data, we consider the following data matrix $\boldsymbol{X}$.

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p_1} & x_{1,p_1+1} & \cdots & x_{1p} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{N_1 1} & x_{N_1 2} & \cdots & x_{N_1 p_1} & x_{N_1,p_1+1} & \cdots & x_{N_1 p} \\ x_{N_1+1,1} & x_{N_1+1,2} & \cdots & x_{N_1+1,p_1} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{Np_1} & * & \cdots & * \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}'_{11} & \boldsymbol{x}'_{21} \\ \vdots & \vdots \\ \boldsymbol{x}'_{1N_1} & \boldsymbol{x}'_{2N_1} \\ \boldsymbol{x}'_{1,N_1+1} & * \\ \vdots & \vdots \\ \boldsymbol{x}'_{1N} & * \end{pmatrix},$$

where "$*$" indicates a missing observation, $N = N_{(12)} = N_1 + N_2, p = p_{(12)} = p_1 + p_2$, and $N_1 > p$. Then, let

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_1} \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \boldsymbol{x}_{1,N_1+1}, \ldots, \boldsymbol{x}_{1N} \overset{i.i.d.}{\sim} N_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}),$$

where $\boldsymbol{x}_j = \left(\boldsymbol{x}'_{1j}, \boldsymbol{x}'_{2j}\right)',\ j = 1, \ldots, N_1$,

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

We also define the sample mean vectors and Wishart matrices as follows.

$$\begin{aligned}
\overline{\boldsymbol{x}}_1^{(1)} &= \frac{1}{N_1}\sum_{j=1}^{N_1} \boldsymbol{x}_{1j},\ \overline{\boldsymbol{x}}_2^{(1)} = \frac{1}{N_1}\sum_{j=1}^{N_1} \boldsymbol{x}_{2j},\ \overline{\boldsymbol{x}}_1^{(2)} = \frac{1}{N_2}\sum_{j=N_1+1}^{N} \boldsymbol{x}_{1j}, \\
\boldsymbol{W}_{11}^{(1)} &= \sum_{j=1}^{N_1} (\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_1^{(1)})(\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_1^{(1)})', \\
\boldsymbol{W}_{12}^{(1)} &= (\boldsymbol{W}_{21}^{(1)})' = \sum_{j=1}^{N_1} (\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_1^{(1)})(\boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_2^{(1)})', \\
\boldsymbol{W}_{22}^{(1)} &= \sum_{j=1}^{N_1} (\boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_2^{(1)})(\boldsymbol{x}_{2j} - \overline{\boldsymbol{x}}_2^{(1)})', \\
\boldsymbol{W}_{11}^{(2)} &= \sum_{j=N_1+1}^{N} (\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_1^{(2)})(\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}_1^{(2)})' + \frac{N_1 N_2}{N}(\overline{\boldsymbol{x}}_1^{(1)} - \overline{\boldsymbol{x}}_1^{(2)})(\overline{\boldsymbol{x}}_1^{(1)} - \overline{\boldsymbol{x}}_1^{(2)})', \\
\boldsymbol{W}_{22\cdot 1}^{(1)} &= \boldsymbol{W}_{22}^{(1)} - \boldsymbol{W}_{21}^{(1)}(\boldsymbol{W}_{11}^{(1)})^{-1}\boldsymbol{W}_{12}^{(1)}.
\end{aligned}$$

The LR under two-step monotone missing data is given by

$$\lambda = \frac{\left|\frac{1}{N}\left(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}_{11}^{(2)}\right)\right|^{\frac{N}{2}} \left|\frac{1}{N_1}\boldsymbol{W}_{22\cdot 1}^{(1)}\right|^{\frac{N_1}{2}}}{\left\{\frac{1}{Np_1 + N_1 p_2}\left(\mathrm{tr}(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}_{11}^{(2)}) + \mathrm{tr}\boldsymbol{W}_{22}^{(1)}\right)\right\}^{\frac{Np_1+N_1p_2}{2}}}.$$

This LR is essentially coincident with the result in Chang and Richards (2010).

2.2 Genenal-step case

Next, we consider the case of k-step monotone missing data ($k \geq 2$).

For $j = 2, \ldots, k$, let

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_1} \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

$$\boldsymbol{x}_{(1\ldots k-j+1), N_{(1\ldots j-1)}+1}, \ldots, \boldsymbol{x}_{(1\ldots k-j+1), N_{(1\ldots j)}} \overset{i.i.d.}{\sim} N_{p_{(1\ldots k-j+1)}}\left(\boldsymbol{\mu}_{(1\ldots k-j+1)}, \boldsymbol{\Sigma}_{(1\ldots k-j+1)(1\ldots k-j+1)}\right),$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_k \end{pmatrix}, \boldsymbol{\mu}_{(1\ldots k-j+1)} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_{k-j+1} \end{pmatrix},$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \cdots & \boldsymbol{\Sigma}_{1k} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \cdots & \boldsymbol{\Sigma}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}_{k1} & \boldsymbol{\Sigma}_{k2} & \cdots & \boldsymbol{\Sigma}_{kk} \end{pmatrix}, \boldsymbol{\Sigma}_{(1\ldots k-j+1)(1\ldots k-j+1)} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \cdots & \boldsymbol{\Sigma}_{1,k-j+1} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Sigma}_{k-j+1,1} & \cdots & \boldsymbol{\Sigma}_{k-j+1,k-j+1} \end{pmatrix},$$

and

$$\boldsymbol{x}_i = \begin{pmatrix} \boldsymbol{x}_{1i} \\ \vdots \\ \boldsymbol{x}_{ki} \end{pmatrix}, \ i = 1, \ldots, N_1,$$

$$\boldsymbol{x}_{(1\ldots k-j+1),i} = \begin{pmatrix} \boldsymbol{x}_{1i} \\ \vdots \\ \boldsymbol{x}_{k-j+1,i} \end{pmatrix}, \ i = N_{(1\ldots j-1)} + 1, \ldots, N_{(1\ldots j)},$$

$$N = N_{(1\ldots k)} = N_1 + \cdots + N_k, p = p_{(1\ldots k)} = p_1 + \cdots + p_k, N_1 > p.$$

Next, we define $N_{(1\ldots j-1)} = 0$ when $j = 1$, and for $a, b, \ell = 1, \ldots, k-j+1; j = 1, \ldots, k$, let

$$\overline{\boldsymbol{x}}_\ell^{(j)} = \frac{1}{N_j} \sum_{i=N_{(1\ldots j-1)}+1}^{N_{(1\ldots j)}} \boldsymbol{x}_{\ell i}, \overline{\boldsymbol{x}}_\ell^{[j]} = \frac{1}{N_{(1\ldots j)}}(N_1\overline{\boldsymbol{x}}_\ell^{(1)} + \cdots + N_j\overline{\boldsymbol{x}}_\ell^{(j)}), \overline{\boldsymbol{x}}_{[\ell]}^{[j]} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{[j]} \\ \vdots \\ \overline{\boldsymbol{x}}_\ell^{[j]} \end{pmatrix}, \overline{\boldsymbol{x}}_{[\ell]}^{(j)} = \begin{pmatrix} \overline{\boldsymbol{x}}_1^{(j)} \\ \vdots \\ \overline{\boldsymbol{x}}_\ell^{(j)} \end{pmatrix},$$

and

$$\begin{aligned} \boldsymbol{W}^{(1)} &= \sum_{i=1}^{N_1} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{[k]}^{(1)})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{[k]}^{(1)})', \\ \boldsymbol{W}^{(j)} &= \sum_{i=N_{(1\ldots j-1)}+1}^{N_{(1\ldots j)}} (\boldsymbol{x}_{(1\ldots k-j+1),i} - \overline{\boldsymbol{x}}_{[k-j+1]}^{(j)})(\boldsymbol{x}_{(1\ldots k-j+1),i} - \overline{\boldsymbol{x}}_{[k-j+1]}^{(j)})' \\ &\quad + \frac{N_{(1\ldots j-1)}N_j}{N_{(1\ldots j)}}(\overline{\boldsymbol{x}}_{[k-j+1]}^{(j)} - \overline{\boldsymbol{x}}_{[k-j+1]}^{[j-1]})(\overline{\boldsymbol{x}}_{[k-j+1]}^{(j)} - \overline{\boldsymbol{x}}_{[k-j+1]}^{[j-1]})', \ j = 2, \ldots, k, \\ \boldsymbol{W}_{ab}^{[j]} &= \boldsymbol{W}_{ab}^{(1)} + \cdots + \boldsymbol{W}_{ab}^{(j)}, \\ \boldsymbol{W}_{jj\cdot 1\ldots j-1}^{[k-j+1]} &= \boldsymbol{W}_{jj}^{[k-j+1]} - \boldsymbol{W}_{j(1\ldots j-1)}^{[k-j+1]}\{\boldsymbol{W}_{(1\ldots j-1)(1\ldots j-1)}^{[k-j+1]}\}^{-1}\boldsymbol{W}_{(1\ldots j-1)j}^{[k-j+1]}, \ j = 2, \ldots, k. \end{aligned}$$

The LR under k-step monotone missing data is given by

$$\lambda^{(k)} = \frac{\left|\frac{1}{N}\boldsymbol{W}_{11}^{[k]}\right|^{\frac{N}{2}} \prod_{j=2}^{k} \left|\frac{1}{N_{(k-j+1)}}\boldsymbol{W}_{jj\cdot 1\ldots j-1}^{[k-j+1]}\right|^{\frac{N_{(k-j+1)}}{2}}}{\left\{\left(\sum_{j=1}^{k} N_{(1\ldots k-j+1)}p_j\right)^{-1} \sum_{j=1}^{k} \mathrm{tr}\boldsymbol{W}_{jj}^{[k-j+1]}\right\}^{\frac{1}{2}\sum_{j=1}^{k} N_{(1\ldots k-j+1)}p_j}}.$$

This LR is essentially coincident with the result in Batsidis and Zografos (2006).

3 Null moment of LR and test statistic

In this section, we derive the null moment of the LR, and use the distribution in Box (1949) to obtain an asymptotic expansion of the distribution functions of test statistics when H_0 is true.

3.1 Two-step case

We consider a random variable Z with the following moments under two-step monotone missing data ($0 < Z \leq 1$).

$$\mathrm{E}[Z^h] = K \left[\frac{\prod_{m=1}^{r} y_m^{y_m}}{\prod_{\ell=1}^{q_1} z_{1\ell}^{z_{1\ell}} \prod_{\ell=1}^{q_2} z_{2\ell}^{z_{2\ell}}} \right]^h \frac{\prod_{l=1}^{q_1} \Gamma\left[z_{1\ell}(1+h)+\xi_{1\ell}\right] \prod_{\ell=1}^{q_2} \Gamma\left[z_{2\ell}(1+h)+\xi_{2\ell}\right]}{\prod_{m=1}^{r} \Gamma\left[y_m(1+h)+\eta_m\right]}, \tag{1}$$

where

$$\sum_{m=1}^{r} y_m = \sum_{\ell=1}^{q_1} z_{1\ell} + \sum_{\ell=1}^{q_2} z_{2\ell}, \tag{2}$$

and K is a constant such that $\mathrm{E}[Z^0] = 1$. Additionally, $z_{1\ell} = a_{1\ell}N, z_{2\ell} = a_{2\ell}N, y_m = b_m N$, where $a_{1\ell}, a_{2\ell}$ and b_m are constants and N is the asymptotic variable.

Furthermore, using

$$\beta_{1\ell} = (1-\rho)z_{1\ell}, \beta_{2\ell} = (1-\rho)z_{2\ell}, \varepsilon_m = (1-\rho)y_m,$$

where $0 < \rho \leq 1$, the cumulant generating function $\Psi(t)$ of $-2\rho \log Z$ can be written as

$$\Psi(t) = -\frac{1}{2} f \log(1-2it) + \sum_{\alpha=1}^{s} \omega_\alpha(\rho) \left\{(1-2it)^{-\alpha} - 1\right\} + O(N^{-s-1}), \tag{3}$$

where

$$f = -2\left(\sum_{\ell=1}^{q_1} \xi_{1\ell} + \sum_{\ell=1}^{q_2} \xi_{2\ell} - \sum_{m=1}^{r} \eta_m - \frac{1}{2}(q_1+q_2-r) \right), \tag{4}$$

$$\begin{aligned}\omega_\alpha(\rho) = \frac{(-1)^{\alpha+1}}{\alpha(\alpha+1)} \Bigg(& \sum_{\ell=1}^{q_1} \frac{B_{\alpha+1}(\beta_{1\ell}+\xi_{1\ell})}{(\rho z_{1\ell})^\alpha} + \sum_{\ell=1}^{q_2} \frac{B_{\alpha+1}(\beta_{2\ell}+\xi_{2\ell})}{(\rho z_{2\ell})^\alpha} \\ & - \sum_{m=1}^{r} \frac{B_{\alpha+1}(\varepsilon_m+\eta_m)}{(\rho y_m)^\alpha} \Bigg),\end{aligned} \tag{5}$$

and $B_{\alpha+1}(a)$ is the Bernoulli polynomial of degree $(\alpha+1)$. Using (3) with $s=1$ and $B_2(a)=a^2-a+(1/6)$, $\Psi(t)$ can be written as

$$\Psi(t)=-\frac{1}{2}f\log(1-2it)+\omega_1(\rho)\big\{(1-2it)^{-1}-1\big\}+O(N^{-2}),$$

where f is given by (4), and using (5), we have

$$\begin{aligned}\omega_1(\rho)=\frac{1}{2\rho}\Bigg\{&-(1-\rho)f+\sum_{\ell=1}^{q_1}z_{1\ell}^{-1}\left(\xi_{1\ell}^2-\xi_{1\ell}+\frac{1}{6}\right)+\sum_{\ell=1}^{q_2}z_{2\ell}^{-1}\left(\xi_{2\ell}^2-\xi_{2\ell}+\frac{1}{6}\right)\\&-\sum_{m=1}^{r}y_m^{-1}\left(\eta_m^2-\eta_m+\frac{1}{6}\right)\Bigg\}.\end{aligned}$$

Therefore, if the value of ρ is

$$\rho=1-\frac{1}{f}\left\{\sum_{\ell=1}^{q_1}z_{1\ell}^{-1}\left(\xi_{1\ell}^2-\xi_{1\ell}+\frac{1}{6}\right)+\sum_{\ell=1}^{q_2}z_{2\ell}^{-1}\left(\xi_{2\ell}^2-\xi_{2\ell}+\frac{1}{6}\right)-\sum_{m=1}^{r}y_m^{-1}\left(\eta_m^2-\eta_m+\frac{1}{6}\right)\right\},\quad(6)$$

$\omega_1(\rho)=0$. Using (3) with $s=2$, $\omega_1(\rho)=0$ and $B_3(a)=a^3-(3/2)a^2+(1/2)a$, $\Psi(t)$ can be written as

$$\Psi(t)=-\frac{1}{2}f\log(1-2it)+\omega_2(\rho)\big\{(1-2it)^{-2}-1\big\}+O(N^{-3}),$$

where

$$\begin{aligned}\omega_2(\rho)=-\frac{1}{6\rho^2}\Bigg[&3(1-\rho)^2\left(\sum_{\ell=1}^{q_1}\xi_{1\ell}+\sum_{\ell=1}^{q_2}\xi_{2\ell}-\sum_{m=1}^{r}\eta_m\right)\\&+(1-\rho)\Bigg\{\sum_{\ell=1}^{q_1}z_{1\ell}^{-1}\left(3\xi_{1\ell}^2-3\xi_{1\ell}+\frac{1}{2}\right)+\sum_{\ell=1}^{q_2}z_{2\ell}^{-1}\left(3\xi_{2\ell}^2-3\xi_{2\ell}+\frac{1}{2}\right)\\&-\sum_{m=1}^{r}y_m^{-1}\left(3\eta_m^2-3\eta_m+\frac{1}{2}\right)\Bigg\}\\&+\sum_{\ell=1}^{q_1}z_{1\ell}^{-2}\left(\xi_{1\ell}^3-\frac{3}{2}\xi_{1\ell}^2+\frac{1}{2}\xi_{1\ell}\right)+\sum_{\ell=1}^{q_2}z_{2\ell}^{-2}\left(\xi_{2\ell}^3-\frac{3}{2}\xi_{2\ell}^2+\frac{1}{2}\xi_{2\ell}\right)\\&-\sum_{m=1}^{r}y_m^{-2}\left(\eta_m^3-\frac{3}{2}\eta_m^2+\frac{1}{2}\eta_m\right)-\frac{3}{2}(1-\rho)^2(q_1+q_2-r)\Bigg],\end{aligned}$$

so that the distribution function of $-2\rho\log Z$ is given by

$$\Pr(-2\rho\log Z\le x)=G_f(x)+\omega_2(\rho)\big\{G_{f+4}(x)-G_f(x)\big\}+O(N^{-3}),\quad(7)$$

where $G_f(x)$ is the distribution function of the chi-square distribution with f degrees of freedom. The distribution function of $-2\log Z$ can also be written as follows.

$$\begin{aligned}\Pr(-2\log Z \le x) = G_f(x) &+ \omega_1(1)\big\{G_{f+2}(x) - G_f(x)\big\} \\ &+ \omega_2(1)\big\{G_{f+4}(x) - G_f(x)\big\} + O(N^{-3}),\end{aligned} \tag{8}$$

where

$$\omega_1(1) = \frac{1}{2}\left\{\sum_{\ell=1}^{q_1} z_{1\ell}^{-1}\left(\xi_{1\ell}^2 - \xi_{1\ell} + \frac{1}{6}\right) + \sum_{\ell=1}^{q_2} z_{2\ell}^{-1}\left(\xi_{2\ell}^2 - \xi_{2\ell} + \frac{1}{6}\right) - \sum_{m=1}^{r} y_m^{-1}\left(\eta_m^2 - \eta_m + \frac{1}{6}\right)\right\},$$

$$\begin{aligned}\omega_2(1) = -\frac{1}{6}\Bigg\{&\sum_{\ell=1}^{q_1} z_{1\ell}^{-2}\left(\xi_{1\ell}^3 - \frac{3}{2}\xi_{1\ell}^2 + \frac{1}{2}\xi_{1\ell}\right) + \sum_{\ell=1}^{q_2} z_{2\ell}^{-2}\left(\xi_{2\ell}^3 - \frac{3}{2}\xi_{2\ell}^2 + \frac{1}{2}\xi_{2\ell}\right) \\ &- \sum_{m=1}^{r} y_m^{-2}\left(\eta_m^3 - \frac{3}{2}\eta_m^2 + \frac{1}{2}\eta_m\right)\Bigg\}.\end{aligned}$$

We now provide a theorem for the h-th null moment of λ.

<u>Theorem 1</u>

For $h = 0, 1, 2, \ldots,$ the h-th null moment of λ is

$$\begin{aligned}\mathrm{E}[\lambda^h] = &\frac{(Np_1 + N_1p_2)^{\frac{(Np_1+N_1p_2)h}{2}}}{N^{\frac{Np_1h}{2}} N_1^{\frac{N_1p_2h}{2}}} \frac{\Gamma_{p_1}\left[\frac{1}{2}(Nh + N - 1)\right]\Gamma_{p_2}\left[\frac{1}{2}(N_1h + N_1 - p_1 - 1)\right]}{\Gamma_{p_1}\left[\frac{1}{2}(N-1)\right]\Gamma_{p_2}\left[\frac{1}{2}(N_1 - p_1 - 1)\right]} \\ &\times \frac{\Gamma\left[\frac{1}{2}\big\{(N-1)p_1 + (N_1-1)p_2\big\}\right]}{\Gamma\left[\frac{1}{2}\big\{(N-1)p_1 + (N_1-1)p_2\big\} + \frac{1}{2}(Np_1 + N_1p_2)h\right]},\end{aligned}$$

where the multivariate gamma function is defined by

$$\Gamma_{p_j}[a] = \pi^{\frac{p_j(p_j-1)}{4}} \prod_{\ell=1}^{p_j} \Gamma\left[a - \frac{1}{2}(\ell - 1)\right].$$

Theorem 1 was proposed in Chang and Richards (2010, p.618, Theorem 4.7). Transforming Theorem 1, the h-th null moment of λ can be written as

$$\mathrm{E}[\lambda^h] = K \left[\frac{\left\{ \frac{1}{2}(Np_1 + N_1p_2) \right\}^{\frac{Np_1+N_1p_2}{2}}}{\prod_{\ell=1}^{p_1} \left(\frac{1}{2}N\right)^{\frac{N}{2}} \prod_{\ell=1}^{p_2} \left(\frac{1}{2}N_1\right)^{\frac{N_1}{2}}} \right]^h$$
$$\times \frac{\prod_{\ell=1}^{p_1} \Gamma\left[\frac{1}{2}N(1+h) - \frac{1}{2}\ell\right] \prod_{\ell=1}^{p_2} \Gamma\left[\frac{1}{2}N_1(1+h) - \frac{1}{2}(\ell + p_1)\right]}{\Gamma\left[\frac{1}{2}(Np_1 + N_1p_2)(1+h) - \frac{1}{2}p\right]},$$

where K is a constant and

$$K = \frac{\Gamma\left[\frac{1}{2}\{(N-1)p_1 + (N_1-1)p_2\}\right]}{\prod_{\ell=1}^{p_1} \Gamma\left[\frac{1}{2}(N-1) - \frac{1}{2}(\ell-1)\right] \prod_{\ell=1}^{p_2} \Gamma\left[\frac{1}{2}(N_1 - p_1 - 1) - \frac{1}{2}(\ell-1)\right]}.$$

Comparing this h-th null moment of the deformed λ with (1),

$$\begin{aligned} r &= 1, q_1 = p_1, q_2 = p_2, \\ z_{1\ell} &= \frac{1}{2}N, \xi_{1\ell} = -\frac{1}{2}\ell, \ell = 1, \ldots, p_1, \\ z_{2\ell} &= \frac{1}{2}N_1, \xi_{2\ell} = -\frac{1}{2}(\ell + p_1), \ell = 1, \ldots, p_2, \\ y_1 &= \frac{1}{2}(Np_1 + N_1p_2), \eta_1 = -\frac{1}{2}p, \end{aligned}$$

where all of the above equations satisfy (2). Therefore, the distribution functions of the LRT statistic and modified LRT statistic can be obtained by substituting the above values into (8) and (7), respectively. Then, we obtain the following theorem.

Theorem 2

When H_0 is true and $\gamma_1 = N_1/N \to \delta_1 \in (0,1)$ $(N_1, N \to \infty)$, the distribution function of the LRT statistic is given as follows.

$$\begin{aligned} \Pr(-2\log\lambda \le x) = G_f(x) &+ \frac{\beta_{(2)}}{N}\left\{G_{f+2}(x) - G_f(x)\right\} \\ &+ \frac{\gamma_{(2)}}{N^2}\left\{G_{f+4}(x) - G_f(x)\right\} + O(N^{-3}), \end{aligned} \tag{9}$$

where

$$\begin{aligned}
\beta_{(2)} &= N\omega_1(1) = \frac{1}{24}\Bigg[p_1(2p_1^2 + 9p_1 + 11) \\
&\quad + \frac{1}{\gamma_1}\big\{p_2(2p_2^2 + 9p_2 + 11) + 6p_1p_2(p+3)\big\} - \frac{2}{p_1 + \gamma_1 p_2}(3p^2 + 6p + 2)\Bigg], \\
\gamma_{(2)} &= N^2\omega_2(1) \\
&= \frac{1}{48}\Bigg[p_1(p_1+1)(p_1+2)(p_1+3) + \frac{1}{\gamma_1^2}\Bigg(p_2(p_2+1)(p_2+2)(p_2+3) \\
&\quad + 2p_1p_2\big\{(p_2+1)(2p+p_1+7) + 2(p_1+1)(p_1+2)\big\}\Bigg) \\
&\quad - \frac{4}{(\gamma_1 p_2 + p_1)^2}p(p+1)(p+2)\Bigg].
\end{aligned}$$

Further, when H_0 is true and $M = \rho N$, the distribution function of the modified LRT statistic can be expressed as follows.

$$\Pr(-2\rho\log\lambda \leq x) = G_f(x) + \frac{\gamma_{(2)}^*}{M^2}\big\{G_{f+4}(x) - G_f(x)\big\} + O(M^{-3}), \tag{10}$$

where

$$\begin{aligned}
f &= \frac{1}{2}(p+2)(p-1), \\
\rho &= 1 - \frac{1}{6(p+2)(p-1)N}\Bigg[p_1(2p_1^2 + 9p_1 + 11) \\
&\quad + \frac{1}{\gamma_1}\big\{p_2(2p_2^2 + 9p_2 + 11) + 6p_1p_2(p+3)\big\} - \frac{2}{p_1 + \gamma_1 p_2}(3p^2 + 6p + 2)\Bigg], \\
\gamma_{(2)}^* &= M^2\omega_2(\rho) = \frac{1}{288}\Bigg[-\frac{1}{(p+2)(p-1)}\Bigg(p_1(2p_1^2 + 9p_1 + 11) \\
&\quad + \frac{1}{\gamma_1}\big\{p_2(2p_2^2 + 9p_2 + 11) + 6p_1p_2(p+3)\big\} - \frac{2}{p_1 + \gamma_1 p_2}(3p^2 + 6p + 2)\Bigg)^2 \\
&\quad + 6p_1(p_1+1)(p_1+2)(p_1+3) + \frac{1}{\gamma_1^2}\Bigg(6p_2(p_2+1)(p_2+2)(p_2+3) \\
&\quad + 12p_1p_2\big\{(p_2+1)(2p+p_1+7) + 2(p_1+1)(p_1+2)\big\}\Bigg) \\
&\quad - \frac{24}{(p_1 + \gamma_1 p_2)^2}p(p+1)(p+2)\Bigg].
\end{aligned}$$

The value of f is the same as that in Choi (2005). From (9) and (10), the upper 100α percentiles of $-2\log\lambda$ and $-2\rho\log\lambda$ are, respectively,

$$q_{(2)}(\alpha) = \chi_f^2(\alpha) + \frac{1}{N}\frac{2\beta_{(2)}}{f}\chi_f^2(\alpha) + \frac{1}{N^2}\frac{2\gamma_{(2)}}{f(f+2)}\chi_f^2(\alpha)\{\chi_f^2(\alpha)+f+2\} + O(N^{-3}),$$
$$q^*_{(2)}(\alpha) = \chi_f^2(\alpha) + \frac{1}{M^2}\frac{2\gamma^*_{(2)}}{f(f+2)}\chi_f^2(\alpha)\{\chi_f^2(\alpha)+f+2\} + O(M^{-3}),$$

where $\chi_f^2(\alpha)$ is the upper 100α percentile of the chi-square distribution with f degrees of freedom.

3.2 General-step case

Next, we consider a random variable Z with the following moments under k-step monotone missing data $(0 < Z \leq 1, k \geq 2)$.

$$\mathrm{E}[Z^h] = K\left[\frac{\prod_{m=1}^{r} y_m^{y_m}}{\prod_{j=1}^{k}\prod_{\ell=1}^{q_j} z_{j\ell}^{z_{j\ell}}}\right]^h \frac{\prod_{j=1}^{k}\prod_{\ell=1}^{q_j}\Gamma\left[z_{j\ell}(1+h)+\xi_{j\ell}\right]}{\prod_{m=1}^{r}\Gamma\left[y_m(1+h)+\eta_m\right]}, \tag{11}$$

where

$$\sum_{m=1}^{r} y_m = \sum_{j=1}^{k}\sum_{\ell=1}^{q_j} z_{j\ell}, \tag{12}$$

and K is a constant such that $\mathrm{E}[Z^0] = 1$. Moreover, $z_{j\ell} = a_{j\ell}N, y_m = b_m N$, where $a_{j\ell}$ and b_m are constants and N is the asymptotic variable.

Using the calculation as described in Section 3.1, the distribution functions of $-2\log Z$ and $-2\rho\log Z$ are, respectively,

$$\begin{aligned}\Pr(-2\log Z \leq x) &= G_f(x) + \omega_1(1)\big\{G_{f+2}(x) - G_f(x)\big\} \\ &\quad + \omega_2(1)\big\{G_{f+4}(x) - G_f(x)\big\} + O(N^{-3}), \\ \Pr(-2\rho\log Z \leq x) &= G_f(x) + \omega_2(\rho)\big\{G_{f+4}(x) - G_f(x)\big\} + O(N^{-3}),\end{aligned}$$

where

$$f = -2\left\{\sum_{j=1}^{k}\sum_{\ell=1}^{q_j}\xi_{j\ell} - \sum_{m=1}^{r}\eta_m - \frac{1}{2}\left(\sum_{j=1}^{k} q_j - r\right)\right\},$$

$$\rho = 1 - \frac{1}{f}\left\{\sum_{j=1}^{k}\sum_{\ell=1}^{q_j} z_{j\ell}^{-1}\left(\xi_{j\ell}^2 - \xi_{j\ell} + \frac{1}{6}\right) - \sum_{m=1}^{r} y_m^{-1}\left(\eta_m^2 - \eta_m + \frac{1}{6}\right)\right\},$$

$$\omega_1(1) = \frac{1}{2}\left\{\sum_{j=1}^{k}\sum_{\ell=1}^{q_j} z_{j\ell}^{-1}\left(\xi_{j\ell}^2 - \xi_{j\ell} + \frac{1}{6}\right) - \sum_{m=1}^{r} y_m^{-1}\left(\eta_m^2 - \eta_m + \frac{1}{6}\right)\right\},$$

$$\omega_2(1) = -\frac{1}{6}\left\{\sum_{j=1}^{k}\sum_{\ell=1}^{q_j} z_{j\ell}^{-2}\left(\xi_{j\ell}^3 - \frac{3}{2}\xi_{j\ell}^2 + \frac{1}{2}\xi_{j\ell}\right) - \sum_{m=1}^{r} y_m^{-2}\left(\eta_m^3 - \frac{3}{2}\eta_m^2 + \frac{1}{2}\eta_m\right)\right\},$$

$$\begin{aligned}\omega_2(\rho) = -\frac{1}{6\rho^2}\Bigg[&3(1-\rho)^2\left(\sum_{j=1}^{k}\sum_{\ell=1}^{q_j}\xi_{j\ell} - \sum_{m=1}^{r}\eta_m\right)\\ &+ (1-\rho)\left\{\sum_{j=1}^{k}\sum_{\ell=1}^{q_j} z_{j\ell}^{-1}\left(3\xi_{j\ell}^2 - 3\xi_{j\ell} + \frac{1}{2}\right) - \sum_{m=1}^{r} y_m^{-1}\left(3\eta_m^2 - 3\eta_m + \frac{1}{2}\right)\right\}\\ &+ \sum_{j=1}^{k}\sum_{\ell=1}^{q_j} z_{j\ell}^{-2}\left(\xi_{j\ell}^3 - \frac{3}{2}\xi_{j\ell}^2 + \frac{1}{2}\xi_{j\ell}\right) - \sum_{m=1}^{r} y_m^{-2}\left(\eta_m^3 - \frac{3}{2}\eta_m^2 + \frac{1}{2}\eta_m\right)\\ &- \frac{3}{2}(1-\rho)^2\left(\sum_{j=1}^{k} q_j - r\right)\Bigg].\end{aligned}$$

As in Theorem 1 (see Chang and Richards (2010) for the proof of Theorem 1), we obtain Theorem 3.

Theorem 3

For $h = 0, 1, 2, \ldots,$ the h-th null moment of $\lambda^{(k)}$ is

$$\begin{aligned}\mathrm{E}[(\lambda^{(k)})^h] = &\frac{\left(\sum_{j=1}^{k} N_{(1\ldots k-j+1)}p_j\right)^{\frac{\left(\sum_{j=1}^{k} N_{(1\ldots k-j+1)}p_j\right)h}{2}}}{\prod_{j=1}^{k}(N_{(1\ldots k-j+1)})^{\frac{N_{(1\ldots k-j+1)}p_j h}{2}}}\\ &\times\left(\prod_{j=1}^{k}\frac{\Gamma_{p_j}\left[\frac{1}{2}(N_{(1\ldots k-j+1)}h + N_{(1\ldots k-j+1)} - p_{(1\ldots j-1)} - 1)\right]}{\Gamma_{p_j}\left[\frac{1}{2}(N_{(1\ldots k-j+1)} - p_{(1\ldots j-1)} - 1)\right]}\right)\\ &\times\frac{\Gamma\left[\frac{1}{2}\sum_{j=1}^{k}(N_{(1\ldots k-j+1)} - 1)p_j\right]}{\Gamma\left[\frac{1}{2}\sum_{j=1}^{k}(N_{(1\ldots k-j+1)} - 1)p_j + \frac{1}{2}\sum_{j=1}^{k}N_{(1\ldots k-j+1)}p_j h\right]},\end{aligned}$$

where $p_{(1\dots j-1)} = 0$ *when* $j = 1$.

By transforming Theorem 3, we obtain the following equation.

$$\mathrm{E}[(\lambda^{(k)})^h] = K \left[\frac{\left(\frac{1}{2}\sum_{j=1}^{k} N_{(1\dots k-j+1)} p_j\right)^{\frac{\sum_{j=1}^{k} N_{(1\dots k-j+1)} p_j}{2}}}{\prod_{j=1}^{k}\prod_{\ell=1}^{p_j}\left(\frac{1}{2}N_{(1\dots k-j+1)}\right)^{\frac{N_{(1\dots k-j+1)}}{2}}} \right]^h$$

$$\times \frac{\prod_{j=1}^{k}\prod_{\ell=1}^{p_j}\Gamma\left[\frac{1}{2}N_{(1\dots k-j+1)}(1+h) - \frac{1}{2}(\ell + p_{(1\dots j-1)})\right]}{\Gamma\left[\frac{1}{2}\left(\sum_{j=1}^{k} N_{(1\dots k-j+1)} p_j\right)(1+h) - \frac{1}{2}p\right]},$$

where K is a constant and

$$K = \frac{\Gamma\left[\frac{1}{2}\sum_{j=1}^{k}(N_{(1\dots k-j+1)} - 1)p_j\right]}{\prod_{j=1}^{k}\prod_{\ell=1}^{p_j}\Gamma\left[\frac{1}{2}(N_{(1\dots k-j+1)} - p_{(1\dots j-1)} - 1) - \frac{1}{2}(\ell - 1)\right]}.$$

Comparing this h-th null moment of the deformed $\lambda^{(k)}$ with (11), for $j = 1, \dots, k$,

$$r = 1, q_j = p_j, z_{j\ell} = \frac{1}{2}N_{(1\dots k-j+1)}, \xi_{j\ell} = -\frac{1}{2}(\ell + p_{(1\dots j-1)}), \ell = 1, \dots, p_j,$$
$$y_1 = \frac{1}{2}\sum_{j=1}^{k} N_{(1\dots k-j+1)} p_j, \eta_1 = -\frac{1}{2}p,$$

where all of the above equations satisfy (12). The distribution functions of $-2\log\lambda^{(k)}$ and $-2\rho\log\lambda^{(k)}$ can be obtained by substituting these values into the distribution functions of $-2\log Z$ and $-2\rho\log Z$, respectively.

Theorem 4

When H_0 is true and $M = \rho N, \gamma_j = N_j/N \to \delta_j \in (0,1)$ $(N_j, N \to \infty), \gamma_{(1\ldots k-j+1)} = \gamma_1 + \cdots + \gamma_{k-j+1},\ j = 1, \ldots, k$, the distribution functions of $-2\log\lambda^{(k)}$ and $-2\rho\log\lambda^{(k)}$ are given as follows.

$$\begin{aligned}\Pr(-2\log\lambda^{(k)} \le x) &= G_f(x) + \frac{\beta_{(k)}}{N}\{G_{f+2}(x) - G_f(x)\} \\ &\quad + \frac{\gamma_{(k)}}{N^2}\{G_{f+4}(x) - G_f(x)\} + O(N^{-3}), \qquad (13)\end{aligned}$$

$$\Pr(-2\rho\log\lambda^{(k)} \le x) = G_f(x) + \frac{\gamma^*_{(k)}}{M^2}\{G_{f+4}(x) - G_f(x)\} + O(M^{-3}), \qquad (14)$$

where

$$\begin{aligned} f &= \frac{1}{2}(p+2)(p-1), \\ \rho &= 1 - \frac{1}{6(p+2)(p-1)N}\left[\sum_{j=1}^{k}\frac{1}{\gamma_{(1\ldots k-j+1)}}\{p_j(2p_j^2 + 9p_j + 11) + 6p_{(1\ldots j-1)}p_j(p_{(1\ldots j)} + 3)\}\right. \\ &\quad \left. - \frac{2}{\sum_{j=1}^{k}\gamma_{(1\ldots k-j+1)}p_j}(3p^2 + 6p + 2)\right], \end{aligned}$$

$$\begin{aligned} \beta_{(k)} &= N\omega_1(1) = \frac{1}{24}\left[\sum_{j=1}^{k}\frac{1}{\gamma_{(1\ldots k-j+1)}}\{p_j(2p_j^2 + 9p_j + 11) + 6p_{(1\ldots j-1)}p_j(p_{(1\ldots j)} + 3)\}\right. \\ &\quad \left. - \frac{2}{\sum_{j=1}^{k}\gamma_{(1\ldots k-j+1)}p_j}(3p^2 + 6p + 2)\right], \\ \gamma_{(k)} &= N^2\omega_2(1) = \frac{1}{48}\left[\sum_{j=1}^{k}\frac{1}{\gamma^2_{(1\ldots k-j+1)}}\Big(p_j(p_j+1)(p_j+2)(p_j+3)\right. \\ &\quad + 2p_{(1\ldots j-1)}p_j\{(p_j+1)(2p_{(1\ldots j)} + p_{(1\ldots j-1)} + 7) + 2(p_{(1\ldots j-1)} + 1)(p_{(1\ldots j-1)} + 2)\}\Big) \\ &\quad \left. - \frac{4}{\left(\sum_{j=1}^{k}\gamma_{(1\ldots k-j+1)}p_j\right)^2}p(p+1)(p+2)\right], \\ \gamma^*_{(k)} &= M^2\omega_2(\rho) = \frac{1}{288}\left[-\frac{1}{(p+2)(p-1)}\left(\sum_{j=1}^{k}\frac{1}{\gamma_{(1\ldots k-j+1)}}\{p_j(2p_j^2 + 9p_j + 11)\right.\right. \\ &\quad + 6p_{(1\ldots j-1)}p_j(p_{(1\ldots j)} + 3)\} \end{aligned}$$

$$
- \frac{2}{\sum_{j=1}^{k} \gamma_{(1\ldots k-j+1)} p_j} (3p^2 + 6p + 2) \Big)^2 + \sum_{j=1}^{k} \frac{1}{\gamma^2_{(1\ldots k-j+1)}} \Big(6p_j(p_j+1)(p_j+2)(p_j+3)
$$

$$
+ 12 p_{(1\ldots j-1)} p_j \big\{ (p_j+1)(2p_{(1\ldots j)} + p_{(1\ldots j-1)} + 7) + 2(p_{(1\ldots j-1)}+1)(p_{(1\ldots j-1)}+2) \big\} \Big)
$$

$$
- \frac{24}{\left(\sum_{j=1}^{k} \gamma_{(1\ldots k-j+1)} p_j \right)^2} p(p+1)(p+2) \Bigg].
$$

From (13) and (14), the upper 100α percentiles of $-2\log\lambda^{(k)}$ and $-2\rho\log\lambda^{(k)}$ are, respectively,

$$
q_{(k)}(\alpha) = \chi^2_f(\alpha) + \frac{1}{N}\frac{2\beta_{(k)}}{f}\chi^2_f(\alpha) + \frac{1}{N^2}\frac{2\gamma_{(k)}}{f(f+2)}\chi^2_f(\alpha)\{\chi^2_f(\alpha) + f + 2\} + O(N^{-3}),
$$

$$
q^*_{(k)}(\alpha) = \chi^2_f(\alpha) + \frac{1}{M^2}\frac{2\gamma^*_{(k)}}{f(f+2)}\chi^2_f(\alpha)\{\chi^2_f(\alpha) + f + 2\} + O(M^{-3}).
$$

Therefore, we can propose $q_1(\alpha), q_2(\alpha), q_3(\alpha)$ for the approximate upper 100α percentiles of $-2\log\lambda^{(k)}$ and $q_1(\alpha), q^\dagger(\alpha)$ for the approximate upper 100α percentiles of $-2\rho\log\lambda^{(k)}$, where

$$
q_1(\alpha) = \chi^2_f(\alpha), \tag{15}
$$

$$
q_2(\alpha) = \chi^2_f(\alpha) + \frac{1}{N}\frac{2\beta_{(k)}}{f}\chi^2_f(\alpha), \tag{16}
$$

$$
q_3(\alpha) = \chi^2_f(\alpha) + \frac{1}{N}\frac{2\beta_{(k)}}{f}\chi^2_f(\alpha) + \frac{1}{N^2}\frac{2\gamma_{(k)}}{f(f+2)}\chi^2_f(\alpha)\{\chi^2_f(\alpha) + f + 2\}, \tag{17}
$$

$$
q^\dagger(\alpha) = \chi^2_f(\alpha) + \frac{1}{M^2}\frac{2\gamma^*_{(k)}}{f(f+2)}\chi^2_f(\alpha)\{\chi^2_f(\alpha) + f + 2\}. \tag{18}
$$

4 Unbiased estimator of σ^2

In this section, we propose an unbiased estimator of σ^2 and estimate the MSEs.

4.1 Two-step case

In two-step monotone missing data, the MLE of σ^2 under H_0 can be written as

$$\widetilde{\sigma}^2 = \frac{1}{N(p_1 + \gamma_1 p_2)} \left(\mathrm{tr}(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}_{11}^{(2)}) + \mathrm{tr}\boldsymbol{W}_{22}^{(1)} \right).$$

Since the expectation of $\widetilde{\sigma}^2$ is

$$\mathrm{E}[\widetilde{\sigma}^2] = \left(1 - \frac{p}{N(p_1 + \gamma_1 p_2)} \right) \sigma^2,$$

an unbiased estimator of σ^2 is given by

$$\widetilde{\sigma}^2_{\mathrm{U}} = \left(1 - \frac{p}{N(p_1 + \gamma_1 p_2)} \right)^{-1} \widetilde{\sigma}^2.$$

4.2 General-step case

Proceeding as in Section 4.1 under k-step monotone missing data ($k \geq 2$), the MLE of σ^2 under H_0 is

$$\widetilde{\sigma}^2 = \left(N \sum_{j=1}^{k} \gamma_{(1 \ldots k-j+1)} p_j \right)^{-1} \sum_{j=1}^{k} \mathrm{tr}\boldsymbol{W}_{jj}^{[k-j+1]}.$$

The expectation of $\widetilde{\sigma}^2$ is

$$\mathrm{E}[\widetilde{\sigma}^2] = \left\{ 1 - p \left(N \sum_{j=1}^{k} \gamma_{(1 \ldots k-j+1)} p_j \right)^{-1} \right\} \sigma^2.$$

Therefore, an unbiased estimator of σ^2 is given by

$$\widetilde{\sigma}^2_{\mathrm{U}} = \left\{ 1 - p \left(N \sum_{j=1}^{k} \gamma_{(1 \ldots k-j+1)} p_j \right)^{-1} \right\}^{-1} \widetilde{\sigma}^2.$$

Further, the variances of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ are, respectively,

$$\mathrm{Var}[\widetilde{\sigma}^2] = 2 \left\{ N \left(\sum_{j=1}^{k} \gamma_{(1 \ldots k-j+1)} p_j \right) - p \right\} \left(N \sum_{j=1}^{k} \gamma_{(1 \ldots k-j+1)} p_j \right)^{-2} \sigma^4,$$

$$\mathrm{Var}[\widetilde{\sigma}^2_{\mathrm{U}}] = 2\left\{N\left(\sum_{j=1}^{k}\gamma_{(1\ldots k-j+1)}p_j\right) - p\right\}^{-1}\sigma^4.$$

Thus, the MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ are, respectively,

$$\begin{aligned}\mathrm{MSE}[\widetilde{\sigma}^2] &= \mathrm{Var}[\widetilde{\sigma}^2] + (\mathrm{E}[\widetilde{\sigma}^2] - \sigma^2)^2 \\ &= \left[2\left\{N\left(\sum_{j=1}^{k}\gamma_{(1\ldots k-j+1)}p_j\right) - p\right\} + p^2\right]\left(N\sum_{j=1}^{k}\gamma_{(1\ldots k-j+1)}p_j\right)^{-2}\sigma^4,\end{aligned}$$

$$\begin{aligned}\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}] &= \mathrm{Var}[\widetilde{\sigma}^2_{\mathrm{U}}] + (\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}] - \sigma^2)^2 \\ &= 2\left\{N\left(\sum_{j=1}^{k}\gamma_{(1\ldots k-j+1)}p_j\right) - p\right\}^{-1}\sigma^4.\end{aligned}$$

Considering the relationship between $\mathrm{MSE}[\widetilde{\sigma}^2]$ and $\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$ in terms of the magnitude, we have the following.

$$\begin{cases}\mathrm{MSE}[\widetilde{\sigma}^2] \le \mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}] & \text{if } 2 \le p \le 4, \\ \mathrm{MSE}[\widetilde{\sigma}^2] \ge \mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}] & \text{if } p = 1, p > 4,\end{cases}$$

where the equalities relationship are satisfied at $N \to \infty$.

5 Affine transformation

In this section, we consider affine transformation in the LR. We consider only two-step monotone missing data as it is the simplest case to study.

Let $\boldsymbol{\Lambda}_{11}$ be a $p_1 \times p_1$ positive definite symmetric matrix, $\boldsymbol{\Lambda}_{22}$ a $p_2 \times p_2$ positive definite symmetric matrix, $\boldsymbol{\Lambda}_{21}$ a $p_2 \times p_1$ matrix, $\boldsymbol{\nu}_1$ a $p_1 \times 1$ vector, and $\boldsymbol{\nu}_2$ a $p_2 \times 1$ vector. Then, we consider the following affine transformation:

$$\begin{aligned}\begin{pmatrix}\boldsymbol{x}^*_{1j} \\ \boldsymbol{x}^*_{2j}\end{pmatrix} &= \boldsymbol{\Lambda}\boldsymbol{C}\begin{pmatrix}\boldsymbol{x}_{1j} \\ \boldsymbol{x}_{2j}\end{pmatrix} + \boldsymbol{\nu},\ j = 1, \ldots, N_1, \\ \boldsymbol{x}^*_{1j} &= \boldsymbol{\Lambda}_{11}\boldsymbol{x}_{1j} + \boldsymbol{\nu}_1,\ j = N_1 + 1, \ldots, N,\end{aligned}$$

where

$$\boldsymbol{\Lambda} = \begin{pmatrix}\boldsymbol{\Lambda}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_{22}\end{pmatrix}, \boldsymbol{C} = \begin{pmatrix}\boldsymbol{I}_{p_1} & \boldsymbol{O} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{I}_{p_2}\end{pmatrix}, \boldsymbol{\nu} = \begin{pmatrix}\boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2\end{pmatrix}.$$

Then,

$$\boldsymbol{x}_{1j}^* = \boldsymbol{\Lambda}_{11}\boldsymbol{x}_{1j} + \boldsymbol{\nu}_1,\ j = 1, \ldots, N,$$
$$\boldsymbol{x}_{2j}^* = \boldsymbol{\Lambda}_{22}\boldsymbol{\Lambda}_{21}\boldsymbol{x}_{1j} + \boldsymbol{\Lambda}_{22}\boldsymbol{x}_{2j} + \boldsymbol{\nu}_2,\ j = 1, \ldots, N_1.$$

Let λ^* be the LR after the affine transformation,

$$\lambda^* = \frac{\left|\frac{1}{N}\left(\boldsymbol{W}_{11}^{(1)^*} + \boldsymbol{W}_{11}^{(2)^*}\right)\right|^{\frac{N}{2}} \left|\frac{1}{N_1}\boldsymbol{W}_{22\cdot 1}^{(1)^*}\right|^{\frac{N_1}{2}}}{\left\{\frac{1}{Np_1 + N_1p_2}\left(\mathrm{tr}(\boldsymbol{W}_{11}^{(1)^*} + \boldsymbol{W}_{11}^{(2)^*}) + \mathrm{tr}\boldsymbol{W}_{22}^{(1)^*}\right)\right\}^{\frac{Np_1+N_1p_2}{2}}},$$

where

$$\begin{aligned}
\overline{\boldsymbol{x}}_1^{(1)^*} &= \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{1j}^*,\ \overline{\boldsymbol{x}}_2^{(1)^*} = \frac{1}{N_1}\sum_{j=1}^{N_1}\boldsymbol{x}_{2j}^*,\ \overline{\boldsymbol{x}}_1^{(2)^*} = \frac{1}{N_2}\sum_{j=N_1+1}^{N}\boldsymbol{x}_{1j}^*,\\
\boldsymbol{W}_{11}^{(1)^*} &= \sum_{j=1}^{N_1}(\boldsymbol{x}_{1j}^* - \overline{\boldsymbol{x}}_1^{(1)^*})(\boldsymbol{x}_{1j}^* - \overline{\boldsymbol{x}}_1^{(1)^*})',\\
\boldsymbol{W}_{12}^{(1)^*} &= (\boldsymbol{W}_{21}^{(1)^*})' = \sum_{j=1}^{N_1}(\boldsymbol{x}_{1j}^* - \overline{\boldsymbol{x}}_1^{(1)^*})(\boldsymbol{x}_{2j}^* - \overline{\boldsymbol{x}}_2^{(1)^*})',\\
\boldsymbol{W}_{22}^{(1)^*} &= \sum_{j=1}^{N_1}(\boldsymbol{x}_{2j}^* - \overline{\boldsymbol{x}}_2^{(1)^*})(\boldsymbol{x}_{2j}^* - \overline{\boldsymbol{x}}_2^{(1)^*})',\\
\boldsymbol{W}_{11}^{(2)^*} &= \sum_{j=N_1+1}^{N}(\boldsymbol{x}_{1j}^* - \overline{\boldsymbol{x}}_1^{(2)^*})(\boldsymbol{x}_{1j}^* - \overline{\boldsymbol{x}}_1^{(2)^*})' + \frac{N_1N_2}{N}(\overline{\boldsymbol{x}}_1^{(1)^*} - \overline{\boldsymbol{x}}_1^{(2)^*})(\overline{\boldsymbol{x}}_1^{(1)^*} - \overline{\boldsymbol{x}}_1^{(2)^*})',\\
\boldsymbol{W}_{22\cdot 1}^{(1)^*} &= \boldsymbol{W}_{22}^{(1)^*} - \boldsymbol{W}_{21}^{(1)^*}(\boldsymbol{W}_{11}^{(1)^*})^{-1}\boldsymbol{W}_{12}^{(1)^*}.
\end{aligned}$$

Then,

$$\begin{aligned}
\boldsymbol{W}_{11}^{(1)^*} + \boldsymbol{W}_{11}^{(2)^*} &= \boldsymbol{\Lambda}_{11}(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}_{11}^{(2)})\boldsymbol{\Lambda}_{11},\\
\boldsymbol{W}_{22\cdot 1}^{(1)^*} &= \boldsymbol{\Lambda}_{11}\boldsymbol{W}_{22\cdot 1}^{(1)}\boldsymbol{\Lambda}_{11},\\
\boldsymbol{W}_{22}^{(1)^*} &= \boldsymbol{\Lambda}_{22}\boldsymbol{\Lambda}_{21}\boldsymbol{W}_{11}^{(1)}\boldsymbol{\Lambda}_{21}'\boldsymbol{\Lambda}_{22} + \boldsymbol{\Lambda}_{22}\boldsymbol{\Lambda}_{21}\boldsymbol{W}_{12}^{(1)}\boldsymbol{\Lambda}_{22}\\
&\quad + \boldsymbol{\Lambda}_{22}\boldsymbol{W}_{21}^{(1)}\boldsymbol{\Lambda}_{21}'\boldsymbol{\Lambda}_{22} + \boldsymbol{\Lambda}_{22}\boldsymbol{W}_{22}^{(1)}\boldsymbol{\Lambda}_{22}.
\end{aligned}$$

Let

$$\boldsymbol{\nu} = -\boldsymbol{\Lambda}\boldsymbol{C}\boldsymbol{\mu}, \boldsymbol{\Lambda}\boldsymbol{C} = \begin{pmatrix} \boldsymbol{\Lambda}_{11} & \boldsymbol{O} \\ \boldsymbol{\Lambda}_{22}\boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{pmatrix} = \sigma\begin{pmatrix} \boldsymbol{\Sigma}_{11}^{-\frac{1}{2}} & \boldsymbol{O} \\ -\boldsymbol{\Sigma}_{22\cdot 1}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{\Sigma}_{22\cdot 1}^{-\frac{1}{2}} \end{pmatrix}.$$

Then, the mean vector and variance-covariance matrix after the affine transformation are, respectively,

$$\boldsymbol{\Lambda C \mu} + \boldsymbol{\nu} = \mathbf{0}, \boldsymbol{\Lambda C \Sigma}(\boldsymbol{\Lambda C})' = \sigma^2 \boldsymbol{I}_p.$$

Therefore, given

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

$\boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}_p$ can be obtained without loss of generality for deriving the null distribution of the LRT statistic and modified LRT statistic by transforming using $\boldsymbol{\Lambda C}$. Since $\boldsymbol{\Lambda}_{11} = \boldsymbol{I}_{p_1}, \boldsymbol{\Lambda}_{22} = \boldsymbol{I}_{p_2}, \boldsymbol{\Lambda}_{21} = \boldsymbol{O}$ under H_0,

$$\begin{aligned} \boldsymbol{W}_{11}^{(1)^*} + \boldsymbol{W}_{11}^{(2)^*} &= \boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}_{11}^{(2)}, \\ \boldsymbol{W}_{22\cdot 1}^{(1)^*} &= \boldsymbol{W}_{22\cdot 1}^{(1)}, \\ \boldsymbol{W}_{22}^{(1)^*} &= \boldsymbol{W}_{22}^{(1)}. \end{aligned}$$

Therefore,

$$\lambda^* = \frac{\left| \frac{1}{N} \left(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}_{11}^{(2)} \right) \right|^{\frac{N}{2}} \left| \frac{1}{N_1} \boldsymbol{W}_{22\cdot 1}^{(1)} \right|^{\frac{N_1}{2}}}{\left\{ \frac{1}{Np_1 + N_1 p_2} \left(\mathrm{tr}(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}_{11}^{(2)}) + \mathrm{tr} \boldsymbol{W}_{22}^{(1)} \right) \right\}^{\frac{Np_1 + N_1 p_2}{2}}} = \lambda.$$

From the above, we see that $-2 \log \lambda$ and $-2\rho \log \lambda$ are invariant under H_0 for the affine transformation and λ is independent of σ^2 under H_0. The same is true under k-step monotone missing data.

6 Simulation studies

In this section, for $k \geq 2$, we denote the Type I errors for the approximate upper 100α percentiles of the LRT statistic and modified LRT statistic respectively as follows.

$$\alpha_i = \Pr\{-2\log\lambda^{(k)} > q_i(\alpha)\}, i = 1, 2, 3,$$
$$\alpha_{\chi^2} = \Pr\{-2\rho\log\lambda^{(k)} > q_1(\alpha)\},$$
$$\alpha^{\dagger} = \Pr\{-2\rho\log\lambda^{(k)} > q^{\dagger}(\alpha)\}.$$

$q_1(\alpha), q_2(\alpha), q_3(\alpha), q^{\dagger}(\alpha)$ are given by (15),(16),(17),(18) in Section 3.2. The upper 100α percentiles of the LRT statistic and modified LRT statistic and the Type I errors for each approximate upper 100α percentile, the expectation and MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ were numerically evaluated using Monte Carlo simulations (with 10^6 runs) for $\alpha = 0.05$. We also assume $\sigma^2 = 1.0$, because the LRs are independent of σ^2 under H_0, as described in the previous section.

In Tables 1-7, the dimensions and sample sizes are set as follows.

- Table 1: two-step case

$$(p_1, p_2) = \begin{cases} (2,2) & (N_1, N_2) = (c, c), (c, 2c), (c, \frac{1}{2}c), \\ (2,4), (4,2) & (N_1, N_2) = (c, c). \end{cases}$$

- Table 2: two-step case

$$(p_1, p_2) = (4, 4), (8, 4),$$
$$(N_1, N_2) = (c, c).$$

- Table 3: three-step case

$$(p_1, p_2, p_3) = (2, 2, 2),$$
$$(N_1, N_2, N_3) = (c, c, c), (2d, d, d), (c, 2c, c), (c, c, 2c), (c, 2c, 2c).$$

- Table 4: three-step case

$$(p_1, p_2, p_3) = (3, 3, 3), (2, 2, 4), (2, 4, 2), (4, 2, 2),$$
$$(N_1, N_2, N_3) = (c, c, c).$$

- Table 5: five-step case

$$(p_1, p_2, p_3, p_4, p_5) = (2, 2, 2, 2, 2),$$
$$(N_1, N_2, N_3, N_4, N_5) = (e, e, e, e, e), (2c, c, c, c, c), (e, 2e, e, e, e),$$
$$(e, e, 2e, e, e), (e, e, e, 2e, e), (e, e, e, e.2e).$$

- Table 6: five-step case

$$(p_1, p_2, p_3, p_4, p_5) = (3, 3, 3, 3, 3), (2, 2, 2, 2, 4), (2, 2, 2, 4, 2),$$
$$(N_1, N_2, N_3, N_4, N_5) = (e, e, e, e, e).$$

- Table 7: five-step case

$$(p_1, p_2, p_3, p_4, p_5) = (2, 2, 4, 2, 2), (2, 4, 2, 2, 2), (4, 2, 2, 2, 2),$$
$$(N_1, N_2, N_3, N_4, N_5) = (e, e, e, e, e),$$

where

$$c = 10, 20, 40, 50, 80, 100, 200, 400, d = 5, 10, 20, 40, 50, 80, 100, 200,$$
$$e = 20, 40, 50, 80, 100, 200, 400,$$

and Tables 1-7 satisfy $N_1 > p$. Moreover, the closest value to α among $\alpha_1, \alpha_2, \alpha_3, \alpha_{\chi^2}$, and $\alpha^\dagger$ are marked in bold in each row.

Table 1: The upper percentiles of $-2\log\lambda$, $-2\rho\log\lambda$ and Type I errors and the expectations and MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ for $(p_1, p_2) = (2,2), (2,4), (4,2)$, $\sigma^2 = 1.0$ and $\alpha = 0.05$.

Sample Size		Upper Percentile					Type I Error					Expectation		Mean Squared Eroor	
N_1	N_2	$q_2(\alpha)$	$q_3(\alpha)$	$-2\log\lambda$	$q^\dagger(\alpha)$	$-2\rho\log\lambda$	α_1	α_2	α_3	α_{χ^2}	$\alpha^\dagger$	$\mathrm{E}[\widetilde{\sigma}^2]$	$\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$	$\mathrm{MSE}[\widetilde{\sigma}^2]$	$\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$
							$(p_1, p_2) = (2,2)$								
10	10	20.90	22.37	22.75	17.39	17.39	.180	.076	.055	.058	**.050**	.933	1.000	.03554	.03568
20	20	18.91	19.28	19.27	17.01	17.00	.095	.055	**.050**	.051	**.050**	.967	1.000	.01719	.01720
40	40	17.92	18.01	18.01	16.94	16.95	.069	.052	**.050**	.051	**.050**	.983	1.000	.00846	.00846
50	50	17.72	17.77	17.78	16.93	16.94	.065	.051	**.050**	**.050**	**.050**	.987	1.000	.00676	.00677
80	80	17.42	17.44	17.46	16.92	16.94	.059	.051	**.050**	**.050**	**.050**	.992	1.000	.00421	.00421
100	100	17.32	17.33	17.34	16.92	16.93	.057	**.050**	**.050**	**.050**	**.050**	.993	1.000	.00336	.00336
200	200	17.12	17.12	17.12	16.92	16.92	.053	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00167	.00167
400	400	17.02	17.02	17.01	16.92	16.91	.051	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00084	.00084
10	20	20.81	22.35	22.66	17.44	17.46	.178	.077	.055	.059	**.050**	.950	1.000	.02627	.02634
20	40	18.86	19.22	19.22	17.02	17.01	.094	.055	**.050**	.052	**.050**	.975	1.000	.01283	.01284
40	80	17.89	17.98	17.97	16.94	16.94	.068	.051	**.050**	**.050**	**.050**	.987	1.000	.00633	.00633
50	100	17.70	17.75	17.75	16.93	16.93	.064	.051	**.050**	**.050**	**.050**	.990	1.000	.00505	.00505
80	160	17.40	17.43	17.43	16.92	16.93	.059	**.050**	**.050**	**.050**	**.050**	.994	1.000	.00314	.00314
100	200	17.31	17.32	17.31	16.92	16.92	.057	**.050**	**.050**	**.050**	**.050**	.995	1.000	.00251	.00251
200	400	17.11	17.12	17.12	16.92	16.92	.053	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00125	.00125
400	800	17.02	17.02	17.00	16.92	16.90	.051	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00063	.00063
10	5	21.02	22.52	22.92	17.33	17.37	.184	.077	.055	.057	**.051**	.920	1.000	.04329	.04355
20	10	18.97	19.34	19.33	17.00	16.99	.096	.055	**.050**	.051	**.050**	.960	1.000	.02077	.02080
40	20	17.94	18.04	18.04	16.94	16.94	.070	.051	**.050**	**.050**	**.050**	.980	1.000	.01019	.01019
80	40	17.43	17.45	17.47	16.92	16.94	.059	.051	**.050**	**.050**	**.050**	.990	1.000	.00505	.00505
100	50	17.33	17.34	17.33	16.92	16.91	.057	**.050**	**.050**	**.050**	**.050**	.992	1.000	.00403	.00403
160	80	17.18	17.18	17.17	16.92	16.91	.054	**.050**	**.050**	**.050**	**.050**	.995	1.000	.00252	.00252
200	100	17.12	17.13	17.11	16.92	16.91	.053	**.050**	**.050**	**.050**	**.050**	.996	1.000	.00201	.00201
400	200	17.02	17.02	17.02	16.92	16.92	.052	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00100	.00100
							$(p_1, p_2) = (2,4)$								
10	10	41.21	45.83	49.23	33.35	33.88	.432	.149	.081	.083	**.056**	.925	1.000	.02877	.02708
20	20	36.31	37.47	37.62	31.73	31.76	.158	.065	.052	.054	**.050**	.963	1.000	.01345	.01301
40	40	33.86	34.15	34.16	31.48	31.50	.090	.053	**.050**	.051	**.050**	.981	1.000	.00648	.00636
50	50	33.37	33.55	33.54	31.45	31.45	.080	.052	**.050**	**.050**	**.050**	.985	1.000	.00515	.00508
80	80	32.63	32.71	32.70	31.43	31.42	.067	.051	**.050**	**.050**	**.050**	.991	1.000	.00318	.00316
100	100	32.39	32.44	32.43	31.42	31.42	.063	**.050**	**.050**	**.050**	**.050**	.992	1.000	.00253	.00252
200	200	31.90	31.91	31.91	31.41	31.41	.056	**.050**	**.050**	**.050**	**.050**	.996	1.000	.00126	.00125
400	400	31.66	31.66	31.66	31.41	31.41	.053	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00063	.00063
							$(p_1, p_2) = (4,2)$								
10	10	39.72	43.50	46.75	33.47	34.39	.365	.137	.081	.091	**.060**	.940	1.000	.02244	.02130
20	20	35.56	36.51	36.68	31.78	31.83	.137	.062	.052	.055	**.051**	.970	1.000	.01060	.01032
40	40	33.49	33.72	33.72	31.49	31.49	.083	.053	**.050**	.051	**.050**	.985	1.000	.00515	.00507
50	50	33.07	33.22	33.23	31.46	31.47	.075	.052	**.050**	.051	**.050**	.988	1.000	.00410	.00405
80	80	32.45	32.51	32.51	31.43	31.43	.064	.051	**.050**	**.050**	**.050**	.992	1.000	.00253	.00251
100	100	32.24	32.28	32.26	31.42	31.41	.061	**.050**	**.050**	**.050**	**.050**	.994	1.000	.00202	.00201
200	200	31.83	31.84	31.83	31.41	31.41	.055	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00101	.00100
400	400	31.62	31.62	31.61	31.41	31.40	.053	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00050	.00050

Note. $q_1(0.05) = \chi^2_9(0.05) = 16.92$ for $(p_1, p_2) = (2,2)$,
$q_1(0.05) = \chi^2_{20}(0.05) = 31.41$ for $(p_1, p_2) = (2,4), (4,2)$.

Table 2: The upper percentiles of $-2\log\lambda$, $-2\rho\log\lambda$ and Type I errors and the expectations and MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ for $(p_1, p_2) = (4,4), (8,4), \sigma^2 = 1.0$ and $\alpha = 0.05$.

Sample Size		Upper Percentile					Type I Error					Expectation		Mean Squared Eroor	
N_1	N_2	$q_2(\alpha)$	$q_3(\alpha)$	$-2\log\lambda$	$q^\dagger(\alpha)$	$-2\rho\log\lambda$	α_1	α_2	α_3	α_{χ^2}	$\alpha^\dagger$	$\mathrm{E}[\widetilde{\sigma}^2]$	$\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$	$\mathrm{MSE}[\widetilde{\sigma}^2]$	$\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$
							$(p_1, p_2) = (4, 4)$								
10	10	67.47	77.58	96.39	56.44	62.20	.801	.393	.211	.217	**.103**	.933	1.000	.01997	.01780
20	20	58.63	61.16	61.90	50.82	50.93	.252	.082	.056	.062	**.051**	.967	1.000	.00918	.00864
40	40	54.22	54.85	54.88	50.01	50.01	.115	.056	**.050**	.052	**.050**	.983	1.000	.00437	.00423
50	50	53.33	53.74	53.72	49.93	49.90	.098	.054	**.050**	.051	**.050**	.987	1.000	.00346	.00337
80	80	52.01	52.17	52.17	49.85	49.85	.077	.051	**.050**	.051	**.050**	.992	1.000	.00213	.00210
100	100	51.57	51.67	51.68	49.83	49.84	.071	.051	**.050**	**.050**	**.050**	.993	1.000	.00170	.00168
200	200	50.69	50.71	50.71	49.81	49.81	.059	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00084	.00084
400	400	50.24	50.25	50.23	49.80	49.79	.054	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00042	.00042
							$(p_1, p_2) = (8, 4)$								
20	20	119.74	127.77	133.80	103.15	104.92	.598	.174	.089	.107	**.062**	.970	1.000	.00577	.00517
40	40	109.11	111.12	111.49	99.38	99.46	.204	.067	.052	.057	**.051**	.985	1.000	.00269	.00254
50	50	106.99	108.27	108.43	99.03	99.07	.156	.060	.051	.054	**.050**	.988	1.000	.00212	.00202
80	80	103.80	104.30	104.31	98.68	98.68	.104	.054	**.050**	.052	**.050**	.992	1.000	.00130	.00126
100	100	102.74	103.06	103.11	98.61	98.66	.090	.053	**.050**	.051	**.050**	.994	1.000	.00103	.00100
200	200	100.61	100.69	100.72	98.51	98.54	.067	.051	**.050**	**.050**	**.050**	.997	1.000	.00051	.00050
400	400	99.55	99.57	99.58	98.49	98.51	.058	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00025	.00025

Note. $q_1(0.05) = \chi^2_{35}(0.05) = 49.80$ for $(p_1, p_2) = (4, 4)$,
$q_1(0.05) = \chi^2_{77}(0.05) = 98.48$ for $(p_1, p_2) = (8, 4)$.

Table 3: The upper percentiles of $-2\log\lambda^{(3)}$, $-2\rho\log\lambda^{(3)}$ and the Type I Errors and the expectations and the MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ for $(p_1,p_2,p_3)=(2,2,2)$, $\sigma^2=1.0$ and $\alpha=0.05$.

Sample Size			Upper Percentile					Type I Error					Expectation		Mean Squared Eroor	
N_1	N_2	N_3	$q_2(\alpha)$	$q_3(\alpha)$	$-2\log\lambda^{(3)}$	$q^\dagger(\alpha)$	$-2\rho\log\lambda^{(3)}$	α_1	α_2	α_3	α_{χ^2}	$\alpha^\dagger$	$\mathrm{E}[\widetilde{\sigma}^2]$	$\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$	$\mathrm{MSE}[\widetilde{\sigma}^2]$	$\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$
$(p_1,p_2,p_3)=(2,2,2)$																
10	10	10	39.62	43.38	46.63	33.53	34.44	.362	.137	.081	.091	**.060**	.950	1.000	.01836	.01756
20	20	20	35.52	36.46	36.65	31.79	31.86	.136	.063	.052	.056	**.051**	.975	1.000	.00876	.00856
40	40	40	33.46	33.70	33.71	31.49	31.50	.082	.053	**.050**	.051	**.050**	.987	1.000	.00427	.00422
50	50	50	33.05	33.20	33.19	31.46	31.46	.075	.052	**.050**	.051	**.050**	.990	1.000	.00340	.00336
80	80	80	32.44	32.50	32.53	31.43	31.46	.064	.051	**.050**	.051	**.050**	.994	1.000	.00211	.00209
100	100	100	32.23	32.27	32.27	31.42	31.42	.061	**.050**	**.050**	**.050**	**.050**	.995	1.000	.00168	.00167
200	200	200	31.82	31.83	31.81	31.41	31.39	.055	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00084	.00084
400	400	400	31.62	31.62	31.60	31.41	31.39	.052	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00042	.00042
10	5	5	40.20	44.20	47.47	33.36	34.18	.385	.140	.081	.088	**.059**	.933	1.000	.02517	.02379
20	10	10	35.81	36.81	36.98	31.75	31.80	.143	.063	.052	.055	**.051**	.967	1.000	.01186	.01150
40	20	20	33.61	33.86	33.86	31.48	31.49	.085	.053	**.050**	.051	**.050**	.983	1.000	.00573	.00565
80	40	40	32.51	32.57	32.59	31.43	31.45	.065	.051	**.050**	**.050**	**.050**	.992	1.000	.00282	.00280
100	50	50	32.29	32.33	32.34	31.42	31.43	.062	**.050**	**.050**	**.050**	**.050**	.993	1.000	.00225	.00223
160	80	80	31.96	31.98	31.99	31.41	31.43	.057	**.050**	**.050**	**.050**	**.050**	.996	1.000	.00140	.00139
200	100	100	31.85	31.86	31.88	31.41	31.43	.056	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00112	.00111
400	200	200	31.63	31.63	31.63	31.41	31.41	.053	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00056	.00056
10	20	10	39.09	42.69	46.03	33.72	34.78	.345	.137	.082	.097	**.062**	.963	1.000	.01344	.01299
20	40	20	35.25	36.15	36.35	31.84	31.91	.130	.063	.052	.056	**.051**	.981	1.000	.00648	.00636
40	80	40	33.33	33.56	33.58	31.50	31.53	.080	.053	**.050**	.051	**.050**	.991	1.000	.00319	.00316
50	100	50	32.95	33.09	33.07	31.47	31.46	.073	.051	**.050**	.051	**.050**	.992	1.000	.00254	.00252
80	160	80	32.37	32.43	32.44	31.43	31.44	.063	.051	**.050**	**.050**	**.050**	.995	1.000	.00158	.00157
100	200	100	32.18	32.21	32.22	31.42	31.44	.061	.051	**.050**	**.050**	**.050**	.996	1.000	.00126	.00126
200	400	200	31.79	31.80	31.81	31.41	31.42	.055	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00063	.00063
400	800	400	31.60	31.60	31.63	31.41	31.41	.052	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00031	.00031
10	10	20	39.59	43.34	46.64	33.56	34.50	.361	.137	.081	.092	**.061**	.957	1.000	.01553	.01494
20	20	40	35.50	36.44	36.62	31.80	31.85	.136	.063	.052	.055	**.051**	.978	1.000	.00745	.00729
40	40	80	33.45	33.69	33.70	31.49	31.51	.082	.053	**.050**	.051	**.050**	.989	1.000	.00364	.00360
50	50	100	33.05	33.20	33.22	31.46	31.49	.075	.052	**.050**	.051	**.050**	.991	1.000	.00291	.00288
80	80	160	32.43	32.49	32.48	31.43	31.42	.064	.051	**.050**	**.050**	**.050**	.995	1.000	.00180	.00179
100	100	200	32.23	32.27	32.25	31.42	31.41	.061	**.050**	**.050**	**.050**	**.050**	.996	1.000	.00144	.00143
200	200	400	31.82	31.83	31.82	31.41	31.31	.055	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00072	.00072
400	400	800	31.61	31.62	31.62	31.41	31.41	.052	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00036	.00036
10	20	20	39.07	42.67	45.98	33.74	34.77	.344	.136	.082	.097	**.062**	.967	1.000	.01185	.01149
20	40	40	35.24	36.14	36.31	31.84	31.88	.129	.062	.052	.056	**.050**	.983	1.000	.00573	.00563
40	80	80	33.33	33.55	33.56	31.50	31.51	.080	.053	**.050**	.051	**.050**	.992	1.000	.00282	.00280
50	100	100	32.94	33.09	33.10	31.47	31.48	.073	.052	**.050**	.051	**.050**	.993	1.000	.00225	.00224
80	160	160	32.37	32.42	32.43	31.43	31.44	.063	.051	**.050**	**.050**	**.050**	.996	1.000	.00140	.00139
100	200	200	32.18	32.21	32.21	31.42	31.43	.060	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00112	.00111
200	400	400	31.79	31.80	31.81	31.41	31.43	.055	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00056	.00056
400	800	800	31.60	31.60	31.61	31.41	31.42	.052	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00028	.00028

Note. $q_1(0.05)=\chi^2_{20}(0.05)=31.41$.

Table 4: The upper percentiles of $-2\log\lambda^{(3)}$, $-2\rho\log\lambda^{(3)}$ and the Type I errors and the expectations and MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ for $(p_1, p_2, p_3) = (2, 2, 2)$, $(2, 2, 4), (2, 4, 2), (4, 2, 2)$, $\sigma^2 = 1.0$ and $\alpha = 0.05$.

Sample Size			Upper Percentile					Type I Error					Expectation		Mean Squared Eroor	
N_1	N_2	N_3	$q_2(\alpha)$	$q_3(\alpha)$	$-2\log\lambda^{(3)}$	$q^\dagger(\alpha)$	$-2\rho\log\lambda^{(3)}$	α_1	α_2	α_3	α_{χ^2}	$\alpha^\dagger$	$\mathrm{E}[\widetilde{\sigma}^2]$	$\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$	$\mathrm{MSE}[\widetilde{\sigma}^2]$	$\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$
$(p_1, p_2, p_3) = (3, 3, 3)$																
10	10	10	81.26	93.92	144.13	70.69	94.62	.925	.644	.439	.467	**.261**	.950	1.000	.01304	.01169
20	20	20	70.87	74.03	75.43	62.08	62.48	.288	.091	.060	.069	**.053**	.975	1.000	.00605	.00570
40	40	40	65.67	66.47	66.51	60.81	60.80	.123	.057	.050	.053	**.050**	.987	1.000	.00290	.00281
50	50	50	64.64	65.14	65.20	60.68	60.72	.104	.055	.050	.052	**.050**	.990	1.000	.00230	.00224
80	80	80	63.08	63.28	63.31	60.56	60.59	.079	.052	**.050**	.051	**.050**	.994	1.000	.00142	.00140
100	100	100	62.56	62.69	62.70	60.53	60.54	.072	.051	**.050**	.051	**.050**	.995	1.000	.00113	.00112
200	200	200	61.52	61.55	61.56	60.49	60.50	.060	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00056	.00056
400	400	400	61.00	61.01	61.02	60.48	60.50	.055	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00028	.00028
$(p_1, p_2, p_3) = (2, 2, 4)$																
10	10	10	67.39	77.48	96.16	56.52	62.20	.800	.393	.211	.218	**.102**	.943	1.000	.01672	.01512
20	20	20	58.59	61.12	61.93	50.84	51.00	.252	.082	.057	.062	**.051**	.971	1.000	.00776	.00736
40	40	40	54.20	54.83	54.84	50.01	50.00	.116	.056	**.050**	.052	**.050**	.986	1.000	.00373	.01169
50	50	50	53.32	53.72	53.73	49.93	49.93	.098	.054	**.050**	.051	**.050**	.989	1.000	.00296	.00289
80	80	80	52.00	52.16	52.17	49.85	49.87	.077	.052	**.050**	.051	**.050**	.993	1.000	.00183	.00180
100	100	100	51.56	51.66	51.65	49.83	49.83	.070	.051	**.050**	**.050**	**.050**	.994	1.000	.00146	.00144
200	200	200	50.68	50.71	50.69	49.81	49.79	.059	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00072	.00072
400	400	400	50.24	50.25	50.23	49.80	49.79	.054	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00036	.00036
$(p_1, p_2, p_3) = (2, 4, 2)$																
10	10	10	64.44	72.26	89.34	55.37	63.08	.699	.338	.199	.225	**.125**	.950	1.000	.01437	.01314
20	20	20	57.12	59.08	59.73	50.76	50.95	.205	.075	.055	.062	**.052**	.975	1.000	.00673	.00642
40	40	40	53.46	53.95	54.01	50.00	50.04	.102	.055	**.050**	.052	**.050**	.987	1.000	.00324	.00316
50	50	50	52.73	53.04	53.06	49.93	49.94	.089	.053	**.050**	.051	**.050**	.990	1.000	.00258	.00253
80	80	80	51.63	51.75	51.77	49.85	49.87	.071	.051	**.050**	.051	**.050**	.994	1.000	.00159	.00157
100	100	100	51.27	51.34	51.34	49.83	49.83	.066	.051	**.050**	**.050**	**.050**	.995	1.000	.00127	.00126
200	200	200	50.53	50.55	50.56	49.81	49.82	.058	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00063	.00063
400	400	400	50.17	50.17	50.14	49.80	49.77	.053	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00031	.00031
$(p_1, p_2, p_3) = (4, 2, 2)$																
10	10	10	64.00	71.69	88.85	55.55	63.52	.689	.337	.200	.232	**.126**	.955	1.000	.01260	.01162
20	20	20	56.90	58.82	59.54	50.80	51.05	.200	.076	.056	.063	**.052**	.978	1.000	.00592	.00567
40	40	40	53.35	53.83	53.87	50.01	50.03	.100	.055	**.050**	.052	**.050**	.989	1.000	.00287	.00281
50	50	50	52.64	52.95	52.92	49.93	49.90	.087	.053	**.050**	.051	**.050**	.991	1.000	.00228	.00224
80	80	80	51.58	51.70	51.70	49.85	49.85	.071	.051	**.050**	.051	**.050**	.994	1.000	.00141	.00140
100	100	100	51.22	51.30	51.32	49.83	49.85	.066	.051	**.050**	**.050**	**.050**	.996	1.000	.00113	.00112
200	200	200	50.51	50.53	50.55	49.81	49.83	.058	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00056	.00056
400	400	400	50.16	50.16	50.17	49.80	49.81	.054	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00028	.00028

Note. $q_1(0.05) = \chi^2_{44}(0.05) = 60.48$ for $(p_1, p_2, p_3) = (3, 3, 3)$,
$q_1(0.05) = \chi^2_{35}(0.05) = 49.80$ for $(p_1, p_2, p_3) = (2, 2, 4), (2, 4, 2), (4, 2, 2)$.

Table 5: The upper percentiles of $-2\log\lambda^{(5)}$, $-2\rho\log\lambda^{(5)}$ and Type I errors and the expectations and MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ for $(p_1, p_2, p_3, p_4, p_5) = (2, 2, 2, 2, 2)$, $\sigma^2 = 1.0$ and $\alpha = 0.05$.

Sample Size					Upper Percentile					Type I Error					Expectation		Mean Squared Eroor	
N_1	N_2	N_3	N_4	N_5	$q_2(\alpha)$	$q_3(\alpha)$	$-2\log\lambda^{(5)}$	$q^{\dagger}(\alpha)$	$-2\rho\log\lambda^{(5)}$	α_1	α_2	α_3	α_{χ^2}	$\alpha^{\dagger}$	$\mathrm{E}[\widetilde{\sigma}^2]$	$\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$	$\mathrm{MSE}[\widetilde{\sigma}^2]$	$\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$
$(p_1, p_2, p_3, p_4, p_5) = (2, 2, 2, 2, 2)$																		
20	20	20	20	20	83.15	86.57	88.55	74.25	75.06	.294	.097	.064	.077	**.057**	.983	1.000	.00356	.00339
40	40	40	40	40	77.65	78.51	78.64	72.59	72.65	.123	.058	.051	.054	**.050**	.992	1.000	.00172	.00168
50	50	50	50	50	76.55	77.10	77.17	72.43	72.47	.103	.055	.051	.053	**.050**	.993	1.000	.00137	.00134
80	80	80	80	80	74.90	75.12	75.11	72.26	72.25	.079	.052	**.050**	.051	**.050**	.996	1.000	.00085	.00084
100	100	100	100	100	74.35	74.49	74.51	72.22	72.24	.072	.051	**.050**	.051	**.050**	.997	1.000	.00068	.00067
200	200	200	200	200	73.25	73.29	73.28	72.17	72.17	.060	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00034	.00033
400	400	400	400	400	72.70	72.71	72.69	72.16	72.14	.055	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00017	.00017
20	10	10	10	10	84.61	88.46	90.43	74.18	74.82	.334	.101	.064	.075	**.055**	.975	1.000	.00549	.00512
40	20	20	20	20	78.38	79.34	79.48	72.57	72.62	.136	.059	.051	.054	**.050**	.988	1.000	.00263	.00253
80	40	40	40	40	75.27	75.51	75.51	72.25	72.26	.083	.052	**.050**	.051	**.050**	.994	1.000	.00128	.00126
100	50	50	50	50	74.64	74.80	74.83	72.21	72.25	.075	.051	**.050**	.051	**.050**	.995	1.000	.00102	.00100
160	80	80	80	80	73.71	73.77	73.78	72.18	72.19	.064	.051	**.050**	**.050**	**.050**	.997	1.000	.00063	.00063
200	100	100	100	100	73.40	73.44	73.44	72.17	72.17	.061	**.050**	**.050**	**.050**	**.050**	.997	1.000	.00050	.00050
400	200	200	200	200	72.78	72.79	72.79	72.16	72.16	.055	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00025	.00025
800	400	400	400	400	72.46	72.47	72.47	72.15	72.16	.053	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00013	.00013
20	40	20	20	20	82.06	85.23	87.27	74.32	75.30	.267	.096	.065	.079	**.058**	.987	1.000	.00277	.00267
40	80	40	40	40	77.10	77.90	77.98	72.62	72.62	.114	.057	.051	.054	**.050**	.993	1.000	.00135	.00133
50	100	50	50	50	76.11	76.62	76.68	72.44	72.47	.097	.055	**.050**	.053	**.050**	.995	1.000	.00108	.00106
80	160	80	80	80	74.63	74.83	74.84	72.26	72.27	.075	.052	**.050**	.051	**.050**	.997	1.000	.00067	.00066
100	200	100	100	100	74.13	74.26	74.27	72.22	72.24	.069	.051	**.050**	.051	**.050**	.997	1.000	.00053	.00053
200	400	200	200	200	73.14	73.18	73.19	72.17	72.19	.059	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00026	.00026
400	800	400	400	400	72.65	72.66	72.67	72.16	72.17	.054	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00013	.00013
20	20	40	20	20	82.85	86.24	88.21	74.32	75.13	.287	.097	.064	.078	**.057**	.986	1.000	.00293	.00282
40	40	80	40	40	77.50	78.35	78.49	72.61	72.67	.121	.058	.051	.054	**.051**	.993	1.000	.00143	.00140
50	50	100	50	50	76.43	76.98	77.06	72.44	72.49	.101	.055	.051	.053	**.050**	.994	1.000	.00114	.00112
80	80	160	80	80	74.83	75.04	75.07	72.26	72.29	.078	.052	**.050**	.051	**.050**	.996	1.000	.00070	.00070
100	100	200	100	100	74.29	74.43	74.46	72.22	72.25	.071	.051	**.050**	.051	**.050**	.997	1.000	.00056	.00056
200	200	400	200	200	73.22	73.26	73.26	72.17	72.17	.060	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00028	.00028
400	400	800	400	400	72.69	72.70	72.71	72.16	72.17	.055	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00014	.00014
20	20	20	40	20	83.08	86.50	88.47	74.27	75.07	.293	.097	.064	.077	**.056**	.985	1.000	.00312	.00299
40	40	40	80	40	77.62	78.47	78.60	72.60	72.64	.122	.058	.051	.054	**.050**	.993	1.000	.00151	.00148
50	50	50	100	50	76.52	77.07	77.16	72.43	72.48	.103	.055	.051	.053	**.050**	.994	1.000	.00120	.00118
80	80	80	160	80	74.89	75.10	75.11	72.26	72.27	.078	.052	**.050**	.051	**.050**	.996	1.000	.00075	.00074
100	100	100	200	100	74.34	74.48	74.50	72.22	72.24	.072	.051	**.050**	.051	**.050**	.997	1.000	.00059	.00059
200	200	200	400	200	73.25	73.28	73.31	72.17	72.20	.060	.051	**.050**	**.050**	**.050**	.999	1.000	.00030	.00030
400	400	400	800	400	72.70	72.71	72.75	72.16	72.20	.055	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00015	.00015
20	20	20	20	40	83.14	86.57	88.49	74.25	75.01	.295	.097	.064	.077	**.056**	.984	1.000	.00331	.00316
40	40	40	40	80	77.65	78.50	78.61	72.60	72.62	.123	.058	.051	.054	**.050**	.992	1.000	.00161	.00158
50	50	50	50	100	76.55	77.10	77.17	72.43	72.47	.103	.055	.051	.053	**.050**	.994	1.000	.00128	.00125
80	80	80	80	160	74.90	75.11	75.12	72.26	72.26	.078	.052	**.050**	.051	**.050**	.996	1.000	.00079	.00078
100	100	100	100	200	74.35	74.49	74.47	72.22	72.21	.072	.051	**.050**	**.050**	**.050**	.997	1.000	.00063	.00063
200	200	200	200	400	73.25	73.29	73.31	72.17	72.19	.060	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00032	.00031
400	400	400	400	800	72.70	72.71	72.70	72.16	72.15	.055	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00016	.00016

Note. $q_1(0.05) = \chi^2_{54}(0.05) = 72.15$.

Table 6: The upper percentiles of $-2\log\lambda^{(5)}$, $-2\rho\log\lambda^{(5)}$ and Type I errors and the expectations and MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ for $(p_1,p_2,p_3,p_4,p_5) = (3,3,3,3,3), (2,2,2,2,4), (2,2,2,4,2)$, $\sigma^2 = 1.0$ and $\alpha = 0.05$.

Sample Size					Upper Percentile					Type I Error					Expectation		Mean Squared Eroor	
N_1	N_2	N_3	N_4	N_5	$q_2(\alpha)$	$q_3(\alpha)$	$-2\log\lambda^{(5)}$	$q^{\dagger}(\alpha)$	$-2\rho\log\lambda^{(5)}$	α_1	α_2	α_3	α_{χ^2}	$\alpha^{\dagger}$	$\mathrm{E}[\widetilde{\sigma}^2]$	$\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$	$\mathrm{MSE}[\widetilde{\sigma}^2]$	$\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$
$(p_1,p_2,p_3,p_4,p_5) = (3,3,3,3,3)$																		
20	20	20	20	20	176.05	189.24	208.44	155.26	164.60	.823	.338	.174	.228	**.112**	.983	1.000	.00247	.00227
40	40	40	40	40	160.76	164.05	165.16	147.37	147.79	.264	.077	.056	.065	**.052**	.992	1.000	.00117	.00112
50	50	50	50	50	157.70	159.81	160.26	146.63	146.78	.195	.065	.052	.058	**.051**	.993	1.000	.00093	.00089
80	80	80	80	80	153.11	153.93	154.07	145.89	145.97	.119	.055	.051	.053	**.050**	.996	1.000	.00057	.00056
100	100	100	100	100	151.58	152.11	152.12	145.73	145.72	.101	.053	**.050**	.052	**.050**	.997	1.000	.00045	.00045
200	200	200	200	200	148.52	148.65	148.63	145.52	145.51	.071	.051	**.050**	**.050**	**.050**	.998	1.000	.00022	.00022
400	400	400	400	400	146.99	147.02	146.99	145.48	145.44	.059	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00011	.00011
$(p_1,p_2,p_3,p_4,p_5) = (2,2,2,2,4)$																		
20	20	20	20	20	118.96	126.84	132.95	103.44	105.31	.583	.174	.090	.112	**.063**	.981	1.000	.00342	.00319
40	40	40	40	40	108.72	110.69	111.11	99.45	99.57	.197	.067	.053	.058	**.051**	.991	1.000	.00164	.00158
50	50	50	50	50	106.67	107.93	108.07	99.08	99.09	.151	.060	.051	.054	**.050**	.993	1.000	.00130	.00126
80	80	80	80	80	103.60	104.10	104.10	98.70	98.69	.101	.053	**.050**	.051	**.050**	.995	1.000	.00080	.00078
100	100	100	100	100	102.58	102.89	102.89	98.62	98.61	.088	.052	**.050**	.051	**.050**	.996	1.000	.00064	.00063
200	200	200	200	200	100.53	100.61	100.61	98.52	98.52	.066	.051	**.050**	**.050**	**.050**	.998	1.000	.00031	.00031
400	400	400	400	400	99.51	99.53	99.51	98.49	98.48	.058	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00016	.00016
$(p_1,p_2,p_3,p_4,p_5) = (2,2,2,4,2)$																		
20	20	20	20	20	115.44	121.26	125.98	102.18	104.28	.455	.138	.081	.100	**.065**	.982	1.000	.00320	.00299
40	40	40	40	40	106.96	108.42	108.69	99.24	99.33	.161	.062	.052	.056	**.051**	.991	1.000	.00153	.00148
50	50	50	50	50	105.27	106.20	106.34	98.95	99.02	.128	.057	.051	.054	**.050**	.993	1.000	.00122	.00118
80	80	80	80	80	102.72	103.09	103.07	98.66	98.63	.090	.052	**.050**	.051	**.050**	.996	1.000	.00075	.00074
100	100	100	100	100	101.88	102.11	102.08	98.59	98.57	.080	.051	**.050**	.051	**.050**	.996	1.000	.00060	.00059
200	200	200	200	200	100.18	100.24	100.28	98.51	98.55	.064	.051	**.050**	**.050**	**.050**	.998	1.000	.00030	.00029
400	400	400	400	400	99.33	99.35	99.33	98.49	98.48	.056	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00015	.00015

Note. $q_1(0.05) = \chi^2_{119}(0.05) = 145.46$ for $(p_1,p_2,p_3,p_4,p_5) = (3,3,3,3,3)$,
$q_1(0.05) = \chi^2_{77}(0.05) = 98.48$ for $(p_1,p_2,p_3,p_4,p_5) = (2,2,2,2,4), (2,2,2,4,2)$.

Table 7: The upper percentiles of $-2\log\lambda^{(5)}$, $-2\rho\log\lambda^{(5)}$ and Type I errors and the expectations and MSEs of $\widetilde{\sigma}^2$ and $\widetilde{\sigma}^2_{\mathrm{U}}$ for $(p_1, p_2, p_3, p_4, p_5) = (2, 2, 4, 2, 2), (2, 4, 2, 2, 2), (4, 2, 2, 2, 2)$, $\sigma^2 = 1.0$ and $\alpha = 0.05$.

Sample Size					Upper Percentile					Type I Error					Expectation		Mean Squared Eroor	
N_1	N_2	N_3	N_4	N_5	$q_2(\alpha)$	$q_3(\alpha)$	$-2\log\lambda^{(5)}$	$q^\dagger(\alpha)$	$-2\rho\log\lambda^{(5)}$	α_1	α_2	α_3	α_{χ^2}	$\alpha^\dagger$	$\mathrm{E}[\widetilde{\sigma}^2]$	$\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$	$\mathrm{MSE}[\widetilde{\sigma}^2]$	$\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$
$(p_1, p_2, p_3, p_4, p_5) = (2, 2, 4, 2, 2)$																		
20	20	20	20	20	114.68	120.29	125.00	102.23	104.44	.436	.137	.081	.102	**.066**	.983	1.000	.00301	.00283
40	40	40	40	40	106.58	107.99	108.34	99.26	99.43	.155	.062	.052	.057	**.051**	.992	1.000	.00145	.00140
50	50	50	50	50	104.96	105.86	106.01	98.96	99.04	.124	.057	.051	.054	**.050**	.993	1.000	.00115	.00112
80	80	80	80	80	102.53	102.88	102.85	98.66	98.63	.088	.052	**.050**	.051	**.050**	.996	1.000	.00071	.00070
100	100	100	100	100	101.72	101.95	101.95	98.60	98.60	.079	.052	**.050**	.051	**.050**	.997	1.000	.00056	.00056
200	200	200	200	200	100.10	100.16	100.15	98.51	98.50	.063	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00028	.00028
400	400	400	400	400	99.29	99.31	99.30	98.49	98.48	.056	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00014	.00014
$(p_1, p_2, p_3, p_4, p_5) = (2, 4, 2, 2, 2)$																		
20	20	20	20	20	114.47	120.04	124.78	102.28	104.53	.430	.136	.081	.103	**.066**	.984	1.000	.00284	.00267
40	40	40	40	40	106.48	107.87	108.13	99.27	99.36	.152	.062	.052	.057	**.051**	.992	1.000	.00137	.00133
50	50	50	50	50	104.88	105.77	105.91	98.97	99.04	.122	.057	.051	.054	**.050**	.994	1.000	.00109	.00106
80	80	80	80	80	102.48	102.83	102.88	98.67	98.70	.088	.053	**.050**	.052	**.050**	.996	1.000	.00067	.00066
100	100	100	100	100	101.68	101.90	101.92	98.60	98.61	.078	.052	**.050**	.051	**.050**	.997	1.000	.00053	.00053
200	200	200	200	200	100.08	100.14	100.14	98.51	98.52	.063	**.050**	**.050**	**.050**	**.050**	.998	1.000	.00027	.00026
400	400	400	400	400	99.28	99.30	99.30	98.49	98.50	.056	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00013	.00013
$(p_1, p_2, p_3, p_4, p_5) = (4, 2, 2, 2, 2)$																		
20	20	20	20	20	114.41	119.98	124.63	102.30	104.48	.430	.135	.080	.102	**.065**	.985	1.000	.00268	.00254
40	40	40	40	40	106.45	107.84	108.16	99.28	99.42	.153	.062	.052	.057	**.051**	.992	1.000	.00130	.00126
50	50	50	50	50	104.85	105.75	105.89	98.97	99.04	.122	.057	.051	.054	**.051**	.994	1.000	.00103	.00101
80	80	80	80	80	102.47	102.81	102.80	98.67	98.65	.087	.052	**.050**	.051	**.050**	.996	1.000	.00064	.00063
100	100	100	100	100	101.67	101.89	101.94	98.60	98.64	.079	.052	**.050**	.051	**.050**	.997	1.000	.00051	.00050
200	200	200	200	200	100.08	100.13	100.15	98.51	98.53	.063	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00025	.00025
400	400	400	400	400	99.28	99.29	99.24	98.49	98.44	.056	**.050**	**.050**	**.050**	**.050**	.999	1.000	.00013	.00013

Note. $q_1(0.05) = \chi^2_{77}(0.05) = 98.48$.

7 Numerical examples

In this section, we provide examples to illustrate the use of test statistics and the approximate upper percentiles proposed in this paper.

7.1 Two-step case

We use two-step monotone missing data pattern from Choi (2005, p.475, Table 1). The data are obtained by discarding the values of the fourth variable as indicated by Little and Rubin (1987) and rearranging the remaining variables. In the data,

$$p_1 = 2, p_2 = 2, N_1 = 9, N_2 = 4, \sigma^2 = 1.0, \alpha = 0.05.$$

Then, the LRT statistic and modified LRT statistic under two-step monotone missing data are

$$-2 \log \lambda = 63.55, -2\rho \log \lambda = 46.36.$$

Additionally, the approximate upper percentiles are, respectively,

$$q_1(0.05) = 16.92, q_2(0.05) = 21.50, q_3(0.05) = 23.36, q^\dagger(0.05) = 17.46.$$

Thus, H_0 is rejected for the two test statistics and their respective upper percentiles.

7.2 Three-step case

The data are obtained by reordering the variables as indicated by Little and Rubin (1987). The parameters are

$$p_1 = 2, p_2 = 2, p_3 = 1, N_1 = 6, N_2 = 3, N_3 = 4, \sigma^2 = 1.0, \alpha = 0.05.$$

Then, the LRT statistic and modified LRT statistic under three-step monotone missing data are

$$-2 \log \lambda^{(3)} = 115.58, -2\rho \log \lambda^{(3)} = 71.57.$$

Additionally, the approximate upper percentiles are, respectively,

$$q_1(0.05) = 23.68, q_2(0.05) = 32.70, q_3(0.05) = 38.40, q^\dagger(0.05) = 27.44.$$

Thus, as with the two-step monotone missing data, H_0 is rejected for the two test statistics and their respective upper percentiles.

8 Conclusions

We discussed the sphericity test under monotone missing data. Specifically, we asymptotically expanded the distribution functions of the LRT statistic and modified LRT statistic under H_0 and derived the upper percentiles of these test statistics. We also proposed an unbiased estimator of σ^2: $\widetilde{\sigma}^2_{\mathrm{U}}$. Simulations were performed for two-step, three-step, and five-step cases, and it can be said that modifying the LRT statistic improves the approximation accuracy. We also confirmed that $\mathrm{E}[\widetilde{\sigma}^2]$ approaches σ^2 as each sample size increases, and that $\mathrm{E}[\widetilde{\sigma}^2_{\mathrm{U}}]$ is always σ^2. For MSE, when $(p_1, p_2) = (2, 2)$, $\mathrm{MSE}[\widetilde{\sigma}^2]$ is greater than or equal to $\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$. In other cases, $\mathrm{MSE}[\widetilde{\sigma}^2]$ is less than or equal to $\mathrm{MSE}[\widetilde{\sigma}^2_{\mathrm{U}}]$. Furthermore, it was observed that the proposed approximation of $q^{\dagger}(\alpha)$ among the upper 100α percentiles of the approximation is effective even when each sample size is small.

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