Jorgensen’s picture of punctured torus groups and its refinement

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Dedicated to the sixtieth birthday of Professor Shin’ichi Suzuki

Abstract

This is a sequel of the paper (J(BbibitemAkiyoshi-Sakuma. We give a description of Jorgensen’s theorem on the Ford domains of punctured torus groups from the 3-dimensional view point, and propose conjectures which refine his theorem and relate it to the bending laminations of the convex core boundaries of the quotient hyperbolic manifolds. We also present partial results and experimental results supporting the conjectures.

1 Introduction

Let $M(\Gamma) = \mathbb{H}^d/\Gamma$ be a cusped hyperbolic manifold. In (J(BciteAkiyoshi-Sakuma we defined a decomposition $\Delta(\Gamma)$ of (a certain subspace of) the convex core $M_0(\Gamma) = \mathscr{C}(\Lambda(\Gamma))/\Gamma$ of $M(\Gamma)$ using the convex hull construction of Epstein and Penner [8], and studied its relation with the structure of the boundary of the convex core. In particular we described the relation between $\Delta(\Gamma)$ and the bending lamination $|pl(\Gamma)|$ in the 3-dimensional case, and showed that $\Delta(\Gamma)$ determines $|pl(\Gamma)|$ if Γ is a quasifuchsian punctured torus group, i.e., a Kleinian group satisfying the following conditions.

1. Γ is freely generated by two isometries whose commutator is parabolic.
2. The domain of discontinuity Ω consists of exactly two simply connected components $\Omega^{\pm}$, whose quotients $\Omega^{\pm}/\Gamma$ are each homeomorphic to a punctured torus T.

In this paper, we study the following problem.

Problem 1.1 *Does $|pl(\Gamma)|$ determine the combinatorial structure of $\Delta(\Gamma)$ for a punctured torus group Γ?*

Jorgensen's work [10] which clarifies the beautiful structure of the Ford domains of quasifuchsian punctured torus groups gives us the starting point, because the Ford domain of a Kleinian group Γ with parabolic transformations can be regarded as a geometric dual to a certain subcomplex $\Delta_{\mathbb{E}}(\Gamma)$ of $\Delta(\Gamma)$ (see Section 2 and [3, Section 10]). The works of Keen and Series [14],[15],[16] which give detailed studies of $pl(\Gamma)$ for quasifuchsian punctured torus groups Γ also hold the key to the study of the above problem. The problem may be regarded as a generalization of the problem to compare the works of Jorgensen with those of Keen and Series.

This paper is organized as follows. In Section 2, we recall the definition of the Ford domain, $Ph(\Gamma)$, and describe the duality between $Ph(\Gamma)$ and the ideal polyhedral complex $\Delta_{\mathbb{E}}(\Gamma)$ which arises from the Epstein-Penner convex hull construction in the Minkowski space. We also recall the definition of the EPH-decomposition $\Delta(\Gamma)$ as a natural extension of $\Delta_{\mathbb{E}}(\Gamma)$. In Sections 3-6, we explain Jorgensen's results and related results from our view point. After giving a quick review to the works of Keen and Series on bending laminations of quasifuchsian punctured torus groups in Section 7, we propose a few conjectures expecting a positive answer to the problem in Section 8. In the last Section 9, we present partial results and experimental results supporting the conjectures.

Though this paper is a sequel of [3], it is self-contained and can be read independently. In particular, it is our pleasure if this article is helpful for those readers who want to understand the beautiful work of Jorgensen [10], which is included in this proceedings. Since the proofs of the results in this paper are based on the arguments developed in the work announced in [4], they will be included in the forthcoming paper [5] which gives the proofs to the announced work.

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2 Punctured torus groups, Ford domains and EPH-decompositions

Let T be the topological (once) punctured torus. A *marked punctured torus group* is the image of a discrete faithful representation $\rho : \pi_1(T) \to PSL(2,\mathbb{C})$ satisfying the following condition:

- If ω is represented by a loop around the puncture, then $\rho(\omega)$ is parabolic.

Two marked punctured torus groups $\Gamma = \rho(\pi_1(T))$ and $\Gamma' = \rho'(\pi_1(T))$ are *equivalent* if ρ is conjugate to ρ' by an element of $PSL(2,\mathbb{C})$.

A *marked quasifuchsian punctured torus group* is a marked punctured torus group Γ such that the domain of discontinuity $\Omega(\Gamma)$ consists of exactly two simply connected components $\Omega^{\pm}(\Gamma)$, whose quotients $\Omega^{\pm}(\Gamma)/\Gamma$ are each homeomorphic to T. We employ a sign convention so that there is an orientation-preserving homeomorphism f from $T\times[-1,1]$ to the quotient manifold $\overline{M}(\Gamma)=(\mathbb{H}^3\cup\Omega(\Gamma))/\Gamma$ such that $f(T\times\{\pm 1\})=\Omega^{\pm}(\Gamma)/\Gamma$ and that the isomorphism $f_*:\pi_1(T\times[-1,1])=\pi_1(T)\to\pi_1(M(\Gamma))=G<PSL(2,\mathbb{C})$ is identified with ρ. Since such a homeomorphism is unique up to isotopy, we can identify the topological triple $(\overline{M}(\Gamma),\Omega^-(\Gamma)/\Gamma,\Omega^+(\Gamma)/\Gamma)$ with $(T\times[-1,1],T\times\{-1\},T\times\{1\})$. The *quasifuchsian punctured torus space* $\mathcal{QF}$ is the space of the equivalence classes of marked quasifuchsian punctured torus groups. As a consequence of Minsky's celebrated theorem [19], the space of marked punctured torus groups is equal to the closure $\overline{\mathcal{QF}}$ of $\mathcal{QF}$ in the $PSL(2,\mathbb{C})$-representation space of $\pi_1(T)$.

We now recall the definition of the Ford domain of a punctured torus group $\Gamma=\rho(\pi_1(T))$, and introduce a notion of the *Ford complex*. To this end, we normalize the group Γ by a conjugation in $PSL(2,\mathbb{C})$ so that the stabilizer Γ_∞ in Γ of the point ∞ of the upper-half space model is the infinite cyclic group generated by a parabolic transformation $\rho(\omega)$, where ω is represented by a puncture. Then for each element $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$ of $\Gamma-\Gamma_\infty$, we have $A(\infty)\neq\infty$, or equivalently, $c\neq 0$, and the *isometric circle* $I(A)$ of A is defined by

$$I(A)=\{z\in\mathbb{C}\mid |A'(z)|=1\}=\{z\in\mathbb{C}\mid |cz+d|=1\}.$$

$I(A)$ is the circle in the complex plane whose center is $-d/c=A^{-1}(\infty)$ and has radius $1/|c|$. The *isometric hemisphere* $Ih(A)$ is the hyperplane of the upper half-space $\mathbb{H}^3$ bounded by $I(A)$. We denote by $P(\Gamma)$ (resp. $Ph(\Gamma)$) the subset of the complex plane (resp. the upper half-space) which consists of all points lying exterior to each of the isometric circles (resp. isometric hemispheres) defined by Γ. These symbols are slightly different from those in [10], where the same sets are denoted by $\widetilde{P}(\Gamma)$ and $\widetilde{Ph}(\Gamma)$ respectively.

Definition 2.1 (1) We call $Ph(\Gamma)$ the *Ford domain* of Γ.

(2) The *Ford complex*, $\mathrm{Ford}(\Gamma)$, of Γ is the subcomplex of the quotient manifold $\overline{M}(\Gamma)$ obtained as the closure of the image of $\partial Ph(\Gamma)$.

In order to describe a geometric meaning of the Ford domain, we pick a small horoball, H_∞, centered at ∞ which projects to a horospherical neighborhood of the (main) cusp in the quotient hyperbolic manifold $M(\Gamma)$. Then H_∞ is precisely invariant by (Γ,Γ_∞), that is, for any element $A\in\Gamma$, $A(H_\infty)\cap H_\infty\neq\emptyset$ if and only if $A\in\Gamma_\infty$. Then for each element $A\in\Gamma-\Gamma_\infty$, the isometric hemisphere $Ih(A)$ is equal to the set of points in $\mathbb{H}^3$ which are equidistant from H_∞ and $A^{-1}(H_\infty)$. This implies that $Ph(\Gamma)$ can be regarded as the "Dirichlet domain of Γ centered at ∞", because

$$Ph(\Gamma)=\{x\in\mathbb{H}^3\,|\,d(x,H_\infty)\leq d(x,AH_\infty)\text{ for any }A\in\Gamma\}.$$

Thus $Ph(\Gamma)$ is a "fundamental domain of Γ modulo Γ_∞" in the sense that the intersection of $Ph(\Gamma)$ with a fundamental domain of Γ_∞ is a fundamental domain of Γ. As is noted in [7, Section 4], it is more natural to work with the quotient $Ph(\Gamma)/\Gamma_\infty$ in $\mathbb{H}^3/\Gamma_\infty$. In fact the hyperbolic manifold $M(\Gamma)$ is obtained from $Ph(\Gamma)/\Gamma_\infty$ by identifying pairs of faces by isometries.

The above geometric description of the Ford domain implies that the Ford complex is equal to the cut locus of a horospherical neighborhood of the cusp of $M(\Gamma)$, that is, $Ford(\Gamma) \cap M(\Gamma)$ consists of the points in $M(\Gamma)$ which have more than two shortest geodesics to a fixed horospherical neighborhood. Here is an intuitive description: Take a horospherical neighborhood of the cusp of $M(\Gamma)$ and let it expand. We regard the horospherical neighborhood as a balloon which is gently expanded, coming to rest where it meets itself. Then the collision locus is equal to $Ford(\Gamma) \cap M(\Gamma)$.

The above description of the Ford complex $Ford(\Gamma)$ yields an ideal polyhedral complex, $\Delta_{\mathbb{E}}(\Gamma)$, dual to $Ford(\Gamma)$ as follows. Let $\widetilde{Ford}(\Gamma)$ be the 2-dimensional complex in the hyperbolic space obtained as the inverse image of $Ford(\Gamma) \cap M(\Gamma)$. Let p be a vertex of $\widetilde{Ford}(\Gamma)$. Then p is the intersection of at least three isometric hemispheres, and hence it is equidistant from at least four horoballs in the orbit ΓH_∞. We regard the ideal polyhedron spanned by the centers of these horoballs as the geometric *dual* to the vertex p. Similarly, for each edge (resp. face) of $\widetilde{Ford}(\Gamma)$, we can associate an ideal polygon (an ideal edge) as its geometric dual. The family of these ideal polyhedra, ideal polygons and ideal edges dual to the cells of $\widetilde{Ford}(\Gamma)$ compose a Γ-invariant ideal polyhedral complex embedded in $\mathbb{H}^3$. This ideal polyhedral complex descends to an ideal polyhedral complex embedded in $M(\Gamma)$; this is the desired $\Delta_{\mathbb{E}}(\Gamma)$.

Following the argument of Epstein and Penner [7, Section 10], we explain that $\Delta_{\mathbb{E}}(\Gamma)$ arises from the Epstein-Penner convex hull construction in the Minkowski space. Let $\mathbb{E}^{1,3}$ be the 4-dimensional Minkowski space with the Minkowski product

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Then

$$\mathbb{H}^3 = \{x \in \mathbb{E}^{1,3} \mid \langle x, x \rangle = -1,\ x_0 > 0\}$$

together with the restriction of the Minkowski product to the tangent space gives a hyperboloid model of the 3-dimensional hyperbolic space. Any horoball H in this model is represented by a vector, v, in the positive light cone (i.e., $\langle v, v \rangle = 0$ and $v_0 > 0$) as

$$H = \{x \in \mathbb{H}^3 \mid \langle v, x \rangle \geq -1\}.$$

The center of the horoball H corresponds to the ray thorough v, and as v moves away from the origin along the ray, the horoball contracts towards the center of the horoball. Let v_∞ denote the light-like vector representing the horoball H_∞. Then its orbit Γv_∞ is the set of light-like vectors corresponding to the horoballs in ΓH_∞.

Let $\mathscr{C}$ be the closed convex hull of Γv_∞ in $\mathbb{E}^{1,3}$. Now consider the ideal polyhedron $\langle x_0, x_1, \cdots, x_k\rangle$ in $\mathbb{H}^3$ which is dual to a vertex p of $\widetilde{Ford}(\Gamma)$. Then there are horoballs $H_{x_0}, H_{x_1}, \cdots, H_{x_k}$ in the orbit ΓH_∞, such that:

1. H_{x_i} is centered at x_i.
2. $d(p, H_{x_0}) = d(p, H_{x_1}) = \cdots = d(p, H_{x_k}) = d(p, \Gamma H_\infty)$.
3. $d(p, H') > d(p, \Gamma H_\infty)$ for any horoball H' in $\Gamma H_\infty - \{H_{x_0}, H_{x_1}, \cdots, H_{x_k}\}$.

Let v_{x_i} be the light-like vector representing the horoball H_{x_i}. After coordinate change, we may assume the vertex p corresponds to the vector $(1,0,0,0)$ in the hyperboloid model. Then the points $v_{x_0}, v_{x_1}, \cdots, v_{x_k}$ lie in a horizontal hyperplane $W : x_0 =$ constant, because of the second condition. Moreover, by the third condition, all points in $\Gamma v_\infty - \{v_{x_0}, v_{x_1}, \cdots, v_{x_k}\}$ lie above the hyperplane W. Hence we see that the hyperplane W is a support plane of the convex hull $\mathscr{C}$ (i.e., $\mathscr{C}$ is contained in one of the two closed half-space bounded by W), and $W \cap \partial\mathscr{C}$ in the polyhedron $\langle v_{x_0}, v_{x_1}, \cdots, v_{x_k}\rangle$. In other words, $\langle v_{x_0}, v_{x_1}, \cdots, v_{x_k}\rangle$ is a (top-dimensional) face of $\partial\mathscr{C}$. Moreover, it is *Euclidean* in the sense that the restriction of the Minkowski product to the hyperplane W is positive-definite. The ideal polyhedron $\langle x_0, x_1, \cdots, x_k\rangle$ dual to p is equal to the image of the Euclidean face $\langle v_{x_0}, v_{x_1}, \cdots, v_{x_k}\rangle$ of $\partial\mathscr{C}$ by the radial projection form the origin to $\mathbb{H}^3$. In conclusion, the ideal polyhedral complex $\Delta_{\mathbb{E}}(\Gamma)$ is obtained as follows. Consider the collection of faces of $\partial\mathscr{C}$ which has a Euclidean support plane, that is, the collection of the subset of $\mathbb{E}^{1,3}$ which is of the form $W \cap \mathscr{C}$ for some Euclidean support plane W of $\mathscr{C}$. Then their images by the radial projection compose a Γ-invariant ideal polyhedral complex embedded in $\mathbb{H}^3$, and $\Delta_{\mathbb{E}}(\Gamma)$ is equal to its image in $M(\Gamma)$ (see [3, Section 10]).

Though $\Delta_{\mathbb{E}}(\Gamma)$ has the nice geometric meaning that it is dual to the Ford complex, its underlying space $|\Delta_{\mathbb{E}}(\Gamma)|$ looks far from nice. In general, it is not convex and is strictly smaller than the convex core $M_0(\Gamma)$. This is because we take only Euclidean faces of $\partial\mathscr{C}$ into account in the construction of $\Delta_{\mathbb{E}}(\Gamma)$. If the group Γ were a finitely generated Kleinian group of *cofinite volume* with parabolic transformations, then as is proved by Epstein and Penner [7], the above construction gives a finite ideal polyhedral decomposition of the whole quotient hyperbolic manifold. Moreover, every face of $\partial\mathscr{C}$ is Euclidean and hence each piece of the decomposition admits a natural Euclidean structure. Thus it is called the *Euclidean decomposition*. However, in general case, $\partial\mathscr{C}$ can have non-Euclidean faces. So, it is natural to try to construct an ideal polyhedral complex by taking all faces of $\partial\mathscr{C}$ into account. This was made explicit in [3], and we call it the EPH-decomposition and denote it by $\Delta(\Gamma)$. Here the letters E, P and H, respectively, stand for Euclidean (or elliptic), parabolic and hyperbolic.

Finally, we give a brief description of $\Delta(\Gamma)$. (Those who are interested only in Jorgensen's work can skip this part.) To this end, we point out the following troublesome phenomena which we must be careful about.

1. The cellular structure of $\partial\mathscr{C}$ is not necessarily locally finite, and we must be careful about the definition of a face (see Remark [3, Remark 2.12]). This forces us to introduce the notion of a *facet* by refining the notion of a face (see [3, Definition 2.11]).
2. Some part of $\partial\mathscr{C}$ may be "invisible" from the origin, that is, there may be a point of $\partial\mathscr{C}$ such that the line segment between the origin and the point contains some other points of $\partial\mathscr{C}$. So, we need to consider only the *visible* facets of $\partial\mathscr{C}$, i.e., those facets whose affine hulls do not contain the origin (see [3, Definition 4.9 and Lemma 4.10]).

The *EPH*-decomposition of $\Delta(\Gamma)$ is defined to be the image of visible open facets of $\partial\mathscr{C}$ in $M(\Gamma)$ (see [3, Definition 4.18]). We call a member of $\Delta(\Gamma)$ an *open facet* of $\Delta(\Gamma)$. The support $|\Delta(\Gamma)|$ is the union of the open facet of $\Delta(\Gamma)$. Then the following are proved in [3, Proposition 4.15 and Corollary 1.1].

1. $|\Delta(\Gamma)|$ is the disjoint union of the open facets of $\Delta(\Gamma)$.
2. $|\Delta(\Gamma)|$ is equal to the convex core $M_0(\Gamma)$ minus the support of the bending lamination (if Γ is a quasifuchsian punctured torus group).

At the end of this section, we note that $Ford(\Gamma)\cap M(\Gamma)$ is a *spine* of $M(\Gamma)$, that is, it is is a strong deformation retract of $M(\Gamma)$. Moreover, this spine is *canonical* in the sense that it is uniquely determined from the cusped hyperbolic manifold $M(\Gamma)$. We can apply the same construction to every cusped hyperbolic manifold, and in the special case when the manifold is of finite volume and has only one cusp (e.g. the complement of a hyperbolic knot), the combinatorial structure of the Ford complex is a complete invariant of the underlying topological 3-manifold by virtue of the Mostow rigidity theorem. So, the study of the Ford complex has an important meaning for the 3-manifold theory and the knot theory, too.

3 Jorgensen's theorem for quasifuchsian punctured torus groups (I)

In this section, we explain Jorgensen's theorem which describes the combinatorial structures of the Ford domains of quasifuchsian punctured torus groups [10]. Before presenting the precise statement of Jorgensen's theorem, we give a brief intuitive description of the idea. Let Γ be a quasifuchsian punctured torus group. Then the quotient manifold $\overline{M}(\Gamma)$ is identified with $T\times[-1,1]$. Since $P(\Gamma)\subset\mathbb{C}$ is a fundamental domain of the action of Γ on $\Omega(\Gamma)=\Omega^-(\Gamma)\cup\Omega^+(\Gamma)$, modulo Γ_∞, $P(\Gamma)$ is a disjoint union of $P^-(\Gamma)$ and $P^+(\Gamma)$ where $P^\pm(\Gamma)=P(\Gamma)\cap\Omega^\pm(\Gamma)$. The the first assertion of Jorgensen's theorem is that each $P^\pm(\Gamma)$ is simply connected. This implies that the image of $\partial P^\pm(\Gamma)$ in $T\times\{\pm1\}$ is a spine of the punctured torus T. If Γ is fuchsian, then these two spines are identical, and the Ford complex $Ford(\Gamma)$ is equal to the product of the spine with the interval $[-1,1]$. In general, these two spines are not isotopic to each other. However, they are related by a canonical sequence of Whitehead moves. (This is the main point why the punctured torus is so special.) The "trace" of the canonical sequence of Whitehead moves form a spine of $T\times[-1,1]$. The main assertion of Jorgensen's theorem is that this spine is isotopic to the Ford complex $Ford(\Gamma)$. Thus we can say that $Ford(\Gamma)$ records the history of how the two boundary spines evolved.

Now let's give the precise statement. We begin by recalling basic topological facts on the punctured torus T. To this end, we identify T with the quotient space $(\mathbb{R}^2-\mathbb{Z}^2)/\mathbb{Z}^2$. A simple loop in T is said to be *essential*, if it bounds neither a disk nor a once-punctured disk. Similarly, a simple arc in T having the puncture as end points is said to be *essential*, if it does not cut off a disk with a point on the boundary removed. Then the isotopy classes of essential simple loops (resp. essential simple arcs) in T are in one-to-one correspondence with $\hat{\mathbb{Q}}:=\mathbb{Q}\cup\{1/0\}$: A representative of the isotopy class corresponding to $r\in\hat{\mathbb{Q}}$ is the projection of a line in $\mathbb{R}^2$ (the line being disjoint from $\mathbb{Z}^2$ for the loop case, and intersecting $\mathbb{Z}^2$ for the arc case). The element $r\in\hat{\mathbb{Q}}$ associated to a circle or an arc is called its *slope*. The representative of the isotopy class of an essential arc of slope r is denoted by β_r.

Consider the ideal triangle in the hyperbolic plane $\mathbb{H}^2=\{z\in\mathbb{C}\mid \Im(z)>0\}$ spanned by the ideal vertices $\{0/1,1/1,1/0\}$. Then the translates of this ideal triangle by the action of $SL(2,\mathbb{Z})$ form a tessellation of $\mathbb{H}^2$. This is called the *modular diagram* or the *Farey tessellation* and is denoted by $\mathscr{D}$. The abstract simplicial complex having the combinatorial structure of $\mathscr{D}$ is also denoted by the same symbol. The set of (ideal) vertices of $\mathscr{D}$ is equal to $\hat{\mathbb{Q}}$, and a typical (ideal) triangle σ of $\mathscr{D}$ is spanned by $\{\frac{p_1}{q_1},\frac{p_1+p_2}{q_1+q_2},\frac{p_2}{q_2}\}$ where $\begin{pmatrix}p_1 & p_2\\ q_1 & q_2\end{pmatrix}\in SL(2,\mathbb{Z})$.

Let $\sigma=\langle r_0,r_1,r_2\rangle$ be a triangle of $\mathscr{D}$. Then the essential arcs β_{r_0}, β_{r_1}, β_{r_2} are mutually disjoint, and their union determines a *topological ideal triangulation* $\mathrm{trg}(\sigma)$ of T, in the sense that T cut open along $\beta_{r_0}\cup\beta_{r_1}\cup\beta_{r_2}$ is the disjoint union of two 2-simplices with all vertices deleted. Let $\mathrm{spine}(\sigma)$ be a 1-dimensional cell complex embedded in T which is dual to the 1-skeleton of $\mathrm{trg}(\sigma)$. Then $\mathrm{spine}(\sigma)$ consists of two vertices and three edges γ_i $(i=0,1,2)$, such that γ_i intersects the 1-skeleton of $\mathrm{trg}(\sigma)$ transversely precisely at a point of $\mathrm{int}\beta_{r_i}$. Note that $\mathrm{spine}(\sigma)$ is a deformation retract of T and hence is a *spine* of T. We define the *slope* of an edge γ_i of $\mathrm{spine}(\sigma)$ to be $r_i\in\hat{\mathbb{Q}}$, the slope of the ideal edge β_{r_i} of $\mathrm{trg}(\sigma)$ dual to γ_i.

Let $\tau=\langle r_0,r_1\rangle$ be an edge of $\mathscr{D}$. Then the union $\beta_{r_0}\cup\beta_{r_1}$ determines a *topological ideal polygonal decomposition* of T, in the sense that T cut open along it is homeomorphic to a quadrilateral with all vertices deleted. Let $\mathrm{spine}(\tau)$ be a 1-dimensional cell complex embedded in T which is dual to the 1-skeleton of $\mathrm{trg}(\sigma)$. Then $\mathrm{spine}(\tau)$ consist of a single vertex and two edges, and it is also a spine of T. The slope of an edge of $\mathrm{spine}(\tau)$ is also defined as explained in the preceding paragraph.

Let $\mathscr{D}^{(i)}$ denote the set of i-simplices of $\mathscr{D}$. Then we have the following well-known fact.

Lemma 3.1 *For any spine C of T, there is a unique element δ of $\mathscr{D}^{(1)} \cup \mathscr{D}^{(2)}$ such that C is isotopic to* $\mathrm{trg}(\delta)$.

If $\tau = \langle r_0, r_1 \rangle$ is an edge of a triangle $\sigma = \langle r_0, r_1, r_2 \rangle$ of $\mathscr{D}$, then $\mathrm{spine}(\tau)$ is obtained from $\mathrm{spine}(\sigma)$ by collapsing the edge γ_2 of $\mathrm{spine}(\sigma)$ of slope r_2 to a point (see Figure 1). By an *elementary transformation*, we mean this transformation or its converse.

Let (δ^-, δ^+) be a pair of elements of $\mathscr{D}^{(1)} \cup \mathscr{D}^{(2)}$. Then, since the 1-skeleton of the dual to $\mathscr{D}$ is a tree, there is a unique sequence $\delta^- = \delta_0, \delta_1, \delta_2, \cdots, \delta_m = \delta^+$ in $\mathscr{D}^{(1)} \cup \mathscr{D}^{(2)}$ satisfying the following conditions.

1. For each $i \in \{0, 1, \cdots, m-1\}$, either δ_i is an edge of δ_{i+1} or δ_{i+1} is an edge of δ_i.
2. $\delta_i \neq \delta_j$ whenever $i \neq j$.

Thus we obtain a canonical sequence of elementary transformations

$$\mathrm{spine}(\delta^-) = \mathrm{spine}(\delta_0) \mapsto \mathrm{spine}(\delta_1) \mapsto \cdots \mapsto \mathrm{spine}(\delta_m) = \mathrm{spine}(\delta^+).$$

Regard the sequence as a continuous family $\{C_t\}_{t \in [-1,1]}$ of spines of T, and set

$$\mathrm{Spine}(\delta^-, \delta^+) = \cup_{t \in [-1,1]} C_t \subset T \times [-1, 1].$$

Then $\mathrm{Spine}(\delta^-, \delta^+)$ is a 2-dimensional subcomplex of $T \times [-1, 1]$ satisfying the following conditions.

1. $\mathrm{Spine}(\delta^-, \delta^+) \cap (T \times \{\varepsilon 1\}) = \mathrm{spine}(\delta^\varepsilon) \times \{\varepsilon 1\}$ for each $\varepsilon = \pm$.
2. There is a level-preserving deformation retraction from $T \times [-1, 1]$ to $\mathrm{Spine}(\delta^-, \delta^+)$.

Figure 1 illustrates $\mathrm{Spine}(\delta^-, \delta^+)$, where δ^- and δ^+ are elements of $\mathscr{D}^{(2)}$ sharing a common edge. We note that it has a natural cellular structure, consisting of a unique inner-vertex, four inner-edges and six 2-dimensional faces.

The following theorem paraphrases Jorgensen's results [11, Theorems 1-3], and describes the combinatorial structures of the Ford domains of quasifuchsian punctured torus groups (see [21, Section 3] for another beautiful exposition).

Theorem 3.2 (Jorgensen) *For any quasifuchsian punctured torus group Γ, the following hold:*

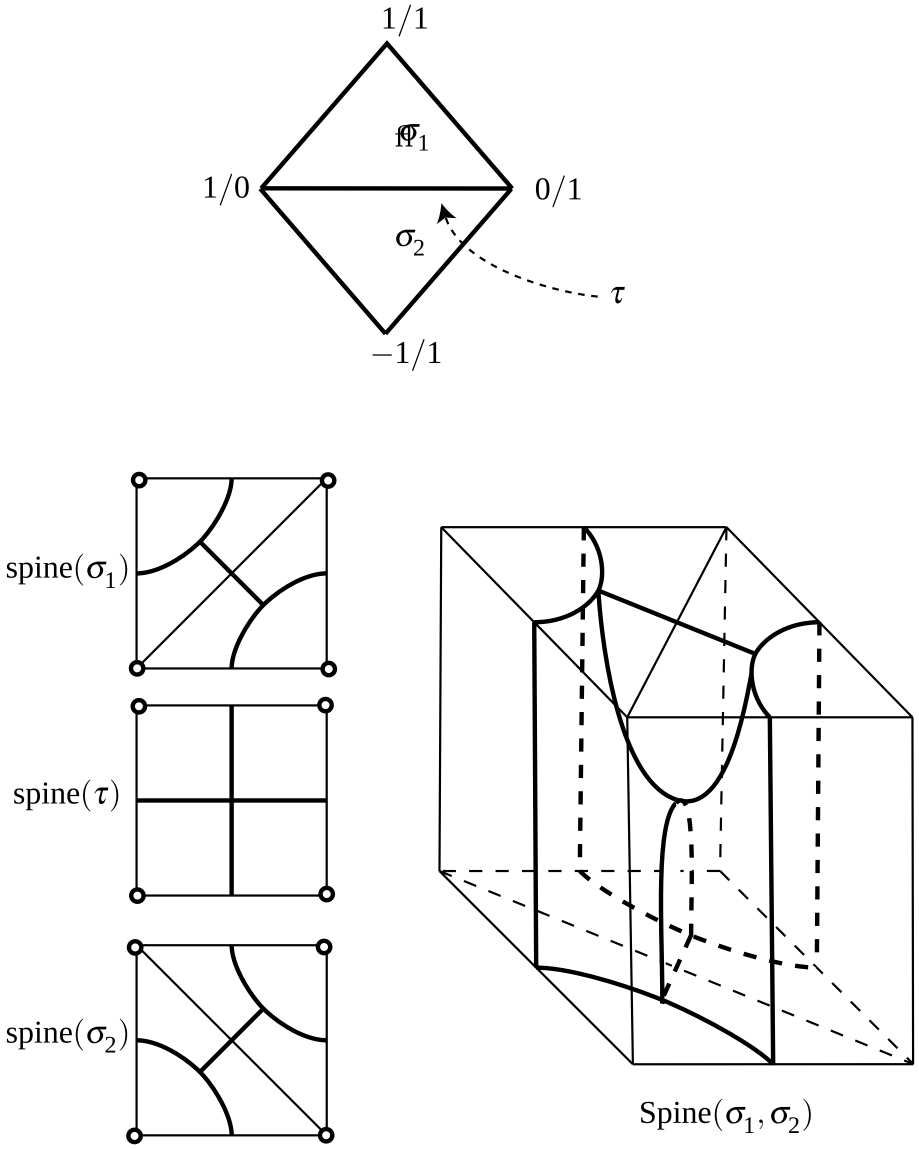


Figure 1

1. *$P(\Gamma)$ consists of two simply connected components $P^{\pm}(\Gamma) \subset \Omega^{\pm}(\Gamma)$. In particular, for each $\varepsilon = \pm$, $P^{\varepsilon}(\Gamma)$ is a fundamental domain of the action of Γ on $\Omega^{\varepsilon}(\Gamma)$ modulo Γ_{∞}, and the image of $\partial P^{\varepsilon}(\Gamma)$ in $\Omega^{\varepsilon}(\Gamma)/\Gamma$ is a spine of T, which we denote by* $\mathrm{spine}^{\varepsilon}(\Gamma)$.

2. *Let δ^{ε} be the element of $\mathscr{D}^{(1)} \cup \mathscr{D}^{(2)}$ such that* $\mathrm{spine}^{\varepsilon}(\Gamma)$ *is isotopic to* $\mathrm{spine}(\delta^{\varepsilon})$. *Then the Ford complex* $\mathrm{Ford}(\Gamma)$ *is isotopic to* $\mathrm{Spine}(\delta^{-}, \delta^{+})$.

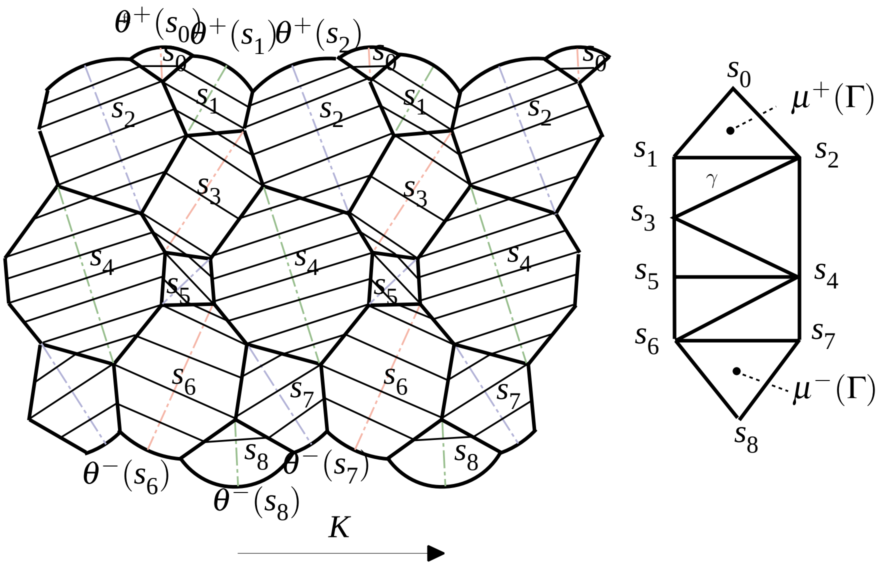
The computer program OPTi [23] made by the third named author visualizes the above theorem: we can see in real time how the Ford domain $Ph(\Gamma)$ and the limit set $\Lambda(\Gamma)$ vary according to deformation of a quasifuchsian punctured torus group Γ. Figure 2(a), which was drawn by using OPTi, illustrates a typical example of the Ford domain of a quasifuchsian punctured torus group Γ. We can observe the following (see [10],[12],[4]).

1. Each face F of the Ford domain $Ph(\Gamma)$ is preserved by an elliptic transformation, P_F, of order 2.
2. The transformations $\{P_F\}$ where F runs over the faces of the $Ph(\Gamma)$ generate a Kleinian group $\tilde{\Gamma}$ which contains Γ as a normal subgroup of index 2. In fact Γ is identified with the orbifold fundamental group of the 2-dimensional orbifold which is the quotient of T by an involution with three fixed points.
3. There is a parabolic transformation, K, of $\tilde{\Gamma}$ such that $K(\infty) = \infty$ and K^2 is the element of Γ corresponding to a peripheral loop of T.
4. If F is a face of $Ph(\Gamma)$, then $F' = K(F)$ is also a face of $Ph(\Gamma)$ and the transformation $K \circ P_F$ is the element of Γ which sends F to F'.
5. There is a continuous family of bi-infinite periodic broken lines $\{L_t\}_{t\in(-1,1)}$ contained in the projection of $Ph(\Gamma)$ in $\mathbb{C}$ each of which is orthogonal to the projection of $F \cap \mathrm{Axis}(P_F)$ whenever they intersect. Let S_t be the intersection of the broken vertical planes above L_t with $Ph(\Gamma)$. Then S_t projects to a torus, T_t, in $M(\Gamma) = T \times (-1,1)$ isotopic to a fiber, and the image, C_t, of ∂S_t forms a spine of T_t. So $\{C_t\}$ gives a continuous family of spines of T. Moreover, C_t is generic or non-generic according as L_t is disjoint from the projections of the vertices of $Ph(\Gamma)$. This family $\{C_t\}$ (with certain modification near $t = \pm 1$) realizes the canonical sequence of elementary moves transforming $\mathrm{spine}^-(\Gamma)$ to $\mathrm{spine}^+(\Gamma)$.

Figure 2(b) illustrates the cross section of $\tilde{\Delta}_{\mathbb{E}}(\Gamma)$ along a small horosphere H_∞ centered at ∞, where $\tilde{\Delta}_{\mathbb{E}}(\Gamma)$ is the Γ-invariant ideal polyhedral complex in $\mathbb{H}^3$ obtained as the inverse image of $\Delta_{\mathbb{E}}(\Gamma)$. It is also regarded as the projection of $\tilde{\Delta}_{\mathbb{E}}(\Gamma) \cap \partial H_\infty$ to the complex plane $\mathbb{C}$. Then the vertices are identified with the centers of the isometric hemispheres supporting faces of the Ford domain $Ph(\Gamma)$. To be more explicit, if v is a projection of a vertex of $\tilde{\Delta}_{\mathbb{E}}(\Gamma) \cap \partial H_\infty$, then the vertical geodesic $\overline{v\infty}$ joining v to ∞ is an edge of $\tilde{\Delta}_{\mathbb{E}}(\Gamma)$, and v is the center of the isometric hemisphere supporting a face of $Ph(\Gamma)$ dual to the edge $\overline{v\infty}$ of $\tilde{\Delta}_{\mathbb{E}}(\Gamma)$. Similarly, each triangle $\langle x_0, x_1, x_2 \rangle$ of $\tilde{\Delta}_{\mathbb{E}}(\Gamma) \cap \partial H_\infty$ is a horospherical cross section of an ideal tetrahedron in $\tilde{\Delta}_{\mathbb{E}}(\Gamma)$ dual to the vertex of $Ph(\Gamma)$ obtained as the intersection of the isometric hemispheres centered at x_0, x_1 and x_2.

Let $\tilde{\Delta}(\Gamma)$ be the inverse image in $\mathbb{H}^3$ of the EPH-decomposition of $\Delta(\Gamma)$. Then $\tilde{\Delta}(\Gamma) \cap \partial H_\infty$ gives a (not necessarily locally finite) decomposition of the infinite

(a) $\mathrm{Ford}(\Gamma) = \mathrm{Spine}(\mu^-(\Gamma), \mu^+(\Gamma)) = \mathrm{Spine}(\Sigma(\Gamma))$



(b) $\Delta_{\mathbb{E}}(\Gamma)$

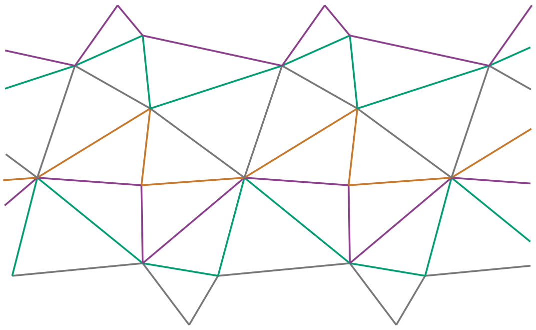


Figure 2

strip which arises as the intersection of the convex hull of the limit set with ∂H_∞, because the underlying space $|\Delta(\Gamma)|$ is equal to the convex core minus the bending laminations by [3, Corollary 1.1].

4 Jorgensen's theorem for quasifuchsian punctured torus groups (II)

In this section, we explain Jorgensen's theorem [10, Theorem 4] which refines Theorem 3.2. A *weighted spine* of T is a spine of T with an assignment of a positive real number to each edge, which we call the *weight* on the edge, such that the sum of the weights is equal to 1. We call such an assignment a *weight system* on the spine. By regarding the weight on an edge as a weight on the slope of the edge, a weight system on a spine is regarded as a barycentric coordinate of a point in

$|\mathscr{D}| - |\mathscr{D}^{(0)}|$. By fixing a $PSL(2,\mathbb{Z})$-equivariant injective continuous map from the underlying space $|\mathscr{D}|$ (of the abstract simplicial complex $\mathscr{D}$) onto $\mathbb{H}^2 \cup \hat{\mathbb{Q}} \subset \overline{\mathbb{H}}^2$, we regard a point in $|\mathscr{D}| - |\mathscr{D}^{(0)}|$ as a point in $\mathbb{H}^2$. Thus each weighted spine corresponds to a unique point of $\mathbb{H}^2$. For each $\nu \in \mathbb{H}^2$, we denote by $\mathrm{spine}(\nu)$ the weighted spine of T corresponding to ν.

Let Γ be a marked quasifuchsian punctured torus group. For each $\varepsilon = \pm$ and for each edge e of $\mathrm{spine}^{\varepsilon}(\Gamma)$, let $t^{\varepsilon}(e)$ be $1/\pi$ times the angle, $\theta^{\varepsilon}(e)$, of a circular arc component of the inverse image of e in $\partial P^{\varepsilon}(\Gamma)$ (see Figure 2). Then we have the following (see [10, Section 4]):

Lemma 4.1 *The sum of* $t^{\varepsilon}(e)$ *where e runs over the edges of* $\mathrm{spine}^{\varepsilon}(\Gamma)$ *is equal to* 1.

Thus $\mathrm{spine}^{\varepsilon}(\Gamma)$ has the structure of a weighted spine of T such that the weight of an edge e is $t^{\varepsilon}(e)$. Let $\nu^{\varepsilon}(\Gamma)$ be the point of $\mathscr{D}$ corresponding to the weighted spine $\mathrm{spine}^{\varepsilon}(\Gamma)$, and put $\nu(\Gamma) = (\nu^-(\Gamma), \nu^+(\Gamma))$. We call it the *side parameter* of Γ following [10]. The following theorem is a refinement of Theorem 3.2(1) (see [10, Theorem 4]):

Theorem 4.2 **(Jorgensen)** *The map* $\nu : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ *is continuous and onto.*

Remark 4.3 (1) Though [10, Theorem 1] asserts that ν is also injective, we have not been able to confirm the assertion.

(2) We do not know any relation between the side-parameter $\nu(\Gamma)$ and the usual end invariant of Γ (see [19]), which records the conformal structure $(\Omega^-(\Gamma)/\Gamma, \Omega^+(\Gamma)/\Gamma) \in Teich(T) \times Teich(T) = \mathbb{H}^2 \times \mathbb{H}^2$.

To combine Theorems 3.2 and 4.2, we introduce the following concept:

Definition 4.4 (1) A *weighted relative spine* of $T \times [-1,1]$ is a 2-dimensional subcomplex C of $T \times [-1,1]$ satisfying the following conditions.

1. There is a level-preserving deformation retraction from $T \times [-1,1]$ to $\mathrm{Spine}(\delta^-, \delta^+)$.
2. A weight system is specified on each of $\partial^{\pm} C := C \cap T \times \{\pm 1\}$.

Two weighted relative spines are *equivalent*, if the underlying relative spines are isotopic and the weight systems coincide (after the isotopy).

(2) For $\nu = (\nu^-, \nu^+) \in \mathbb{H}^2 \times \mathbb{H}^2$, $\mathrm{Spine}(\nu)$ denotes the weighted relative spine, such that the underlying relative spine is $\mathrm{Spine}(\delta^-(\nu), \delta^+(\nu))$, where $\delta^{\varepsilon}(\nu)$ denotes the triangle or edge of $\mathscr{D}$ whose interior contains ν^{ε}, and the weight system on $\partial^{\pm}\mathrm{Spine}(\nu)$ is given by $\nu^{\pm}$.

Then we can summarize Theorems 3.2 and 4.2 as follows:

Theorem 4.5 **(Jorgensen)** *For each* $\Gamma \in \mathcal{QF}$*, the Ford complex* $\mathrm{Ford}(\Gamma)$ *has a natural structure of weighted relative spine of* $T \times [-1,1]$. *Moreover, there is a continuous onto map* $\nu : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$*, such that the weighted spine* $\mathrm{Ford}(\Gamma)$ *is equivalent to* $\mathrm{Spine}(\nu(\Gamma))$ *for any* $\Gamma \in \mathcal{QF}$.

5 The topological ideal polyhedral complex $\mathrm{Trg}(\nu)$ dual to $\mathrm{Spine}(\nu)$

As explained in Section 2, the Ford complex $\mathrm{Ford}(\Gamma)$ is a dual to the subcomplex $\Delta_{\mathbb{E}}(\Gamma)$ of $\Delta(\Gamma)$ consisting of the Euclidean (or elliptic) facets. In this section, we describe the structure of $\Delta_{\mathbb{E}}(\Gamma)$ following the exposition by Floyd-Hatcher [8] of Jorgensen's ideal triangulation of punctured torus bundles over S^1.

For each element $\nu = (\nu^-, \nu^+)$ of $(\mathscr{D} - \mathscr{D}^{(0)}) \times (\mathscr{D} - \mathscr{D}^{(0)})$, we construct a topological ideal triangulation $\mathrm{Trg}(\nu)$ as follows. Let $\Sigma(\nu) = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ ($n \geq 0$) be the triangles of $\mathscr{D}$ whose interiors intersect the oriented geodesic segment joining ν^- with ν^+ in this order. Note that $n = 0$ if and only if $\nu^\pm$ is contained in a single edge τ of $\mathscr{D}$. In this case we redefine $\Sigma(\nu) = \{\tau\}$.

Case 1. $n \geq 2$. Let $\widetilde{\mathrm{trg}}(\sigma_i)$ be the ideal triangulation of $\mathbb{R}^2 - \mathbb{Z}^2$ obtained as the lift of $\mathrm{trg}(\sigma_i)$. By superimposing $\widetilde{\mathrm{trg}}(\sigma_{i+1})$ upon $\widetilde{\mathrm{trg}}(\sigma_i)$, we obtain an array of ideal tetrahedra whose bottom faces compose $\widetilde{\mathrm{trg}}(\sigma_i)$ and whose top faces compose $\widetilde{\mathrm{trg}}(\sigma_{i+1})$. We denote this array of ideal tetrahedra by $\widetilde{\mathrm{Trg}}(\sigma_i, \sigma_{i+1})$. By stacking $\widetilde{\mathrm{Trg}}(\sigma_1, \sigma_2), \cdots, \widetilde{\mathrm{Trg}}(\sigma_{n-1}, \sigma_n)$ up in order, we obtain a set of layers whose bottom faces form $\widetilde{\mathrm{trg}}(\sigma_1)$ and whose top faces form $\widetilde{\mathrm{trg}}(\sigma_n)$. The covering transformation group $\mathbb{Z}^2$ of the covering $\mathbb{R}^2 - \mathbb{Z}^2 \to T$ naturally acts on the above topological ideal simplicial complex, and we define $\mathrm{Trg}(\nu)$ to be the quotient topological ideal simplicial complex.

Case 2. $n = 1$. Then $\mathrm{Trg}(\nu)$ is defined to be the 2-dimensional topological ideal triangulation $\mathrm{trg}(\sigma_1)$.

Case 3. $n = 0$. Then $\delta^-(\nu) = \delta^+(\nu)$ is an edge of $\mathscr{D}$. We define $\mathrm{Trg}(\nu)$ to be the 2-dimensional topological ideal triangulation $\mathrm{Trg}(\delta^\pm(\nu))$.

Note that the underlying space $|\mathrm{Trg}(\nu)|$ is homeomorphic to the the quotient space of $T \times [-1, 1]$ by an equivalence relation $\sim$ such that $(x, s) \sim (y, t)$ only if $x = y$. In particular, $|\mathrm{Trg}(\nu)|$ is homotopy equivalent to T and has a natural embedding into $T \times (-1, 1)$. Then we have the following theorem by Theorem 3.2.

Theorem 5.1 *For any* $\Gamma \in \mathcal{QF}$, $\Delta_{\mathbb{E}}(\Gamma)$ *is isotopic to* $\mathrm{Trg}(\nu(\Gamma))$ *in the convex core* $M_0(\Gamma)$ *of* $M(\Gamma)$.

Figure 2(b) illustrates the cross section of $\Delta_{\mathbb{E}}(\Gamma)$ along a horosphere centered at ∞ for the quasifuchsian punctured torus group Γ in Figure 2(a).

6 Generalization of Jorgensen's theorem to the groups in $\overline{\mathcal{QF}}$

We first generalize the constructions of $\mathrm{Spine}(\nu)$ and $\mathrm{Trg}(\nu)$ in the previous sections. Let $\nu = (\nu^-, \nu^+)$ be an element of $\nu \in \overline{\mathbb{H}}^2 \times \overline{\mathbb{H}}^2 - \mathrm{diag}(\partial \mathbb{H}^2)$. Let $\Sigma(\nu) =$

$\{\cdots, \sigma_i, \sigma_{i+1}, \cdots\}$ be the possibly (bi-)infinite sequence of triangles of the modular diagram $\mathscr{D}$ whose interior intersect the oriented geodesic joining ν^- with ν^+ in this order. Then we construct $\mathrm{Spine}(\nu)$ and $\mathrm{Trg}(\nu)$ as in Sections 3 and 4 by using $\Sigma(\nu)$, where we introduce the following modification in the construction of $\mathrm{Spine}(\nu)$:

- Suppose ν^ε is equal to a rational point $s_0^\varepsilon \in \hat{\mathbb{Q}} \subset \partial\mathbb{H}^2$. Let $\sigma^\varepsilon = \langle s_0^\varepsilon, s_1^\varepsilon, s_2^\varepsilon\rangle$ be the triangle in $\Sigma(\nu)$ having s_0^ε as a vertex. Consider the loop α in $\mathrm{spine}(\sigma^\varepsilon)$ obtained as the union of the edges of slopes s_1^ε and s_2^ε. (Note that (i) the slope of α in T is s_0^ε and that (ii) the assumption $\nu^\varepsilon = s_0^\varepsilon$ means that the element of Γ corresponding to α is parabolic.) Then we shrink α to a point and remove it. (See [10, Figure 6] for the reason of this modification.)

Then the following gives a generalization of Theorems 3.2, 4.2, and 4.5.

Theorem 6.1 *The side parameter map* $\nu : \mathscr{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ *is extended to a map* $\nu : \overline{\mathscr{QF}} \to \overline{\mathbb{H}}^2 \times \overline{\mathbb{H}}^2 - \mathrm{diag}(\partial\mathbb{H}^2)$ *which satisfies the following conditions.*

1. *For any* $\Gamma \in \overline{\mathscr{QF}}$, $\mathrm{Ford}(\Gamma)$ *and* $\Delta_{\mathbb{E}}(\Gamma)$ *are isotopic to* $\mathrm{Spine}(\nu(\Gamma))$ *and* $\mathrm{Trg}(\nu(\Gamma))$*, respectively.*
2. $\nu^\varepsilon(\Gamma) \in \partial\mathbb{H}^2$ *if and only if the end invariant of the* ε*-end lies in* $\partial\mathbb{H}^2$*. In this case, the end invariant is equal to* $\nu^\varepsilon(\Gamma)$*.*

We continue to call the extended ν the *side parameter* map. The second assertion of the above theorem implies the following:

1. $\Omega^\varepsilon(\Gamma)/\Gamma$ is a triply punctured sphere if and only if $\nu^\varepsilon(\Gamma)$ is a rational point of $\partial\mathbb{H}^2$. In this case, the simple loop $\nu^\varepsilon(\Gamma)$ corresponds to the accidental parabolic transformation.
2. $\Omega^\varepsilon(\Gamma) = \emptyset$, i.e., the ε-end is degenerate, if and only if $\nu^\varepsilon(\Gamma)$ is an irrational point of $\partial\mathbb{H}^2$. In this case, $\nu^\varepsilon(\Gamma)$ is equal to the ending lamination of the ε-end.

For geometrically finite boundary groups, the above theorem had been obtained by Jorgensen (cf. [10, Theorem 5]). In [10], he also studied the Ford domains of geometrically infinite punctured torus groups. Moreover, Jorgensen-Marden [12] gave an explicit construction of the Ford domains for the fiber groups of the two simplest punctured torus bundles over S^1. Their construction was generalized by Helling [9] to an explicit construction of the Ford domains of a certain infinite family of punctured torus bundles over S^1. For the general hyperbolic punctured torus bundles over S^1, Parker [20] gave a geometric description of the canonical decompositions, and Lackenby [18] gave a purely topological proof to the description of their canonical decompositions due to Jorgensen (cf. [8]). For a proof of Theorem 6.1, please see the first author's announcement [1] and his forthcoming paper.

7 Pleating invariants for punctured torus groups

In this section, we give a quick review of the works of Keen and Series [14], [15], [16] on bending laminations of quasifuchsian punctured torus groups. Let Γ be a marked quasifuchsian punctured torus group. Then the boundary of the convex core $M_0(\Gamma)=\mathscr{C}(\Lambda(\Gamma))/\Gamma$ of $M(\Gamma)=\mathbb{H}^3/\Gamma$ has two components, the component $\partial^+M_0(\Gamma)$ facing $\Omega^+(\Gamma)/\Gamma$ and the component $\partial^-M_0(\Gamma)$ facing $\Omega^-(\Gamma)/\Gamma$. Then $\partial^\pm M_0(\Gamma)$ has a structure of a complete hyperbolic punctured torus bent along a measured geodesic lamination, $pl^\pm(\Gamma)$, called the *bending measured lamination* (see Thurston [22] and Epstein-Marden [6]). Set $pl(\Gamma)=(pl^-(\Gamma),pl^+(\Gamma))$, and let $[pl(\Gamma)]$ be the pair of projective measured laminations $([pl^-(\Gamma)],[pl^+(\Gamma)])$. For a pair $\mu=(\mu^-,\mu^+)$ of distinct projective measured laminations on T, set

$$\mathscr{P}(\mu)=\{\Gamma\in\mathscr{QF}-\mathscr{F}\,|\,[pl(\Gamma)]=\mu\}.$$

Then the following results have been proved by Keen-Series [16] (cf. [14], [15]).

1. Let $\lambda_{\mu^\pm}:\mathscr{QF}\to\mathbb{C}$ be the complex length function (see [14, Section 6.2]). Then $\lambda_{\mu^\pm}$ is real valued on $\mathscr{P}(\mu)$ and $\lambda_{\mu^-}\times\lambda_{\mu^+}:\mathscr{P}(\mu)\to\mathbb{R}_+\times\mathbb{R}_+$ is a diffeomorphism onto the region bounded by the two positive axes in $\mathbb{R}_+\times\mathbb{R}_+$ and the graph of a continuous function $f_\mu:\mathbb{R}_+\to\mathbb{R}_+$. The function f_μ is monotone decreasing, $\lim_{t\to+0}f_\mu(t)=+\infty$ and $\lim_{t\to+\infty}f_\mu(t)=0$.

2. The three components of the boundary of the image of $\mathscr{P}(\mu)$ in $\mathbb{R}_+\times\mathbb{R}_+$ correspond to three distinct parts of the closure $\overline{\mathscr{P}(\mu)}$ in $\overline{\mathscr{QF}}$. The component corresponding to the graph of f_μ represents groups on the Kerckhoff's line of minima in the Fuchsian space $\mathscr{F}$. For the groups corresponding to the axis $\lambda_{\mu^\pm}=0$, the component $\Omega^\pm$ is degenerated and the support $|\mu^\pm|$ of $\mu^\pm$ is an ending lamination. The boundary point $(0,0)$ represents a doubly degenerate group, unique by [19], with the two real bending lamination μ^- and μ^+.

3. The complement of the image of $\mathscr{P}(\mu)$ in $\mathbb{R}_+\times\mathbb{R}_+$ corresponds to Fuchsian groups. (We thank Y. Komori and H. Miyachi for informing us of this fact.)

8 Does $pl(\Gamma)$ determine $\Delta(\Gamma)$?

In this section, we propose the following conjecture, and explain its refinements and related conjectures.

Conjecture 8.1 *The combinatorial structure of the EPH-decomposition of a quasifuchsian punctured torus group* Γ *is determined by the pair* $\mu=(\mu^-,\mu^+)$ *of its projective bending measured laminations.*

In the following, we explain the conjectural picture of the EPH-decomposition $\Delta(\Gamma)$ of a group Γ in the pleating variety $\mathscr{P}(\mu)$, which we denote by $\mathrm{Trg}(\mu,\emptyset)$. (The symbol $\emptyset$ in the above notation represents the fact that both ends of $M(\Gamma)$ are geometrically finite and have no accidental parabolics.) Note that $\mu \in \partial\mathbb{H}^2 \times \partial\mathbb{H}^2 - \mathrm{diag}(\partial\mathbb{H}^2)$ and hence we have the topological ideal simplicial complex $\mathrm{Trg}(\mu)$, which we introduced in Section 6. The desired complex $\mathrm{Trg}(\mu,\emptyset)$ is obtained from $\mathrm{Trg}(\mu)$ by attaching a new piece, $Q(\mu^+)$ and $Q(\mu^-)$, to the $(+)$-side and $(-)$-side, respectively, as explained below. For simplicity, we explain only the (generic) case where $\Sigma(\mu)$ contains a triangle (see the second paragraph in Section 5).

Case 1. μ^ε is rational. Let $\sigma^\varepsilon = \langle s_0^\varepsilon, s_1^\varepsilon, s_2^\varepsilon\rangle$ be the triangle in $\Sigma(\mu)$ such that s_0^ε is the slope of μ^ε. Consider a triangular prism, one of the quadrangular face, A, is triangulated into two triangles, and let $Q(\mu^\varepsilon)$ be the space obtained from it by applying the following operations (see Figure 3 (a)).

1. Remove the vertices and the ridge line opposite to the triangulated quadrangular face A.

2. Identify the two triangular faces of the triangular prism through a translation. Note that $A - \{\text{vertices}\}$ projects to an annulus, A', with one point removed from each boundary component, and the triangulation of A induces a topological ideal triangulation of A'.

3. Identify the two ideal edges of A' so as to obtain a punctured torus. Identify the induced topological ideal triangulation on the punctured torus with $\mathrm{trg}(\sigma^\varepsilon)$ so that the edge of slope s_0^ε corresponds to the image of the two ideal edges of A'.

The cellular structure of the triangular prism induces the "cellular structure" on $Q(\mu^\varepsilon)$ with

1. one 3-dimensional (non-simply connected) facet,

2. four 2-dimensional facets, two of which are topological ideal triangles of A and the remainders are homeomorphic to $S^1 \times (0,1]$ with one point in $S^1 \times 1$ removed (note that we do not regard the images of the two triangular faces of the triangular prism as facets), and

3. three ideal edges.

Note that $\mathrm{trg}(\sigma^\varepsilon)$ is the boundary component of $\mathrm{Trg}(\mu)$ on the ε-side. We attach $Q(\mu^\varepsilon)$ to $\mathrm{Trg}(\mu)$ along $\mathrm{trg}(\sigma^\varepsilon)$. We note that the removed ridge line of the triangular prism corresponds the axis of the purely hyperbolic transformation representing the rational bending locus μ^ε.

Case 2. μ^ε is irrational. Fix a complete hyperbolic structure on a punctured torus, and continue to denote the hyperbolic punctured torus by T. Then $Q(\mu^\varepsilon) = T - |\mu^\varepsilon|$, that is, the complement of the underlying geodesic lamination $|\mu^\varepsilon|$, and its cellular structure consists of

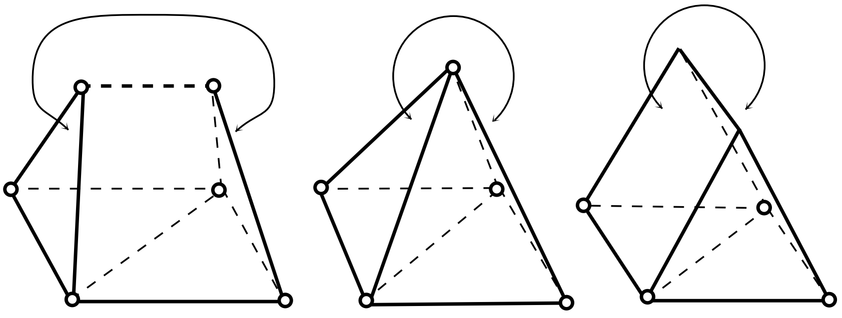

(a) (b) (c)

Figure 3

1. two edges each joining the puncture and one of the two ends of the open once-punctured bigon $T - |\mu^\varepsilon|$ with path metric, and

2. two 2-dimensional facets obtained as the connected components of the complements of the above two edges.

In the following, we explain the way the 2-dimensional piece $Q(\mu^\varepsilon)$ is attached to the ε-end of $\mathrm{Trg}(\mu)$. For simplicity, we assume $\varepsilon = +$. Pick a triangle σ_0 in $\Sigma(\mu)$ and consider the triangles $\sigma_0, \sigma_1, \sigma_2, \cdots$ of $\Sigma(\mu)$ starting from σ_0. Let f be a surjective continuous map from $T \times [0,\infty)$ to the subcomplex $\cup_{i\geq 0}\mathrm{Trg}(\sigma_i, \sigma_{i+1})$ of $\mathrm{Trg}(\mu)$ which satisfy the following conditions.

1. $f(T \times [i, i+1]) = \mathrm{Trg}(\sigma_i, \sigma_{i+1})$.

2. The restriction of f to $T \times i$ is a homeomorphism onto $\mathrm{trg}(\sigma_i) = \mathrm{Trg}(\sigma_{i-1}, \sigma_i) \cap \mathrm{Trg}(\sigma_i, \sigma_{i+1})$ such that the inverse image of each edge of $\mathrm{trg}(\sigma_i)$ is the geodesic of the hyperbolic punctured torus $T \times i$ of the same slope.

Let g be the homeomorphism $g : T \times [0,1) \to T \times [0,\infty)$ defined by $g(x,t) = (x, t/(1-t))$, and set $h = f \circ g : T \times [0,1) \to \cup_{i\geq 0}\mathrm{Trg}(\sigma_i, \sigma_{i+1})$. Let $\sim$ be the equivalence relation on $T \times [0,1]$ such that $x \sim y$ if and only if (i) $x = y$ or (ii) $x, y \in T \times [0,1)$ and $h(x) = h(y)$. Then we can identify $\cup_{i\geq 0}\mathrm{Trg}(\sigma_i, \sigma_{i+1})$ with $T \times [0,1)/\sim$, which is embedded in $(T \times [0,1] - (T - |\mu^+|) \times 1)/\sim$. This gives the way to attach $Q(\mu^\varepsilon) = T - |\mu^+|$ to $\mathrm{Trg}(\mu)$ along the $(+)$-end of $\mathrm{Trg}(\mu)$.

The following conjecture is a refinement of Conjecture 8.1.

Conjecture 8.2 *Let Γ be a quasifuchsian punctured torus group in the pleating variety $\mathscr{P}(\mu)$. Then the EPH-decomposition $\Delta(\Gamma)$ is isotopic to* $\mathrm{Trg}(\mu, \emptyset)$.

To extend the above conjecture to that for all punctured torus groups, recall that the map $\Gamma \mapsto \mu = (\mu^-, \mu^+)$ from $\mathscr{QF}$ to $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 - \mathrm{diag}(\partial\mathbb{H}^2)$ has a natural extension to a map from the closure $\overline{\mathscr{QF}}$. Namely, if $\Omega^\varepsilon(\Gamma)/\Gamma$ is not a punctured torus, then we define $\mu^\varepsilon(\Gamma) = \mu^\varepsilon \in \partial\mathbb{H}^2$ to be the end invariant of the ε-end (see [19]), which is equal to the side parameter $\nu^\varepsilon(\Gamma)$ by Theorem 6.1. For each group $\Gamma \in \overline{\mathcal{QF}}$, let $\iota(\Gamma)$ be the subset of $\{-,+\}$ defined by

$$\begin{aligned}\iota(\Gamma) &= \{\varepsilon \,|\, \mu^\varepsilon(\Gamma) = \nu^\varepsilon(\Gamma) \in \partial\mathbb{H}^2\}\\ &= \{\varepsilon \,|\, \partial^\varepsilon M(\Gamma) \text{ is a triply punctured sphere or the empty set}\}.\end{aligned}$$

For each subset $\iota \subset \{-,+\}$, we construct yet another topological ideal polyhedral complex $\mathrm{Trg}(\mu,\iota)$ from $\mathrm{Trg}(\mu)$, by generalizing the construction of $\mathrm{Trg}(\mu,\emptyset)$, as follows. Let $\varepsilon \in \{-,+\}$.

Case 1. $\varepsilon \notin \iota$. Then we attach the piece $Q(\mu^\varepsilon)$ to $\mathrm{Trg}(\mu)$ as in the construction of $\mathrm{Trg}(\mu,\emptyset)$.

Case 2. $\varepsilon \in \iota$ and μ^ε is rational. Then we also attach the 3-dimensional piece $Q(\mu^\varepsilon)$ to $\mathrm{Trg}(\mu)$ as in the construction of $\mathrm{Trg}(\mu,\emptyset)$. However, we should regard that the construction starts with a pyramid which is obtained from the triangular prism in Figure 3(a) by shrinking the removed ridge line to a point (and remove it) (see Figure 3(b)). The removed point corresponds to the parabolic fixed point of the accidental parabolic transformation representing the rational lamination μ^ε.

Case 3. $\varepsilon \in \iota$ and μ^ε is irrational. In this case $Q^\varepsilon(\mu,\iota)$ is the empty set, i.e., we leave the ε-end of $\mathrm{Trg}(\mu)$ as it is.

The following conjecture is a generalization of Conjecture 8.1.

Conjecture 8.3 *Let Γ be a punctured torus group which lies in the closure $\overline{\mathcal{P}}(\mu)$ of the pleating variety $\mathcal{P}(\mu)$. Then the EPH-decomposition $\Delta(\Gamma)$ is isotopic to* $\mathrm{Trg}(\mu,\iota(\Gamma))$.

Recall that, for each pair $\mu = (\mu^-,\mu^+)$, the distinct projective laminations on the punctured torus T, $\Sigma(\mu)$ denotes the set $\{\cdots,\sigma_i,\sigma_{i+1},\cdots\}$ of possibly (bi-)infinite sequence of triangles of the modular diagram $\mathcal{D}$ whose interior intersect the oriented geodesic joining μ^- with μ^+ in this order. Let $|\Sigma(\mu)|$ be the union of triangles in $\Sigma(\mu)$. Since the Ford domain is dual to the subcomplex $\Delta_{\mathbb{E}}(\Gamma)$ of $\Delta(\Gamma)$, the above conjecture implies the following conjecture on the Ford domain of Γ, which relates the works of Jorgensen with those of Keen-Series.

Conjecture 8.4 *Let Γ be a punctured torus group which lies in the closure $\overline{\mathcal{P}}(\mu)$. Then $\nu^\pm(\Gamma) \in |\Sigma(\mu)|$.*

We obtain a partition of $\overline{\mathcal{P}}(\mu)$ according to the combinatorial structures of $\mathrm{Ford}(\Gamma)$. The above conjecture is refined to a conjecture on the structure of this partition. For simplicity, we state the conjecture only for the rational case.

Conjecture 8.5 *Let $\mu = (\mu^-,\mu^+)$ be a pair of distinct rational projective measured laminations on T, and let $\sigma_1,\cdots,\sigma_n$ be the members of $\Sigma(\mu)$. For each pair (σ_i,σ_j) $(1 \le i \le j \le n)$, set*

$$\overline{\mathcal{P}}(\mu;\sigma_i,\sigma_j) = \{\Gamma \in \overline{\mathcal{P}}(\mu) \,|\, \nu^-(\Gamma) \in \sigma_i,\ \nu^+(\Gamma) \in \sigma_j\}.$$

Then $\{\overline{\mathscr{P}}(\mu;\sigma_i,\sigma_j)\,|\,1\le i\le j\le n\}$ *gives a partition of* $\overline{\mathscr{P}}(\mu)$ *as illustrated in Figure 4. Here the region labeled* (i,j) *corresponds to* $\overline{\mathscr{P}}(\mu;\sigma_i,\sigma_j)$.

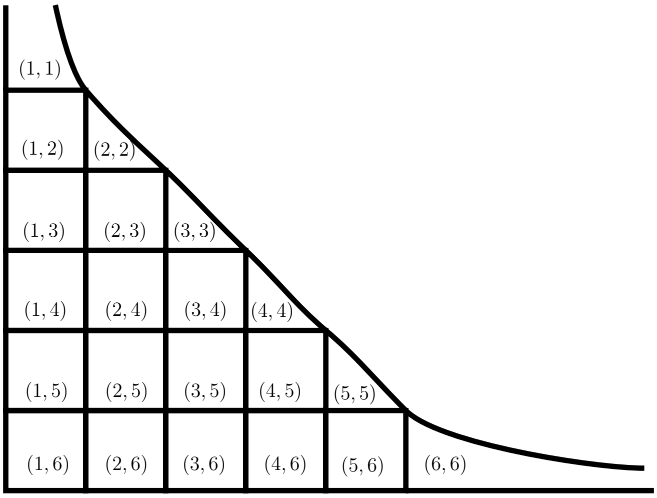


Figure 4

Let us explain the meaning of the above conjecture. If Γ corresponds to a point on the diagonal edge, then Γ is fuchsian, and the side parameter satisfies $\nu^-(\Gamma)=\nu^+(\Gamma)$. So the label for this group is (i,i) for some i (provided that Conjecture 8.4 is true). We expect that the number i changes monotonically as Γ moves on the diagonal edge. If Γ corresponds to the origin, then Γ is a double cusp group, where the simple loops μ^- and μ^+ are pinched. So, by Theorem 6.1, we have $\nu^\pm(\Gamma)=\mu^\pm$ and hence the label for Γ must be $(1,n)$. If Γ corresponds to a point on the y-axis then Γ lies in the Maskit slice (see [14]) where μ^- is pinched. So, by Theorem 6.1, the label for Γ should be $(1,i)$ for some i (provided that Conjecture 8.4 is true). We conjecture that the number i changes monotonically as Γ moves on the y-axis. Likewise, the label for a group Γ corresponding to a point on the x-axis is (i,n) for some i, and we expect that the number i changes monotonically. In general, Conjecture 8.5 predicts the following: As a group $\Gamma\in\overline{\mathscr{P}}(\mu)$ evolves from a fuchsian group (on the diagonal edge) to the most complicated double cusp group (at the origin), the Ford domain becomes complicated "monotonically". All possible shapes of the Ford domains are determined as long as Γ evolves in the pleating variety $\overline{\mathscr{P}}(\mu)$. Even if Γ is just an infant, i.e., very near to a fuchsian group, one should be able to see the possible shapes of the Ford domains in its future. Because Conjecture 8.3 predicts that the EPH-decomposition $\Delta(\Gamma)$ is uniquely determined by μ and because the Ford complex is dual to the subcomplex $\Delta_{\mathbb{E}}(\Gamma)$ of $\Delta(\Gamma)$. Even a very young Γ should have the same EPH-decomposition as the ultimate (double cusp) group.

Finally, we present yet another conjecture. In the example of a Ford domain illustrated in Figure 2, we see that the axes of the faces incline to the left or right "alternately". To be precise, the axes of the faces in Figure 2(a) corresponding to the vertices on the left side of Figure 2(b) (i.e., s_1, s_3, s_5, s_6) are inclined one way and those corresponding to the vertices on the right (i.e., s_2, s_4, s_7) are inclined the other way. Here by an axis of a face F, we mean (the projection to $\mathbb{C}$) of $F \cap \mathrm{Axis}(P_F)$, where P_F is the order 2 elliptic transformation which maps F to itself (see the paragraph after Theorem [10]). Recall the parabolic transformation K such that $K(\infty) = \infty$ and K^2 corresponds to the puncture of T (see Section 3). Set $A_F = KP_F$. Then $A_F(F) = K(F)$ is also a face of the Ford domain $Ph(\Gamma)$, and hence A_F is a face pairing for $Ph(\Gamma)$. Then we can easily see the following:

$$\arg(\text{the projection of } \mathrm{Axis}(P_F) \text{ to } \mathbb{C}) \equiv \pi - \arg(\mathrm{tr} A_F) \pmod{\pi}.$$

So the following conditions are equivalent.

1. A_F is a right screw motion, i.e., $\arg(\mathrm{tr} A_F) \in (0, \pi/2) \cup (\pi, 3\pi/2)$.
2. The axis of F inclines to the left, i.e., $\arg(\text{the projection of } \mathrm{Axis}(P_F)) \in (\pi/2, \pi) \cup (3\pi/2, 2\pi)$.

The above observation was brought to us by Jorgensen through his lecture in Osaka [11], where he suggested that it would be an interesting and challenging problem to study this phenomenon. The following conjecture formulates his proposal in conjunction with the bending laminations.

Conjecture 8.6 *Let Γ be an element of $\overline{\mathscr{P}}(\mu)$. Let $\Sigma(\mu)^{(0)}_L$ (resp. $\Sigma(\mu)^{(0)}_R$) be the subset of the vertex set $\Sigma(\mu)^{(0)}$ of $\Sigma(\mu)$ which lies on the left (resp. right) of the oriented geodesic $\ell(\mu)$ joining μ^- to μ^+. Then for any element s of $\Sigma(\mu)^{(0)}_L$ (resp. $\Sigma(\mu)^{(0)}_R$), A_s is a right (resp. left) screw motion. Here A_s denotes an element of Γ corresponding to a simple loop on T of slope s.*

9 Partial positive answers and experimental results

Theorem 6.1 implies that Conjectures 8.3 and 8.4 are valid for doubly degenerate groups. Moreover, [3, Theorem 11.2] shows that Conjecture 8.3 holds for the restriction of $\Delta(\Gamma)$ to the boundary of the convex core $M_0(\Gamma)$ for a quasifuchsian punctured torus group Γ. In addition to these partial results, we have the following partial result.

Theorem 9.1 *Suppose μ is rational, and let Γ_0 be the double cusp group in $\overline{\mathscr{P}}(\mu)$. Then there is a neighborhood U of Γ_0 in $\overline{\mathscr{P}}(\mu)$, such that Conjectures 8.1–8.4 are valid for any group in U.*

We note that an analogy of Conjecture 8.3 for the groups on the outside of the quasifuchsian space has already been established by [2] as follows. If μ is rational, the $\mathscr{P}(\mu)$ has a natural extension to the outside of the quasifuchsian space and each group in the extension can be regarded as the holonomy group of a hyperbolic cone-manifold with a cusp. Moreover, it has been proved that the "Ford complex" of the cusped hyperbolic cone-manifold is dual to the complex constructed as in Cases 1 and 2 in Section 8 with the following minor modification:

1. We use the triangular prism which is obtained from the pyramid in Case 2 by expanding the peak vertex (that was obtained by shrinking the ridge line in Case 1 to a point) to an edge in a direction perpendicular to the original ridge line (see Figure 3 (c)). This new ridge line corresponds to the axis of an elliptic transformation A_{s^ε}, and hence it is a component of the cone axis.
2. We do not delete the new ridge line.

For the full description of this result, please see the announcement [4]. For the proof of this result and the results announced in this paper, please see the forthcoming paper [5].

For Conjecture 8.6, we have the following partial result.

Proposition 9.2 *Suppose μ^ε is rational for some $\underline{\varepsilon} = \pm$, and let s^ε be the rational number corresponding to μ^ε. Then for any $\Gamma \in \overline{\mathscr{P}}(\mu)$, the following holds. Let s be a rational number such that s and s^ε span an edge of $\mathscr{D}$. Then A_s is a right or left screw motion according as s lies on the left or right of the oriented geodesic, $\ell(\mu)$, joining μ^- to μ^+. In particular, Conjecture 8.6 is valid for the elements s of $\Sigma(\mu)^{(0)}$ such that s and s^ε span an edge of $\mathscr{D}$.*

Finally, we explain some experimental results towards Conjectures 8.5 and 8.6. Suppose that μ is rational. Then the diffeomorphism $L_\mu = \lambda_{\mu^-} \times \lambda_{\mu^+}$ in Section 7 is defined by

$$\lambda_{\mu^\pm}(\Gamma) = \text{the translation length of } A^\pm,$$

where $A^\pm$ denote the elements of Γ represented by the rational lamination $|\mu^\pm|$. Since the translation length is determined by the trace, we may replace L_μ with the map $T_\mu = \mathrm{tr}_{\mu^-} \times \mathrm{tr}_{\mu^+} : \mathscr{P}(\mu) \to (2,\infty) \times (2,\infty)$ defined by $\mathrm{tr}_{\mu^\pm}(\Gamma) = \mathrm{tr}(\tilde{A}^\pm)$, where $\tilde{A}^\pm$ are the element of $SL(2,\mathbb{C})$ with positive real trace projecting to the elements $A^\pm$ of Γ. (Recall that $A^\pm$ are purely hyperbolic by [14, Lemma 4.6].) T_μ extends to a map $\overline{\mathscr{P}}(\mu) \to [2,\infty) \times [2,\infty)$, and for every point $(x,y) \in [2,\infty) \times [2,\infty)$ there is a punctured torus group Γ in $\overline{\mathscr{P}}(\mu) \cup \mathscr{F}$ such that $T_\mu(\Gamma) = (x,y)$: moreover, such a group is unique if $(x,y) \in T_\mu(\overline{\mathscr{P}}(\mu))$. Since there is an effective method to determine the Ford complex of Γ (cf. [10],[4],[17]), we can do an experimental study on Conjecture 8.5. The computer experiments made by the last named author support the following results:

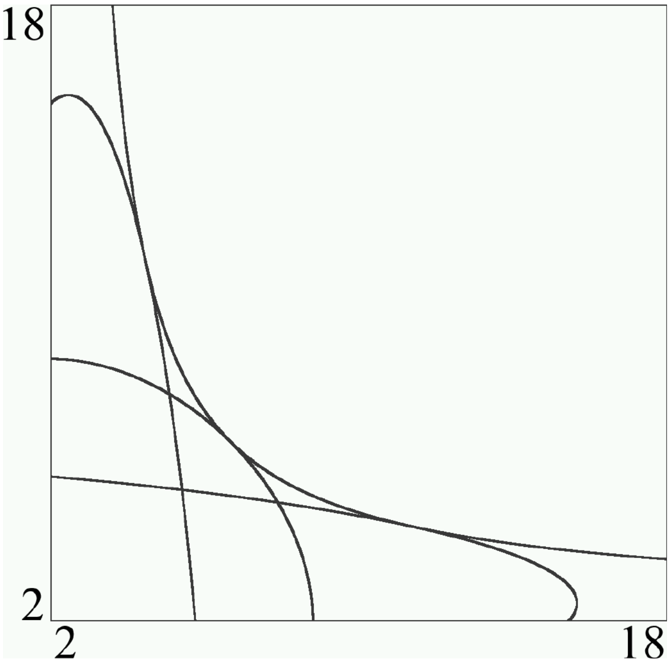


Figure 5 (a)

- Conjectures 8.5 and 8.6 are valid for $\mu = (1/0, \mu^+)$ with
$$\mu^+ \in \{0/1, 1/2, 1/3, 1/4, 2/5, 3/7, 3/8\}.$$

We note that Conjectures 8.5 and 8.6 for $\mu = (1/0, \mu^+)$ with $\mu^+ = 0/1, 1/2$ can be easily proved. Figure 5 (a) and (b), respectively, are the output for $\mu = (1/0, 1/4)$ and $(1/0, 2/5)$.

The idea to compare the two convex hull constructions for punctured torus groups lead us to a refinement of McShane's identity [2], which seems to have some relation with Conjecture 8.6.

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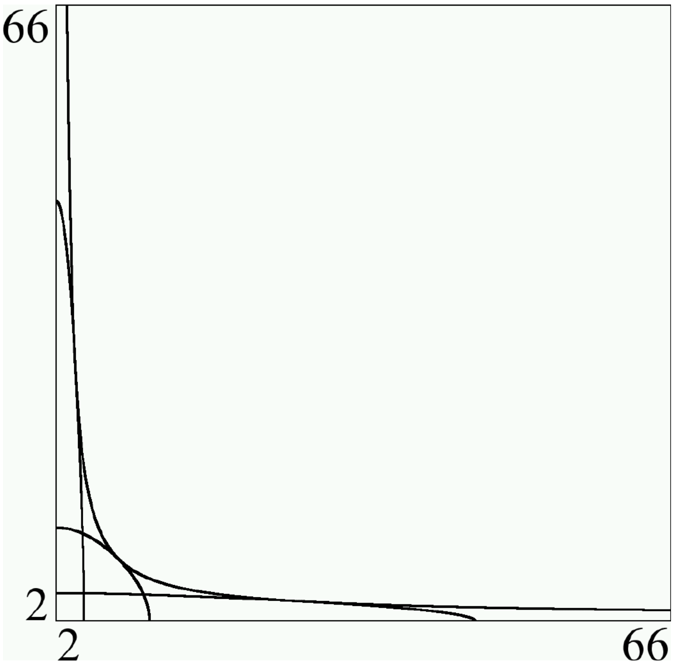

Figure 5 (b)

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