Low dimensional topology and complex analysis (1)

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contents

This lecture

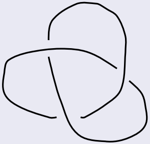
- will not be very ambitious (if the title may sound so),
- but will only show, by some examples, the close relationship between the two disciplines,
- and present my recent results on moduli space of Riemann surfaces.

The first result of modern knot thoery

In 1908, H. Tietze (1884 – 1964) published the first result in modern knot theory:

Tietze (Monatshefte für Math. und Physik, 1908, §18.)

The knot group of a trefoil is generated by s, t and has a relation $sts = tst$. This group is not infinite cyclic, thus trefoil is knotted.



No hint of the proof is given.

M. Epple pointed out $\cdots$ (Historia Mathematica, 1995.)

A closer reading of Tietze's paper indicates W. Wirtinger's influence:

Wilhelm Wirtinger (1865 – 1945, Austria. Tietze's advisor) gave a talk on "Branching about functions of two variables" in 1905.

He did not publish this talk, but Epple claims that the reconstruction is possible from several available documents.

Wirtinger's work (according to Epple)

Wirtinger (and his contemporaries) viewed algebraic functions of two variables as branched coverings of $\mathbb{C}^2$:

$$\{(x,y,z) \in \mathbb{C}^3 \mid f(x,y,z) = 0\} \xrightarrow{p} \mathbb{C}^2, \quad (x,y,z) \mapsto (x,y).$$

Wirtinger claims : Local monodromy group of the covering of a neighbourfood of a singular point (x_0, y_0) characterizes the branch point (topologically).

Wirtinger's example

Algebraic function z of two variables (x, y):

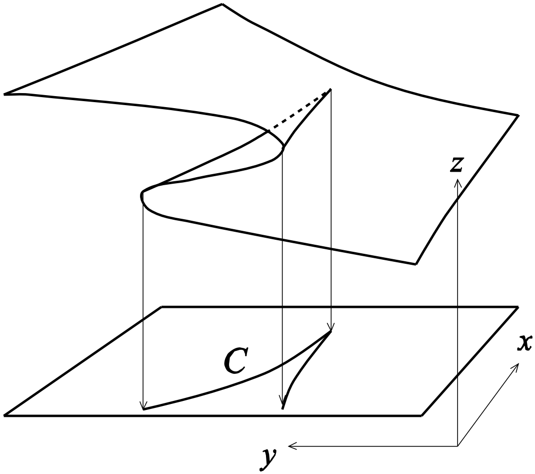
$$z^3 + 3xz + 2y = 0.$$

The equation of the branching curve C:

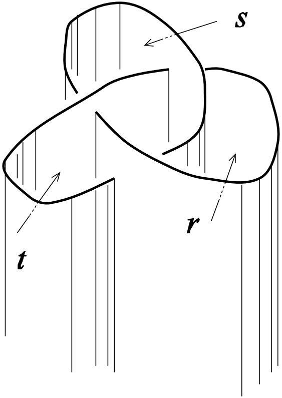
$$x^3 + y^2 = 0.$$

Wirtinger saw that the intersection $C \cap S^3$ was a trefoil knot!

Algebraic function as branched covering



Wirtinger cut the covering into three 1-connected sheets.



$r,\ s,\ t$: **sheet permutaions. He found the relation**

$$1 = rsr^{-1}t^{-1} = sts^{-1}r^{-1} = trt^{-1}s^{-1}.$$

It is immediately seen

by substituting $r = sts^{-1}$ in other relations, that

$$sts = tst,$$

which Tietze wrote in his 1908 paper without any proof.

Epple's viewpoint

It is evident which has the priority of computation of the (monodromy or knot) group.
But Epple claims that Tietze's papar is a striking example of "context-elimination",i.e.

- Wirtinger worked in the context of algebraic function theory.
- Tietze initiated modern knot theory by having asked the right (properly knot theoretical) questions without specifying the context.

He concludes: p.396.

Knot theory was neither an invention out of thin air nor an application of general topological notions to a particular problem.
It emerged as part of a general process of differentiation out of the theory of algebraic functions.
Another context was the physical one (from which Tait had drived his justification for tabulating knots).

Incidentally · · ·

Epple's observation (on p.386) on Wirtinger's and Artin's work:

- **From the sources, it is not quite clear whether Wirtinger was fully aware of the fact that he had actually developed · · · a presentation of the fundamental group of arbitrary knot complement.**
- **The method became generally known under Wirtinger's name when Artin described it in his widely read article on the braid group (1925).**

Epple's observation (bis)

- Witinger's investigation $\cdots$ also contained the first example of a knotted surface in a manifold of four real dimension. (i.e. the complex branch curve in $\mathbb{C}^2$.)
- This is exactly the position of the complex branch curve in the complex plane $\cdots$ when looked at from the point of view of real manifolds.
- It is evident that Artin had this example in mind when he inaugulated the study of knotted surfaces in his short paper (1925).

Branched coverings date back to Riemann (1).

Riemann wrote

In a region of the plane where there are two or more different prolongations of the function, the surface will have two or more layers; it will be composed of superimposed sheets, one sheet for each branch.

(Collected Papers, p.81, The Theory of Abelian Functions, translated into English by Roger Baker et al. 2004. Kndrick Press, USA)

Branched covering dates back to Riemann (2).

Riemann wrote (bis)

Around a branch point one sheet of the surface continues into the next, and in the neighborhood of the branch point the surface may be considered as a helicoidal surface whose axis goes through the point perpendicular to the (x, y) plane and whose pitch is infinitely small.

(Collected Papers, p.81)

Riemann surfaces were born as branched coverings.

Riemann surfaces

- **Important in the theory of algebraic functions,**
  $\cdots\cdots$ in the second half of 19C.
- **Topological classification (by g)**
  $\cdots\cdots$ in the second half of the 19C.
- **Introduction of the Poincaré metric**
- **Conformal structures on Riemann surfaces**
  $\cdots\cdots$ in the 20C. (O. Teichmüller, L. Ahlfors, L. Bers, $\cdots$)
- **Study on self-homeomorphisms**
  $\cdots\cdots$ in the 20C. (M. Dehn, J. Nielsen, J. Birman, W. Thurston, $\cdots$)

Mapping class group

Σ_g: closed oriented surface of genus g.

Definition.

The mapping class group Γ_g

$$= \{f : \Sigma_g \to \Sigma_g \mid \text{orientation pres. homeom.}\}/\text{isotopy}.$$

$$\Gamma_0 = \{1\}, \quad \Gamma_1 \cong SL_2(\mathbb{Z}).$$

Γ_g has a strong connection with "Teichmüller space".

Teichmüller space $T(\Sigma_g)$

$T_g = T(\Sigma_g)$ **classifies all the conformal sructures on** Σ_g **up to isotopy (or equivalently, up to homotopy).**

Teichmüller space (bis)

More precisely:

(S, w): S a Riemann surface, $w : S \longrightarrow \Sigma_g$ an orientation preserving homeomorphism.

$(S_1, w_1) \sim (S_2, w_2)$: equivalent iff $\exists$ isotopically (or equivalently, homotopically) commutative diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{w_1} & \Sigma_g \\ t \downarrow & & \downarrow = \\ S_2 & \xrightarrow[w_2]{} & \Sigma_g \end{array}$$

where $t : S_1 \longrightarrow S_2$ is a biregular map (a conformal isomorphism).

Definition. $\boxed{T_g = \{(S, w)\}/ \sim}$

Γ_g acts on T_g

Assume $g \geqq 2$. Γ_g acts on T_g:

For $[f] \in \Gamma_g$ and $p = [S, w] \in T_g$, define

$$[f]_*[S, w] = [S, f \circ w]$$

- **T_g is a $(3g-3)$-dimensional complex bounded domain (Ahlfors, Bers), and Γ_g acts holomorphically.**
- **T_g is a metric space (w. "Teichmüller metric"), and Γ_g acts isometrically**

Teichmüller space is useful in topology.

- **Thurston (and Bers) used T_g to classify mapping classes of Σ_g (1970–80).**
- **S. Kerckhoff used T_g in his solution of Nielsen's realization problem, (1983).**

Types of mapping classes

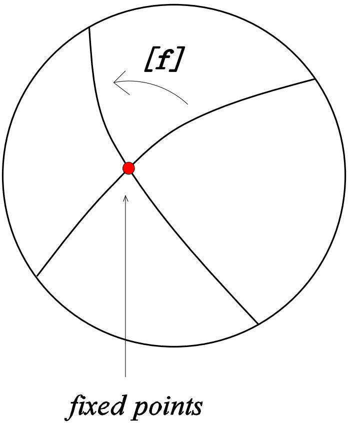
Let me state the classification theorem in Bers' formulation:

Bers Theorem (Acta Math. 1978)

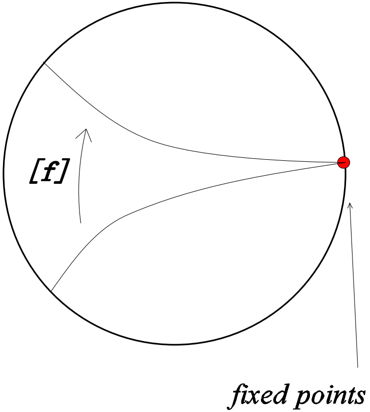
A mapping class $[f] \in \Gamma_g$ is one of the four types:

1. **periodic**: $\exists n(> 0) s.t. [f]^n = 1$.
2. **parabolic**: $[f]$ **is "reduced "by** $\mathcal{C} = \{C_1, \ldots, C_r\}$ **on** Σ_g**, and its component maps are periodic.**
3. **hyperbolic**: **"pseudo-Anosov" in Thurston's sense.**
4. **pseudo-hyperbolic**: **"reducible"but not parabolic.**

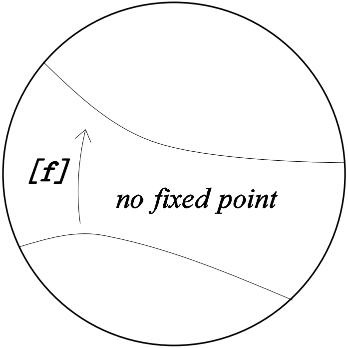
Dynamics of a periodic mapping class on $T(\Sigma_g)$



Dynamics of a parabolic mapping class on $T(\Sigma_g)$



Dynamics of a hyperbolic mapping class on $T(\Sigma_g)$



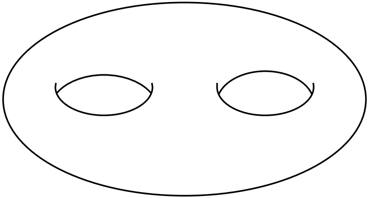
Pseudo-periodic maps

For later use, we would like to define

Definition.

$$[f] \; : \; \text{pseudo-periodic} \iff \begin{cases} [f] : \text{periodic, or} \\ [f] : \text{parabolic} \end{cases}$$

Applications to 4-manifolds

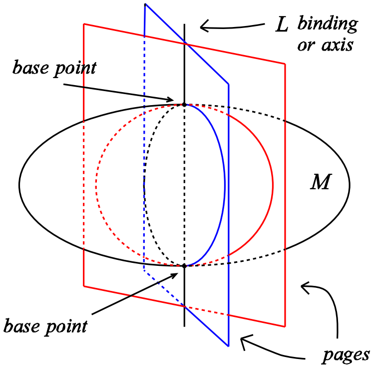


Topology of Riemann surfaces is now well understood.

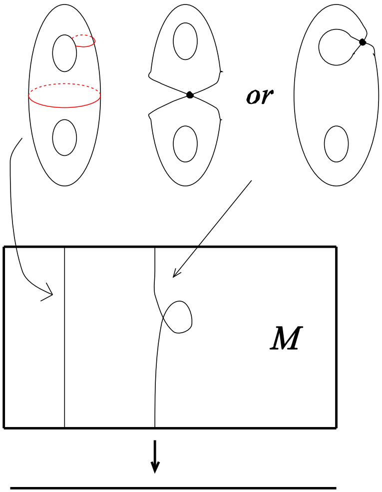
But (smooth) topology of 4-manifolds is still mysterious.

Decomposing 4-manifolds into a collection of Riemann surfaces

Lefschetz pencil (in 1920's), M: Algebraic surface in $\mathbb{C}P^n$. Axis L: $(n-2)$-subspace, and "pages": $(n-1)$-subspaces parametrized by $\mathbb{C}P^1$ ($\cong S^2$).

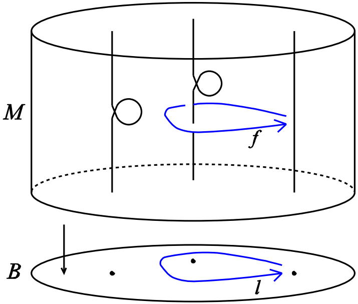


Blow up the base points to get a Lefschetz fibration over $\mathbb{C}P^1$



Monodromy

Moving a fixed smooth fiber $F_0(\cong \Sigma_g)$ along a loop l in the base space B ($= \mathbb{C}P^1$ or more generally, a Riemann surface), we get a self-homeomorphism of Σ_g $f : \Sigma_g \longrightarrow \Sigma_g$ (topological monodromy along l).



Monodromy representation

Given a Lefschetz fibration $\varphi : M \longrightarrow B$, we get a homomorphism (called monodromy representation)

$$\rho : \pi_1(B - \{b_1, \ldots, b_n\}, b_0) \longrightarrow \Gamma_g$$

$$[l] \mapsto [f]$$

where g is the genus of a general fiber, $\{b_1, \ldots, b_n\}$ is the set of critical values, and b_0 is a base point.

C^∞-isomorphism of Lefschetz fibrations

Definition

Lefschetz fibrations $\varphi : M \longrightarrow B$ and $\varphi' : M' \longrightarrow B'$ of the same fiber genus g are C^∞-isomorphic i.e.,the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\exists C^\infty\text{-diffeom.}} & M' \\ \varphi \downarrow & & \downarrow \varphi' \\ B & \xrightarrow[\exists C^\infty\text{-diffeom.}]{} & B', \end{array}$$

Equivalence of monodromy representations

Definition

Two monodromy representations ρ and ρ' are equivalent if there is an orientation preserving homeomorphism

$$h : (B, \{b_1, \ldots, b_n\}, b_0) \longrightarrow (B', \{b'_1, \ldots, b'_n\}, b'_0)$$

s.t. the following diagram commutes:

$$\begin{array}{ccc} \pi_1(B - \{b_1, \ldots, b_n\}, b_0) & \xrightarrow{\ \rho\ } & \Gamma_g \\ h_* \big\downarrow & & \big\downarrow \text{inner auto.} \\ \pi_1(B' - \{b'_1, \ldots, b'_n\}, b'_0) & \xrightarrow[\ \rho'\ ]{} & \Gamma_g \end{array}$$

Monodromy representation classifies Lefschetz fibrations !

Theorem (A. Kas, 1985. M. 1995)

Lefschetz fibrations $\varphi : M \longrightarrow B$ and $\varphi' : M' \longrightarrow B'$ of the same fiber genus g are C^∞-isomorphic iff their monodromy representations are equivalent.

Actual enumeration of Lefschetz fibrations

At present only successful is the case where $g = 1$ ("Kodaira's elliptic surfaces without multiple fibers").

Theorem (Kas, Moishezon, 1977, M. 1986)

Two Lefschetz fibrations of fiber genus 1, M and M' , are C^∞- isomorphic iff $\chi(M) = \chi(M')$ and $\chi(B) = \chi(B')$.

This was proved by a certain combinatorial group theoretic argument.
Recently Kamada's chart theory gives a much simpler proof (Kamada, M., Matumoto, Waki 2005)

I would like to finish here for today.

Tomorrow, I would like to talk about

- More about Lefschetz fibrations
- Orbifolds
- Moduli space of Riemann surfaces
- Compactification
- Orbifold structure of the compactified moduli space, etc.

Thank you!