Global structure of the stationary solutions for the limiting system of an Chemotaxis-Growth Model

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We consider the following system as the limiting one of the stationary problem for an Chemotaxis-Growth Model [1]:

$$\begin{cases} dv_{xx} + g(v,\ E) = 0, & x \in (0,\ 1), \\ v_x(0) = v_x(1) = 0 \end{cases} \tag{1}$$

with the nonlocal constraint

$$\int_0^1 f(Ee^{\alpha v})\ dx = 0, \tag{2}$$

where $g(v,\ E) = Ee^{\alpha v} - \gamma v$, $f(u) = u(1-u)$ and d, α, γ and E are positive constants.

By the numerical computations for the original Chemotaxis-Growth model, stationary and dynamical Turing patterns induced the Chemotaxis have been investigated in [2], [3]. But we do not show the global structure of the stationary solutions of this mode. In order to find clues for the structure, we consider the limiting system (1), (2) in the case of the large diffusion coefficients for this model.

Lemma 1 *There exists a positive constant* $\hat{E}$ *such that for any* $0 < E < \hat{E}$*, there are two positive roots* $v_*(E)$, $v^*(E)$ *of* $g(v,\ E) = 0$ *which satisfies* $v_*(E) < 1/\alpha < v^*(E)$, $\lim_{E\to 0} v_*(E) = 0$, $\lim_{E\to 0} v^*(E) = \infty$, $\lim_{E\to \hat{E}} v_*(E) = 1/\alpha = \lim_{E\to \hat{E}} v_*(E)$*. On the other hand, there is no root of* $g(v,\ E) = 0$ *for* $\hat{E} < E$*.*

Therefore, this becomes a monostable system. On the other hand, we already study the bistable type of the similar limiting system as (1), (2) in [4].

Hereafter, we only consider a monotone increasing solution $v(x,\ d,\ E)$ of (1), (2) because all oscillating solution can be constructed by connecting rescaling parts of monotone solutions. We have solutions of (1) as the solution bifurcated from the constant solution $v^*(E)$ with $d = d^*(E)$.

Theorem 2 *For any* $0 < E < \hat{E}$*, there exists* $d^*(E) > 0$*, which satisfies* $\lim_{E\downarrow 0} d^*(E) = \infty$ *and* $\lim_{E\uparrow \hat{E}} d^*(E) = 0$*, such that there is a monotone increasing solution* $v(x,\ d,\ E)$ *of* (1) *for* $0 < d < d^*(E)$ *such that*

$$\lim_{d\downarrow 0} v(x,\ d,\ E) = \begin{cases} v_*(E) & 0 \le x < 1 \\ \\ \overline{v}(E) & x = 1, \end{cases}$$

where $\overline{v}(E) > v^*(E)$ *is given by* $\displaystyle\int_{v_*(E)}^{\overline{v}(E)} g(v,\ E)\ dv = 0.$

Moreover, we can prove the direction of the bifurcation brunch around the bifurcation point $d^*(E)$ by Theorem 2.7 in [5]. Therefore, we show

Remark 1 *Set* $D = \{(E,\ d) \mid 0 < E,\ 0 < d\ \}$ *and* $D^* = \{(E,\ d) \mid 0 < E < \hat{E},\ 0 < d^*(E)\ \}$*. There does not exist any nonconstant solution of* (1) *in* $D \setminus D^*$*.*

Next, we obtain the solution of (1) satisfying the integral constraint (2).

Case $1 < \gamma/\alpha$ **:**

Lemma 3 *If* $1 < \gamma/\alpha$*, there is a constant* $0 < E_* < \hat{E}$ *such that for any* $E_* < E < \hat{E}$*, there does not exist any nonconstant solution of* (1), (2) *with* $0 < d < d^*(E)$*.*

Theorem 4 *If* $1 < \gamma/\alpha$*, there exists a continuous function* $d(E)$ *such that there is a solution* $(d(E),\ v(x,\ d(E),\ E)$ *of* (1), (2)*, which satisfies that* $0 < d(E) < d^*(E)$ *for* $0 < E < E_*$ *and* $\lim_{E \to E_*} d(E) = 0$*.*

Case $1 > \gamma/\alpha > 0$ **:**

Theorem 5 *If* $1 > \gamma/\alpha > 0$*. there exist a constant* $0 < E^* < \hat{E}$ *and a continuous function* $d(E)$ *with* $0 < E < E^*$ *such that there is a solution* $(d(E),\ v(x,\ d(E),\ E)$ *of* (1), (2)*, which satisfies* $0 < d(E) < d^*(E)$ *for* $0 < E < E^*$ *and* $\lim_{E \to E^*} d(E) = d^*(E)$*.*

We introduce the global structure of the stationary solutions by using the numerical simulations and the difference between monostable and bistable cases.

References

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