RIMS Workshop

Mathematical Analysis of Pattern Formation Arising in Nonlinear Phenomena

Organizers : Yoshihito Oshita (Okayama University)
Kazuhiro Takimoto (Hiroshima University)
Ken-Ichi Nakamura (Kanazawa University)

Date : October 30 (Wed.) 13:40 – November 1 (Fri.) 13:50, 2013
Place : Room 420, Research Institute for Mathematical Sciences, Kyoto University
Webpage : `http://www.math.sci.hiroshima-u.ac.jp/~takimoto/rims2013_e.html`

Program

October 30 (Wed.)

13:40 – 14:30 **Masaharu Taniguchi** (Okayama University)
An $(N-2)$-dimensional surface with positive principal curvatures gives an N-dimensional traveling front in the Allen-Cahn equation

14:50 – 15:40 **Takeshi Ohtsuka** (Gunma University)
Growth rate of a crystal surface with a co-rotating pair of spiral steps evolving by an eikonal-curvature flow

16:00 – 16:50 **Yoshihiro Tonegawa** (Hokkaido University)
A time-global existence of mean curvature flow via reaction diffusion approximation

October 31 (Thu.)

10:00 – 10:50 **Hideki Murakawa** (Kyushu University)
Cross-diffusion systems: RDS approximation and Numerical analysis

11:10 – 12:00 **Kunimochi Sakamoto** (Hiroshima University)
Destabilization/Stabilization of Diffusion Systems by Diffusion and Boundary Flux

13:40 – 14:30 **Takashi Sakamoto** (Meiji University)
Double zero degeneracy in the presence of 0:1:2 resonance

14:50 – 15:40 **Tohru Tsujikawa** (University of Miyazaki)
Global structure of the stationary solutions for the limiting system of an Chemotaxis-Growth Model

16:00 – 16:50 **Yong Jung Kim** (KAIST)
Starvation driven dispersal and pattern formations

18:30 – Banquet

November 1 (Fri.)

10:00 – 10:50 **Kanako Suzuki** (Ibaraki University)
Instability and blowup phenomena induced by diffusion in some reaction-diffusion-ODE

11:10 – 12:00 **Xiaofeng Ren** (George Washington University)
A double bubble solution in a ternary system with inhibitory long range interaction

13:00 – 13:50 **Cyrill Muratov** (New Jersey Institute of Technology)
Front propagation in geometric and phase field models of stratified media

This workshop is partially supported by

- Research Institute for Mathematical Sciences, Kyoto University
- Yoshihito Oshita, Grant-in-Aid for Young Scientists (B) (No. 23740079)

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An $(N-2)$-dimensional surface with positive principal curvatures gives an N-dimensional traveling front in the Allen-Cahn equation

Masaharu Taniguchi

Multi-dimensional traveling fronts have been studied by mathematicians for bistable reaction-diffusion equations in the whole space. V-form fronts are studied by Ninomiya and myself (2005). Cylindrically symmetric traveling fronts have been studied by Hamel, Monneau and Roquejoffre (2005). Pyramidal traveling fronts are studied by myself (2007) and Kurokawa and myself (2011). In this work, we show that an $(N-2)$-dimensional surface with positive principal curvatures gives an N-dimensional traveling front in the Allen-Cahn equation.

Growth rate of a crystal surface with a co-rotating pair of spiral steps evolving by an eikonal-curvature flow

Takeshi Ohtsuka
Faculty of Science and Technology, Gunma University

In this talk we introduce a simple level set formulation for a co-rotating pair of spirals evolving by an eikonal-curvature flow, and a simple way to reconstruct a crystal surface with spiral steps proposed by screw dislocations. We apply this method to simulations of the evolution of a crystal surface by a co-rotating pair of spiral steps, and present some numerical results and observations on the growth rate and the critical distance of a co-rotating pair.

Burton, Cabrera and Frank proposed a theory of crystal growth with aid of screw dislocations in 1951. According to the theory a crystal surface with screw dislocations evolves in vertical direction of the surface with rotating spiral steps. The spiral steps evolve in the holizontal direction with an eikonal-curvature flow

$$V = v_\infty(1 - \rho_c\kappa), \tag{1}$$

where V is the velocity of the spiral steps, which are regarded as spiral curves on the plane, κ is the curvature of the spiral steps with the opposite direction of V, v_∞ and ρ_c are constants.

Burton et al also proposed some speculations on the growth rate of a crystal surface by a co-rotating pair of single spiral steps by approximating spirals with an Archimedean spiral $r = 2\rho_c\xi$, where (r, ξ) is the polar coordinate. They pointed out that, for a pair of co-rotating spiral centers a_1 and a_2, the growth rate (they call it "activity") of the pair is indistinguishable from that of a single spiral if $|a_1 - a_2| > 2\pi\rho_c$. If $|a_1 - a_2| \ll \rho_c$, then the activity of the pair should be twice of that of a single spiral. There is no estimate of the activity of the pair which lies between 1 and 2 times of the single one if $|a_1 - a_2| < 2\pi\rho_c$. The critical distance $\tilde{d}_c = 2\pi\rho_c$ is derived from the profile of the approximating Archimedean spiral.

In this talk we show more accurate critical distance of the co-rotating pair is farther than $2\pi\rho_c$ from numerical results. To obtain the critical distance we crudely estimate of the growth rate by a co-rotating pair, and numerically demonstrate that the growth rate R_p by the co-rotating pair is given as

$$R_p = \frac{2}{1 + |a_1 - a_2|\omega_1/(\pi\rho_c)} R_S \tag{2}$$

if $|a_1 - a_2| < \pi\rho_c/\omega_1$, where $\omega_1 = 0.330958061$ is the coefficient of the angle velocity $\omega = \omega_1 v_\infty/\rho_c$ for a single rotating spiral by (1) which is obtained by Ohara-Reid in 1973, and R_S is the growth rate by a single spiral step evolving with (1). From the view point of the growth rate we propose a new definition of the critical distance as the distance where $R_p = R_S$ in (2), and thus the more accurate critical distance is $d_c = \pi\rho_c/\omega_1$. Note that the approximation by the Archimedean spiral $r = 2\rho_c\xi$ yields that $\omega_1 = 1/2$. Thus our definition with the speculation by Burton et al yields the critical distance $\tilde{d}_c = 2\pi\rho_c$ by Burton et al.

This is the joint work with Y.-H. R. Tsai and Y. Giga.

A time-global existence of mean curvature flow via reaction diffusion approximation

Yoshihiro Tonegawa
Hokkaido University

Given a compact C^1 hypersurface in $\mathbb{R}^n$ ($n \geq 2$) and a vector field $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ which belongs to $L^q_{loc}([0, \infty); W^{1,p}(\mathbb{R}^n))$, $2 < q < \infty$, $\frac{nq}{2(q-1)} < p$ ($\frac{4}{3} \leq p$ additionally in case of $n = 2$), we prove some existence and regularity results for evolving hypersurfaces whose velocity is given by its mean curvature plus u. For the existence, we take a singular perturbation limit of the Allen-Cahn equation with additional transport term. The hypersurfaces remain C^1 for a short time, and they are C^1 almost everywhere away from the so called higher multiplicity region. This is a joint work with K. Takasao and the results have appeared in arXiv:1307.6629.

Cross-diffusion systems: RDS approximation and Numerical analysis

Hideki Murakawa
Faculty of Mathematics, Kyushu University, Japan.

This talk is concerned with the following nonlinear diffusion problem: Find $\boldsymbol{z} = (z_1, \ldots, z_M) : \overline{\Omega} \times [0,T) \to \mathbb{R}^M$ ($M \in \mathbb{N}$) such that

$$\begin{cases} \dfrac{\partial \boldsymbol{z}}{\partial t} = \Delta \boldsymbol{\beta}(\boldsymbol{z}) + \boldsymbol{f}(\boldsymbol{z}) & \text{in} \quad Q := \Omega \times (0,T), \\ \dfrac{\partial \boldsymbol{\beta}(\boldsymbol{z})}{\partial \nu} = \boldsymbol{0} & \text{on} \quad \partial\Omega \times (0,T), \\ \boldsymbol{z}(\cdot, 0) = \boldsymbol{z}_0 & \text{in} \quad \Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) is a bounded domain with smooth boundary $\partial\Omega$, T is a positive constant, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_M)$, $\boldsymbol{f} = (f_1, \ldots, f_M)$: $\mathbb{R}^M \to \mathbb{R}^M$ and $\boldsymbol{z}_0 = (z_{01}, \ldots, z_{0M})$: $\Omega \to \mathbb{R}^M$ are given functions, ν is the unit outward normal vector to the boundary $\partial\Omega$. We note that the diffusivity β_i of the ith component depends not only on the ith variable but also on the jth ($j \neq i$) variables in general. This mixture of diffusion terms is called cross-diffusion. Numerous problems of this type have been proposed in the literature, especially in the area of population ecology. A typical example of nonlinear cross-diffusion system is the well known Shigesada-Kawasaki-Teramoto cross-diffusion system [3].

We introduce a reaction-diffusion system approximation to the cross-diffusion system [1, 2]. The solutions of the nonlinear diffusion problems can be approximated by those of semilinear reaction-diffusion systems. This indicates that the mechanism of nonlinear-diffusion might be captured by reaction-diffusion interaction. Furthermore, we deal with nonlinear and linear discrete-time algorithms for the cross-diffusion systems. The nonlinear scheme corresponds to backward differences in time. The linear scheme is derived by discretizing the reaction-diffusion system approximation [1]. After discretizing the linear scheme in space, we obtain an unconditionally stable and very easy to implement numerical scheme. In this talk, we give recent results of analysis. Uniqueness and regularity results of weak solutions of the cross-diffusion systems are established. We derive convergence rates of the reaction-diffusion system approximation and the discrete-time schemes. We note that convergence rates of the nonlinear and the linear discrete-time schemes are the same. Moreover, these orders are optimal.

References

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Destabilization/Stabilization of Diffusion Systems by Diffusion and Boundary Flux

Kunimochi Sakamoto
Hiroshima University

We study linear diagonal diffusion systems under linear non-diagonal Robin boundary conditions. “ Non-diagonal ” means that the flux of each component on the boundary depends linearly on the values of (possibly) all components consisting of the system. Our purpose is to show how the stability of the trivial solution depends on the diagonal diffusion matrix and the eigenvalues of the matrix (mass transfer matrix) which specifies the aforementioned linear dependence on the boundary. There are two simplifying situations where the stability and instability of the system are characterized in rather easy ways. One is the case in which the mass transfer matrix is symmetric, and the other is the case in which the diffusion rates are all equal. Investigations in these situations enable us to identify stable and unstable systems in terms of the stability/instability of the mass transfer matrix. Outside these two cases, to describe the critical eigenvalues by using the properties of the diffusion and mass transfer matrices is a subtle issue, and this leads to a Turing type instability. In contrast to closed reaction-diffusion systems, our system also exhibits a phenomenon called anti- Turing mechanism, in which originally unstable systems for equal diffusion rates are stabilized by biased (unequal) diffusion effects.

Double zero degeneracy in the presence of 0:1:2 resonance

Takashi Sakamoto
Meiji University

We deal with a reaction-diffusion system with spatially nonlocal effect under Neumann boundary conditions. The system provides triply degenerate points for two spatially non-uniform modes and uniform one (zero mode). We focus our attention on the 0:1:2-mode interaction in the reaction-diffusion system. At the such degenerate point, the reduced equation on the center manifold is obtained. The normal form on the center manifold has double zero degeneracy (Bogdanov-Takens type degeneracy) as the secondary bifurcation point. We analyze the bifurcation structures around them. We also present the chaotic behavior numerically. These results are based on the joint works with Toshiyuki Ogawa.

Global structure of the stationary solutions for the limiting system of an Chemotaxis-Growth Model

Tohru TSUJIKAWA (University of Miyazaki)

We consider the following system as the limiting one of the stationary problem for an Chemotaxis-Growth Model [1]:

$$\begin{cases} dv_{xx} + g(v,\ E) = 0, & x \in (0,\ 1), \\ v_x(0) = v_x(1) = 0 \end{cases} \tag{1}$$

with the nonlocal constraint

$$\int_0^1 f(Ee^{\alpha v})\ dx = 0, \tag{2}$$

where $g(v,\ E) = Ee^{\alpha v} - \gamma v$, $f(u) = u(1-u)$ and d, α, γ and E are positive constants.

By the numerical computations for the original Chemotaxis-Growth model, stationary and dynamical Turing patterns induced the Chemotaxis have been investigated in [2], [3]. But we do not show the global structure of the stationary solutions of this mode. In order to find clues for the structure, we consider the limiting system (1), (2) in the case of the large diffusion coefficients for this model.

Lemma 1 *There exists a positive constant* $\hat{E}$ *such that for any* $0 < E < \hat{E}$*, there are two positive roots* $v_*(E)$*,* $v^*(E)$ *of* $g(v,\ E) = 0$ *which satisfies* $v_*(E) < 1/\alpha < v^*(E)$*,* $\lim_{E\to 0} v_*(E) = 0$*,* $\lim_{E\to 0} v^*(E) = \infty$*,* $\lim_{E\to\hat{E}} v_*(E) = 1/\alpha = \lim_{E\to\hat{E}} v_*(E)$*. On the other hand, there is no root of* $g(v,\ E) = 0$ *for* $\hat{E} < E$*.*

Therefore, this becomes a monostable system. On the other hand, we already study the bistable type of the similar limiting system as (1), (2) in [4].

Hereafter, we only consider a monotone increasing solution $v(x,\ d,\ E)$ of (1), (2) because all oscillating solution can be constructed by connecting rescaling parts of monotone solutions. We have solutions of (1) as the solution bifurcated from the constant solution $v^*(E)$ with $d = d^*(E)$.

Theorem 2 *For any* $0 < E < \hat{E}$*, there exists* $d^*(E) > 0$*, which satisfies* $\lim_{E\downarrow 0} d^*(E) = \infty$ *and* $\lim_{E\uparrow\hat{E}} d^*(E) = 0$*, such that there is a monotone increasing solution* $v(x,\ d,\ E)$ *of* (1) *for* $0 < d < d^*(E)$ *such that*

$$\lim_{d\downarrow 0} v(x,\ d,\ E) = \begin{cases} v_*(E) & 0 \le x < 1 \\ \\ \overline{v}(E) & x = 1, \end{cases}$$

where $\overline{v}(E) > v^*(E)$ *is given by* $\displaystyle\int_{v_*(E)}^{\overline{v}(E)} g(v,\ E)\ dv = 0.$

Moreover, we can prove the direction of the bifurcation brunch around the bifurcation point $d^*(E)$ by Theorem 2.7 in [5]. Therefore, we show

Remark 1 *Set* $D = \{(E,\ d) \mid 0 < E,\ 0 < d\ \}$ *and* $D^* = \{(E,\ d) \mid 0 < E < \hat{E},\ 0 < d^*(E)\ \}$*. There does not exist any nonconstant solution of* (1) *in* $D \setminus D^*$*.*

Next, we obtain the solution of (1) satisfying the integral constraint (2).

Case $1 < \gamma/\alpha$ **:**

Lemma 3 *If* $1 < \gamma/\alpha$*, there is a constant* $0 < E_* < \hat{E}$ *such that for any* $E_* < E < \hat{E}$*, there does not exist any nonconstant solution of* (1), (2) *with* $0 < d < d^*(E)$*.*

Theorem 4 *If* $1 < \gamma/\alpha$*, there exists a continuous function* $d(E)$ *such that there is a solution* $(d(E),\ v(x,\ d(E),\ E)$ *of* (1), (2)*, which satisfies that* $0 < d(E) < d^*(E)$ *for* $0 < E < E_*$ *and* $\lim_{E \to E_*} d(E) = 0$*.*

Case $1 > \gamma/\alpha > 0$ **:**

Theorem 5 *If* $1 > \gamma/\alpha > 0$*. there exist a constant* $0 < E^* < \hat{E}$ *and a continuous function* $d(E)$ *with* $0 < E < E^*$ *such that there is a solution* $(d(E),\ v(x,\ d(E),\ E)$ *of* (1), (2)*, which satisfies* $0 < d(E) < d^*(E)$ *for* $0 < E < E^*$ *and* $\lim_{E \to E^*} d(E) = d^*(E)$*.*

We introduce the global structure of the stationary solutions by using the numerical simulations and the difference between monostable and bistable cases.

References

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Starvation driven dispersal and pattern formations

Yong Jung Kim
KAIST

Starvation driven dispersal models a non-uniform random dispersal of a biological species that increases on a starvation or a similar event. The non-uniformity of the dispersal or motility jumps on certain environmental changes may produce oscillations, patterns, and traveling waves. The cross diffusion and advection appear naturally from the starvation driven diffusion, which are main reasons for pattern formation.

Instability and blowup phenomena induced by diffusion in some reaction-diffusion-ODE systems

Kanako Suzuki
Ibaraki University

We consider mathematical models of a pattern formation arising in processes described by a system of a single reaction-di?usion equation coupled with an ordinary differential equation. This type of models exhibits the di?usion-driven instability, and it is expected that non-constant stationary solutions exist and some spatially inhomogeneous solutions converge toward them.

First, we shall discuss the instability of inhomogeneous stationary solutions. It will be shown that a certain natural (autocatalysis) property of a system leads to instability of all inhomogeneous stationary solutions. Next, we shall discuss a possible large time behavior of solutions. The system can be reduced to a simple model, so called the shadow system, and we will see that space inhomogeneous solutions of the shadow system become unbounded in either finite or infinite time, even if space homogeneous solutions are bounded uniformly in time.

These are joint works with Anna Marciniak-Czochra (University of Heidelberg) and Grzegorz Karch(University of Wroclaw).

A double bubble solution in a ternary system with inhibitory long range interaction

Xiaofeng Ren
The George Washington University

We consider a ternary system of three constituents, a model motivated by the triblock copolymer theory. The free energy of the system consists of two parts: an interfacial energy coming from the boundaries separating the three constituents, and a longer range interaction energy that functions as an inhibitor to limit micro domain growth. We show that a perturbed double bubble exists as a stable solution of the system. Each bubble is occupied by one constituent. The third constituent fills the complement of the double bubble. Two techniques are developed. First one defines restricted classes of perturbed double bubbles. Each perturbed double bubble in a restricted class is obtained from a standard double bubble by a special perturbation. The second technique is the use of the so called internal variables. The advantage of the internal variables is that they are only subject to linear constraints, and perturbed double bubbles in each restricted class represented by internal variables are elements of a Hilbert space. A local minimizer of the free energy in each restricted class is found as a fixed point of a nonlinear equation. This perturbed double bubble satisfies three of the four equations for critical points of the free energy. The unsolved equation is the 120 degree angle condition at triple junction points. Perform another minimization among the local minimizers from all restricted classes. A minimum of minimizers emerges and solves all the equations for critical points.

Front propagation in geometric and phase field models of stratified media

Cyrill Muratov
New Jersey Institute of Technology

We study front propagation problems for forced mean curvature flows and their phase field variants that take place in stratified media, i.e., heterogeneous media whose characteristics do not vary in one direction. We consider phase change fronts in infinite cylinders whose axis coincides with the symmetry axis of the medium. Using the recently developed variational approaches, we provide a convergence result relating asymptotic in time front propagation in the diffuse interface case to that in the sharp interface case, for suitably balanced nonlinearities of Allen-Cahn type. The result is established by using arguments in the spirit of Γ-convergence, to obtain a correspondence between the minimizers of an exponentially weighted Ginzburg-Landau-type functional and the minimizers of an exponentially weighted area-type functional. These minimizers yield the fastest moving traveling waves in the respective models and determine the asymptotic propagation speeds for front-like initial data. We further show that generically these fronts are the exponentially stable global attractors for this kind of initial data and give sufficient conditions under which complete phase change occurs via the formation of the considered fronts.