Reflexivity of
Locally Convex Spaces
over Local Fields

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0 Introduction

For any Hilbert space $\mathscr{H}$, the Hermit inner product induces an anti $\mathbb{C}$-linear isometric isomorphism $\mathscr{H} \cong \mathscr{H}^\vee$, and in particular, the canonical $\mathbb{C}$-linear homomorphism $\mathscr{H} \to \mathscr{H}^{\vee\vee}$ is an isometric isomorphism.

Example 0.1. For a set I, we put

$$\ell^2(I) \ := \ \left\{ f \colon I \to \mathbb{C} \ \middle| \ \sum_{n=0}^{\infty} |f(\iota_n)|^2 < \infty, {}^\forall \iota \colon \mathbb{N} \hookrightarrow I \right\}$$

$$\| \cdot \| \colon \ell^2(I) \to [0, \infty) \colon f \mapsto \sqrt{\sum_{i \in I} |f(i)|^2}.$$

Then $(\ell^2(I), \| \cdot \|)$ is a Banach $\mathbb{C}$-vector space admitting a unique structure of a Hilbert space. On the other hand, every Hilbert space is isometrically isomorphic to $(\ell^2(I), \| \cdot \|)$ as a Banach $\mathbb{C}$-vector space for some set I.

Let k be a local field. There are several non-Archimedean analogues of a Hilbert space. One is a strictly Cartesian Banach k-vector space, and another one is a compact Hausdorff flat linear topological O_k-module.

$$(V, \|\cdot\|) \quad ; \quad \begin{array}{l}\text{a Banach } k\text{-vector space, i.e.} \\ \text{a } k\text{-vector space + a complete non-Archimedean norm}\end{array}$$

$$(V, \|\cdot\|) \text{ is } \textit{strictly Cartesian}. \overset{\text{def}}{\Leftrightarrow} V = O \text{ or } \|V\| = |k|$$

Example 0.2. For a set I, we put

$$\begin{aligned} \mathrm{C}_0(I,k) &:= k^{\hat{\oplus} I} = k \otimes_{O_k} \varprojlim_{r \in \mathbb{N}} O_k^{\oplus I}/\varpi_k^r \\ &\cong \left\{ f \colon I \to k \,\middle|\, \lim_{n \to \infty} f(\iota_n) = 0, {}^\forall \iota \colon \mathbb{N} \hookrightarrow I \right\} \end{aligned}$$

$$\|\cdot\| \colon \mathrm{C}_0(I,k) \to [0,\infty) \colon f \mapsto \max_{i \in I} |f(i)| .$$

Then $(\mathrm{C}_0(I,k), \|\cdot\|)$ is a strictly Cartesian Banach k-vector space. On the other hand, every strictly Cartesian Banach k-vector space is isometrically isomorphic to $(\mathrm{C}_0(I,k), \|\cdot\|)$ for some set I.

Remark 0.3. Every Banach k-vector space is homeomorphically (not necessarily isometrically) isomorphic to a strictly Cartesian Banach k-vector space.

$$\begin{aligned} \mathrm{Ban}(k) &:= \begin{pmatrix} \text{the category of Banach } k\text{-vector spaces} \\ \text{and continuous } k\text{-linear homomorphisms} \end{pmatrix} \\ \mathrm{Ban}(O_k) &:= \begin{pmatrix} \text{the category of strictly Cartesian Banach } k\text{-vector} \\ \text{spaces and submetric } k\text{-linear homomorphisms} \end{pmatrix} \end{aligned}$$

$$M \quad ; \quad \text{a topological } O_k\text{-module}$$

$$\begin{array}{cl} & M \text{ is } \textit{linear}. \\ \overset{\text{def}}{\Leftrightarrow} & \left[\begin{array}{l} \text{The set of open } O_k\text{-submodules of } M \text{ forms} \\ \text{a fundamental system of neighbourhoods of } 0. \end{array} \right] \end{array}$$

$$\begin{array}{cl} & M \text{ is a } \textit{chflt } O_k\textit{-module}. \\ \overset{\text{def}}{\Leftrightarrow} & \left[\begin{array}{l} M \text{ is a compact Hausdorff flat linear} \\ \text{topological } O_k\text{-module.} \end{array} \right] \end{array}$$

Example 0.4. For a set I, O_k^I is a chflt O_k-module. On the other hand, every chflt O_k-module is homeomorphically isomorphic to O_k^I for some set I.

$$\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{fl}}(O_k) \;:=\; \left(\begin{array}{c} \text{the category of chflt } O_k\text{-modules and} \\ \text{continuous } O_k\text{-linear homomorphisms} \end{array} \right)$$

$$(V, \|\cdot\|) \quad ; \quad \text{a Banach } k\text{-vector space}$$

$$\begin{aligned} V(1) &:= \{v \in V \mid \|v\| \leq 1\} \\ (V, \|\cdot\|)^{\mathrm{D}} &:= \left\{m \colon V \to k \,\middle|\, \|m(v)\| \leq \|v\|, {}^{\forall} v \in V\right\} \\ &\cong \mathrm{Hom}_{O_k}(V(1), O_k) \end{aligned}$$

We endow $(V, \|\cdot\|)^{\mathrm{D}}$ with the relative topology of $O_k^{V(1)}$ through the embedding

$$(V, \|\cdot\|)^{\mathrm{D}} \hookrightarrow O_k^{V(1)} \colon m \mapsto (m(v))_{v \in V(1)},$$

with respect to which $(V, \|\cdot\|)^{\mathrm{D}}$ is a chflt O_k-module.

$$M \quad ; \quad \text{a chflt } O_k\text{-module}$$

$$M^{\mathrm{D}} := \left\{v \in \mathrm{Hom}_{O_k}(M, k) \,\middle|\, v \text{ is continuous.}\right\}$$

We endow M^{D} with the norm

$$\|\cdot\| \colon M^{\mathrm{D}} \to [0, \infty) \colon v \mapsto \max_{m \in M} |v(m)|,$$

with respect to which M^{D} is a strictly Cartesian Banach k-vector space.

Theorem 0.5 (Iwasawa-type duality for the trivial group, by Schikhof, Schneider, and Teitelbaum)**.**

(i) For any strictly Cartesian Banach k-vector space $(V, \|\cdot\|)$*, the canonical k-linear homomorphism* $(V, \|\cdot\|) \to (V, \|\cdot\|)^{\mathrm{DD}}$ *is an isometric isomorphism.*

(ii) For any chflt O_k*-module M, the canonical* O_k*-linear homomorphism* $M \to M^{\mathrm{DD}}$ *is a homeomorphic isomorphism.*

(iii) For any Banach k-vector space $(V, \|\cdot\|)$*, the canonical k-linear homomorphism* $(V, \|\cdot\|) \to (V, \|\cdot\|)^{\mathrm{DD}}$ *is a homeomorphic isomorphism.*

(iv) The correspondence D *gives contravariant* O_k*-linear equivalences* $\mathrm{Ban}(O_k) \cong \mathrm{Mod}^{\mathrm{ch}}_{\mathrm{fl}}(O_k)$ *and* $\mathrm{Ban}(k) \cong k \otimes_{O_k} \mathrm{Mod}^{\mathrm{ch}}_{\mathrm{fl}}(O_k)$.

Example 0.6. For a set I, the canonical pairing

$$O_k^I \times \mathrm{C}_0(I, k) \to k \colon (\mu, f) \mapsto \int f d\mu := \sum_{i \in I} \mu(i) f(i)$$

yields an isometric k-linear isomorphism $\mathrm{C}_0(I, k) \cong (O_k^I)^{\mathrm{D}}$ and a homeomorphic O_k-linear isomorphism $O_k^I \cong (\mathrm{C}_0(I, k), \|\cdot\|)^{\mathrm{D}}$.

Question 0.7. Is there a category $\mathscr{C}$ containing $\mathrm{Ban}(O_k)$ and $\mathrm{Mod}^{\mathrm{ch}}_{\mathrm{fl}}(O_k)$ on which D extends to a contravariant automorphism $\mathbb{D}$?

We construct an explicit example of a pair $(\mathscr{C}, \mathbb{D})$.

1 Locally Convex Spaces

$$W \quad ; \quad \text{a topological } k\text{-vector space}$$

$$\begin{aligned} & W \text{ is a } \textit{locally convex } k\textit{-vector space}. \\ \overset{\text{def}}{\Leftrightarrow} \quad & W \text{ is linear as a topological } O_k\text{-module}. \end{aligned}$$

Example 1.1. For a Banach k-vector space $(V, \|\cdot\|)$, the underlying topological k-vector space of $(V, \|\cdot\|)$ is a complete locally convex k-vector space.

$$\begin{aligned} W \quad ; \quad & \text{a locally convex } k\text{-vector space} \\ L \quad ; \quad & \text{an } O_k\text{-submodule of } W \end{aligned}$$

$$\begin{aligned} & L \text{ is a } \textit{lattice of } W. \\ \overset{\text{def}}{\Leftrightarrow} \quad & L \text{ is a bounded closed subset generating } W \text{ as a } k\text{-vector space}. \end{aligned}$$

Example 1.2. For a Banach k-vector space $(V, \|\cdot\|)$, $V(1)$ is a lattice of the underlying complete locally convex k-vector space of $(V, \|\cdot\|)$.

Example 1.3. For a chflt O_k-module M, $k \otimes_{O_k} M$ admits a canonical topology with respect to which $k \otimes_{O_k} M$ is a complete locally convex k-vector space and the natural embedding $M \hookrightarrow k \otimes_{O_k} M$ is a homeomorphic O_k-linear isomorphism onto a lattice.

Remark 1.4. A locally convex k-vector space does not necessarily admit a lattice. For example, $k^{\mathbb{N}}$ is a complete locally convex k-vector space admitting no lattice.

$$L \quad ; \quad \text{a topological } O_k\text{-module}$$

L is *adically bounded*.

$\overset{\text{def}}{\Leftrightarrow}$ [Every open subset of L is open with respect to the ϖ_k-adic topology on the underlying O_k-module.]

L is a *locally convex O_k-module*.

$\overset{\text{def}}{\Leftrightarrow}$ [L is adically bounded linear, and the scalar multiplication $L \to L \colon l \mapsto \varpi_k l$ is a homeomorphism onto the closed image.]

Example 1.5.

(i) For a Banach k-vector space $(V, \|\cdot\|)$, $V(1)$ is a locally convex O_k-module.

(ii) Every chflt O_k-module is a locally convex O_k-module.

Example 1.6. For a locally convex k-vector space W, every lattice of W is a locally convex O_k-module. On the other hand, for any locally convex O_k-module L, $k \otimes_{O_k} L$ admits a canonical topology with respect to which $k \otimes_{O_k} L$ is a locally convex k-vector space and the natural embedding $L \hookrightarrow k \otimes_{O_k} L$ is a homeomorphic O_k-linear isomorphism onto a lattice.

$$L \;\;;\;\; \text{a topological } O_k\text{-module}$$

$$\begin{aligned} \mathrm{K}(L) &:= \text{the set of compact } O_k\text{-submodules of } L \\ L^{\mathbb{D}} &:= \left\{\lambda \in \mathrm{Hom}_{O_k}(L, O_k) \;\middle|\; \lambda \text{ is continuous.}\right\} \end{aligned}$$

We endow $L^{\mathbb{D}}$ with the topology generated by the set

$$\left\{ \left\{ \lambda \in L^{\mathbb{D}} \;\middle|\; \lambda(l) - \lambda_0(l) \in \varpi_k^r O_k, {}^{\forall} l \in K \right\} \;\middle|\; (\lambda_0, K, r) \in L^{\mathbb{D}} \times \mathrm{K}(L) \times \mathbb{N} \right\},$$

with respect to which $L^{\mathbb{D}}$ is a Hausdorff flat linear topological O_k-module.

Proposition 1.7. *For any locally convex O_k-module L, $L^{\mathbb{D}}$ is also a locally convex O_k-module. In particular, $\mathbb{D}$ gives a contravariant endomorphism on the category of locally convex O_k-modules and continuous O_k-linear homomorphisms.*

We remark that $\mathbb{D}$ is an extension of D. We construct a full subcategory of the category of locally convex O_k-modules closed under $\mathbb{D}$ on which $\mathbb{D}$ is a contravariant automorphism.

2 Hahn–Banach Theorem

Theorem 2.1 (Hahn–Banach theorem for a seminormed $\mathbb{C}$-vector space, by Banach, Hahn, Helly, and Riesz)**.** *Let W be a seminormed $\mathbb{C}$-vector space, and $W_0 \subset W$ a $\mathbb{C}$-vector subspace. Then every continuous $\mathbb{C}$-linear homomorphism $W_0 \to \mathbb{C}$ extends to a continuous $\mathbb{C}$-linear homomorphism $W \to \mathbb{C}$.*

Theorem 2.2 (Hahn–Banach theorem for a locally convex k-vector space, by Perez-Garcia, Schikhof, and Schneider)**.** *Let W be a locally convex k-vector space, and $W_0 \subset W$ a k-vector subspace. Then every continuous k-linear homomorphism $W_0 \to k$ extends to a continuous k-linear homomorphism $W \to k$.*

Hahn–Banach theorem plays an important role in duality theory. We establish Hahn–Banach theorem for a locally convex O_k-module.

$$\begin{aligned} L \ &; \ \text{a topological } O_k\text{-module} \\ L_0 \ &; \ \text{an } O_k\text{-submodule of } L \end{aligned}$$

$$L_0 \text{ is adically saturated in } L. \overset{\text{def}}{\Leftrightarrow} L_0 \cap \varpi_k L = \varpi_k L_0$$

Theorem 2.3 (Hahn–Banach theorem for a locally convex O_k-module)**.** *Let L be a Hausdorff locally convex O_k-module, and $K \subset L$ a compact adically saturated O_k-submodule. Then the restriction map $L^{\mathbb{D}} \to K^{\mathbb{D}}$ is surjective.*

Corollary 2.4. *Let L be a Hausdorff locally convex O_k-module. Then the canonical O_k-linear homomorphism $L \to L^{\mathbb{D}\mathbb{D}}$ is injective.*

Proof. Let $l \in L \backslash \{0\}$. The smallest adically saturated O_k-submodule $L_0 \subset L$ containing l is a Hausdorff free O_k-module of rank 1. In particular, L_0 is a compact adically saturated O_k-submodule of L such that the canonical O_k-linear homomorphism $L_0 \to L_0^{\mathbb{D}\mathbb{D}}$ is a homeomorphic isomorphism. Applying Theorem 2.3 to L_0, we obtain the surjectivity of the restriction map $L^{\mathbb{D}} \to L_0^{\mathbb{D}}$. Therefore the composite $L_0 \cong L_0^{\mathbb{D}\mathbb{D}} \to L^{\mathbb{D}\mathbb{D}}$ is injective. Thus the image of $l \in L_0 \subset L$ in $L^{\mathbb{D}\mathbb{D}}$ is non-trivial. □

3 Compactly Generated Modules

$$L \quad ; \quad \text{a Hausdorff linear topological } O_k\text{-module}$$

L is *compactly generated*.

$$\overset{\text{def}}{\Leftrightarrow} \left[\begin{array}{l} \text{The canonical continuous bijective } O_k\text{-linear homomorphism} \\ \varinjlim_{K \in \mathrm{K}(L)} K \to L \text{ is a homeomorphism.} \end{array} \right]$$

Example 3.1.

(i) Every first countable complete linear topological O_k-module is compactly generated. In particular, for a Banach k-vector space $(V, \|\cdot\|)$, $V(1)$ is compactly generated.

(ii) Every chflt O_k-module is compactly generated.

Lemma 3.2. *Let L be a compactly generated Hausdorff linear topological O_k-module. Then a subset of $L^{\mathbb{D}}$ is totally bounded if and only if it is equicontinuous.*

Theorem 3.3. *For any compactly generated Hausdorff locally convex O_k-module L, the canonical O_k-linear homomorphism $L \to L^{\mathbb{D}\mathbb{D}}$ is a homeomorphism onto the image.*

Proof. The continuity of $L \to L^{\mathbb{D}\mathbb{D}}$ follows from the equicontinuity of a compact subsets of $L^{\mathbb{D}}$ by the definition of the topology of $L^{\mathbb{D}\mathbb{D}}$. The openness onto the image follows from the Iwasawa-type duality $\mathrm{Ban}(O_k) \cong \mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O_k)$ because every Hausdorff linear topological O_k-module embeds into the direct product of Banach k-vector spaces. □

Theorem 3.4. *For any first countable complete locally convex O_k-module L, the canonical O_k-linear homomorphism $L \to L^{\mathbb{D}\mathbb{D}}$ is a homeomorphism.*

Proof. Since L is first countable and complete, it is compactly generated by Example 3.1 (i), and hence the given homomorphism is a homeomorphism onto the image by Theorem 3.3. The surjectivity follows from Iwasawa-type duality $\mathrm{Ban}(O_k) \cong \mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O_k)$. Indeed, let $\ell \in L^{\mathbb{D}\mathbb{D}}$. Then $\ell \colon L^{\mathbb{D}} \to O_k$ factors through the restriction map $L^{\mathbb{D}} \to K^{\mathbb{D}}$ for some $K \in \mathrm{K}(L)$ by the definition of the topology of $L^{\mathbb{D}}$. Therefore the Iwasawa-type duality $\mathrm{Ban}(O_k) \cong \mathrm{Mod}_{\mathrm{fl}}^{\mathrm{ch}}(O_k)$ ensures that there is an $l \in K \subset L$ whose image in $L^{\mathbb{D}\mathbb{D}}$ coincides with ℓ. $\square$

4 Main Result

The remaining problem is that $\mathbb{D}$ does not preserve the first countability. For this reason, we introduce the dual notion of the first countability.

$$L \quad ; \quad \text{a topological } O_k\text{-module}$$

$$\begin{aligned} & L \text{ is } \textit{reductively } \sigma\textit{-compact}. \\ \overset{\text{def}}{\Leftrightarrow} \quad & L/\varpi_k L \text{ is } \sigma\text{-compact}. \end{aligned}$$

Example 4.1.

(i) For a separable Banach k-vector space $(V, \|\cdot\|)$, $V(1)$ is a first countable reductively σ-compact complete locally convex O_k-module.

(ii) Every separable chflt O_k-module is a first countable reductively σ-compact complete locally convex O_k-module.

(iii) For a separable Banach k-vector space $(V, \|\cdot\|)$, $\mathrm{End}_{O_k}(V(1))$ is a first countable reductively σ-compact complete locally convex O_k-module with respect to the topology of strong convergence (not Banach or chflt).

(iv) For a chflt O_k-module M, $\mathrm{End}^{\mathrm{cont}}_{O_k}(M)$ is a first countable reductively σ-compact complete locally convex O_k-module with respect to the topology of uniform convergence (not Banach or chflt).

Lemma 4.2. *Let L be a complete locally convex O_k-module. If L is first countable, then $L^{\mathbb{D}}$ is reductively σ-compact and complete. In addition if L is reductively σ-compact, then $L^{\mathbb{D}}$ is first countable.*

Theorem 4.3 (Extended Iwasawa-type duality)**.** *The full subcategory $\mathscr{C}$ of the category of locally convex O_k-modules and continuous O_k-linear homomorphisms consisting of first countable reductively σ-compact complete locally convex O_k-modules is closed under $\mathbb{D}$, and the restriction of $\mathbb{D}$ to $\mathscr{C}$ is a contravariant automorphism.*