Low dimensional topology and complex analysis (3)

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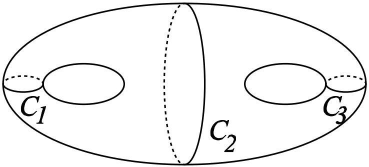
Hiroshima University, January 12, 2011

contents

some details : pants decomposition

(geodesic) pants decomposition: pants $= D^2 \setminus 3 -$ **disks**



$(C_1, C_2, \cdots, C_{3g-3}) \longleftarrow$ **closed geodesics**

Fenchel-Nielsen Coordinates

$$T(\Sigma_g) \overrightarrow{\cong} (\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3}$$

$$[S] \mapsto (l(C_i), \theta(C_i))$$

$$\nearrow \qquad \nwarrow$$

"geodesic length" *"twisting angle"*

Basic lemmas (cf. Abikoff's Lecture Notes, LNM 820, pp.95)

Lemma A

∃ a universal const. M_0 s.t. simple closed geodesics C_1, C_2 with

$$l(C_1), l(C_2) < M_0 \Longrightarrow C_1 \cap C_2 = \emptyset$$

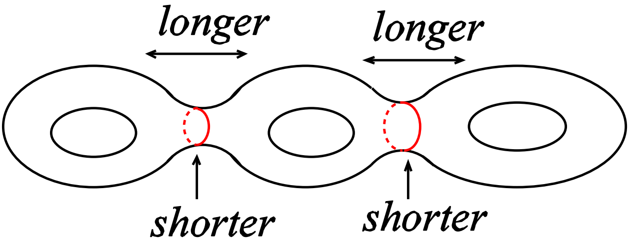
"short simple closed geodesics do not intersect" (figure)

Lemma B

∃ a universal constant M_1 s.t. every Riemann surface S ($\cong \Sigma_g$) has a pants decomposition with curves $\{C_i\}$ of length $l(C_i) < M_1$.

explanation of Lemma A

If the red curves become shorter, transverse curves become longer.



Compactificatin process of $M_g (= T_g(\Sigma_g)/\Gamma_g)$

Given a set of infinite # of points $\{p_i\} \subset T(\Sigma_g)$,

by the action of Γ_g, we may assume $\swarrow$ **($3g - 3$ factors)**

$$\exists \text{ infinite } \# \text{ of points } \{p_i\} \subset (0, M_1] \times \cdots \times (0, M_1] \times [-K, K] \times \cdots \times [-K, K]$$

$\nwarrow$ **($3g - 3$ factors)**

(w.r.t. a certain pants decomposition; Fenchel-Nielsen coordinates)

Thus either

1. $\exists$ convergent subsequence $\longrightarrow$ a point $\in T(\Sigma_g)$

or

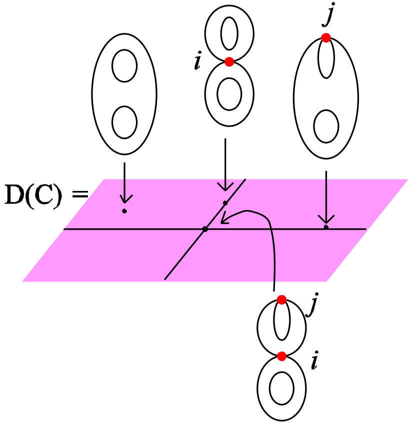
2. $\exists$ $\{C_i\}$ $l(C_i) \to 0$ (nodes)

Bers' deformation space (1)

To describe the second case, Bers introduced

"deformation space" $D(\mathcal{C})$, $\mathcal{C} = \{C_{i_1}, C_{i_2}, \cdots, C_{i_p}\}$.

$\dim_{\mathbb{C}} D(\mathcal{C}) = 3g - 3.$



Deformation space (2)

The difinition of $D(C)$ is similar to that of $T(\Sigma_g)$, but starts with the pair (S, u) with

- S a Riemann surface or a Riemann surface with nodes
- $u : S \to \Sigma_g/\mathcal{C}$: "deformation"
- $(S_1, u_1) \equiv (S_2, u_2)$ iff $\exists$ a homotopy commmutative diagram

$$\begin{array}{ccc} S_1 & \xrightarrow[u_1]{} & \Sigma_g/\mathcal{C} \\ \downarrow \text{isom.} & & \downarrow = \\ S_2 & \xrightarrow{u_2} & \Sigma_g/\mathcal{C} \end{array}$$

$$D(\mathcal{C}) = \{(S, u)\}/\equiv .$$

Deformation space (3)

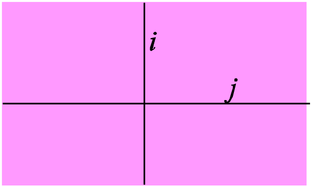
$D(\mathcal{C})$ **parametrizes Riemann surfaces with nodes.**

Let $\Gamma(\mathcal{C}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ **be the free abelian group generated by Dehn twists** $\tau_{C_{i_1}}, \tau_{C_{i_2}}, \cdots, \tau_{C_{i_p}}$, $\quad \mathcal{C} = \{C_{i_1}, C_{i_2}, \cdots, C_{i_p}\}$.

$D(\mathcal{C})$ **is isomorphic to**

$$D(\mathcal{C}) = \text{completion of } T(\Sigma_g)/\Gamma(\mathcal{C})$$

Explanation of $D(\mathcal{C})$

$D(C)=$ 

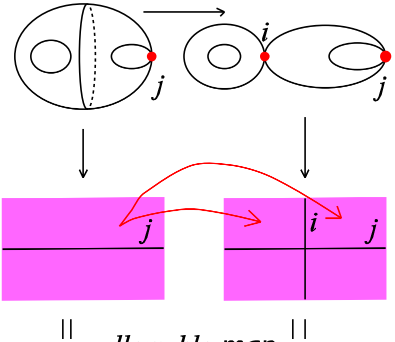
$T(\Sigma_g)/\Gamma(\mathcal{C}) \;=\;$ **"off axis" part**

and

π_1(**"off axis" part**) $\cong \Gamma(\mathcal{C})$.

Allowable map (Bers)

f : deformation



allowable map

$$D(C_j) \longrightarrow D(C_i, C_j)$$

$\cdot/\Gamma(C_i)$ **"infinite cyclic covering"**

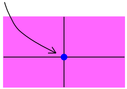
Further quotient of $D(\mathcal{C})$

To obtain $M(\Sigma_g)$, we must further take "quotient" of $D(\mathcal{C})$.

But we cannot "see" the action of $\Gamma(\Sigma_g)$ on $D(\mathcal{C})$, because the action of $\Gamma(\Sigma_g)$ is not well-defined on $D(\mathcal{C})$.

Def. $N\Gamma(\mathcal{C}) :=$ normalizer of $\Gamma(\mathcal{C})$ in $\Gamma(\Sigma_g)$
$W(\mathcal{C}) := N\Gamma(\mathcal{C})/\Gamma(\mathcal{C})$
$W(\mathcal{C})$ acts on $D(\mathcal{C})$ biholomorphically.

$T(\mathcal{C})$ = Teichmüller space of Riemann surfaces with nodes $\leftrightarrow \mathcal{C}$



$\dim_{\mathbb{C}} T(\mathcal{C}) = 3g - 3 - \#\mathcal{C}$

Subdeformation space (1)

Let ε be a sequence

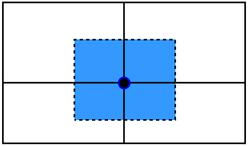
$$0 < \varepsilon_1 < \eta_1 < \varepsilon_2 < \eta_2 < \cdots < \varepsilon_{3g-3} < \eta_{3g-3} < M_0,$$

where M_0 is the number of Lemma A.
Let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\} \subset \Sigma_g$ $(k \leqq 3g-3)$, then define

$$D_\varepsilon(\mathcal{C}) = \{[S,u] \in D(\mathcal{C}) \mid l(\hat{C}_i) < \varepsilon_k, i = 1, 2, \ldots, k,$$
$$l(\text{other geodesics in } S) > \eta_k\}$$

Subdeformation space (2)

$D_\varepsilon(\mathcal{C}) = \epsilon -$ **neighborhood of** $T(\mathcal{C})$**in** $D(\mathcal{C})$.



Action of $W(\mathcal{C})$ **preserves** $D_\varepsilon(\mathcal{C})$.
If f **is parabolic, reduced by** $\mathcal{C}$, **then** $[f]$ **(**$\in W(\mathcal{C})$**) acts on** $D_\varepsilon(\mathcal{C})$ **periodically.**

orbifold structure

We can construct the compactificatin $\overline{M_g}$ as an orbifold.

Folding charts:

$$\begin{cases} (T(\Sigma_g), \Gamma(\Sigma_g)) \quad \text{and} \\ (D_\varepsilon(\mathcal{C}), W(\mathcal{C})) \end{cases}$$

Type 1 Singular fibers over $T(\Sigma_g)/\Gamma(\Sigma_g)$ have periodic monodromy.

Type 2 $\exists$ family of Riemann surfaces with nodes on $D_\varepsilon(\mathcal{C})$ (cf. I. Kra, 1990),
but on $D_\varepsilon(\mathcal{C})/W(\mathcal{C})$, we have singular fibers with pseudo-periodic monodromy.

Orbifolds

Orbifolds were introduced by I. Satake ("V-manifolds" 1956), and W. Thurston (ca. 1977). See also F. Bonahon and L. Siebenmann (1985).

A (complex) orbifold X is a Hausdorff space covered by an atlas of folding charts.

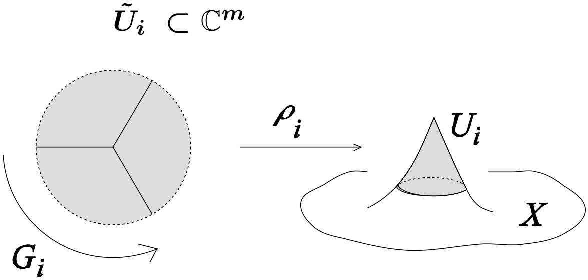
$$\Downarrow$$

$\{(\tilde{U}_i, G_i, \rho_i, U_i)\}_{i\in I}$: $\tilde{U}_i \subset \mathbb{C}^m$,

G_i a finite group acting on $\tilde{U}_i$,

$\rho_i : \tilde{U}_i \longrightarrow \tilde{U}_i / G_i = U_i (\subset X)$, quotient map

Folding chart $(\tilde{U}_i, G_i, \rho_i, U_i)$



An orbifold is a locally uniformizable space (hence a normal complex analytic space).

A typical example

M: a complex manifold

G: a discrete group acting on M holomorphically and properly discontinuously

M/G: has a structure of an orbifold

Orbifold map (locally uniformizable map)

X, Y: orbifolds of possibly different dimensions.
A holomorphic map $f : X \longrightarrow Y$ is an orbifold map
if for $\forall p \in X$,
$\exists(\tilde{U}_i, G_i, \rho_i, U_i)$ of X, and $\exists(\tilde{V}_k, H_k \sigma_k, V_k)$ of Y s.t.
$p \in U_i$, $h(U_i) \subset V_k$, and $h|U_i : U_i \longrightarrow V_k$ is “lifted” to a holomorphic map $h_{ki} : \tilde{U}_i \longrightarrow \tilde{V}_k$.

$$
\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{h_{ki}} & \tilde{V}_k \\
\rho_i \downarrow & & \downarrow \sigma_k \\
U_i & \xrightarrow[h|U_i]{} & V_k
\end{array}
$$

Generic orbifold map

An orbifold map $h : X \to Y$ is generic, if for each pair of folding charts $(\tilde{U}_i, G_i, \rho_i, U_i)$ of X and $(\tilde{V}_k, H_k, \sigma_k, V_k)$ of Y s.t. $h(U_i) \subset V_k$, we have

$$h(U_i) \cap (V_k - \Sigma(Y)) \neq \emptyset,$$

where $\Sigma(Y) =$"cone point set" of Y.

Lemma 1.

For a generic map, $\exists$ a homomorphism $h^\flat_{ki} : G_i \to H_k$ w.r.t.which $h_{ki} : \tilde{U}_i \to \tilde{V}_k$ is an equivariant map.

Fiber spaces over orbifolds

X: a (complex) orbifold,

E: a (not necessarily normal) complex analytic space.

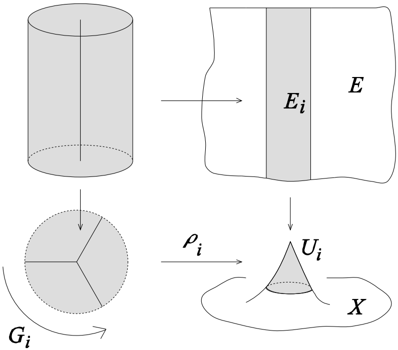
Definition.

$\varphi : E \longrightarrow X$ is a fiber space over an orbifold, if

- φ : a surjective, proper holomorphic map,
- $\dim$ of fibers are constant,
- $\varphi : E \longrightarrow X$ is covered by an atlas of fibered folding charts $\{(\tilde{\varphi}_i : \tilde{E}_i \longrightarrow \tilde{U}_i, G_i, \tilde{\rho}_i, \rho_i, U_i)\}_{i \in I}$, where $(\tilde{U}_i, G_i, \rho_i, U_i)$ is a folding chart of X.

Fibered folding chart $(\tilde{\varphi}_i : \tilde{E}_i \to \tilde{U}_i, G_i, \tilde{\rho}_i, \rho_i, U_i)$

$$\tilde{E}_i \xrightarrow{\tilde{\rho}_i} E_i = \varphi^{-1}(U_i)$$



G_i acts on the fiber space $\tilde{\varphi}_i : \tilde{E}_i \to \tilde{U}_i$.

The quotient is $\varphi_i : E_i \to U_i$.

Orbifold pull-back diagram

Lemma 2.

If $h : X \to Y$ is a generic orbifold map, then any fiber space over Y can be pulled back by h, and we have the orbifold pull-back diagram.

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{h}} & E \\ \varphi' \downarrow & & \downarrow \varphi \\ X & \xrightarrow[h]{} & Y. \end{array}$$

Orbifold fiber space

Definition.

A fiber space over an orbifold $\varphi : E \longrightarrow X$ is an orbifold fiber space, if E and X are orbifolds, and φ is an orbifold map.

Caution: A pull-back of an orbifold fiber space is not always an orbifold fiber space.

Main Theorem

Theorem

1. **The compactified fiber space**

$$\overline{Y(\Sigma_g)} \to \overline{M(\Sigma_g)}$$

is an orbifold fiber space.

2. **For $g \geqq 3$, the compactified fiber space is the universal orbifold family (parametrizing virtually all types of degenerate Riemann surfaces).**

Ashikaga's precise stable reduction theorem (1)

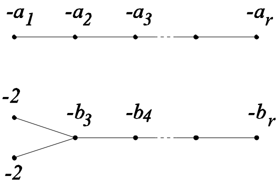
Preliminaries:

Blowing up a relatively minimal degenerating family $\varphi : M \to \Delta$, we get a normally minimal family

$$\varphi' : M' \to \Delta$$

Contracting linear or Y-shaped configurations of spheres in M', we get an (orbifold) fiber space

$$\varphi_{\#} : M_{\#} \to \Delta.$$



Ashikaga's precise stable reduction theorem (2)

Le N be the pseudo-period of $\varphi : M \to \Delta$, i.e., the smallest N s.t. $[f]^N =$ a product of integral Dehn twists.

Theorem (T. Ashikaga), Comment. Math. Helv., 2010

(1) There exists a 'stable reduction'diagram, where $M^{(N)}$ has 'mild cyclic quotient singularities'.

$$\begin{array}{ccc} M^{(N)} & \xrightarrow{\tilde{\rho}} & M_{\#} \\ {\scriptstyle \varphi^{(N)}}\downarrow & & \downarrow{\scriptstyle \varphi_{\#}} \\ \Delta^{(N)} & \xrightarrow[\rho=(z \mapsto z^N)]{} & \Delta \end{array}$$

(2) $\exists$ an action of $\mathbb{Z}/N$ on $M^{(N)}$ s.t. $M^{(N)}/(\mathbb{Z}/N) = M_{\#}$.

$\varphi_\# : M_\# \to \Delta$ becomes an orbifold fiber space.

By the diagram

$$\begin{array}{ccc} M^{(N)} & \xrightarrow{\tilde{\rho}} & M_\# \\ \varphi^{(N)} \downarrow & & \downarrow \varphi_\# \\ \Delta^{(N)} & \xrightarrow[\rho=(z\mapsto z^N)]{} & \Delta \end{array}$$

we see that

$$(\varphi^{(N)} : M^{(N)} \to \Delta^{(N)}, \mathbb{Z}/N, \tilde{\rho}, \rho, \Delta)$$

is a fibered folding chart for $\varphi_\# : M_\# \to \Delta$. Δ becomes an orbifold with the isotropic group $\mathbb{Z}/N$ at the center.

Applying Ashikaga's theorem, we have $\cdots$

Given a relatively minimal degenerating family of Riemann surfaces

$$\varphi : M \to B,$$

by blowing up we get a normally minimal family

$$\varphi' : M' \to B.$$

By contracting linear or Y-shaped configurations of spheres in M', we get an orbifold fiber space (uniquely determined by $\varphi : M \to B$)

$$\varphi_{\#} : M_{\#} \to B.$$

The universality of $\overline{Y(\Sigma_g)} \to \overline{M(\Sigma_g)}$

S is **asymmetric** $\iff$ $\mathrm{Aut}(S) = \{1\}$.

$\varphi : M \to B$ is **almost asymmetric** $\iff$ $\exists$ a set of isolated points $Symm \subset B$ s.t. the fiber over $p \in B - Symm$ is asymmetric.

Precise version of (2) of Main Theorem

If $\varphi : M \to B$ is almost asymmetric with $g \geqq 3$, there exists an orbifold pull-back diagram:

$$\begin{array}{ccc} M_{\#} & \xrightarrow{\text{orbifold pull-back}} & \overline{Y(\Sigma_g)} \\ {\scriptstyle \varphi_{\#}}\downarrow & & \downarrow \\ B & \xrightarrow[b_{\#}]{} & \overline{M(\Sigma_g)} \end{array}$$

Meaning of the universality

Starting from a relatively minimal almost asymmetric family $\varphi : M \to B$, we get the diagram

$$\begin{array}{ccccccc} M & \longleftarrow & M' & \longrightarrow & M_{\#} & \longrightarrow & \overline{Y(\Sigma_g)} \\ \varphi\downarrow & & \varphi'\downarrow & & \varphi_{\#}\downarrow & & \downarrow \\ B & \underset{=}{\longleftarrow} & B & \underset{=}{\longrightarrow} & B & \longrightarrow & \overline{M(\Sigma_g)} \end{array}$$

To prove the main theorem · · ·

We have only to apply Y. Imayoashi's pull-back method of singular fibers (given in his 1981 paper, Ann. Math. Studies, *Riemann surfaces and related topics***), and interprete it locally on a level of fibered folding charts.**
It "automatically" gives an orbifold pull-back diagram by the orbifold formalism.

Thank You !