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Mathematical Biosciences 201 (2006) 90–100

Mathematical Biosciences

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Pulse replications and spatially differentiated structure formation in one-dimensional lattice dynamical system

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Received 22 July 2004; received in revised form 14 July 2005; accepted 3 December 2005
Available online 3 February 2006

Abstract

Replication and differentiation of pulses are studied through a simple dynamical system on a one-dimensional lattice. Depending on the parameter values, our proposed system exhibits the following types of pulse replications: (I) each pulse replicates by itself, (II) each pulse replicates if two or more pulses collide each other, and (III) pulses replicate by the organization of pulse generators. Moreover, replicated pulses construct the spatial–temporal pattern when the number of pulses is large. By selecting local objects in such spatial–temporal patterns and replanting them numerically, it is found that there are two types differentiated local structures: one that continues the production of pulses and the other that does not.

1. Introduction

Recently, simple chemical reaction–diffusion systems have been studied extensively. Some of these models are known to exhibit the pulse replication [1–9], as has been observed in experiments [10].

In each model system studied to this time, only a single type of pulses appears, and there is no differentiation to create different types of pulses or different types of local structures. To search for

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0025-5564/$ - see front matter
doi:10.1016/j.mbs.2005.12.013

possibility in complex pattern formation as in biological systems, it is interesting to study a simple reaction–diffusion system displaying not only the replication but also differentiation of local pulse patterns. We consider a simple dynamical system on a one-dimensional lattice, with one chemical component. The model shows rather complex dynamics. In particular, it is found that the spatial–temporal pattern of pulses has two distinct local structures with different dynamical behaviors. We study how this differentiation of pulse structures appears.

2. Model

As a simple lattice dynamical system [8], we study the following model, possessing only one component,

$$\dot{u}_i = v_i u_i^2 - u_i + D(u_{i+1} - 2u_i + u_{i-1}), \tag{1}$$

$$v_i = F(u_i, u_{i-1}, u_{i+1}, u_{i-2}, u_{i+2}, u_{i-3}, u_{i+3}, \ldots). \tag{2}$$

Here, u_i denotes the concentration of the chemical component at the ith site of the lattice. The diffusion constant is denoted by D.

This model is inspired from Gray–Scott model [1,2], given by,

$$\dot{U} = VU^2 - BU + D_U \nabla_x^2 U, \tag{3}$$

$$\dot{V} = -VU^2 + A(1 - V) + D_V \nabla_x^2 V. \tag{4}$$

Here, U and V denote the concentration of the chemical components, D_U and $D_V (> D_U)$ are diffusion constants for each chemical component, with A and B as constant parameters. Now, we consider the case that the relaxation of V is fast enough compared to the variation of U. By neglecting the spatial variation, we obtain $V = A/(A + U^2)$ from Eq. (4) by setting $\dot{V} = 0$. If we also consider the effect of the diffusion, V is expected to be given by the form $V(x) = \int dx' W(x' - x) A/(A + U(x')^2)$, where $W(x)$ is an unknown weight function that decays to zero as $|x| \to \infty$. Then, $V(x)$ is approximated by some kinds of spatial average of a decrease function of $U(x)$. Thus, it is natural to choose $F(\ldots)$ so that it gives some spatial average of a decrease function of u_i. As a simple choice, we employ a following form

$$v_i = \frac{1}{2L+1} \sum_{n=i-L}^{i+L} a(1 - u_n) \tag{5}$$

in this paper. Hence, we obtain the following equation represented only by u_i

$$\dot{u}_i = \left\{ \frac{1}{2L+1} \sum_{n=i-L}^{i+L} a(1 - u_n) \right\} u_i^2 - u_i + D(u_{i+1} - 2u_i + u_{i-1}), \tag{6}$$

where L gives a range for the spatial averaging of u_i. This model is found to show a rich behavior with the change of parameter a or D, in particular for the case with $L = 2$.

As the initial condition, we choose mainly the following two conditions: (A) $u_i = u^c$ is given for a site $i = j$ and $u_i = 0$ are given for sites $i \neq j$, (B) $u_i = u_i^c$ are given for some sites $i = j_1$, $i = j_2 \ldots$ located in a small area compared to the system's space and $u_i = 0$ are given for other sites. Here, u^c

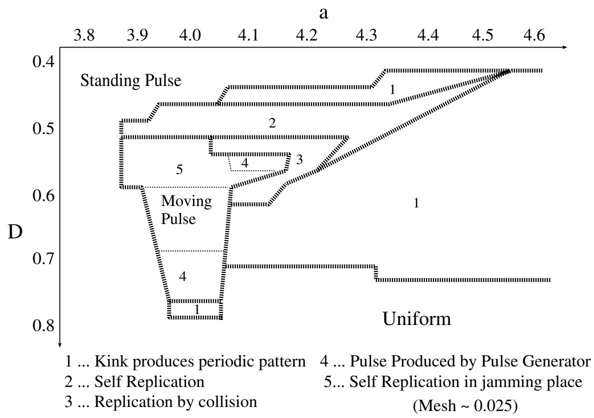

Fig. 1. Rough phase diagram in the D–a plane. Details of each area are given in the text.

is a random number between $((2L+1)a-((2L+1)^2a^2-4a(2L+1)(2D+1))^{0.5})/2a$ and $((2L+1)a+((2L+1)^2a^2-4a(2L+1)(2D+1))^{0.5})/2a$. By the initial condition (A), the states with only one pulse are prepared in initial, and the states with some pulses are prepared in initial by (B). We employ the periodic boundary condition for all simulations.[1]

3. Phase diagram and typical behaviors

The model shows a variety of spatial–temporal pattern with pulse replications. Here, we classify the types of the behaviors by focusing on the case with $L = 2$. Some characteristic behaviors such as annihilating pulses observed in the original Gray–Scott model are not found in this system, but instead, some types of novel pulse replication are observed that are not observed in the Gray–Scott model. By a straightforward analysis for the fixed point solutions, spatially uniform solutions with $u_i \neq 0$ exist for $a \geqslant 4$, while only the solution with $u_i = 0$ is allowed as the spatially uniform fixed point solution for $a < 4$. Now, we focus on the study of dynamics of the model mostly with the parameter region around $a \sim 4$.

Fig. 1 shows a rough phase diagram of this system with $L = 2$ in the D–a plane. 'Standing Pulse' is a state with fixed peaks of u_i, 'Moving Pulse' is a state with moving peaks of u_i without replication and annihilation, and 'Uniform' is a spatially uniform state with $u_i \neq 0$.

In other (a,D) region, the replication of pulses of u_i is observed. In practice, we found several types of pulse replicating behaviors as follows.

(1) *Gray–Scott like replication*. In region 1, the pulse replicates itself when the density of pulse is low. This state is nothing but that observed in the simulation of Gray–Scott model [1,2,7]. In this case, the pulse replication ceases and the system becomes stationary forming a spatially periodic pattern of u_i when pulses fill throughout the entire system. Fig. 2 shows a typical spatial–temporal

[1] We calculate Eq. (6) by Eular's scheme with the time step $\delta t = 0.01$. The same results can be obtained with $\delta t = 0.025$ or $\delta t = 0.003$ are used.

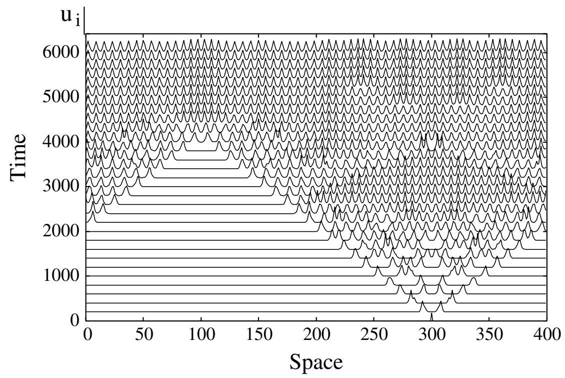

Fig. 2. Spatial–temporal evolution of u_i at region 2 in Fig. 1, where $D = 0.5$, $a = 4.0$.

evolution of u_i at region 2 in the phase diagram ($D = 0.5$, $a = 4.0$). In this case, the pulse replicates itself not only at a region with low pulse density but also at a region with high pulse density, in contrast to the Gray–Scott model. Similarly to the case with the region 1, the system becomes stationary when pulses fill throughout the entire system. However, the spatial configuration of pulses in this case is not regular.

(2) *Replication by collision*. Fig. 3 shows a typical spatial–temporal evolution of u_i at the region 3 in the phase diagram ($D = 0.55$, $a = 4.2$). In this case, the pulse cannot replicate itself but can replicate when two or more pulses collide. (Similar pulse replication is observed recently for a partial differential equations system[4].) As in the region 2, the system becomes stationary, forming an irregular pulse configuration pattern when pulses fill throughout the entire system.

Now, we discuss initial condition dependence of the final pattern. The pulse replications in the regions 1, 2 and 3 in the phase diagram are observed independently of the selections of the initial conditions from (A) or (B). On the other hand in the region 4 and 5 of the phase diagram, the pulse replication is observed only if the condition (B) is prepared as the initial condition. If we prepare the condition (A) as the initial condition, only moving pulses without replications are observed. Later, we only employ the condition (B) as the initial condition.

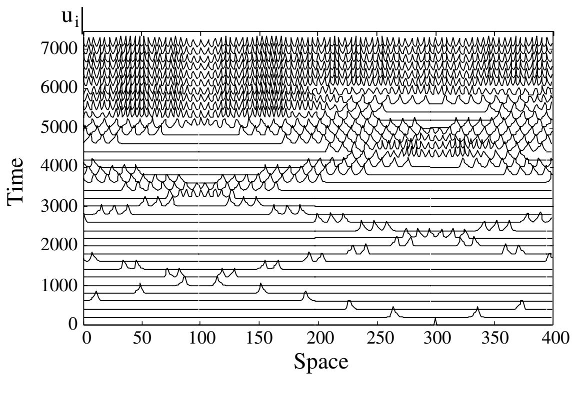

Fig. 3. Spatial–temporal evolution of u_i at region 3 in Fig. 1, where $D = 0.55$, $a = 4.2$.

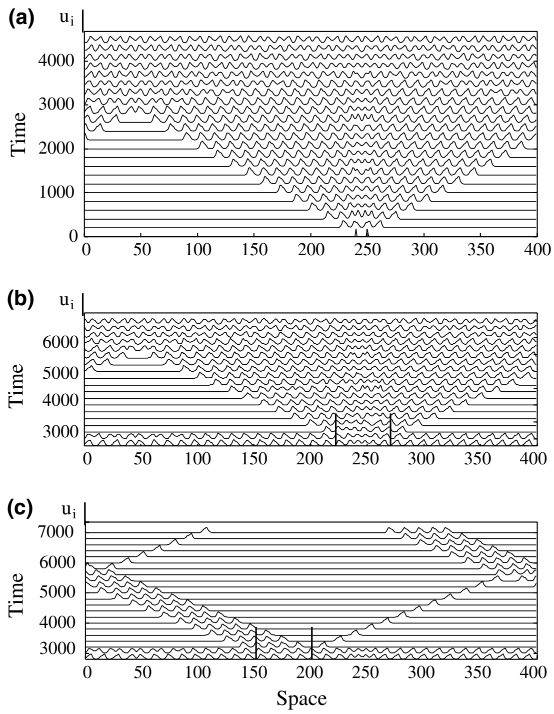

Fig. 4. Spatial–temporal evolution of u_i at region 4 in Fig. 1, where $D = 0.7$, $a = 4.0$.

(3) *Replication by pulse generator*. Fig. 4(a) shows the typical spatial–temporal evolution of u_i at the region 4 in the phase diagram ($D = 0.7$, $a = 4.0$). In this case, the following two types of characteristic pulses appear: the oscillating pulses which are the pulses with the temporal oscillations of u_i without spatial moving, and propagating pulses which move spatially in a direction with keeping their forms. Here, the oscillating pulses are generated around the area where $u_i \neq 0$ points exist initially, and propagating pulses are generated around the oscillating pulses. The propagating pulses are produced one after another by oscillating pulses. The characteristics of these two types of pulses are different, which will be shown later.

(4) *Replication by specific pulses*. Fig. 5(a) and (b) show the spatial–temporal evolution of u_i and that of peak points of each pulse, for the region 5 in the phase diagram ($D = 0.55$, $a = 4.0$). In Fig. 5(b), circles are located when a pulse is replicated. As shown in the figure, two types of pulses appear: one that continues the replication and the other that replicates only a few times. The former pulses are originated from the inner area where $u_i \neq 0$ points exist initially.

The appearance of the coexistence of two types of pulses is influenced by the initial condition. Here, we note that for some pulse patterns, its wavelength is a multiple of lattice sites, while for some others, the pattern is incommensurate. In Fig. 6(a) and (b), we show two typical initial conditions: (a) the area is occupied mostly by commensurate pulse pattern as shown by arrows, and (b) the area is occupied by incommensurate patterns. Here, the initial condition in Fig. 6(a) leads to evolution of u_i as in Fig. 6(c), while Fig. 6(b) gives an initial condition for the spatial–temporal pattern as in Fig. 5. As shown in Fig. 5, the pulse replication continues until pulses fill throughout the

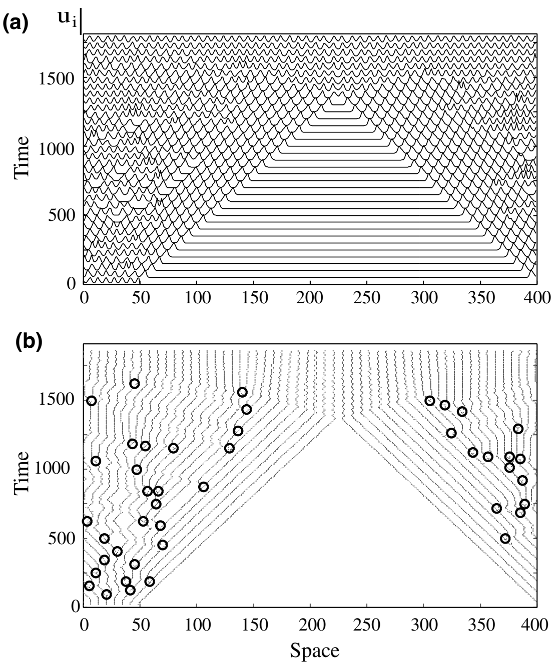

Fig. 5. Spatial–temporal evolution of u_i at region 5 in Fig. 1, where $D = 0.55$, $a = 4.0$. (a) The evolution of each site. (b) The evolution of tops of each pulse.

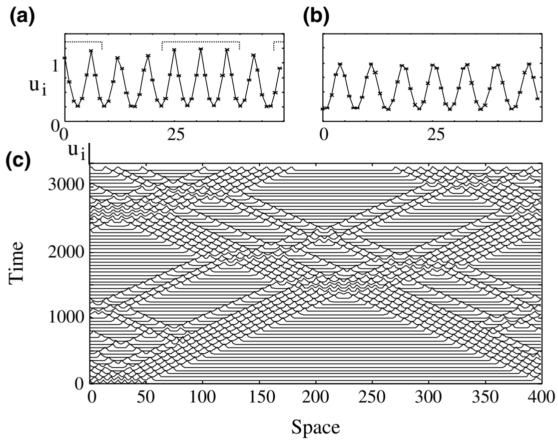

Fig. 6. (a) and (b) give typical initial conditions, where (b) gives initial condition of the spatial–temporal evolution of u_i in Fig. 5, and (a) gives a initial condition of the other type of the spatial–temporal evolution of u_i in (c), where the parameters are identical with those of Fig. 5.

entire system from the initial condition with the area mostly filled by incommensurate pulse patterns. On the other hand, as in Fig. 6 (c), pulses only propagate without replication eventually when started from the initial condition constructed only by commensurate patterns. In Section 5, we will give a qualitative explanation why incommensurate patterns can produce a large number of pulses.

4. Characteristics of replicating pulses

Now, we focus on the characteristic differences between pulses which generate new pulses and those which cannot. Fig. 7 shows two types of typical pulses observed in the region 5 of the phase diagram. The pulses in the figure propagate with a positive velocity, while only the left pulse replicates. In Fig. 7, the tail of the left pulse oscillates slightly. From the oscillation, a new pulse is produced. The pulses observed in regions 2 and 3 of the phase diagram have the same property as this left pulse.

This tail oscillation is caused by the following fact. Here, we focus on the behavior of one pulse. When the diffusion of u_i is slow, compared to the increase of u_i, u_i at the pulse site and its neighboring sites become large. In such a case, v_i of the region around the pulse site decreases because the local average of u_i increases. Then, u_i of a site $i = j$ (u_j) at the middle of this region decreases and this pules is divided into two pulses.

If the diffusion is actually slow as in the region 2, the above scenario is valid for the pulse replication. On the other hand, if the diffusion is a little larger as in the case with the regions 3 and 5, u_i diffuses a little farther before the decrease of v_j is enough to create a new pulse. In such case, u_j increases again little by little in a little while. Thus, the pulse cannot be divided but some parts of this pulse oscillate for a long time. In the traveling pulse as observed in region 3 or 5, such oscillating parts shift to the tail of the pulse.

In the region 5, these tail oscillations are usually damped slowly with some iterations of replications. However, these oscillations are sometime amplified, for example by collisions between pulses, and a new pulse is created. In the region 3, not only the diffusion but also the increase of u_i gets larger. Accordingly, this tail oscillation is sustained. When two or more pulses collide with each other, this oscillation is amplified to create a new pulse.

5. Differentiations of local patterns

With the increase of pulses, there appear some types of spatial–temporal patterns with regards to the pulse concentrations. In particular, at the regions 4 and 5 of the phase diagram, these patterns of the pulse configuration consist of different types of local pulse structures. In other words,

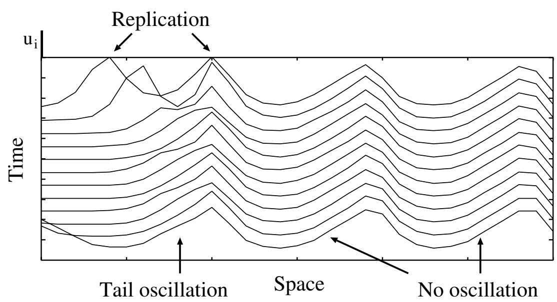

Fig. 7. Temporal evolution of pulses with and without tail oscillations. $D = 0.55$, $a = 4.2$. This figure is plotted by taking a moving coordinate with the same velocity as the propagating velocity of pulses to the right-hand side.

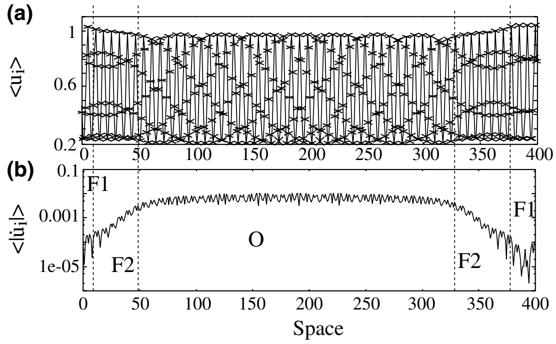

Fig. 8. (a) Time average of u_i, after the total system is filled out by pulses. The same parameter as Fig. 5 are chosen and (b) shows the time averaged $|\dot{u}_i|$ corresponding to (a). O, F1 and F2 indicate the Oscillating, F1-type area, and F2-type areas, respectively.

there are different attracting patterns. Then, it is interesting to study if temporal evolutions from each of these local pulse structure may lead to different pulse patterns or not. In order to see the difference in the attractive to each pattern clearly, we introduce the following 'replant' experiment. First, we choose a local structure from the pattern and 'replant' it into the homogeneous pattern with $u_i = 0$. In other words, we restart the simulation from the initial configuration with a given local structure and $u_i = 0$ otherwise.

First, we classify local structures in the region 4, using this replanting experiment. There appear two types of characteristic pulses: the oscillating pulses and propagating pulses (Fig. 4(a)), distinguishable by replanting experiment. When an area containing more than about 20 sites with two or more oscillating pulses is replanted, the pulse replications as in Fig. 4(b) are observed. On the other hand, only traveling pulses without replications are observed at all times as in Fig. 4(c), when the replanted area does not include at least two oscillating pulses. Thus, a region including two oscillating pulses behaves like a pulse generator studied in [5,6]. Furthermore, the pattern of u_i, that appears when the system is filled by pulses has different characteristics. When an area is replanted from this final pattern, only the traveling pulses without replications are observed.[2]

Second, we discuss the differentiation of pulses in region 5, in a little detail. In this region, when the system is filled out by pulses, there appear three types of areas. To see the difference between these types, we plotted short time averaged spatial profile of u_i and $|\dot{u}_i|$ in Fig. 8(a) and (b) after the total system is filled out by pulses. Here, Fig. 8 are obtained by choosing the same parameter and the same initial condition as Fig. 5. At the area around the middle of Fig. 8(a) and (b), $\langle|\dot{u}_i|\rangle_{\rm time}$ takes a finite and almost constant value, and u_i oscillates periodically in time. We name this area as an oscillating (O) area. In this area, the propagating pulses fill the space and collide with each other frequently. The time averaged profile of u_i is obtained spatially periodic. The remaining area is called frozen area, because $\langle|\dot{u}_i|\rangle_{\rm time}$ is much smaller than that at the oscillating area, and the pattern u_i is almost frozen. The frozen area is the jam of pulses which is originated by the frequent

[2] The behavior explained here as a characteristics of region 4, indeed is observed in the middle part of the region, while the behavior around the lower boundary of the region is slightly different.

pulse replications. It is noted that this area tends to appear at the space where a group of pulses as described in Fig. 6(b) is put in initial because pulses replicate frequently around this region. The frozen area is further divided into two areas: that with spatially periodic pattern, having higher peaks of u_i than that in the oscillating area (which we call F1 type.), and the transition area between the O and the F1 areas, where the pattern changes spatially as shown in Fig. 8 (which is termed F2 area.). F1 area is occupied by commensurate pulse patterns. On the other hand, as in Fig. 8(a), F2 type area lies between two areas with different spatial patterns. It does not have spatially periodic pattern. The F2 type area contains incommensurate pulse patterns.

Now, we study the characteristic behavior of O, F1, F2 areas, by replanting local structures in each area. Fig. 9(a)–(c) show spatial–temporal evolutions obtained from the replanting from each area taken from the pattern of Fig. 8(a). The pulse replication similar to that in Fig. 5(a) appears in Fig. 9(a), by starting from the initial condition with F2 area. On the other hand, by starting from the initial condition with O or F1 area, as in Fig. 9(b) and (c), only the pulse propagations are observed at all times except a few initial replications caused by some disturbance of replanting.

The F2 area consists of incommensurate pulse patterns as in Fig. 6(b). In such incommensurate patterns, the density of u_i can be larger at some part, where v_i decreases, leading to the decrease of

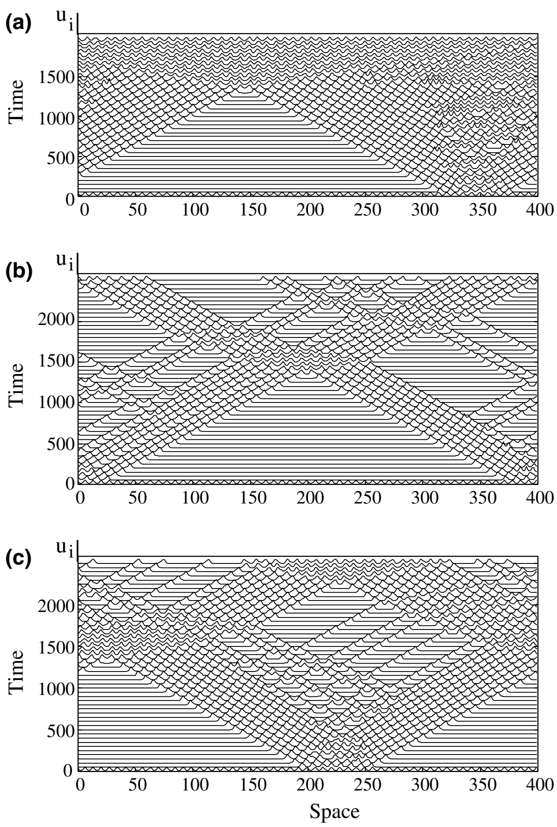

Fig. 9. Typical spatial–temporal evolution of u_i after replanting a local structure F2 (a), F1 (b), and O areas (c). The same parameters are adopted as in Fig. 5.

u_i. Hence oscillation of pattern may be produced. Indeed, such incommensurate pulse patterns have oscillatory instability at the tail, when it is replanted. Although the oscillations at the tail damps slowly with time as a single pulse by itself, they may be amplified by frequent collisions between pulses, as shown around the middle of Fig. 9(a). By repetition of the collisions, the replication of pulses continues as in Fig. 5. Note that, in order to keep such frequent collisions, we need to prepare a large enough area, at least including about 50 sites, with more than 7–8 incommensurate peaks.

To sum up the result of the region 5, only an incommensurate pulse pattern maintains potentiality of replication of pulses. It is interesting to note that two local patterns are distinguishable with regards to the potentiality of replication of pulses.

At the region without the region 4 or 5, the pulse replications can realize independently of the initial pulse configurations as mentioned in section 3. Then, we always obtain the same results by the replant experiments in these cases.

6. Summary and discussion

In this paper, we have studied a simple one-dimensional lattice dynamical system with one variable, that shows replication of pulses as well as differentiation of pulse patterns.

By changing parameter values, several phases are classified, with regards to the propagation and (self-) replication of pulses. Dynamics of pulse patterns are studied at each phase where the replication of pulses is possible. We find that distinct spatial–temporal structure of pulses is formed as the number of pulses increases through the replication. Then, we introduced replant experiment, to see the initial condition dependence of patterns. This replant experiment is carried out by choosing a local structures of the original pattern and taking an initial condition of the local structure around the origin, and 0 otherwise. Here we have found that there are two types of local structures of pulses: one that keeps replication of pulses and the other that does not. The origin of this difference in replication is discussed in terms of oscillatory dynamics of pulses.

In considering biology-oriented applications of chemical reaction systems, self-reproduction of a specific structure is important. In contrast to crystal growth, biological replication is possible only if specific initial conditions of biochemical systems (for a cell) are selected. Hence, it is interesting to study the selection of initial conditions allowing for self-replication. Although the present model is very simple, it may provide a first-step to study such problem in terms of dynamical systems.

In the present model, we have chosen the diffusive coupling up to the next-nearest-neighbor sites, i.e., $L = 2$. For $L = 1$ (the case with the nearest neighbor coupling), the pulse replication is not observed, and only the standing pulse is observed (besides the uniform state). For $L > 4$, on the other hand, self-replication of pulses is not observed, and only the growth of pulse region from the boundaries (i.e., the phase 1 of Fig. 1) is observed, as in the Gray–Scott equation. The self-replications and the differentiation of pulses, reported here, are observed only if the system has the specific spatial discreteness as the cases $2 \leqslant L \leqslant 4$. Indeed, for $L = 3$ or 4, the similar phenomena are observed as that reported here for $L = 2$.

Mathematical analysis of several pulse replicating behaviors as well as an extension of the present model to include more than one chemical components will be of importance in future.

Acknowledgments

The author A.A. is grateful to H. Takagi, Y. Hayase and K. Fujimoto for useful discussions. This work is supported by Grant-in-Aids for Scientific Research from the Ministry of Education, Science and Culture of Japan (11CE2006).

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