

# THE BRAID INDEX IS NOT ADDITIVE FOR THE CONNECTED SUM OF 2-KNOTS

SEIICHI KAMADA, SHIN SATOH, AND MANABU TAKABAYASHI

ABSTRACT. Any 2-dimensional knot  $K$  can be presented in a braid form and its braid index,  $\text{Braid}(K)$ , is defined. For the connected sum  $K_1 \# K_2$  of 2-knots  $K_1$  and  $K_2$ , it is easily seen that  $\text{Braid}(K_1 \# K_2) \leq \text{Braid}(K_1) + \text{Braid}(K_2) - 1$  holds. Birman and Menasco proved that the braid index (minus one) is additive for the connected sum of 1-dimensional knots; the equality holds for 1-knots. We prove that the equality does not hold for 2-knots unless  $K_1$  or  $K_2$  is a trivial 2-knot. We also prove that the 2-knot obtained from a granny knot by Artin's spinning is of braid index 4 and there are infinitely many 2-knots of braid index 4.

## 1. INTRODUCTION

By the Alexander theorem [1], any 1-dimensional knot  $k$  can be presented in a braid form. The *braid index* of  $k$  is the minimum number among the degrees of all closed braids that are equivalent to  $k$ , which is denoted by  $\text{Braid}(k)$ . When 1-knots  $k_1$  and  $k_2$  are presented by an  $m_1$ -braid  $b_1$  and an  $m_2$ -braid  $b_2$  respectively, one can construct an  $(m_1 + m_2 - 1)$ -braid  $b_1 \# b_2$  which presents the connected sum  $k_1 \# k_2$ . Thus there is an obvious inequality

$$\text{Braid}(k_1 \# k_2) \leq \text{Braid}(k_1) + \text{Braid}(k_2) - 1. \quad (1)$$

Actually, J. S. Birman and W. Menasco [4] proved that the braid index (minus one) is additive under the connected sum:

**Theorem 1** (Birman-Menasco[4]). *For the connected sum  $k_1 \# k_2$  of 1-knots  $k_1$  and  $k_2$ ,*

$$\text{Braid}(k_1 \# k_2) = \text{Braid}(k_1) + \text{Braid}(k_2) - 1. \quad (2)$$

In this paper we consider an analogous problem for 2-knots. The notion of a 2-dimensional braid was introduced by O. Ya. Viro [29] and it is proved in [12] that any 2-knot (or any closed oriented 2-submanifold of  $\mathbb{R}^4$ ) can be presented in a braid form. A similar notion, called a braided surface, was studied by L. Rudolph [25, 26, 27]. J. S. Carter and M. Saito have been studying the braid presentations of 2-knots, [5, 6, 7, 8]. Refer to [10, 11, 12, 13, 16, 18, 19] for the first author's research in this field. Another kind of braid presentation of a 2-knot is studied by F. González-Acuña [9]. 2-dimensional braids are also related to the braid monodromies of stable branch curves in complex geometry [22, 23].

The *braid index* of a 2-knot  $K$  is defined to be the minimum number among the degrees of all simple closed 2-dimensional braids that are equivalent to  $K$ , which is

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denoted by  $\text{Braid}(K)$ . The braid index is 1 if and only if  $K$  is a trivial 2-knot. There exist no 2-knots of braid index 2 (cf. [10]). It is proved in [10] that if the braid index of a 2-knot  $K$  is 3, then  $K$  is a ribbon 2-knot, that is, it can be obtained from some mutually disjoint trivial 2-spheres in  $\mathbb{R}^4$  by surgery along some 1-handles, cf. [20]. The converse is not true; for example, the 2-knot obtained from a figure eight knot by Artin's spinning construction [2] is ribbon and the braid index is not 3, [16].

For a 2-dimensional  $m_1$ -braid  $S_1$  and a 2-dimensional  $m_2$ -braid  $S_2$ , we can construct a 2-dimensional  $(m_1 + m_2 - 1)$ -braid  $S_1 \# S_2$  such that the closure in  $\mathbb{R}^4$  is equivalent to the connected sum of the closures of  $S_1$  and  $S_2$ . The following is just an analogy of the inequality (1).

**Lemma 2** ([17]). *For 2-knots  $K_1$  and  $K_2$ ,*

$$\text{Braid}(K_1 \# K_2) \leq \text{Braid}(K_1) + \text{Braid}(K_2) - 1, \quad (3)$$

where  $K_1 \# K_2$  is the connected sum of  $K_1$  and  $K_2$ .

It is obvious that

$$\text{Braid}(K_1 \# K_2) = \text{Braid}(K_1) + \text{Braid}(K_2) - 1 \quad (4)$$

provided that  $K_1$  or  $K_2$  is a trivial 2-knot. Surprisingly, this is the only case where the equality of (3) holds for the connected sum of 2-knots. The following theorem is our main result.

**Theorem 3.** *If neither  $K_1$  nor  $K_2$  is a trivial 2-knot, then*

$$\text{Braid}(K_1 \# K_2) < \text{Braid}(K_1) + \text{Braid}(K_2) - 1. \quad (5)$$

There exist infinitely many ribbon 2-knots of braid index 3 (cf. Lemma 15 or [16]). Using Theorem 3, we prove the following.

**Corollary 4.** *There exist an infinite series of ribbon 2-knots of braid index 4, including the 2-knot obtained from a granny knot by Artin's spinning.*

This paper is organized as follows. In Sections 2 and 3, we review the notions of a 2-dimensional braid and the chart presentation. In Sections 4 and 5, we prove Theorem 3 by using the chart presentation. A proof of Corollary 4 is given in Section 6.

## 2. SIMPLE 2-DIMENSIONAL BRAIDS

Let  $D_1$  and  $D_2$  be 2-disks and  $X_m$  a fixed set of  $m$  distinct interior points of  $D_1$ , where  $m$  is a positive integer. Let  $\pi : D_1 \times D_2 \rightarrow D_2$  be the projection map. A 2-dimensional  $m$ -braid (or a 2-dimensional braid of degree  $m$ ) is a compact oriented surface  $S$  properly embedded in  $D_1 \times D_2$  such that

- (i) the restriction map  $\pi|_S : S \rightarrow D_2$  of the projection  $\pi$  to  $S$  is an  $m$ -fold branched covering map of  $D_2$ , and
- (ii) the boundary  $\partial S$  is the trivial closed  $m$ -braid  $X_m \times \partial D_2$  in the solid torus  $D_1 \times \partial D_2$ .

Moreover, if an additional condition that

- (iii) the branched covering  $\pi|_S : S \rightarrow D_2$  is simple, that is,  $\#(S \cap \pi^{-1}(x)) = m - 1$  or  $m$  for each  $x \in D_2$ ,

is satisfied, then the 2-dimensional  $m$ -braid  $S$  is said to be *simple*.

Two simple 2-dimensional braids  $S$  and  $S'$  are *equivalent* if they are ambiently isotopic by an ambient isotopy  $\{h_t\}_{t \in [0,1]}$  of  $D_1 \times D_2$  such that for each  $t \in [0, 1]$ ,  $h_t(S)$  is a simple 2-dimensional braid. This definition is different from that in [10]. However these definitions are equivalent, [14]. Moreover, it is proved in [14] that simple 2-dimensional  $m$ -braids  $S$  and  $S'$  are equivalent if and only if there exists a one-parameter family  $\{S_t\}_{t \in [0,1]}$  of simple 2-dimensional  $m$ -braids with  $S_0 = S$  and  $S_1 = S'$ . We usually regard equivalent 2-dimensional braids as the same.

Divide the 2-disk  $D_2$  into two 2-disks  $D_2^{(1)}$  and  $D_2^{(2)}$  by a properly embedded arc in  $D_2$ . For simple 2-dimensional  $m$ -braids  $S_1$  and  $S_2$ , we define the product  $S_1 \cdot S_2$  to be the simple 2-dimensional  $m$ -braid such that the restriction to  $D_1 \times D_2^{(i)}$  is a copy of  $S_i$  for each  $i \in \{1, 2\}$ . This product is uniquely determined by  $S_1$  and  $S_2$  up to equivalence.

Let  $S$  be a simple 2-dimensional  $m$ -braid. Identify  $D_2$  with the product  $I_1 \times I_2$  of the unit intervals  $I_1$  and  $I_2$ . For each  $t \in I_2 = [0, 1]$ , we denote the cylinder  $D_1 \times I_1 \times \{t\} \subset D_1 \times D_2$  by  $(D \times I)_t$  and put  $b_t = S \cap (D \times I)_t$ . Then  $b_t$  is a 1-dimensional  $m$ -braid in  $(D \times I)_t$  for all  $t$  but a finite number of exceptional values. For each exceptional value  $t$ , the singular braid  $b_t$  has double points in its strands, see Figure 1. We call the one-parameter family  $\{b_t\}_{t \in [0,1]}$  a *motion picture* of  $S$ . Note that both of  $b_0$  and  $b_1$  are trivial  $m$ -braids, since  $\partial S = X_m \times \partial D_2 \subset D_1 \times D_2$ .

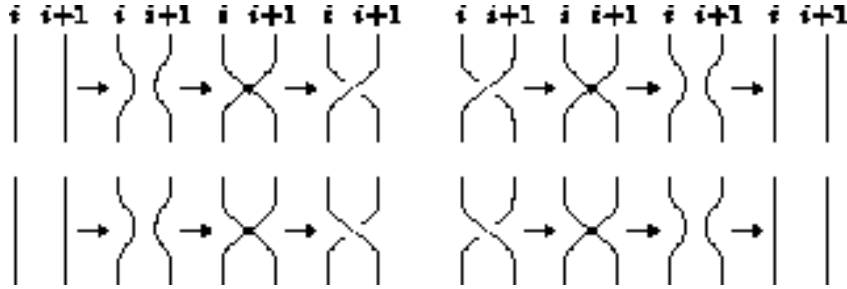


FIGURE 1

When  $\{b_t^{(1)}\}_{t \in [0,1]}$  and  $\{b_t^{(2)}\}_{t \in [0,1]}$  are motion pictures of  $S_1$  and  $S_2$  respectively, we have a motion picture of the product  $S_1 \cdot S_2$  of  $S_1$  and  $S_2$  by taking the usual product  $b_t^{(1)} \cdot b_t^{(2)}$  as braids for each  $t \in [0, 1]$ .

Embed the 4-disk  $D_1 \times D_2$  in  $\mathbb{R}^4$ . Let  $S$  be a 2-dimensional  $m$ -braid. The boundary of  $S$  is a trivial link in the 3-sphere  $\partial(D_1 \times D_2)$ . Attaching  $m$  sheets of 2-disks along the boundary of  $S$  in  $\mathbb{R}^4 \setminus \text{int}(D_1 \times D_2)$  in the obvious way, we have a closed oriented surface embedded in  $\mathbb{R}^4$ . It is called the *closure* of  $S$  in  $\mathbb{R}^4$  and denoted by  $\widehat{S}$ .

When  $S$  is described by a motion picture  $\{b_t\}_{t \in [0,1]}$ , we have a motion picture of the closure  $\widehat{S}$  as follows: For each  $t \in [0, 1]$ , assume that  $b_t$  is lying in the hyperplane  $\mathbb{R}^3 \times \{t\} \subset \mathbb{R}^3 \times \mathbb{R}^1 = \mathbb{R}^4$  and consider the closure  $\widehat{b}_t$  of  $b_t$  in the hyperplane. The trace of the closed braids forms a properly embedded surface in  $\mathbb{R}^3 \times [0, 1]$ . Since  $b_0$  and  $b_1$  are trivial  $m$ -braids,  $\widehat{b}_0$  and  $\widehat{b}_1$  are trivial links in  $\mathbb{R}^3 \times \{0\}$  and  $\mathbb{R}^3 \times \{1\}$  respectively. Attaching  $m$  trivial disks in the lower side and  $m$  trivial disks in the upper side, we have a closed surface in  $\mathbb{R}^4$ , which is the closure of  $S$ .

**Theorem 5** (Viro [29]/ [12]). *Any 2-knot (or closed oriented 2-submanifold of  $\mathbb{R}^4$ ) is equivalent to the closure of a simple 2-dimensional  $m$ -braid for some  $m$ .*

**Definition 6.** The *braid index* of a 2-knot  $K$ , denoted by  $\text{Braid}(K)$ , is the minimum number among the degrees of all simple 2-dimensional braids whose closures are equivalent to  $K$ .

The following theorem corresponds to the Markov theorem for 1-knots [3, 21].

**Theorem 7** ([13, 18, 19]). *Let  $S$  and  $S'$  be simple 2-dimensional braids. The closure of  $S$  is equivalent to that of  $S'$  if and only if there exists a finite sequence*

$$S = S_0, S_1, \dots, S_n = S'$$

*of simple 2-dimensional braids such that each  $S_i$ ,  $i = 1, 2, \dots, n$ , is obtained from  $S_{i-1}$  by a braid ambient isotopy, a conjugation move, a stabilization move or a destabilization move.*

Here, two simple 2-dimensional  $m$ -braids  $S$  and  $S'$  are *braid ambient isotopic* if they are ambiently isotopic by an ambient isotopy  $\{h_t\}_{t \in [0,1]}$  of  $D_1 \times D_2$  such that for each  $t \in [0, 1]$ ,  $h_t(S)$  is a (possibly non-simple) 2-dimensional  $m$ -braid. If  $S$  and  $S'$  are equivalent, then they are braid ambient isotopic. It is unknown whether the converse holds or not. Conjugation, stabilization and destabilization moves are explained in the next section in terms of the chart presentation.

### 3. THE CHART PRESENTATION

In this section, we recall the chart presentation of a simple 2-dimensional braid (cf. [8, 10, 19]). An  $m$ -chart is a finite graph in the interior of a 2-disk  $D_2$ , which may be empty or have *hoops* (that are closed edges without vertices), satisfying the following conditions:

- (i) Every vertex has valency 1, 4, or 6.
- (ii) Every edge is oriented and labeled by an integer from  $\{1, 2, \dots, m-1\}$ .
- (iii) Around each 4-valent vertex, diagonal edges have the same label and are oriented coherently, and the labels  $i$  and  $j$  of the diagonals satisfy  $|i-j| > 1$ .
- (iv) Around each 6-valent vertex, three consecutive edges are oriented inward and the others are oriented outward. These six edges are labeled  $i$  and  $i+1$  alternately around the vertex for some  $i$ .

See Figure 2. We do not distinguish two  $m$ -charts if they are ambiently isotopic in  $D_2$ . A *free edge* means a single edge whose endpoints are 1-valent vertices.

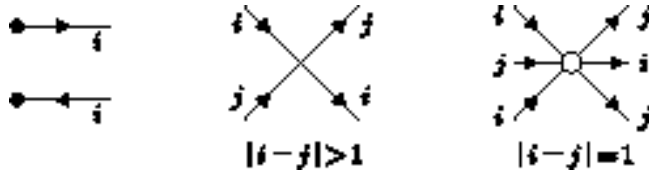


FIGURE 2

Divide  $D_2$  into two 2-disks  $D_2^{(1)}$  and  $D_2^{(2)}$  by a properly embedded arc in  $D_2$  as in Section 2. For two charts  $\Gamma_1$  and  $\Gamma_2$  in  $D_2$ , we denote by  $\Gamma_1 \cdot \Gamma_2$  the chart that is the union of a copy of  $\Gamma_1$  in  $D_2^{(1)}$  and a copy of  $\Gamma_2$  in  $D_2^{(2)}$ .

Let  $\Gamma$  be an  $m$ -chart in  $D_2 = I_1 \times I_2$ . Put  $\ell_t = I_1 \times \{t\}$  for each  $t \in I_2 = [0, 1]$ . Modifying  $\Gamma$  by an isotopy of  $D_2$ , we assume that for all  $t$  but a finite number of exceptional values, there are no vertices of  $\Gamma$  on  $\ell_t$  and the intersection of  $\ell_t$  and  $\Gamma$  consists of transverse double points; and that for each exceptional value  $t$ , the intersection of  $\ell_t$  and  $\Gamma$  consists of transverse double points and a single point that is a vertex of  $\Gamma$  or a maximal or minimal point of an edge of  $\Gamma$ .

For a regular value  $t$ , assign each point of the intersection  $\ell_t \cap \Gamma$  a letter  $\sigma_i$  (or  $\sigma_i^{-1}$ ) if its intersecting edge of  $\Gamma$  is labeled  $i$  and oriented from left to right (or right to left). We read all these letters along  $\ell_t$  to obtain a word on the standard generators  $\sigma_1, \dots, \sigma_{m-1}$  of the  $m$ -braidgroup. We denote this braid word by  $w_\Gamma(\ell_t)$  and call it the *intersection braid word* along  $\ell_t$  with respect to the chart  $\Gamma$ . It is proved in [10] that there is a simple 2-dimensional  $m$ -braid  $S_\Gamma$  whose motion picture  $\{b_t\}_{t \in [0,1]}$  satisfies the condition that for each regular value  $t$ , the  $m$ -braid  $b_t = S_\Gamma \cap (D \times I)_t$  is presented by the braid word  $w_\Gamma(\ell_t)$ . It is also proved that the equivalence class of  $S_\Gamma$  is uniquely determined from (the ambient isotopy class of) the chart  $\Gamma$ . Conversely, any simple 2-dimensional  $m$ -braid  $S$  is equivalent to  $S_\Gamma$  for some (not unique)  $m$ -chart  $\Gamma$ . In this situation, we say that  $\Gamma$  is a *chart presentation* of  $S$ , or  $S$  is presented by the chart  $\Gamma$ . Refer to [8, 10, 19] for more details.

**Example 8.** The top of Figure 3 shows an example of a 4-chart, and the 2-dimensional braid presented by this chart is depicted in the middle and bottom of the figure by the motion picture, where  $t_1, \dots, t_4$  are exceptional values.

Local operations on  $m$ -charts listed below are called CI-, CII-, and CIII-*moves*, respectively. A *C-move* is one of them or its inverse. Two  $m$ -charts are *C-move equivalent* if they are related by a sequence of C-moves (up to ambient isotopy in  $D_2$ ). Let  $\Gamma$  be an  $m$ -chart in  $D_2$ .

- (CI) For a 2-disk  $E$  in  $D_2$  such that  $\Gamma \cap E$  has no 1-valent vertices, change  $\Gamma \cap E$  arbitrarily as long as it has no 1-valent vertices.
- (CII) Suppose that an edge  $e$  connects a 4-valent vertex  $v^4$  and a 1-valent vertex  $v^1$ . Remove  $e$  and  $v^4$ , attach  $v^1$  to the edge of  $v^4$  opposite to  $e$ , and connect the other edges in a natural way (see Figure 4).
- (CIII) Let a 1-valent vertex  $v^1$  and a 6-valent vertex  $v^6$  be connected by an edge  $e$ . Suppose that  $e$  is not the middle edge of a set of three consecutive edges attaching to  $v^6$  which are oriented in the same direction. Then, remove  $e$  and  $v^6$ , attach  $v^1$  to the edge of  $v^6$  opposite to  $e$ , and connect the other edges in a natural way (see Figure 4).

Figure 5 shows two examples of CI-moves. The first one in the figure is called a *channel change move*.

**Theorem 9** ([10, 15]). *Two  $m$ -charts  $\Gamma$  and  $\Gamma'$  are C-move equivalent if and only if the simple 2-dimensional  $m$ -braids presented by  $\Gamma$  and  $\Gamma'$  are equivalent.*

We consider two more operations on charts. Let  $\Gamma$  be an  $m$ -chart in  $D_2$ .

- (MI) Add some hoops (simple loops) parallel to  $\partial D_2$  surrounding  $\Gamma$ . Each of the hoops is oriented and labeled by an integer from  $\{1, \dots, m-1\}$  arbitrarily.
- (MII) Insert a free edge labeled  $m$  near  $\partial D_2$  and regard the union of  $\Gamma$  and the free edge as an  $(m+1)$ -chart.

Let  $\Gamma$  be a chart presentation of a simple 2-dimensional  $m$ -braid  $S$ . We say that a simple 2-dimensional  $m$ -braid  $S'$  is obtained from  $S$  by a *conjugation move* if  $S'$  has

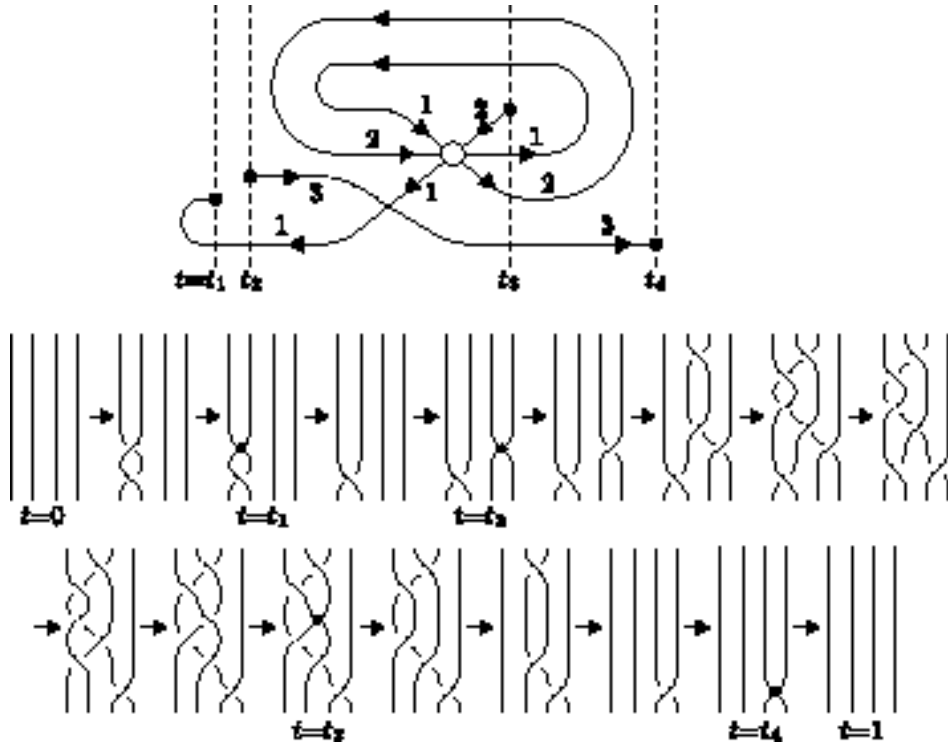


FIGURE 3

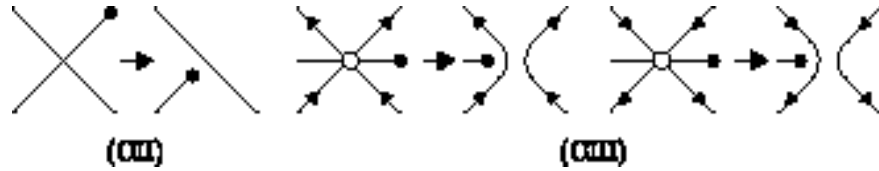


FIGURE 4

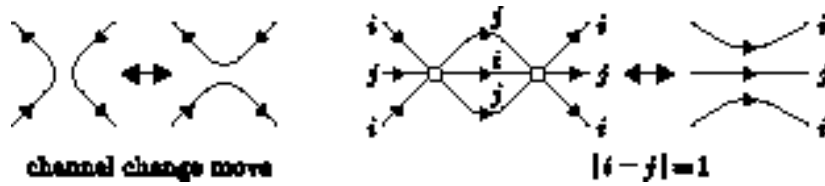


FIGURE 5

a chart presentation that is obtained from  $\Gamma$  by an MI-move or its inverse. A simple 2-dimensional  $(m+1)$ -braid  $S'$  is obtained from  $S$  by a *stabilization move* if  $S'$  has a chart presentation that is obtained from  $\Gamma$  by an MII-move. A *destabilization move* is the inverse operation of a stabilization move. See Figures 6 and 7 for

the motion pictures. These definitions stated in terms of the chart presentation [17] are compatible with the original definitions in [13] stated geometrically, and the definitions in [18] stated in terms of braid systems. Refer to [19] for details and for a complete proof of Theorem 7. Therefore, if two charts are related by a sequence of C-moves, MI-moves and MII-moves, then the closures of their presenting 2-dimensional braids are equivalent.

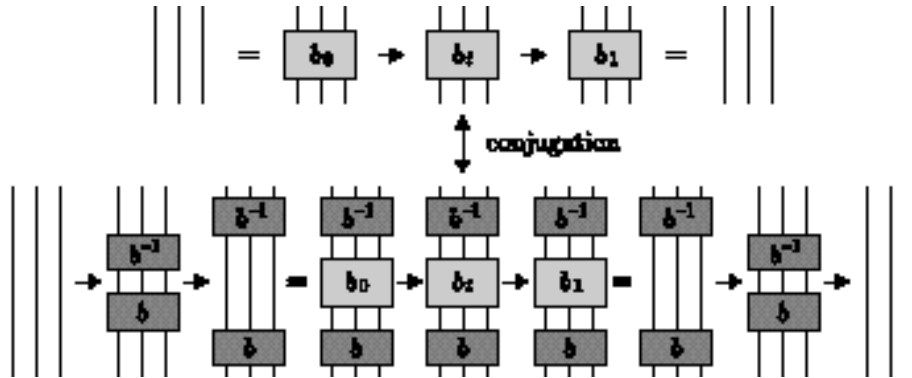


FIGURE 6

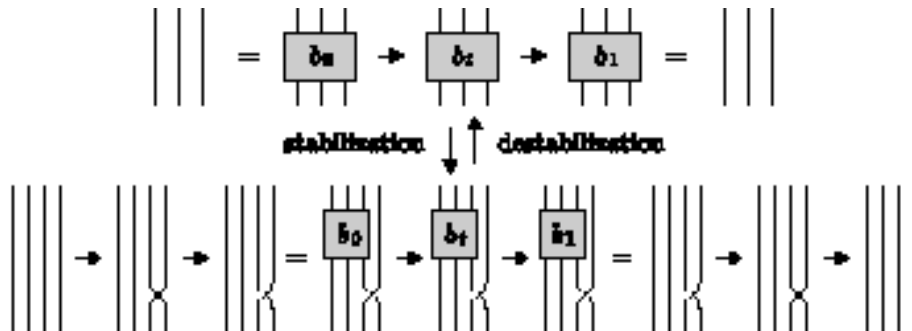


FIGURE 7

#### 4. THE COMPOSITE OF 2-DIMENSIONAL BRAIDS

Let  $p$  and  $q$  be non-negative integers. For a (1-dimensional)  $m$ -braid  $b$ , we denote by  $\iota_p^q(b)$  the  $(m + p + q)$ -braid that is obtained from  $b$  by introducing  $p$  trivial strands on the left side of  $b$  and  $q$  trivial strands on the right side. In other words, when a braid  $b$  is expressed by a word on the standard generators  $\sigma_1, \dots, \sigma_{m-1}$ , the  $(m + p + q)$ -braid  $\iota_p^q(b)$  is obtained by shifting the subscripts  $i$  of  $\sigma_i$  to  $i + p$  ( $i = 1, \dots, m - 1$ ).

The *composite* or the *connected sum* of an  $m_1$ -braid  $b_1$  and an  $m_2$ -braid  $b_2$ , denoted by  $b_1 \# b_2$ , is the  $(m_1 + m_2 - 1)$ -braid

$$\iota_0^{m_2-1}(b_1) \cdot \iota_{m_1-1}^0(b_2)$$

(see Figure 8). The closure of  $b_1 \# b_2$  is a connected sum of the closures of  $b_1$  and  $b_2$ . The dotted circle in the figure indicates a decomposing 2–sphere in  $\mathbb{R}^3$  for the connected sum; namely, it shows the equatorial cross-section of the decomposing 2–sphere with respect to the plane in  $\mathbb{R}^3$  on which the knot diagram is constructed.

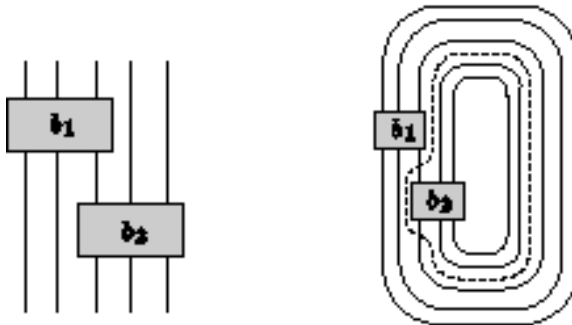


FIGURE 8

The notion of the composite of 1-dimensional braids is generalized to that of 2-dimensional braids as follows: For a 2-dimensional  $m$ -braid  $S$ , we denote by  $\iota_p^q(S)$  the 2-dimensional  $(m+p+q)$ -braid that is obtained from  $S$  by introducing  $p$  trivial 2-disks on the left side of  $S$  and  $q$  trivial 2-disks on the right side. When  $S$  is described by a motion picture  $\{b_t\}_{t \in [0,1]}$ , then the 2-dimensional  $(m+p+q)$ -braid  $\iota_p^q(S)$  is described by the motion picture  $\{\iota_p^q(b_t)\}_{t \in [0,1]}$ . Note that we use the same notation  $\iota_p^q$  for 1- and 2-dimensional braids, but it will make no confusion.

The *composite* or the *connected sum* of a 2-dimensional  $m_1$ -braid  $S_1$  and a 2-dimensional  $m_2$ -braid  $S_2$ , denoted by  $S_1 \# S_2$ , is the 2-dimensional  $(m_1 + m_2 - 1)$ -braid

$$\iota_0^{m_2-1}(S_1) \cdot \iota_{m_1-1}^0(S_2).$$

In other words, a motion picture of the composite  $S_1 \# S_2$  is obtained from the motion pictures  $\{b_t^{(1)}\}_{t \in [0,1]}$  and  $\{b_t^{(2)}\}_{t \in [0,1]}$  of  $S_1$  and  $S_2$  by taking the composite braid  $b_t^{(1)} \# b_t^{(2)}$  for each  $t \in [0,1]$ .

**Lemma 10** ([17]). *The closure of the composite  $S_1 \# S_2$  is a connected sum of the closures of  $S_1$  and  $S_2$ .*

*Proof.* The trace of the decomposing 2–spheres for the closures of  $b_t^{(1)} \# b_t^{(2)}$  in  $\mathbb{R}^3 \times \{t\} \subset \mathbb{R}^4$  for all  $t \in [0,1]$  (as in Figure 8) forms an embedded 3-manifold in  $\mathbb{R}^3 \times [0,1] \subset \mathbb{R}^4$  which is homeomorphic to  $S^2 \times [0,1]$ . Attach a 3-ball in the upper side and a 3-ball in the lower side in an obvious way, we have a decomposing 3-sphere in  $\mathbb{R}^4$  for a connected sum of the closures of  $S_1$  and  $S_2$ .  $\square$

Lemma 2 follows from this lemma immediately.

In terms of the chart presentation, the composite of 2-dimensional braids is stated as follows: For an  $m$ -chart  $\Gamma$  and non-negative integers  $p$  and  $q$ , let  $\iota_p^q(\Gamma)$  be the  $(m+p+q)$ -chart obtained from  $\Gamma$  by adding  $p$  to the label of every edge of  $\Gamma$  and regarding the chart as an  $(m+p+q)$ -chart. This notion,  $\iota_p^q(\Gamma)$ , is compatible with the notion of  $\iota_p^q(S)$  for a 2-dimensional braid  $S$ : If a 2-dimensional  $m$ -braid  $S$  has a chart presentation  $\Gamma$ , then the chart  $\iota_p^q(\Gamma)$  is a chart presentation of the

2-dimensional  $(m + p + q)$ -braid  $\iota_p^q(S)$ . For an  $m_1$ -chart  $\Gamma_1$  and an  $m_2$ -chart  $\Gamma_2$ , we denote by  $\Gamma_1 \# \Gamma_2$  the  $(m_1 + m_2 - 1)$ -chart

$$\iota_0^{m_2-1}(\Gamma_1) \cdot \iota_{m_1-1}^0(\Gamma_2).$$

Then  $\Gamma_1 \# \Gamma_2$  is a chart presentation of  $S_1 \# S_2$ , where  $S_1$  and  $S_2$  are 2-dimensional braids presented by  $\Gamma_1$  and  $\Gamma_2$ .

The following lemma is equivalent to Lemma 10.

**Lemma 11** ([17]). *Let  $K_i$  be a closed oriented surface in  $\mathbb{R}^4$  presented by an  $m_i$ -chart  $\Gamma_i$  ( $i = 1, 2$ ). Then the  $(m_1 + m_2 - 1)$ -chart  $\Gamma_1 \# \Gamma_2$  presents a connected sum  $K_1 \# K_2$ .*

### 5. PROOF OF THEOREM 3

Let  $v$  be a 1-valent vertex of a chart  $\Gamma$  in  $D_2$ . The *label* of  $v$  stands for the label of the edge adjacent to  $v$ . The *sign* of  $v$  is  $+1$  (or  $-1$ ) if the adjacent edge is oriented outward from  $v$  (or toward  $v$ ). We say that  $v$  is *outermost* if we can connect the vertex  $v$  and a point of  $\partial D_2$  by a simple arc  $\alpha$  such that  $\alpha \cap \Gamma = \{v\}$ .

**Lemma 12.** *Let  $\Gamma$  be an  $m$ -chart with 1-valent vertices, where  $m \geq 2$ . For any  $\varepsilon \in \{+1, -1\}$  and  $k \in \{1, \dots, m - 1\}$ , one can transform  $\Gamma$  by C-moves and MI-moves such that there exists an outermost 1-valent vertex with sign  $\varepsilon$  and label  $k$ .*

*Proof.* Note that half of the 1-valent vertices have positive signs and the other half have negative signs, cf. [10]. Choose a 1-valent vertex whose sign is  $\varepsilon$ , say  $v$ . The composition of a CI-move and a CIII-move illustrated in Figure 9 changes the label  $i$  of  $v$  into an integer  $j$  with  $|i - j| = 1$  and  $1 \leq j \leq m - 1$ . Applying this inductively, we may assume that the label of  $v$  is the given integer  $k$ . Using an MI-move and a CI-move, we can make the vertex  $v$  to be outermost as follows: Consider a simple arc  $\alpha$  from  $v$  to a point of  $\partial D_2$  intersecting  $\Gamma$  transversely and read the intersection braid word  $w_\Gamma(\alpha)$  along  $\alpha$  (ignoring the starting point  $v$ ) as in Section 3. Add some hoops parallel to  $\partial D_2$  oriented and labeled such that the intersection braid word of these hoops along  $\alpha$  is the inverse of  $w_\Gamma(\alpha)$ . This is an MI-move. Applying channel change moves along  $\alpha$  as in Figure 10, which is a CI-move, we can remove the intersection of the chart and the arc  $\alpha$ , except the vertex  $v$ .  $\square$

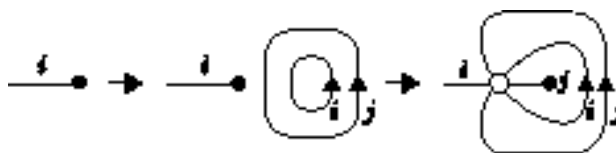


FIGURE 9

Let  $\Gamma$  and  $E$  be an  $m$ -chart and a 2-disk in  $D_2$  respectively such that  $\Gamma \cap \partial E$  is empty or consists of transverse double points missing the vertices of  $\Gamma$ . Then we call the intersection  $\Gamma \cap E$  a *subchart* of  $\Gamma$ .

**Lemma 13.** *Let  $m$  and  $\ell$  be integers with  $1 < m < \ell$ . Let  $\Gamma$  be an  $\ell$ -chart which has a subchart  $\Lambda$  whose edges are labeled by integers in  $\{m, m + 1, \dots, \ell - 1\}$  as shown in the left of Figure 11. Let  $\Gamma'$  be the  $\ell$ -chart obtained from  $\Gamma$  by the replacement*

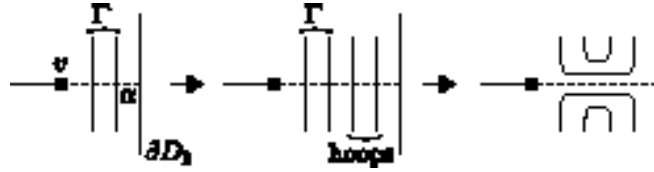


FIGURE 10

illustrated in Figure 11, where  $\Lambda'$  is the subchart obtained from  $\Lambda$  by shifting all of the labels by  $-1$ . Then  $\Gamma$  and  $\Gamma'$  present the same 2-dimensional braid.

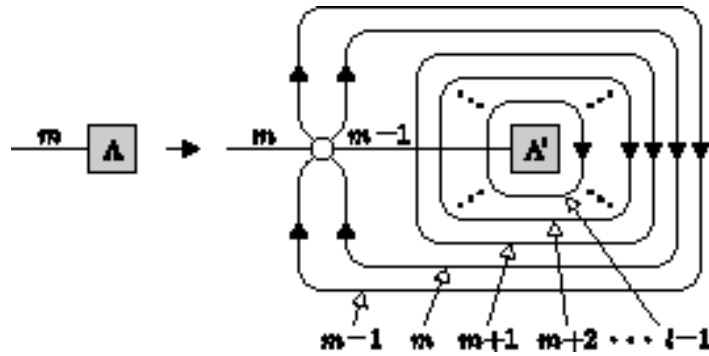


FIGURE 11

**Lemma 14.** Let  $m$  and  $l$  be integers with  $1 < m < l$ . Let  $\Gamma$  be an  $l$ -chart which has a subchart as shown in the left of Figure 12, and let  $\Gamma'$  be the  $l$ -chart obtained from  $\Gamma$  by the replacement illustrated in Figure 12. Then  $\Gamma$  and  $\Gamma'$  present the same 2-dimensional braid.

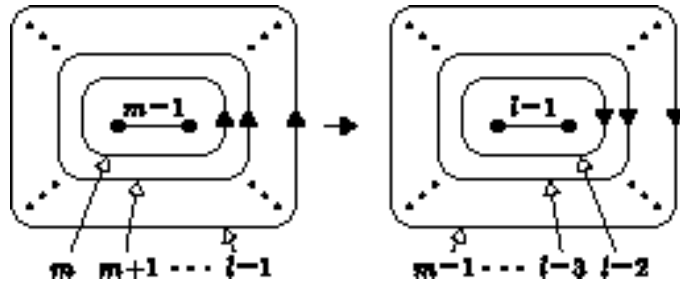


FIGURE 12

*Proofs of Lemmas 13 and 14.* Since the braid monodromies of  $\Gamma$  and  $\Gamma'$  are the same, they present the same (upto equivalence) 2-dimensional braid(see [14, 15, 19] for the definition of the braid monodromy and a proof of the fact that the braid monodromy determines the 2-dimensional braid).  $\square$

The ideas of Lemmas 13 and 14 are schematized as in Figure 13(i) and (ii) respectively.

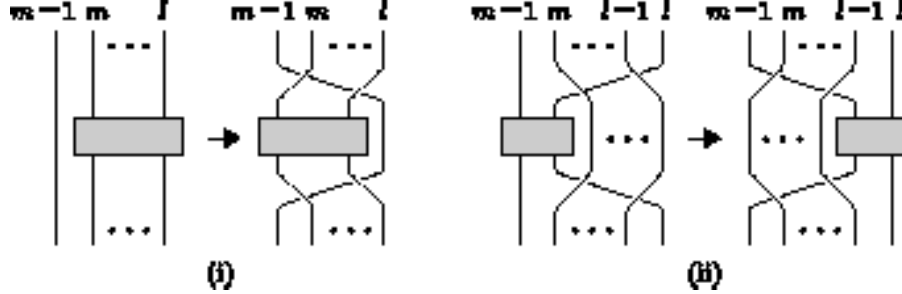


FIGURE 13

*Proof of Theorem 3.* Let  $K_i$  be a non-trivial 2-knot and  $\Gamma_i$  an  $m_i$ -chart presenting  $K_i$  ( $i = 1, 2$ ). It is sufficient to prove that the connected sum  $K_1 \# K_2$  has a chart presentation by an  $(m_1 + m_2 - 2)$ -chart. Note that both of  $\Gamma_1$  and  $\Gamma_2$  have 1-valent vertices. (If  $\Gamma_i$  has no 1-valent vertices, then the 2-dimensional braid presented by  $\Gamma_i$  is trivial [10], and  $K_i$  is the union of  $m_i$  trivial 2-spheres in  $\mathbb{R}^4$ . Since  $K_i$  is a non-trivial 2-knot, this case does not occur.) Using Lemma 12, we may assume that the  $m_1$ -chart  $\Gamma_1$  has an outermost 1-valent vertex with sign  $-1$  and label  $m_1 - 1$  and that the  $m_2$ -chart  $\Gamma_2$  has an outermost 1-valent vertex with sign  $+1$  and label  $1$ . Let  $\Gamma$  be the composite chart  $\Gamma_1 \# \Gamma_2 = \iota_0^{m_2-1}(\Gamma_1) \cdot \iota_{m_1-1}^0(\Gamma_2)$ . This is an  $(m_1 + m_2 - 1)$ -chart presenting the connected sum  $K_1 \# K_2$ . Put  $m = m_1$  and  $\ell = m_1 + m_2 - 1$ . The chart  $\Gamma$  is shown as in the left top of Figure 14, where  $\Lambda_1$  is a subchart whose edges are labeled by integers in  $\{1, \dots, m - 1\}$  and  $\Lambda_2$  is a subchart whose edges are labeled by integers in  $\{m, m + 1, \dots, \ell - 1\}$ . Transform the chart  $\Gamma$  as in Figures 14 and 15: (Step 1) It is a CIII-move; (Step 2) It is a channel change move which is a CI-move; (Step 3) Do the replacement established in Lemma 13; (Step 4) It is a channel change move; (Step 5) It is a CI-move as in Figure 5; (Step 6) It is a combination of C-moves; (Step 7) It is an MI-move with channel change moves; (Step 8) It is the replacement established in Lemma 14; (Step 9) It is an MI-move with channel change moves. Note that after this step, the chart has a free edge of label  $\ell - 1$  and the other edges of the chart are labeled by integers smaller than  $\ell - 1$ . (Step 10) Apply the inverse operation of an MII-move and obtain the  $(\ell - 1)$ -chart in the figure. (Step 11) It is an MI-move. Thus the  $(\ell - 1)$ -chart illustrated in the right bottom of Figure 15 is a chart presentation of  $K_1 \# K_2$ .  $\square$

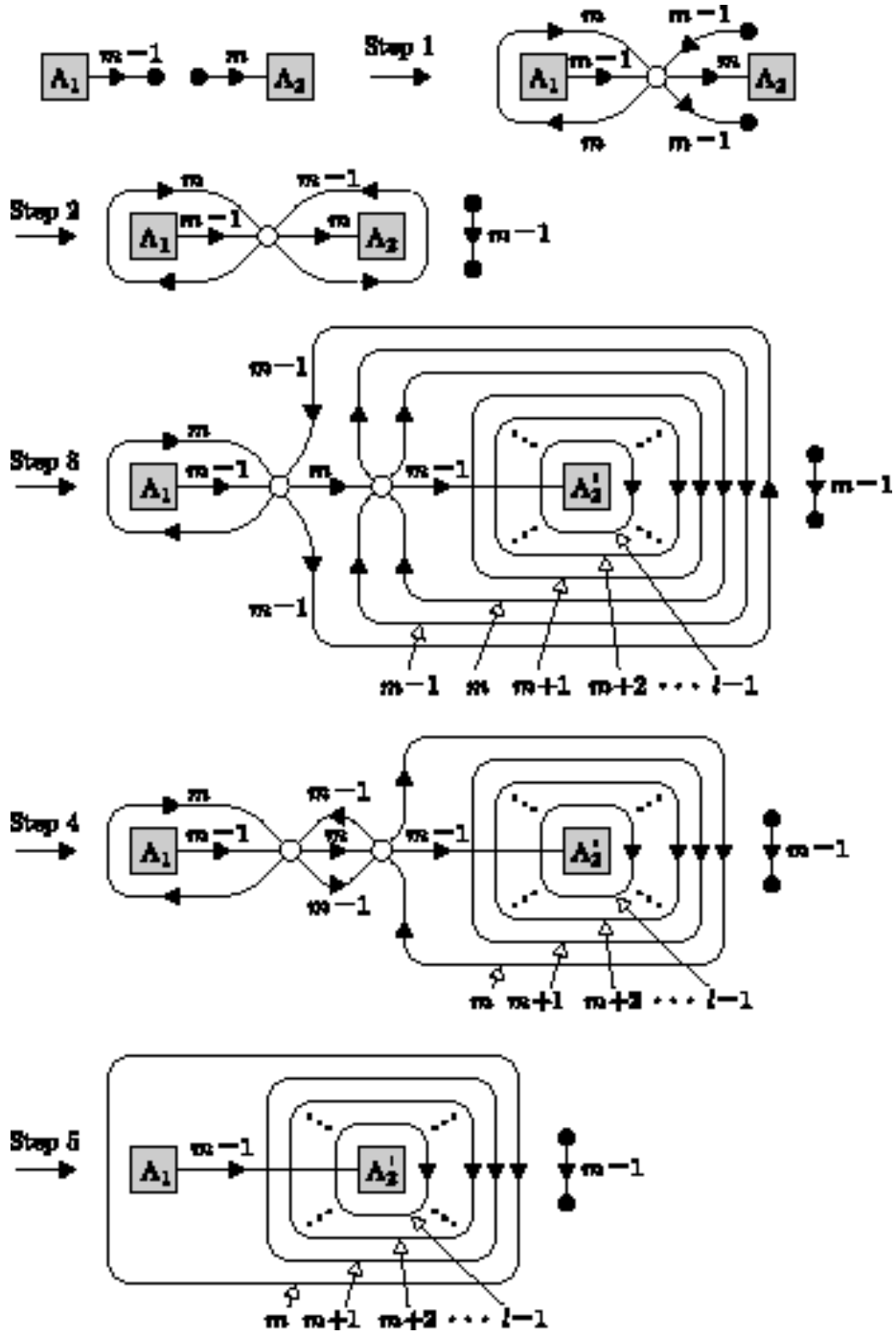


FIGURE 14

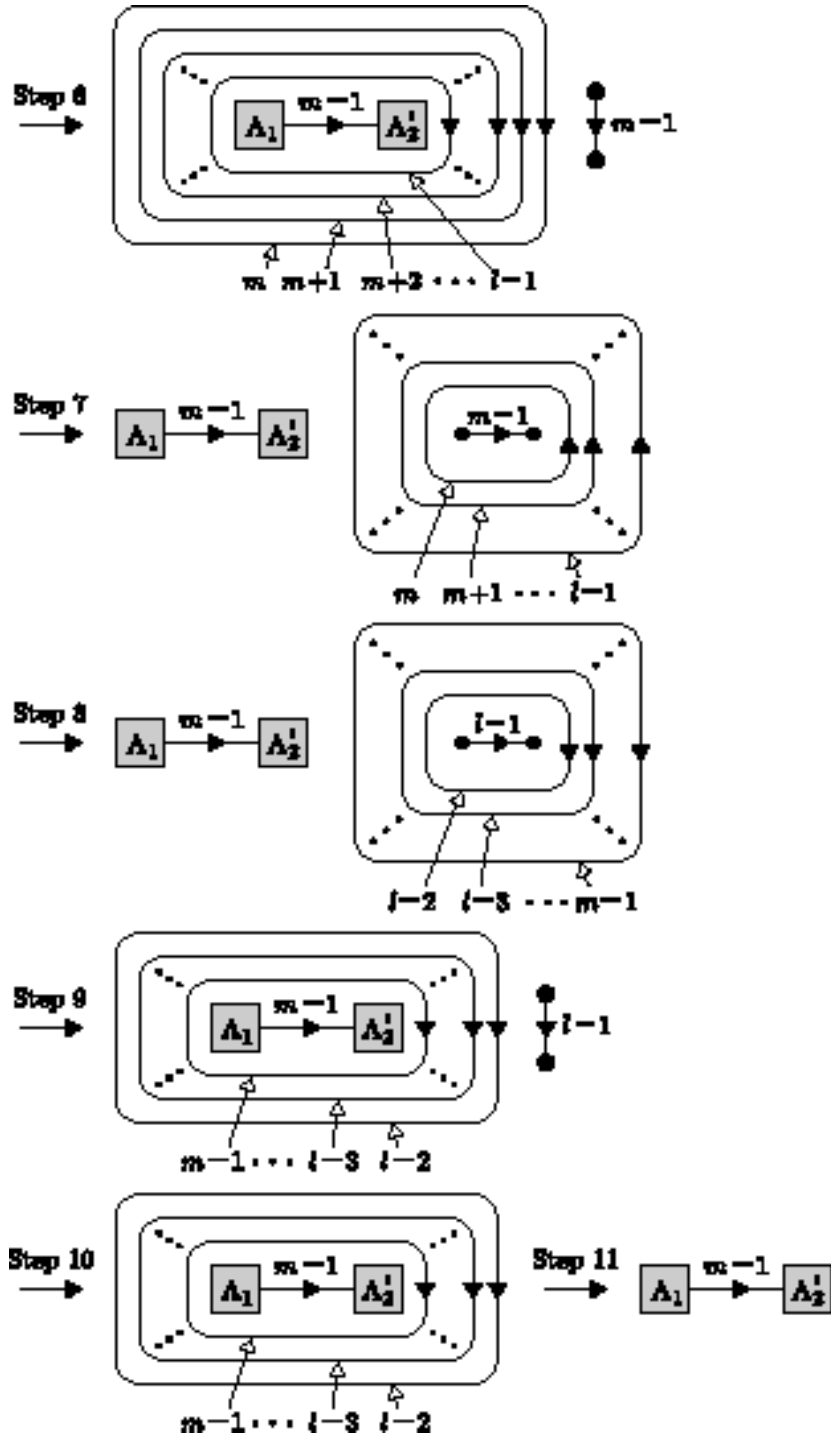


FIGURE 15

## 6. EXAMPLES

Let  $n$  be an odd integer with  $n \geq 3$  and let  $\Gamma_n$  be the 3-chart illustrated in Figure 16 which consists of two free edges with label 1 and  $n$  parallel hoops with label 2 oriented in the same direction surrounding one of the free edges. We denote by  $K_n$  the 2-knot presented by the chart  $\Gamma_n$  (which was denoted by  $S_{n,1}$  in [16]). Note that  $K_n$  is the 2-knot obtained from a  $(2,n)$ -torus knot by Artin's spinning (cf. [8, 10, 16, 19]).

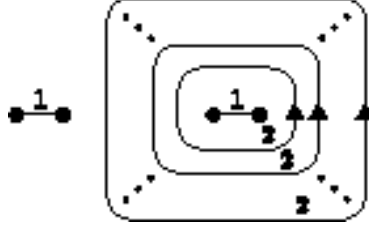


FIGURE 16

**Lemma 15.**  $\text{Braid}(K_n) = 3$  for any odd integer  $n \geq 3$ . If  $n \neq n'$ , then  $K_n$  is not equivalent to  $K_{n'}$ . Hence, there exist an infinite series of 2-knots of braid index 3.

*Proof.* The first Alexander module, the  $\Lambda = \mathbb{Z}[t, t^{-1}]$ -module of the homology  $H_1(\tilde{E}; \mathbb{Z})$  of the universal abelian covering  $\tilde{E} \rightarrow E$  of the knot exterior  $E$ , of the 2-knot  $K_n$  is isomorphic to that of a  $(2,n)$ -torus knot. Thus, we see that  $K_n$  is nontrivial, and that  $K_n$  is not equivalent to  $K_{n'}$  if  $n \neq n'$ . Since any nontrivial 2-knot has braid index greater than 2, we have the result.  $\square$

**Lemma 16.**  $\text{Braid}(K_n \# K_n) = 4$  for any odd integer  $n \geq 3$ .

*Proof.* By Theorem 3 and Lemma 15, we have  $\text{Braid}(K_n \# K_n) \leq 4$ . Using the standard form of a 3-braid 2-knot [16], it is seen that any 3-braid 2-knot is a 1-fusion ribbon 2-knot, that is, it is obtained from a pair of trivial 2-spheres in  $\mathbb{R}^4$  by surgery along a single 1-handle. So its first Alexander module must be cyclic. On the other hand, the first Alexander module of  $K_n \# K_n$  is the direct sum of two copies of  $\Lambda/(f_n)$ , which is not cyclic, where  $f_n$  is the Alexander polynomial of a  $(2,n)$ -torus knot. Hence, we have  $\text{Braid}(K_n \# K_n) > 3$ .  $\square$

*Proof of Corollary 4.* Consider a family  $\{K_n \# K_n\}$  for odd integers  $n \geq 3$ . If  $n \neq n'$ , then  $K_n \# K_n$  and  $K_{n'} \# K_{n'}$  are not equivalent, since their first Alexander modules are not isomorphic. In particular,  $K_3 \# K_3$  is the 2-knot obtained from a granny knot by Artin's spinning.  $\square$

**Question.** Do there exist 2-knots  $K_1$  and  $K_2$  with

$$\text{Braid}(K \# K') < \text{Braid}(K_1) + \text{Braid}(K_2) - 2?$$

In particular, do there exist 2-knots  $K_1$  and  $K_2$  with

$$\text{Braid}(K_1 \# K_2) = \text{Braid}(K_1) = \text{Braid}(K_2) = 3?$$

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