WAFOM with parameter for higher QMC: Revenge of the algebraic geometry code, Part II.

`(An Introduction to Tylsonian Civilization.)`

Makoto Matsumoto, Hiroshima University

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The point sets are available from Ohori's GitHub;

`http://majiang.github.io/qmc/index.html`

Usually, the speaker should thank to the organizers, but instead I APOLOGIZE for giving such a strange talk.

The talk below is a (Super-) Science (non)-Fiction; non-real if you don't want to believe. (And true if you want to believe.)

(A talk celebrating Makoto Matsumoto's 50th birthday)

1. Tylsonia Planet and Tylsonian: Kazuya Kato (a pure mathematician, arithmetic geometer famous for his study on p-adic Hodge theory) taught me something on Tylsonian civilazation.

A Tylsonian: a tribe similar to prime periodical cicadas (copyright Tsuburaya Pro.): they consider $\mathbb{F}_p := \{0, 1, \ldots, p-1\}$ more natural than $[0, 1)$ for a prime p. Remark: p depends on each tribe of Tylsonian.
For each prime p, called ***p*-adic Tylsonian** tribe.

Tylsonian Mathematics versus Terrestrian. For simplicity, we assume that $p = 2$: explain on 2-adic Tylsonian mathematics.

Common thing: An infinite sequence $b_1, b_2, \ldots, \in \{0, 1\} = \mathbb{F}_2$ is used to denote a "quantity" with infinite precision.

Terrestrian:

$$(b_1, b_2, \ldots) \mapsto 0.b_1 b_2 \cdots = \sum_{i=1}^{\infty} b_i 2^{-i} \in [0, 1).$$

I.e., $\mathbb{F}_2^{\mathbb{N}} \to [0, 1)$. Almost one-to-one (negl. a meaure-0 subset).

Tylsonian:

$$(b_1, b_2, \ldots) \mapsto 0.b_1 b_2 \cdots := \sum_{i=1}^{\infty} b_i t^{-i} \in \mathbb{F}_2[[t^{-1}]].$$

I.e., $\mathbb{F}_2^{\mathbb{N}} \to \mathbb{F}_2[[t^{-1}]]$. Completely one-to-one.

Testing Your understanding on Tylsonian Mathematics (甲).

Q. Tell the difference between the Tylsonian meaning and the Terrestrian meaning of one same notation

$$0.b_1 b_2 b_3 \cdots .$$

A.

In Tylsonian $0.b_1 b_2 b_3 \cdots \in \mathbb{F}_2[[t^{-1}]]$.

In Terrestrian $0.b_1 b_2 b_3 \cdots \in [0, 1)$.

Thus, Tylsonian meaning of "quantity" is in $\mathbb{F}_2[[t^{-1}]]$, while Terrestrian in $[0, 1)$.

Tylsonian Mathematics (2) No carry, no borrow

Tylsonian arithmetics is polynoimial (or formal power series). Thus, Tylsonians are so generous: On Tylsonian planet,

- $1+1=0$. $1=1+(1-1)=(1+1)-1=0-1=-1$. (Note: this is a "physical" law on Tylsonian planet.)
- $-1=1$. When a Tylsonian borrows some money, he/she does not worry to return (better to say, they have NO notion on borrow nor debt nor keeping money in bank, since $1+1=0$).
- No carries among digits in addition in $\mathbb{F}_2[[t^{-1}]]$.

Testing Your understanding on (b-adic) Tylsonian Mathematics (乙).

Q. Describe the transformation

$$T_b : 0.b_1 b_2 b_3 \cdots \mapsto 0.b_2 b_3 b_4 \cdots + 0.b_1 b_1 b_1 \cdots$$

defined by b-adic Tylsonian addition in $\mathbb{F}_2[[t^{-1}]]$, in terms of Terrestrian (mis-)interpretation in $[0, 1)$.

A 甲. It is known as the tent function if $b = 2$.

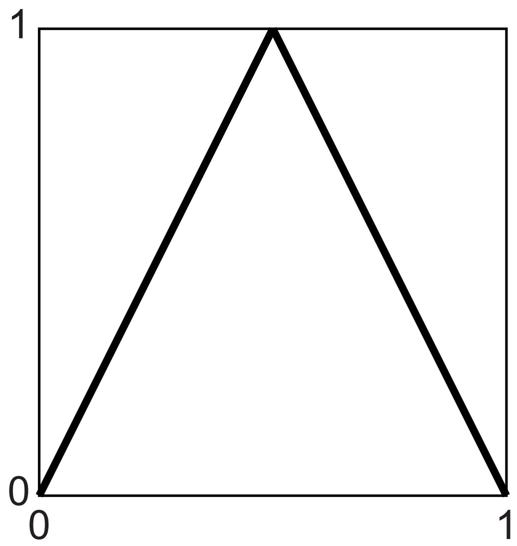


図 1: 2-adic Tent Function

A 甲=Figure 1 : 2-adic Tylsonian linear transformation

$$T_b : 0.b_1b_2b_3\cdots \mapsto 0.b_2b_3b_4\cdots + 0.b_1b_1b_1\cdots$$

when a Terrestrian observes (or better to say “misundersdand” $\mathbb{F}_2[[t^{-1}]]$ as $[0,1)$).

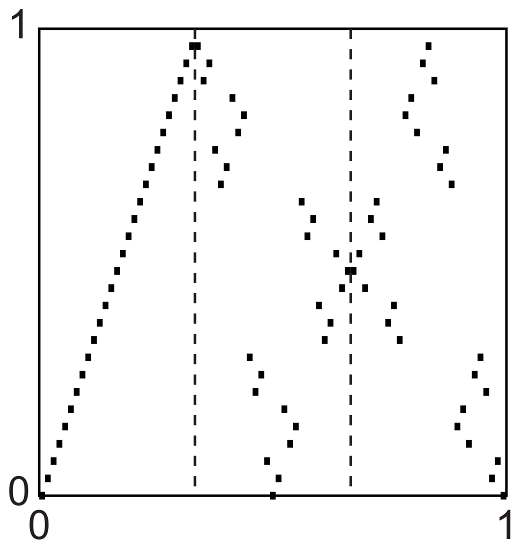


図 2: 3-adic Tent Function

A 乙=Figure 2 : 3-adic Tylsonian linear transformation

$$T_b : 0.b_1 b_2 b_3 \cdots \mapsto 0.b_2 b_3 b_4 \cdots + 0.b_1 b_1 b_1 \cdots$$

when a Terrestrian observes (or better to say "misundersdand" $\mathbb{F}_3[[t^{-1}]]$ as $[0,1)$).

Intermission: Japanimation Madoka-Magica


Two contradicting "feelings" of me in my mind, when I told on Tylsonian civilization to a Terrestrian, and he/she didn't understand.

Magicicada= Prime periodic cicada, implies that the Japanimation PROJECT is Tylsonian.

IMPORTANT REMARK: Tylsonian civilization reached to the notion of the real numbers $\mathbb{R}$ and/or $[0,1)$ and $\mathbb{C}$ and

$$T := \{z \in \mathbb{C} \mid |z| = 1\} \overset{\exp(2\pi\sqrt{-1}x)}{\cong} \mathbb{R}/\mathbb{Z} = [0,1),$$

the Pontryagin duality etc., $[0,1)$ is used to approximate $\mathbb{F}_2[[t^{-1}]]$.

A "Tylsonian Walsh" versus a "Terrestrian Fourier."

Terrestrian Fourier:

$$e(-|-) : \mathbb{R}/\mathbb{Z} \times \mathbb{Z} \to T, \quad (x,n) \mapsto \exp(2\pi\sqrt{-1}xn).$$

$$\ulcorner f : [0,1) = \mathbb{R}/\mathbb{Z} \to \mathbb{R} \lrcorner \mapsto \ulcorner \hat{f} : \mathbb{Z} \to \mathbb{C} \lrcorner$$

where

$$\hat{f}(n) := \int_{[0,1)} f(x)e(x|n)dx, \quad f(-x) = \sum \hat{f}(n)e(x|n).$$

Tylsonian Walsh:

$$e(-|-) : \mathbb{F}_2[[t^{-1}]] \times \mathbb{F}_2[t] \to \{\pm 1\}, \quad (x,k) \mapsto (-1)^{(x\cdot k)}.$$

$$\ulcorner f : [0,1) \overset{\text{by confusion}}{=} \mathbb{F}_2[[t^{-1}]] \to \mathbb{R} \lrcorner \mapsto \ulcorner \hat{f} : \mathbb{F}_2[t] \to \mathbb{R} \lrcorner$$

where $\hat{f}$ is called *the k-th Walsh coefficient*:

$$\hat{f}(k) := \int_{[0,1)} f(x)e(x|k)dx \overset{\text{by confusion}}{=} \int_{x\in\mathbb{F}_2[[t^{-1}]]} f(x)e(x|k),$$

where

$$\begin{aligned}
x &= 0.b_1b_2b_3\cdots = \textstyle\sum_{i=1}^{\infty} b_i t^{-i} \in \mathbb{F}_2[[t^{-1}]],\\
k &= \cdots b_{-2}b_{-1}b_0 = \textstyle\sum_{i=0}^{\text{finite}} b_{-i}t^{i} \in \mathbb{F}_2[t]\\
&= \mathbb{F}_2[t] \overset{\text{misunderstand}}{=} \mathbb{N}\cup\{0\} \ni \cdots b_{-2}2^2 + b_{-1}2^1 + b_0.\\
x\cdot k &:= \text{inner product} := \textstyle\sum_{i=0}^{\infty} b_{i+1}b_{-i}\\
&= \text{the constant term of } xk \in \mathbb{F}_2((t^{-1})).
\end{aligned}$$

Now you know 0.0000001% of Tylsonian civilization.

We shall come back to the Earth.

What Terrestrian calls "Walsh expansion of $f : [0,1) \to \mathbb{R}$" is:

$$f(x) = \sum_{k \in \mathbb{N} \cup \{0\}} \hat{f}(k) \mathbf{wal}_k(x) \stackrel{\text{inter-universe}}{=} \sum_{k \in \mathbb{F}_2[t]} \hat{f}(k)(-1)^{x \cdot k},$$

where $x \in [0,1) \stackrel{\text{inter-universal identification}}{=} \mathbb{F}_2[[t^{-1}]]$.

I have forgotten the aim of this talk.

The aim is a numerical integration by QMC:

$$I(f) := \int_{[0,1)^s} f(x)dx \sim I(f;P) := \frac{1}{N} \sum_{x \in P} f(x)$$

where $P \subset [0,1)^s$ is a well-chosen finite point set with $N = \#(P)$.

Figure of Merit, or Koksma-Hlawka type inequality
QMC-Integration Error is bounded:

$$|I(f) - I(f; P)| < C_s \cdot V(f) \cdot D(P),$$

so we want to find:

1. A good definition of "Variance" $V(f)$ of f (but I omit them),
2. A good definition of "Discrepancy from the ideal uniformity" $D(P)$ of P, sometimes called a *Figure of Merit* of P, and
3. Point sets P with small $D(P) < O(1/N)$, for various (increasing) N.

Walsh Figure of Merit (WAFOM).

IMPORTANT REMARK:

Please remember "WAFOM" everytime you sneeze (Niesen).

WAFOM(P) is a ridiculously simplified version of Dick's $W_\alpha(P)$.

We should have named it "DIck Figure of Merit=DIFOM," as Owen suggested me. But it seems hard to sneeze "DIFOM."

Theorem 1 *(Dick, M-Saito-Motoba, Yoshiki, Suzuki, ...)*

1. $|I(f) - I(f;P)| < C_s \cdot V_{Dick}(f) \cdot \mathrm{WAFOM}(P)$.

2. $\mathrm{WAFOM}(P) \sim O(N^{-C(\log N)/s})$ *is (easily) achievable.*

(s appeared in $f : [0,1)^s \to \mathbb{R}$*.)*

Definition of WAFOM, and generousity of Tylsonian.

Tylsonian does not care about truncation:

$$[0,1) = \mathbb{F}_2^{\mathbb{N}} \xrightarrow{\text{truncation at } n} \mathbb{F}_2^n, \quad 0.b_1b_2b_3\cdots \mapsto 0.b_1b_2\cdots b_n$$

because it is a homomorphism and thus no accumulation of errors, and it has a pseudo-inverse

$$\mathbb{F}_2^n \to \mathbb{F}_2^{\mathbb{N}} \quad (\to \mathbb{F}_2^n).$$

So I DO identify

$$\mathbb{F}_2^n = \mathbb{F}_2^{\mathbb{N}} = [0,1).$$

Practioners neither care: one uses sigle precision for QMC, that means $n = 24$.

Definition of WAFOM (continued)

By the above identification,

$$P \subset (\mathbb{F}_2^n)^s = [0,1)^s = M_{n,s}(\mathbb{F}_2).$$

For $A \in M_{n,s}(\mathbb{F}_2)$, define its Hamming weight

$$H(A) := \sum a_{ij}, \text{addition in Terrestrian sense.}$$

and its *Dick*-weight

$$\mu(A) := \sum j \cdot a_{ij}, \text{addition in Terrestrian sense.}$$

Definition (Dick, MSM, and Ohori-Yoshiki for parametered)

$$\mathrm{WAFOM}(P) := \sum_{A \in P^\perp - \{0\}} 2^{-\mu(A)}.$$

WAFOM with derivation sensitivity parameter δ by Ohori-Yoshiki:

$$\mathrm{WAFOM}_\delta(P) := \sum_{A \in P^\perp - \{0\}} 2^{-\mu(A) - \delta H(A)}.$$

WAFOM with Derivation Sensitivity Parameter δ.

Ohori-Yoshiki proved:

$$|I(f) - I(f;P)| < C_{s,\delta} \cdot V_\delta(f) \cdot \mathrm{WAFOM}_\delta(P).$$

They found that by increasing δ, one can find a good WAFOM_δ point set for high dimensions such as $s \sim 16$ (gave an algorithm to choose a reasonable δ according to s).

The greater the value of δ, the easier to find a point set, at the cost that the integrand function should be the more smooth ($V_\delta(f)$ is the more sensitive to the norms of higher partial derivatives of f).

t-value by Sobol and Niederreiter

I would have defined t-value of P (which is THE big brother of WAFOM), if I might have used Tylsonian mathematical language. (Note: the famous book by Dick-Pillichshammer has now Tylsonian translation consisting of only 20 pages.)

Remark

- Selection by the t-value works for even non-smooth functions.
- t-value takes only a non-negative integer, in grading point sets.
- WAFOM is finer; it takes a non-negative real number in grading. Can be used to select the best one from those sharing the same t-value. (Harase's idea, but the chosen point sets I refer to as Ohori-WAFOM.)

Experiments on MVN integration

(Remark: MVN=MultiVariate Normal function)

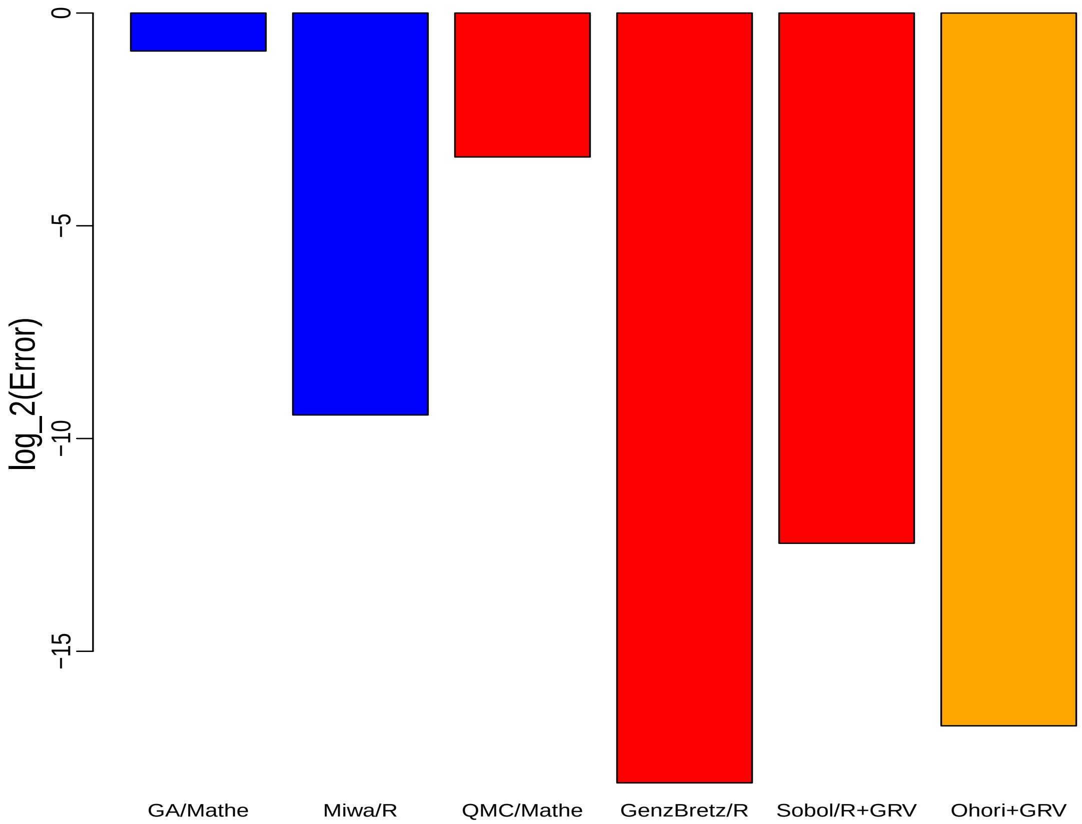
For a positive constant C and a symmetric positive definite $s \times s$ matrix A with diagonals $a_{ii} = 1$, consider the following s-dimensional integration (MVN):

$$I(\mathbf{b}) := \int_{(-\infty, b_1] \times \cdots \times (-\infty, b_s]} \frac{1}{C} \exp\left(-\frac{1}{2}{}^t\mathbf{x}A\mathbf{x}\right) d\mathbf{x}.$$

We chose $b_i := 0$ for simplicity.

We used Gaussian Reduction of Variance (GRV); which seems well-known to the specialists (but we don't know how to refer to): use a Probit transformation to each variable; so that A is replaced with $A - \mathrm{diag}(c_1, \ldots, c_s)$; choose c_i as large as possible, keeping the semi-positivity. This study is due to ダンパ et. al.

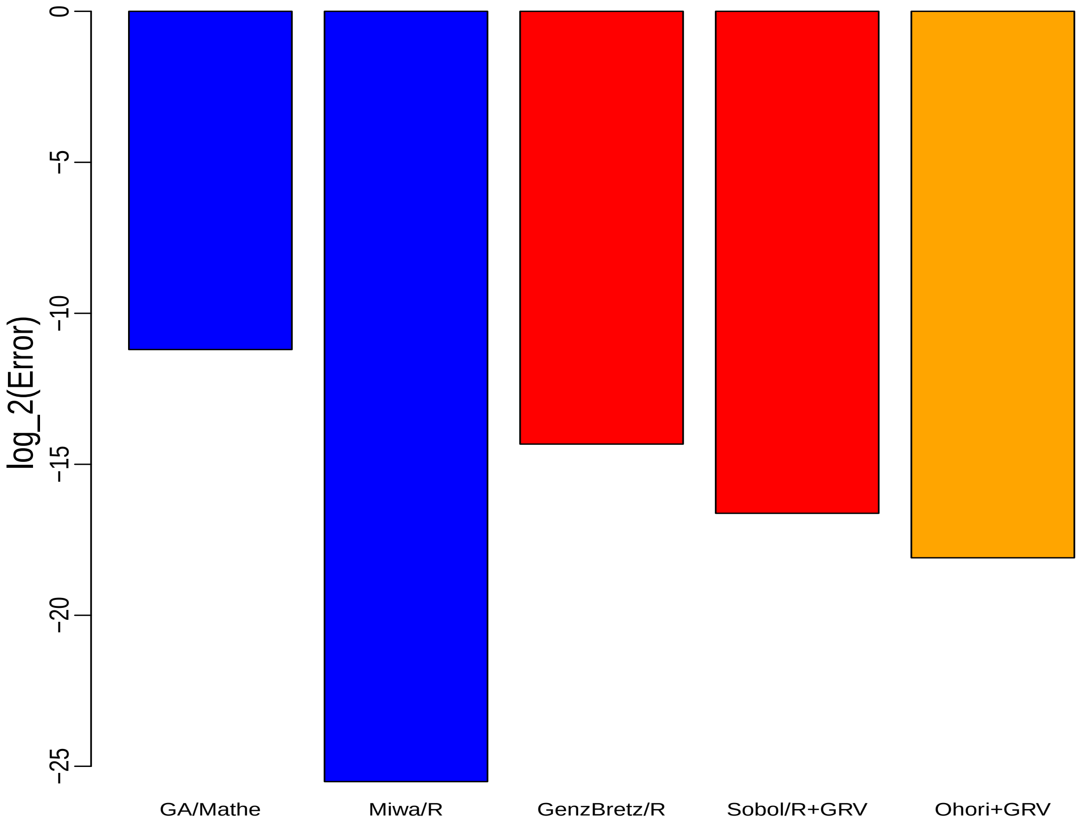
The $\log_2$ of the absolute errors for 6 methods. ($s = 13$, $a_{ij} = \frac{1}{5s}$, 2^{20} points for QMCs.) Ohori-WAFOM is the second best for $s = 13$. We omit the graphs, but often Ohori-WAFOM performs better than GenzBretz for other dimensions $s \neq 13$.



GA/Mathe
Miwa/R
QMC/Mathe
GenzBretz/R
Sobol/R+GRV
Ohori+GRV

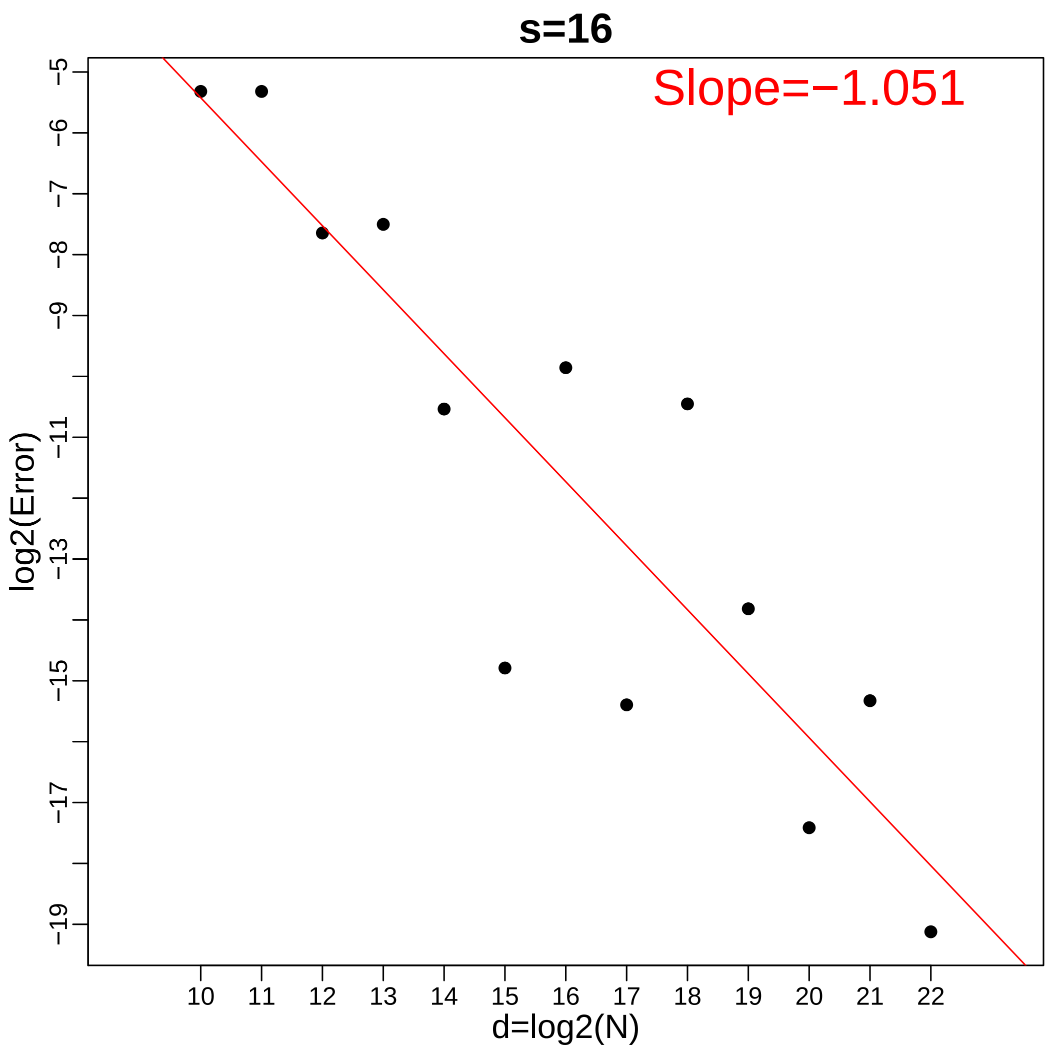
An A from Miwa-Heyter-Kuriki:

The $\log_2$ of the absolute errors for 5 methods. $(s = 8, a_{ij} = -\frac{1}{s})$
Ohori-WAFOM performs better than GenzBretz. Miwa is the best for $s = 8$. Note that Miwa has complexity of $O(s!)$.



Higher Order Convergence of Ohori-WAFOM.

$s = 16$, $a_{ij} = \frac{1}{5s}$



Revenge of the algebraic geometry code Part I: We used Niederreiter-Xing (NX) point sets as a prototype. NX comes from the algebraic geometry code (AGC). Note that AGC has never been used since there is no efficient decoding algorithm (except for the case genus zero).

Revenge of the algebraic geometry code Part II: Harase, Ohori: applied linear scrambling (LS) to Niederreiter-Xing point sets. LS preserves t-value, and varies WAFOM$_\delta$. Choose the best point set w.r.t. WAFOM$_\delta$ by random LSs.

- NX: elites.
- Ohori-WAFOM: elites among elites, high-dimensional.

Concluding remarks

- Japanimation is important to understand Tylsonian mathematics. I recommend you to watch: it takes only 6 hours to see the whole Madoka-Magica story.
- We are not alone in the inter-universal sense.
- 「弥陀の五劫思惟の願をよくよく案ずれば、ひとえに親鸞一人がためなりけり。されば、そくばくの業をもちける身にてありけるを、たすけんとおぼしめしたちける本願のかたじけなさよ」

  from 歎異抄 (13th Century) written by a Japanese monk 唯円: the letters mean "Only (唯) MADOKA （円）saves." I guess that MADOKA is AMIDA-NYORAI, the Buddhism saviour.

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Sorry (or thank you, depending on each audience) for listening.