Cross-diffusion systems: RDS approximation and Numerical analysis

Hideki Murakawa
Faculty of Mathematics, Kyushu University, Japan.

This talk is concerned with the following nonlinear diffusion problem: Find $\boldsymbol{z} = (z_1, \ldots, z_M) : \overline{\Omega} \times [0,T) \to \mathbb{R}^M$ ($M \in \mathbb{N}$) such that

$$\begin{cases} \dfrac{\partial \boldsymbol{z}}{\partial t} = \Delta \boldsymbol{\beta}(\boldsymbol{z}) + \boldsymbol{f}(\boldsymbol{z}) & \text{in} \quad Q := \Omega \times (0,T), \\ \dfrac{\partial \boldsymbol{\beta}(\boldsymbol{z})}{\partial \nu} = \mathbf{0} & \text{on} \quad \partial\Omega \times (0,T), \\ \boldsymbol{z}(\cdot, 0) = \boldsymbol{z}_0 & \text{in} \quad \Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) is a bounded domain with smooth boundary $\partial\Omega$, T is a positive constant, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_M)$, $\boldsymbol{f} = (f_1, \ldots, f_M)$: $\mathbb{R}^M \to \mathbb{R}^M$ and $\boldsymbol{z}_0 = (z_{01}, \ldots, z_{0M})$: $\Omega \to \mathbb{R}^M$ are given functions, ν is the unit outward normal vector to the boundary $\partial\Omega$. We note that the diffusivity β_i of the ith component depends not only on the ith variable but also on the jth ($j \neq i$) variables in general. This mixture of diffusion terms is called cross-diffusion. Numerous problems of this type have been proposed in the literature, especially in the area of population ecology. A typical example of nonlinear cross-diffusion system is the well known Shigesada-Kawasaki-Teramoto cross-diffusion system [3].

We introduce a reaction-diffusion system approximation to the cross-diffusion system [1, 2]. The solutions of the nonlinear diffusion problems can be approximated by those of semilinear reaction-diffusion systems. This indicates that the mechanism of nonlinear-diffusion might be captured by reaction-diffusion interaction. Furthermore, we deal with nonlinear and linear discrete-time algorithms for the cross-diffusion systems. The nonlinear scheme corresponds to backward differences in time. The linear scheme is derived by discretizing the reaction-diffusion system approximation [1]. After discretizing the linear scheme in space, we obtain an unconditionally stable and very easy to implement numerical scheme. In this talk, we give recent results of analysis. Uniqueness and regularity results of weak solutions of the cross-diffusion systems are established. We derive convergence rates of the reaction-diffusion system approximation and the discrete-time schemes. We note that convergence rates of the nonlinear and the linear discrete-time schemes are the same. Moreover, these orders are optimal.

References

[1] H. Murakawa, A linear scheme to approximate nonlinear cross-diffusion systems, *Math. Mod. Numer. Anal.*, **45** (2011), 1141–1161.

[2] H. Murakawa, A relation between cross-diffusion and reaction-diffusion. *Discrete Contin. Dyn. Syst. S.* **5** (2012) 147–158.

[3] N. Shigesada, K. Kawasaki and E. Teramoto, Spatial segregation of interacting species. *J. Theor. Biol.* **79** (1979) 83–99.